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Elimination problems in logic: a brief history

William Craig

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Abstract A common aim of elimination problems for languages of logic is to express the entire content of a set of formulas of the language, or a certain part of it, in a way that is more elementary or more informative. We want to bring out that as the languages for logic grew in expressive power and, at the same time, our knowledge of their expressive limitations also grew, elimination problems in logic underwent some change. For languages other than that for monadic second-order logic, there remain important open problems.

Keywords Elimination of variables and quantifiers · Boolean equations · Monadic second-order logic · Atomless and atomic Boolean algebras · Cardinality quantifiers · Boole · Schröder · Löwenheim · Skolem · Behmann · Ackermann

Elimination problems that arise in pure logic mostly differ from those that arise when tools from logic are used in the study of certain mathematical theories, such as the elementary theory of real numbers. Nevertheless, as we shall see, their history is to some extent intertwined and work on the former provided some guidance, or at least sense of direction, to work on the latter.

Let $A = A(R, S)$ be a sentence, or conjunction of sentences, and let R and S be sets of predicate symbols such that $R \neq \emptyset$, $S \neq \emptyset$, and $R \cap S = \emptyset$, and such that every predicate symbol that occurs in A belongs to $R \cup S$. Quite often it is desirable to find a sentence, or conjunction of sentences, $A^{\nabla} = A^{\nabla}(S)$ that satisfies the following two conditions, where $\vDash$ is the relation of logical consequence.

W. Craig (✉)
Department of Philosophy, University of California,
314 Moses Hall, Berkeley, CA 94720-2930, USA
e-mail: mancosu@socrates.berkeley.edu

(1) (i) $A(R, S) \vDash A^{\nabla}(S)$ and (ii) every predicate symbol that occurs in $A^{\nabla}(S)$ belongs to S (and not to R).
(2) If $C(S)$ is any sentence such that (i) $A(R, S) \vDash C(S)$ and (ii) every predicate symbol that occurs in $C(S)$ belongs to S, then $A^{\nabla}(S) \vDash C(S)$.

Let $C(S)$ be an *S-consequence of* $A(R, S)$ if and only if it satisfies conditions (i) and (ii) in (2). Then (1) is the condition that $A^{\nabla}(S)$ is an S-consequence of $A(R, S)$. Thus, the conjunction of (1) and (2) can be rephrased as follows:

$A^{\nabla}(S)$ axiomatizes the set of S-consequences of $A(R, S)$.

Perhaps Aristotle is the first logician to obtain what, in effect, is an axiomatization of this kind, although he may not have been aware of this. Consider any syllogism. Let $A(R, S)$ be the conjunction of its premises, let R consist of its middle term and S of the other two terms, and let $A^{\nabla}(S)$ be its conclusion. Then in many cases, although not in the case of subalternation, any S-consequence $C(S)$ of $A(R, S)$ is also a consequence of $A^{\nabla}(S)$.

This aspect of syllogisms was recognized and emphasized by George Boole. He devised a system of logic which was intended not only to deal with a much larger and more varied class of arguments but also to allow one, given any argument in that class whose premises form a conjunction of the form $A(R, S)$, to find a conclusion $A^{\nabla}(S)$ that axiomatizes the set of S-consequences of $A(R, S)$.

Boole's discussion, on pages 8–10 Boole (1916), of "the requirements of a general method in logic", begins as follows. "As the conclusion must express a relation among the whole or among a part of the elements involved in the premises, it is requisite that we should possess the means of eliminating those elements which we desire not to appear in the conclusion, and of determining the whole amount of relation implied by the premises among the elements we wish to retain". In Schröder's writings there occur similar passages (e.g. on page 198 of Schröder, vol. II).

Boole obtains his $A^{\nabla}(S)$ from his $A(R, S)$ by a process that ends with eliminating in a certain way the symbols in R (Cf. Chap. VII and VIII in Boole 1916). For many years, many logicians expressed doubts about his argumentation. Although in Burris' recent work (Burris 2001), he seems to have succeeded in clarifying and justifying Boole's line of thought, Schröder's modification of Boole's work seems easier to present and will be followed here.

Schröder's main change from Boole, a change first suggested by Jevons, is from a partial union or sum operation that is only defined for pairs of disjoint arguments to one that is defined for any pair of arguments. Other changes made by Schröder, in particular giving a prominent role to the inclusion relation and making use of its intuitive appeal, were suggested to him by the writings of C. S. Peirce. Like Boole's algebraic theory of logic, Schröder's theory abstracts from sets or classes and from laws that are satisfied by certain operations on these and deals with algebraic structures **B** whose primitive functions satisfy the same formal laws. These structures **B** are complemented distributive lattices (cf. Sects. 1, ..., 17 in Schröder, vol. I) and thus Boolean algebras in the present-day sense.

Henceforth, *BA* will be the class of Boolean algebras in this sense. Thus, a structure **B** in *BA* will be of the form $\langle B, \cdot, +, -, 0, 1\rangle$, where B is a nonempty set and where the five primitive functions $\cdot, +, -, 0, 1$ of **B** operate on 2, 2, 1, 0, 0 arguments respectively. If **B** is a Boolean algebra of sets, then these five functions are the following set-theoretic operations respectively: intersection; union; complementation (with respect to 1); the null set $\emptyset$; the largest set in B (Boole's "universe of discourse").

