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DEDEKIND'S ANALYSIS OF NUMBER: SYSTEMS AND AXIOMS

Treated philosophically,
it [mathematics] becomes a part of philosophy
*Herbart**

1. INTRODUCTION

In 1888 Hilbert made his *Rundreise* from Königsberg to other German university towns. He arrived in Berlin just as Dedekind's *Was sind und was sollen die Zahlen?* had been published. Hilbert reports that in mathematical circles everyone, young and old, talked about Dedekind's essay, but mostly in an opposing or even hostile sense.[1] A year earlier, Helmholtz and Kronecker had published articles on the concept of number in a *Festschrift* for Eduard Zeller. When reading those essays in parallel to Dedekind's and assuming that they reflect accurately more standard contemporaneous views, it is easy to understand how difficult it must have been to grasp and appreciate Dedekind's remarkably novel and thoroughly abstract approach. This is true even for people sympathetic with Dedekind's ways. Consider, for example, the remark Frobenius made in a letter of 23 December 1893 to Dedekind's collaborator and friend Heinrich Weber who was planning to write a book on algebra:

> I hope you often walk on the paths of Dedekind, but avoid the too abstract corners, which he now likes so much to visit. His newest edition contains so many beauties, §173 is highly ingenious, but his permutations are too disembodied, and it is also unnecessary to push abstraction so far.[2]

This remark was made by someone who refers to Dedekind as "our admired friend and master". The use of permutations, i.e., isomorphisms, in Dedekind's algebraic investigations is systematically related to the use of similar mappings in *Was sind und was sollen die Zahlen?* (The introduction of the general concept of mapping and its structure-preserving variety for mathematical investigations is perhaps the methodologically most distinctive and most radical step in Dedekind's work.)

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Dedekind was well aware that such difficulties would arise. In the preface to the first edition of *1888* he writes that anyone with sound common sense can understand his essay and that philosophical or mathematical school knowledge is not needed in the least. He continues, as if anticipating the reproach of having pushed mathematical abstraction and logical analysis too far:

But I know very well that many a reader will hardly recognize his numbers, which have accompanied him as faithful and familiar friends all his life, in the shadowy figures I present to him; he will be frightened by the long series of simple inferences corresponding to our step-by-step understanding, by the sober analysis of the sequence of thoughts on which the laws of numbers depend, and he will become impatient at having to follow proofs for truths which to his supposed inner intuition seem evident and certain from the very beginning.[3]

Dedekind arrived at his approach only after protracted labor as he emphasized in his letter to Keferstein dated 27 February 1890; in this letter Dedekind defended his essay against Keferstein's critical review *1890*. Indeed, Dedekind had started to develop his views concerning numbers in a manuscript, or rather a sequence of manuscripts, written during the period between 1872 and 1878.[4] These intellectual developments are not isolated foundational ruminations, but have to be seen in the context of Dedekind's contemporaneous work on algebraic number theory; cf. Section 4.

The publication of the essays by Helmholtz and Kronecker moved Dedekind finally to sharpen, complete, and publish his considerations. He characterized his views as "being in some respects similar [to those of Helmholtz and Kronecker], but through their grounding essentially different".[5] This is a gentle formulation of sharp mathematical and philosophical differences. The differences emerged in Dedekind's reflections slowly and over a long period of time, but they ultimately resulted in a dramatic shift. The latter can be understood, or so we will argue more explicitly below, as articulating an *axiomatic* approach that is joined with a *genetic* one in a methodologically coherent way. In his essay *Über den Zahlbegriff*, Hilbert distinguished sharply between the axiomatic and genetic method, but did not recognize then the complementary roles they play for the foundations of arithmetic. Dedekind's and Hilbert's investigations have to be seen against the backdrop of the arithmetization of analysis, that is, of the reduction of analysis to number theory. Dedekind's approach is associated with a novel *structuralist* perspective on mathematics and is grounded in *logic broadly conceived*. Hilbert sustains this general perspective in what he later calls *existential axiomatics*, but he gives up a logicist in favor of a finitist grounding of mathematics; of course, that presupposes the formalistic sharpening of the axiomatic method and the syntactic formulation of the consistency problem. For that development, see *Sieg* (*1999*, *2002*).

By tracing its development we provide a view of Dedekind's evolving foundational position that apparently differs from Hilbert's: to our knowledge, Hilbert never considered Dedekind as having used the axiomatic method. The view we provide is definitely in conflict with that of contemporary writers like Ferreirós, Corry, and McCarty. According to Ferreirós, Dedekind is non-modern in logical matters, as he can be viewed as "anti-axiomatic"; according to Corry, Dedekind is non-modern in mathematical matters, as he can't be taken to be a mathematical structuralist; finally, according to McCarty, Dedekind is non-modern in philosophical matters, as he is a thoroughgoing Kantian.[6] The reason for Ferreirós's and Corry's judgments is rooted, ultimately, in a particular understanding of the foundational essays *1872* and *1888*. (That understanding is made explicit by *Ferreirós* on pp. 119–124 and by *Corry* on pp. 71–75.) Our paper should make it very clear that their understanding of Dedekind's (methodology for the) treatment of real and natural numbers is inadequate. This also holds for McCarty. When contrasting *1872* and *1888* in his paper *1995*, McCarty points out that the essay from 1888 contains a categoricity result, whereas that from 1872 does not. McCarty asks on p. 81, why Dedekind does not establish that the geometric straight line and the system of rational cuts are isomorphic. He continues, "To this the short – but by my lights correct – answer is: Dedekind thinks that such an isomorphism would be impossible to establish". Our contrary answer is indicated in the Concluding Remarks, Section 7.