Henceforth, L shall be the first-order language that will be used here in talking about the algebras **B** in *BA*. Since it should be apparent from the context whether a function of **B** or the symbol for it in L is involved, the function symbols of L shall be: $\cdot$, $+$, $-$, 0, 1. The terms of L shall be formed in the usual way from these and from (individual) variables $v, w, x, y, \ldots$ (regarded as ranging over B). The atomic formulas shall be the equalities $t = t'$, where t and t' are terms. Other formulas shall be formed from atomic formulas in the usual way, using $\wedge$, $\vee$, $\neg$, $\forall$, and $\exists$. For brevity or suggestiveness, $t \leq t'$ will often be used instead of $t \cdot t' = t$, and $t \neq t'$ instead of $\neg\, t = t'$. The set of variables of L shall be *Vb*. For any subset S of *Vb*, an *S-formula* shall be any formula C such that any v in *Vb* that has a free occurrence in C belongs to S. Thus, if $S = Vb \cap -\{x\}$, then C is an S-formula if and only if x does not occur free in C.

In changing from everyday language to the language L for Boolean algebras, one replaces the use of predicate symbols referring to sets by the use of individual variables intended to refer to these. The original problem of axiomatizing those consequences of a conjunction A of sentences in which the predicate symbols in R do not occur changes accordingly. Given a truth functional combination of formulas of L, one now wants to axiomatize those Boolean consequences in which certain chosen individual variables, such as $x, y, \ldots$, have no free occurrence.

Consider any **B** in *BA* and any mapping μ of the set *Vb* of variables into the set B of elements of **B**. There is a natural extension of μ to a mapping $\mu_{\mathbf{B}}$ of the set of terms into B. This mapping induces a unique mapping, which will also be denoted by $\mu_{\mathbf{B}}$, of the set of atomic formulas into the set $\{\top, \bot\}$ of truth-values. Specifically, if $\mu_{\mathbf{B}}(t) = \mu_{\mathbf{B}}(t')$ then $\mu_{\mathbf{B}}(t = t') = \top$, and if $\mu_{\mathbf{B}}(t) \neq \mu_{\mathbf{B}}(t')$ then $\mu_{\mathbf{B}}(t = t') = \bot$. This mapping of the atomic formulas into $\{\top, \bot\}$ has a natural extension, which will also be denoted by $\mu_{\mathbf{B}}$, of any formula C of L into $\{\top, \bot\}$. If $\mu_{\mathbf{B}}(C) = \top$ then C shall be *true under* $\mu_{\mathbf{B}}$. For any term t and any formula C, $\mu_{\mathbf{B}}(t)$and $\mu_{\mathbf{B}}(C)$ shall be *the interpretation* (*in* **B**) *of* t or *of* C, respectively.

A formula C shall be a *BA consequence* of a set $\{A_i\colon i \in I\}$ of formulas, symbolically $\{A_i\colon i \in I\} \vDash_{BA} C$, if and only if, for any **B** in *BA* and any interpretation $\mu_{\mathbf{B}}$ in **B**, if every A_i is true under $\mu_{\mathbf{B}}$, then so is C. Also, two formulas A and C shall be *BA equivalent* if and only if $A \vDash_{BA} C$ and $C \vDash_{BA} A$. Further, a formula C shall be *BA valid* if and only if it $\emptyset \vDash_{BA} C$.

For $1 \leq n < \omega$, let A be a conjunction of the equalities in a set $\{t_i = t_i' : 0 \leq i < n\}$. Let R be a subset of *Vb* such that every v in R has a free occurrence in A, and let $S = Vb \cap -R$. To obtain by the Boole-Schröder method an equality that axiomatizes the set $\{C\colon A \vDash_{BA} C, C$ is an S-formula$\}$ one proceeds as follows. First, one replaces every equality $t_i = t_i'$ by the equality $(t_i \cdot -t_i') + (t_i' \cdot -t_i) = 0$, which is *BA* equivalent to it. Next, one replaces the conjunction of these equalities by the following single equality, which is *BA* equivalent to this conjunction

$$F_1\colon (t_0 \cdot -t_0') + (t_0' \cdot -t_0) + \cdots + (t_{n-1} \cdot -t_{n-1}') + (t_{n-1}' \cdot -t_n) = 0.$$

To describe the final steps, let us first assume that R is the singleton $\{x\}$, so that $s = Vb \cap -\{x\}$. Using various laws for transforming terms r into terms r' such that $r = r'$ is BA valid, including de Morgan's law, which allows one to lessen the scope of $-$, and a distributive law, which allows one to factor out the term x and also the term $-x$, one transforms the term t that constitutes the left of the equality F_1 above into the term t' that constitutes the left of the equality F_2 below, where s and s' are terms that do not contain x. The term t' is called by Boole and by Schröder *the development of t (with respect to x)*, while s and s' are called *the coefficient of x or of* $-x$, respectively. Since $t = t'$ is BA valid, therefore F_1 and F_2 are BA equivalent. (Cf. Schröder, vol. I, p. 409, Theorem 44_+.)

$$\begin{aligned} F_2 &: (s \cdot x) + (s' \cdot -x) = 0. \\ F_3 &: s \cdot s' = 0. \\ F_{2,\leq} &: s' \leq x \wedge x \leq -s. \\ F_4 &: (s \cdot s') + (s' \cdot -s') = 0. \\ \exists x F_2 &: \exists x((s \cdot x) + (s' \cdot -x) = 0. \end{aligned}$$

As one can see, F_2 and $F_{2,\leq}$ are BA equivalent and $F_{2,\leq} \vDash_{BA} F_3$. Since x does not occur in F_3, there follows that F_3 is in the set $\{C\colon F_2 \vDash_{BA} C,\ x$ does not occur free in $C\}$. Now consider any C in this set. Since x does not occur free in C and since $F_2 \vDash_{BA} C$, therefore $\exists x F_2 \vDash_{BA} C$. Since $F_3 \vDash_{BA} F_4$ and $F_4 \vDash_{BA} \exists x F_2$, therefore $F_3 \vDash_{BA} C$. There now follows that each of $F_3, F_4, \exists x F_2$ axiomatizes $\{C\colon F_2 \vDash_{BA} C,\ x$ does not occur free in $C\}$. Each of them also axiomatizes $\{C\colon A \vDash_{BA} C,\ x$ does not occur free in $C\}$, since A and F_2 are BA equivalent.

The same conclusions are obtained in Sect. 21 of Schröder, vol. I and in Sect. 130 of Müller (1910) by arguments that are largely similar. The formula F_3 is called there a *full resultant of eliminating x from* F_2.

In the case where, for some $k > 1$, R is a set $\{x_1, \ldots, x_k\}$ of k variables, one can axiomatize the set $\{C\colon A \vDash_{BA} C$, none of $x_1, \ldots, x_k$ occur in $C\}$ by k successive uses of the above process of eliminating one variable. Any two different orders of thus eliminating the variables in $\{x_1, \ldots, x_k\}$ result in $Vb \cap -\{x_1, \ldots, x_k\}$ formulas that are BA equivalent.

The statement that some R are S, where R and S are sets, can be rendered thus: $R \cap S \neq 0$. In order to also take into account statements of this kind, Schröder considers, in Sects. 36, 41, and 49 of Schröder, vol. II, an arbitrary quantifier-free formulas A, so that A is logically equivalent to a disjunction $A_1 \vee \cdots \vee A_j$, where each A_i is a conjunction formed from equalities and negations of equalities.

Concentrating on the case where a single variable x is to be eliminated from A, he observes that since $\exists x A$ is logically equivalent to the disjunction $\exists x A_1 \vee \cdots \vee \exists x A_j$, it is sufficient to consider separately, for each i, $1 \leq i \leq j$, elimination of x from A_i. Moreover, on pages 380–381 of Schröder, vol. II, and also in Sect. 161 of Müller (1910), the problem is reduced to the case where every component of this conjunction A_i is a formula $t \neq t'$. As we saw earlier, $t = t'$ is BA equivalent to a formula $(r \cdot x) + (s \cdot -x) = 0$, where neither r nor s contains x. Also, as one can see, every formula $r' + s' \neq 0$ is BA equivalent to the disjunction $r' \neq 0 \vee s' \neq 0$. Now assume that A_i is a conjunction of n disjunctions of the form $r \cdot x \neq 0 \vee s \cdot -x \neq 0$, where neither

r nor s contains x. Then, from the distributivity of $\wedge$ over $\vee$ there follows that A_i is logically equivalent to a disjunction of 2^n formulas $A_{i,1}, \ldots, A_{i,2^n}$, each of which, for some m such that $0 \leq m \leq n$, is of the form F_5 below, where $r_0, \ldots, r_m, s_{m+1}, \ldots, s_n$ do not contain x. Since $\exists x(A_{i,1} \vee \cdots \vee A_{i,2^n})$ and $\exists x A_{i,1} \vee \cdots \vee \exists x A_{i,2^n}$ are logically equivalent, the problem of eliminating x from a quantifier-free formula A is thus reduced to the problem of eliminating x from formulas of the form F_5.

F_5: $r_1 \cdot x \neq 0 \wedge \cdots \wedge r_m \cdot x \neq 0 \wedge s_{m+1} \cdot -x \neq 0 \wedge \cdots \wedge s_n \cdot -x \neq 0.$
F_6: $r_1 \neq 0 \wedge \cdots \wedge r_m \neq 0 \wedge s_{m+1} \neq 0 \wedge \cdots \wedge s_n \neq 0$.

Let $\exists x F_5$ be the formula that results from F_5 by prefixing $\exists x$. Then $F_5 \vDash \exists x F_5$ and x does not occur free in $\exists x F_5$. Moreover, if $F_5 \vDash_{BA} C$ and x does not occur free in C, then $\exists x F_5 \vDash_{BA} C$. There follows that $\exists x F_5$ axiomatizes the set

$$\{C\colon F_5 \vDash_{BA} C,\ x \text{ does not occur free in } C\}\ .$$

Evidently, F_6 belongs to this set. In Sects. 41 and 49, Schröder shows that there are certain Boolean algebras $\mathbf{B}$ and interpretations $\mu_{\mathbf{B}}$ in $\mathbf{B}$ such that, if F_6 is true under $\mu_{\mathbf{B}}$, so is $\exists x F_5$. He calls F_6 a *rough-and-ready resultant* or *crude resultant* ("Resultante aus dem Rohen") of eliminating x from F_5, to indicate that in some, but not in all, cases F_6 can serve as an axiomatization of the above set. He probably also wanted to suggest that F_6 not only readily comes to mind, but also may serve as a useful first step toward finding a resultant that is full.

Let $\mathbf{B}$ be any Boolean algebra. As usual, an *atom* of $\mathbf{B}$ will be any element a of B such that, for any b in B, if $b \leq a$ and $b \neq a$, then $b = 0$. (In Sect. 47, Schröder calls an atom of $\mathbf{B}$ an indivisible element or *individuum*.) Also, $\mathbf{B}$ will be *atomic* (*atomless*) if and only if, for every $b \neq 0$ in B, there is some a (no a) such that a is an atom and $a \leq b$. An element b of B shall be *small* (*in* $\mathbf{B}$) if and only if either $b = 0$ or there is a finite nonempty set $\{a_1, \ldots, a_j\}$ of atoms of $\mathbf{B}$ such that $b = a_1 + \cdots + a_j$.

Let $\mathbf{B}$ be any Boolean algebra and let $\mu_{\mathbf{B}}$ be any interpretation in $\mathbf{B}$. Let $\mu_{\mathbf{B}}(r_1) = b_1, \ldots, \mu_{\mathbf{B}}(r_m) = b_m, \mu_{\mathbf{B}}(s_{m+1} = c_{m+1}, \ldots, \mu_{\mathbf{B}}(s_n) = c_n$. Then $\exists x F_5$ or F_6 is true under $\mu_{\mathbf{B}}$ if and only if there holds, respectively, $(5)_\exists$ or (6) below.

$(5)_\exists$ There are $b'_1, \ldots, b'_m, c'_{m+1}, \ldots, c'_n$ such that

(i) $0 \neq b'_1 \leq b_1, \ldots, 0 \neq b'_m \leq b_m,\ 0 \neq c'_{m+1} \leq c_{m+1}, \ldots, 0 \neq c'_n \leq c$, and
(ii) $(b'_1 + \cdots + b'_m) \cdot (c'_{m+1} + \cdots + c'_n) = 0$.

(6) $b_1 \neq 0, \ldots, b_m \neq 0,\ c_{m+1} \neq 0, \ldots c_n \neq 0$.

As one can see, if $\mathbf{B}$ is atomless, then (6) implies $(5)_\exists$. More generally, the answer to when (6) implies $(5)_\exists$ depends only on those $b_1, \ldots, b_m, c_{m+1} \ldots c_n$ that are small. Thus, without loss of generality, it will be assumed in what follows that each of $b_1, \ldots, b_m, c_{m+1} \ldots c_n$ is small. If (6) holds, then this assumption is equivalent to the following condition.

(7) There are atoms $a_1, \ldots, a_k$ such that $b_1 + \cdots + b_m + c_{m+1} + \cdots + c_n = a_1 + \cdots + a_k$.

Assume (7). Then $(5)_\exists$ is equivalent to the following condition

$(5)^+_\exists$ There are $a_{j_1}, \ldots, a_{j_m}, a_{j_{m+1}}, \ldots, a_{j_n}$ in $\{a_1, \ldots, a_k\}$ such that

(i) $a_{j_1} \leq b_1, \ldots, a_{j_m} \leq b_m,\ a_{j_{m+1}} \leq c_{m+1}, \ldots, a_{j_n} \leq c_n$, and
(ii) $\{a_{j_1}, \ldots, a_{j_m}\} \cap \{a_{j_{m+1}}, \ldots, a_{j_n}\} = \emptyset$.

If (i) holds, then $\langle a_{j_1}, \ldots, a_{j_m}, a_{j_{m+1}}, \ldots, a_{j_n}\rangle$ shall be an *instantiation (by atoms) of* $\langle\langle b_1, \ldots, b_m\rangle, \langle c_{m+1}, \ldots, c_n\rangle\rangle$. If also (ii) holds, then this instantiation shall satisfy the *disjointness condition*. It may be suggestive, if (i) holds, to think of $a_{j_1}, \ldots, a_{j_m}$, $a_{j_{m+1}}, \ldots, a_{j_n}$ as the chosen *representatives* from the n sets of atoms below $b_1, \ldots, b_m$, $c_{m+1}, \ldots, c_n$ respectively. Then (ii) is a disjointness condition on the sets $\{a_{j_1}, \ldots, a_{j_m}\}$ and $\{a_{j_{m+1}}, \ldots, a_{j_m}\}$ of the chosen representatives.

The question of when the conjunction of (6) and (7) implies $(5)^+_\exists$ can be narrowed further. Evidently, condition (ii) in $(5)^+_\exists$ is trivially satisfied if either $m = 0$ or $m = n$. In what follows, it will therefore be assumed that $0 \neq m \neq n$. Also, whether or not the disjointness condition (ii) in $(5)^+_\exists$ can be satisfied depends only on those $b_{m'}$ such that $b_{m'} \leq c_{m+1} + \cdots + c_n$ and those $c_{n'}$ such that $c_{n'} \leq b_1 + \cdots + b_m$. (Cf. Schröder, vol. II, pp. 391–392.) In what follows it will therefore be assumed that the following hold.

$(7)^+$ There are atoms $a_1, \ldots, a_k$ such that $b_1 + \cdots + b_m = c_{m+1} + \cdots + c_n = a_1 + \cdots + a_k$.

Among the cases where (6) and $(7)^+$ hold but where $(5)^+_\exists$ and hence $(5)_\exists$ does not, so that the rough-and-ready resultant F_6 of eliminating x from F_5 is not a full resultant, are those of the following kind. (Cf. Schröder, vol. II, p. 395.)

(8) Either some $b_{m'}$ is a sum $a_{i_1} + \cdots + a_{i_{k'}}$ of atoms, each of which is some $c_{n'}$, or some $c_{n'}$ is a sum $a_{i_1} + \cdots + a_{i_{k'}}$ of atoms, each of which is some $b_{m'}$.

There are also cases other than those just described where, among the instantiations of $\langle\langle b_1+\cdots+b_m\rangle, \langle c_{m+1}+\cdots+c_n\rangle\rangle$ there is none that satisfies the disjointness condition. On pages 395–396 of Schröder, vol. II, Schröder lists several where $m = n–m = 2$ and $k = 3$, and also several more where $m = 3$, $n-m = 2$, and $k = 4$. He leaves it open whether his list is complete.

As an example where there is no instantiation that satisfies the disjointness condition but where none of $b_1, \ldots, b_m, c_{m+1}, \ldots, c_n$ is an atom, Schröder gives the following

(9) $b_1 = a_1 + a_2, \quad b_2 = a_2 + a_3, \quad b_3 = a_3 + a_4, \quad b_4 = a_1 + a_4 + a_5,$
$c_5 = a_1 + a_3, \quad c_6 = a_1 + a_4, \quad c_7 = a_2 + a_4, \quad c_8 = a_2 + a_3 + a_5.$

One can verify that there is no instantiation satisfying the disjointness condition by verifying that there is none with $a_{j_1} = a_1$ and also none with $a_{j_1} = a_2$.

In order to describe instantiations and also presence or absence of disjoint instantiations, Schröder, in essence, resorts to a 2-sorted language. The 2-sorted language L_2 that will be used here results from the (one-sorted, first-order) language L for Boolean algebras **B** that has hitherto been used by adjoining variables of a disjoint second sort. Each variable of this second sort will have i as a subscript, which may or may not be followed by a second subscript. Thus, they are of the form $v_i, w_i, x_i, v_{i,1}, x_{i,3}, \ldots$. These variables are intended to range over the atoms or individua of the Boolean algebra **B** concerned. (Schröder uses $i, i^1, i^2, \ldots$ for his variables of the second sort.) Since their range is more restricted than the range of the variables of L, they shall be the *restricted variables* of L_2. A quantifier of L_2 shall be *restricted* if and only if its variable is restricted. Also, a formula shall be *restricted* if and only if it is a formula of L_2 in which every quantifier is restricted. The sublanguage of L_2 whose formulas are the restricted formulas shall be $L_{2,r}$. Note that while both L and $L_{2,r}$ are sublanguages of L_2, neither one is a sublanguage of the other.

In order to interpret L_2, $\mu_{\mathbf{B}}$ will now be extended from variables, terms, and formulas of L to those of L_2. This will be done only for those Boolean algebras **B** that are not atomless, since, as was noted earlier, if **B** is atomless, then a formula such as $\exists F_5$ and its rough-and-ready resultant have the same truth-value under $\mu_{\mathbf{B}}$, so that there is no need for going beyond L. Also, this will avoid the problem of what value $\mu_{\mathbf{B}}$ should assign to restricted variables when there are no atoms. Regarding Boolean algebras **B** that are not atomless, it will be assumed that $\mu_{\mathbf{B}}$ has been extended to the terms and formulas of L_2 in the obvious way.

Note that for every formula A of L_2 there is a formula A' of L such that for every $\mu_{\mathbf{B}}$ such that **B** is not atomless, $\mu_{\mathbf{B}}(A)$ and $\mu_{\mathbf{B}}(A')$ have the same truth-value. For example, consider a formula $\exists\, v_i C$ where C results from a formula C' of L by replacing every free occurrence of v by an occurrence of v_i. Then, for any non-atomless **B**, the truth-value under $\mu_{\mathbf{B}}$ of the formula $\exists\, v_i C$ is the same as that of the following formula:

$$\exists\, v(v \neq 0 \wedge \forall w(v \cdot w \neq 0 \vee v \cdot -w \neq 0) \wedge C')\ .$$

Thus, L_2 is in certain respects more versatile than L but, in at least one sense of the phrase, does not have greater expressive power.

As Schröder noted, one can sometimes use a restricted formula to rule out certain cases where (6) and (7) hold but $(5)^{+}_{\exists}$ does not, i.e., cases where the rough-and-ready resultant is not a full resultant. For example, there is a quantifier-free formula F_9 of L_2 among whose variables are $v_{i,1}, \ldots, v_{i,5}$ that expresses condition (9). Let $\exists F_9$ be the restricted formula that results from F_9 by prefixing $\exists v_{i,1} \ldots \exists v_{i,5}$. Then, for any Boolean algebra **B** that is not atomless, if $m = n - m = 4$ and F_5 is true under $\mu_{\mathbf{B}}$, then so is $\neg\exists F_9$ (in addition to F_6). Thus the conjunction $\neg\exists F_9 \wedge F_6$ is also a resultant of eliminating x from F_5 (with respect to Boolean algebras **B** that are not atomless). It "comes closer" than does F_6 to being a full resultant.

Schröder's aim was to obtain for formulas such as F_5 enough restricted formulas K that, like $\neg\exists F_9$, rule out certain cases where for $\langle\langle b_1, \ldots, b_m\rangle, \langle c_{m+1}, \ldots, c_n\rangle\rangle$ there is no instantiation that satisfies the disjointness condition, so that conjunction $K_1 \wedge \cdots \wedge K_p$ of these formulas rules out all such cases and hence $K_1 \wedge \cdots \wedge K_p \wedge F_6 \vDash_{BA} \exists\, x F_5$. To this intended conjunction Schröder applied the term "Klausel" (clause). He made clear that in only a few cases he was able to find one and expressed his hope that others would continue the work.

Despite this lack of success, by providing a deeper and more nuanced understanding of what was to be achieved by solving his elimination problem, Schröder made an important contribution to it and to logic in general. Schröder was well aware that, for any formula A, if C is a formula without free occurrences of x, then $A \vDash C$ if and only if $\exists x A \vDash C$, so that $\exists x A$ axiomatizes $\{C: A \vDash_{BA} C,\ x$ does not occur free in $C\}$. Nevertheless, he seems to have regarded axiomatization by means of $\exists x A$ to be, at best, of limited value and, as we have seen, spent much effort in finding axioms that would be more useful or informative. Specifically, whereas $\exists x A$ quantifies over arbitrary elements of the Boolean algebra **B** under consideration, Schröder tried to find axiomatizations by means of formulas whose quantifiers, if any, are restricted and hence only range over the atoms of **B**. In the case where **B** is a Boolean algebra of sets, the truth-value of $\exists x F_5$, for example, depends on which sets b are elements of B. In contrast, if moreover **B** is atomic and complete, then the atoms of **B** are the singleton sets, so that the restricted quantifiers are essentially individual quantifiers. Schröder's distinction between a 2-sorted language such as L_2 and its restricted sublanguage thus anticipates the distinction between monadic second-order logic with equality and monadic first-order logic with equality.

It should be noted for the record that both Boole and Schröder were interested in elimination of x not only for its own sake, but also for the help they thought it might provide in connection with problems of another kind that they thought to be important, namely, problems of *solving for x*. For example, assuming that there holds the equality F_2 listed earlier, formula $F_{2,\leq}$ seems to solve for x in their sense (Cf. Couturat 1914, Sect. 38 and Müller 1910, Sects. 122–150.). This complex topic in logic will not be discussed further.

The most important elimination result in logic concerns monadic second-order logic with equality. Given any formula A in the language L_1^2 for this logic, one can find a formula A' in its sublanguage L_1^1 for monadic first-order logic with equality such that A and A' are logically equivalent. This result was first sketched by Löwenheim (1915) in Sect. 3. Detailed proofs were given by Skolem (1919) in Sect. 4 and by Behmann (1922). These proofs also yield a decision method for L_1^2 and for L_1^1. Useful discussions can be found in Sect. 5 of volume 1 of Hilbert-Bernays (1934) and in Church's book (1956). A more recent examination of Behmann's work is provided by Richard Zach (2007).

Skolem begins his work on the elimination problem, in the last section of Skolem (1919), with an assertion to the effect that he will work within the framework of Schröder's calculus of classes and relations. To be more precise, the main body of his work is concerned with the subclass BA_{at} of BA that consists of those **B** that are atomic. However, his final theorem, Theorem 18, a reformulation of his main theorem, Theorem 15b, is a statement of what is now known as the elimination theorem for monadic second-order logic with equality.

Skolem's argument that his Theorem 18 is a reformulation of his Theorem 15(b) makes use of the fact that the subject matter of monadic second-order logic with equality may be viewed in two ways. In accord with a tradition originating with the theory of types, we nowadays think of monadic second-order logic as concerned with individual objects and with sets of these, where none of these sets is an individual object. However, one can also think of monadic second-order logic as concerned with sets only. Instead of individual objects, one then considers sets of a special kind, namely, sets that have only one member, that member being an individual object. If one deals with sets as elements of a Boolean algebra, then the sets of this special kind are atoms of the algebra. It it then natural to think of monadic second-order logic as dealing with Boolean algebras that satisfy the further condition of being atomic and complete.

Although, at his time, monadic second-order logic had not yet become a clearly defined topic of investigation, Schröder approached elimination problems with an attitude close to this second point of view. As mentioned earlier, he made use of a 2-sorted language where one sort, that of atoms, forms a subset of the other.

It is important to note that Skolem's proof, outlined below, applies to arbitrary atomic Boolean algebras, not only to those that are both atomic and complete. Thus, the elimination theorem for standard monadic second-order logic also applies to interpretations of second-order logic that admit general models in the sense of Henkin's (1950).

Let **B** be any Boolean algebra that is atomic. For any set X, let $\|X\|$ be the cardinality of X. Then, for any element b of B, $b \neq 0$ if and only if $\|\{a : a \leq b, a \text{ is an atom}\}\| \geq 1$. Also, for any elements b and c of B, the condition that $b \neq c$, which plays a prominent role in Schröder's study of elimination, is equivalent to

$$\|\{a\colon a \leq (b \cdot -c) + (c \cdot -b), a \text{ is an atom}\}\| \geq 1.$$

For any non-negative integer n and for any term t of L_2, Skolem makes use of two formulas $\exists_{\geq n} v_i t$ and $\exists_{\leq n} v_i t$ of $L_{2,r}$ that have the following property: For any Boolean algebra **B** that is atomic and any interpretation $\mu_{\mathbf{B}}$, $\exists_{\geq n} v_i t$ or $\exists_{\leq n} v_i t$, respectively, is true under $\mu_{\mathbf{B}}$ if and only if there holds, respectively,

$$\|\{a\colon a \leq \mu_{\mathbf{B}}(t),\ a \text{ is an atom}\}\| \geq n,$$
$$\|\{a\colon a \leq \mu_{\mathbf{B}}(t),\ a \text{ is an atom}\}\| \leq n.$$

(Löwenheim made use of formulas of $L_{2,r}$ that expressed both a cardinality condition on $\mu_{\mathbf{B}}(t)$ and a cardinality condition on its complement $-\mu_{\mathbf{B}}(t)$. These uses by Löwenheim and by Skolem may be the first use in logic of cardinality quantifiers other than $\exists = \exists_{\geq 1}$ and its dual.)

Let A be a formula of L_2 and let x be a variable that is not restricted. Skolem first shows that the formula $\exists x A$ is BA_{at} equivalent to a disjunction $\exists x A_1 \wedge \cdots \wedge \exists x A_k$ where, for each j, there is some term s_j not containing x such that A_j is a conjunction of formulas, each of which is of one of the four forms $\exists_{\geq n} x(s_j \cdot x)$, $\exists_{\leq n} x(s_j \cdot x)$, $\exists_{\geq n} x(s_j \cdot -x)$, $\exists_{\leq n} x(s_j \cdot -x)$. Moreover, each s_j can be chosen to be a product which, for every variable v other than x that occurs in A, contains as factor either v or $-v$. Skolem then shows that, if A_j is the conjunction $A_{j_1} \wedge \cdots \wedge A_{j_p}$ then $\exists x A_j$ and $\exists x A_{j_1} \wedge \cdots \wedge \exists x A_{j_p}$ are BA_{at} equivalent. If $n \geq n'$, then $\exists_{\geq} x(s_j \cdot x)$, $\exists_{\geq} x(s_j \cdot -x)$, $\exists_{\leq n'} x(s_j \cdot x)$, and $\exists_{\leq n'} x(s_j \cdot -x)$ implies $\exists_{\geq n'} x(s_j \cdot x)$, $\exists_{\geq n'} x(s_j \cdot -x)$, $\exists_{\leq n} x(s_j \cdot x)$, $\exists_{\leq n} x(s_j \cdot -x)$, respectively. Hence every conjunction $A_{j_{p'}}$ is BA_{at} equivalent to a conjunction $C_{j_{p'}}$ of either 4, 3, 2, or 1 factors of one of the above four forms, with no two factors being of the same form.

Skolem verifies that, in any of these four cases, $\exists x C_{j_{p'}}$ is BA_{at} equivalent to a conjunction of formulas that are of the form $\exists_{\geq n} x s_j$ or $\exists_{\leq n} x s_j$. There now follows that $\exists x A$ is BA_{at} equivalent to a truth-functional combination A' of formulas of the form $\exists_{\geq n} x s$ or $\exists_{\leq n} x s$, where s is a product that, for every variable v other than x that occurs in A, contains as factor either v or $-v$. Thus, A' is a formula of $L_{2,r}$. Similarly, the formula $\exists x \neg A'$ is BA_{at} equivalent to a formula A'' of $L_{2,r}$. There follows that

$\neg\exists x \neg A$ and hence also $\forall x A$ is a formula of $L_{2,r}$. There follows by induction that every formula of L_2 is BA_{at} equivalent to a formula of the restricted language $L_{2,r}$.

We can now see that, as logic evolved, so did ideas about elimination. The original problem concerned axiomatizations of a certain kind: Given a set or conjunction A of statements about certain classes, how to axiomatize the set of those logical consequences C that involve only certain designated ones among these classes. Boole saw that there are advantages in an algebraic approach where statements about classes are made by means of equalities between terms that denote these classes. The problem thus became one of axiomatizing, for a given conjunction A of equalities, the set of those consequences C of A that are an equality in which there occur none of the variables $x, y, \ldots$ that serve to denote a non-designated class. Because of syntactical similarities between his procedure for obtaining suitable axioms and procedures that were being used in algebra for solving equations, Boole applied to his procedures the same name: *elimination of variables.*

Boole, Schröder, and others approached their work from two different, to some extent complementary, points of view. On the one hand, at the center of their interest was logic. As a problem in logic, the elimination problem was therefore a certain problem concerning the relation $\models$ of logical consequence. On the other hand, they saw the advantage of sometimes adopting a more abstract point of view and of considering interpretations of their system by mathematical structures that are not algebras of sets but satisfy the same or similar laws. This led them to consider consequence relations such as $\models_{BA}$, where $A \models_{BA} C$ if and only if C is a logical consequence of the conjunction of A and the axioms for the class BA of Boolean algebras. Thus, practically from the start, elimination problems concerning logic were accompanied by related elimination problems concerning certain mathematical theories.

It was all along understood that elimination of x from A should result in an axiomatization of $\{C\colon A \models_{BA} C,\ x$ does not occur free in $C\}$ that is useful and informative. Boole's procedure of eliminating x from a conjunction A of equalities of L yields a formula E that satisfies this condition. As we saw, if $BA^{\setminus}$ is the class of Boolean algebras that are atomless, A is any truth-functional combination of equalities of L, and E is the rough-and-ready resultant of eliminating x from A, then E axiomatizes the set $\{C : A \models_{BA^{\setminus}} C,\ x$ does not occur free in $C\}$. This axiomatization also seems to be reasonably informative. In contrast, as Schröder realized, although $\exists x A$ axiomatizes $\{C : A \models_{BA} C,\ x$ does not occur free in $C\}$, this axiomatization is, in general, not very informative. This is why he tried to find an E that is BA equivalent to $\exists x A$, not only a BA consequence of $\exists x A$, and that, in some sense, provides more insight. The task of eliminating *a* variable from A thus became the task of eliminating a quantifier from $\exists x A$.

This reformulation of the task, made explicit by Löwenheim and Skolem, but restricted from BA to BA_{at}, makes the task more demanding but has certain advantages. In the first place, it provided suggestions of how one could try to obtain E from $\exists x A$ by an inductive argument. More importantly, since $\forall x A$ is logically equivalent to $\neg\exists x \neg A$, therefore from the BA_{at} equivalence of $\exists x \neg A$ and a formula E' of $L_{2,r}$ there follows the BA_{at} equivalence of $\forall x A$ and the formula $\neg E'$ of $L_{2,r}$. Thus, whereas the elimination procedures originally envisaged were intended to apply to those formulas A of L that are quantifier-free, the procedures described by Skolem and Behmann are applicable to any A in the larger class of all formulas of L_2. Among formulas in this larger class are formulas A with free occurrences of restricted variables such as v_r and w_r. Formulas A of this kind are often used to denote a relation, rather than to make a statement. Quantifier elimination applied to A then tends to yield a description of the relation that is simpler. Thus, when one can carry it out, quantifier elimination achieves more than the axiomatization procedures from which it evolved.

It should be emphasized that, in the present case, as in many others, quantifier elimination does not, in general, yield formulas E without quantifiers. In the present case, instead of the unrestricted quantifiers such as $\exists x$ and $\forall y$ in A, one ends up with restricted quantifiers. In many other cases of quantifier elimination, use of quantifiers in an arbitrary context is replaced by use in a more circumscribed context that is more manageable or easier to comprehend.

Whereas quantifier elimination has been successfully applied to a variety of axiomatic theories, there are severe limitations to where it can be applied in logic. There are fairly simple formulas of second-order logic that express an axiom of infinity. One of these is a formula $\exists RA$ that asserts of the universe U under consideration that there is a binary relation R that is single-valued which maps a proper subset of U onto U. Since any first-order consequence of $\exists RA$ is logically equivalent to a first-order formula whose only predicate symbol is one for equality, therefore every finite set of first-order consequence of $\exists RA$ is true in some universe U that is finite. Hence, no finite set of first-order formulas is logically equivalent to $\exists RA$.

Examples of this kind led Ackermann to consider, in Ackermann (1935), weakening the requirements on elimination by widening the notion of resultant. Instead of obtaining a single first-order formula E that is logically equivalent to $\exists RA$, one was to obtain a set $\{E_n\colon 0 \leq n < \omega\}$ of such formulas such that $\exists RA$ and the set are logically equivalent. In Sect. 4, he succeeded with regard to those formulas $\exists RA$ that are of the form $\exists P \forall x_1 \ldots \forall x_n M$, where M is quantifier-free, $x_1, \ldots, x_n$ are individual variables, and P is 1-ary. In Sects. 5 and 6 he obtained resultants in his widened sense for certain further formulas.

If A is a first-order formula, then the class of models of $\exists R_1 \ldots \exists R_m A$ is nowadays called a *pseudo-elementary class*, or PC-*class.* Also, the class of models of a set $\{E_n\colon 0 \leq n < \omega\}$ of first-order formulas is an *elementary class in the wider sense*, or an EC_Δ-*class.* It has been known, at least since 1950, that there are PC-classes that are not an EC_Δ-class. Thus, there are second-order formulas $\exists R_1 \ldots \exists R_m A$ for which there exists no set $\{E_n\colon 0 \leq n < \omega\}$ of first-order formulas that is logically equivalent. For formulas of this kind the elimination problem does not have a solution in Ackermann's sense.

In light of this situation, it makes sense to return to the more modest demands that originated with Boole and that are of the kind sketched at the beginning of this article. Even when $A(R, S)$ is a first-order formula such that no set of first-order formulas is logically equivalent to $\exists RA(R, S)$, there are many times when it would be useful to have available a set $\{E_n\colon 0 \leq n < \omega\}$ that in an informative way axiomatizes the set of those first-order logical consequencese C of $A(R, S)$ that do not contain R.

A method of axiomatizing this set of consequences C of $A(R, S)$ by means of an axiom scheme is described in Craig (1960). However, the notion of scheme employed there is much wider than what would ordinarily be understood by that term. The axiomatizations resulting from this method are unlikely to be useful. Perhaps some future refinements of the method will, in some cases, yield better axiomatizations and perhaps, in some other cases, will show that there exists no axiomatization that is essentially better.

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References

Ackermann, W. (1935). Untersuchungen über das Eliminationsproblem der mathematischen Logik. *Mathematische Annalen, 110*, 390–413.

Behmann, H. (1922). Beiträge zur Algebra der Logik. *Mathematische Annalen, 86*, 163–229.

Boole, G. (1854). *An investigation of the laws of thought.* London: Walton (Also Open Court, 1916).

Burris, S. (Draft 2001). *Contributions of the Logicians, Part I: From Richard Whately to William Stanley Jevons*.

Church, A. (1956). *Introduction to mathematical logic.* Princeton.

Couturat, L. (1914). *The algebra of logic.* Open Court: Chicago and London.

Craig, W. (1960). Bases for first-order theories and subtheories. *Journal of Symbolic Logic, 25*, 97–142.

Henkin, L. (1950). Completeness in the theory of types. *Journal of Symbolic Logic, 15*, 81–91.

Hilbert, D and Bernays, P. (1968). *Grundlagen der Mathematik* (Vol. 1). Springer-Verlag, Berlin, XV+473pp.

Löwenheim, L. (1915). Über Möglichkeiten im Relativkalkül. *Mathematische Annalen, 79*, 447–470 (English translation, In J. van Heijenort (Ed.), *From Frege to Gödel* (pp. 228–251)).

Müller, E. (1910). *Abriss der Algebra der Logik*. Leipzig: Teubner (Also in Schröder, E., *Algebra der Logik* (Vol. III). Chelsea).

Schröder, E. (1890, 1891). *Vorlesungen über die Algebra der Logik* (Vols. I, II). Leipzig: Teubner (Also in NY: Chelsea).

Skolem, Th. (1919). Untersuchungenn über die Axiome des Klassenkalküls (Also In J. E. Fenstad (Ed.), *Selected works in logic by Th. Skolem*, pp. 67–101).

Zach, R. (2007). The decision problem and the development of metalogic. *ASL Annual Meeting, March 2007*.