Our general views are informed by the work of Belna, Dugac, Gray, Mehrtens, Noether, Parsons, Stein, and Tait. However, they have been shaped most importantly by a close reading of manuscripts in Dedekind's Nachlass. It is fair to say that these manuscripts – including *Arithmetische Grundlagen*, the three drafts listed in Section 3.3, *1871/1872*, *1872/1878*, and also *1887* – have not yet been taken into account properly for a detailed analysis of the development of Dedekind's foundational views and its intimate connection to the evolution of his mathematical work. Our essay continues and deepens earlier work in Sieg (1990, 2000) and Schlimm (2000), but focuses almost exclusively on the systematic development of Dedekind's approach to the foundations of the theory of numbers. And what a stunning development it is! In a second essay, Dedekind's general methodological concerns will take center stage.

Let us give a brief orientation of this paper. Section 2 is concerned with the important *Habilitationsrede* of 1854, as it reveals Dedekind's perspective on the classical number systems and some broad methodological issues. We expose a *subtle*, *but pervasive circularity* in Dedekind's considerations, when he connects the *creation* of numbers beyond the naturals with the *extension* of operations. This subtle circularity is addressed, fully and satisfactorily, through the developments described next. Section 3 presents Dedekind's more systematic treatment of numbers around 1872. He introduces the successor function for natural numbers and takes a dramatic step of "analysis" toward their coherent extension to integers and rationals. This step is complemented by a "synthesis" described in Sections 4.1 and 4.2; the central demands underlying these extensions – together with a quite new aspect of abstraction – are emphasized, when we consider the *free creation* of irrational numbers in Section 4.3. This leaves open in *1872* the question how the natural numbers can be characterized. The evolution of Dedekind's theory of chains and the formulation of the Dedekind–Peano axioms for natural numbers are described in Section 5, based on a detailed analysis of *1872/1878*.

Remarkable metamathematical investigations of this axiom system are presented in *1888*. Dedekind's attempt to establish its consistency and his proof of its categoricity are of course crucial here; they are discussed in Section 6. The categoricity result allows him to justify the claim that the numbers can be called a *free creation* of the human mind. "Free creation" is understood here in a different, but related way from "free creation" in *1872*; however, the meaning is radically different from "creation" in *1854*. This central part of Dedekind's foundational work, elucidated by some general reflections, is in accord with the spirit of Herbart's remark we quoted as the motto for our essay: the mathematical work opens up a completely novel and distinctive philosophical perspective on the nature of number and, indeed, of mathematics.

2. EXTENDING OPERATIONS

Richard Dedekind, born in 1831 as a citizen of Braunschweig, finished his dissertation under Gauss in 1852 and gave a talk on the occasion of his Habilitation only two years later. The talk was entitled *Über die Einführung neuer Funktionen in der Mathematik* and was presented on 30 June 1854 to an audience that included Gauss, the classical philologist Hoeck, the historian Waitz, and the physicist Weber. Dedekind had chosen to talk about the general way, "in which new functions, or, as one might also want to say, new *operations*, are added to the chain of already existing ones in the progressive development of this science (i.e., mathematics)".[7] For Dedekind in *1854*, the introduction of new functions was an extremely important component in the development of mathematics; for us now, Dedekind's observations reveal general aspects of his intellectual approach as well as special features of his understanding of the classical number systems.

2.1. *Systematic Reflections*

The concrete analyses of the introduction of some functions are preceded by expansive remarks about the role of functions and concepts in organizing a body of knowledge, in "shaping a system". That role pertains to the law as well as to the sciences and, in particular, to mathematics. Dedekind made these remarks at the age of twenty-three for a particular occasion. Nevertheless, they bring out striking characteristics of his way of thinking and, consequently, of his later mathematical work. Their intrinsic significance is underlined by the fact that he returned to them in *1888*.

In the preface to *1888* Dedekind mentions with some satisfaction that the purpose of his *Habilitationsrede* had been approved by Gauss; he characterizes it then and there as defending the claim that the most significant and most fruitful advances in mathematics and other sciences have been made "by the creation and introduction of new concepts, rendered necessary by the frequent recurrence of complex phenomena, which could be controlled only with difficulty by the old ones".[8] This need to introduce new and more appropriate notions arises for Dedekind, in *1854*, from the fact that human intellectual powers are imperfect; their limitation leads us to frame the object of a science in different forms or systems. To introduce a concept, "as a motive for shaping the system", means in a certain sense to formulate an hypothesis concerning the inner nature of a science, and it is only the further development that determines the real value of such a notion by its efficacy in recognizing general truths. These truths, in turn, affect the formulation of definitions. Dedekind summarizes his considerations in a most revealing way:

So it may very well happen that the concepts, introduced for whatever motive, have to be modified, because they were initially conceived either too narrowly or too broadly; they will require modification so that their efficacy, their import, can be extended to a larger domain. The greatest art of the systematizer lies in carefully turning over definitions for the sake of the discovered laws or truths in which they play a role.[9]

Dedekind turns his attention then to mathematics. Definitions in mathematics are initially of a restricted form, but their generalizations are determined without arbitrariness. Indeed, Dedekind asserts, "they follow with compelling necessity from the earlier narrower ones". I.e., they do follow with necessity, if one applies the principle that some laws holding for the initial definitions are viewed as *generally valid*. These laws become consequently the source of the generalized definitions, when one asks, "How must the general definition be formulated such that the found characteristic law is always satisfied?" Dedekind views this as the distinctive feature of mathematical definitions, and the feature by which mathematics is distinguished from the other sciences. This claim will be taken up below; here we just note that in mathematics the *creation* of new objects may be involved, whereas the objects of the other sciences are presumably given. In order to illustrate this general point, we consider one of Dedekind's mathematical examples – an example that provides furthermore a real insight into his contemporaneous understanding of the classical number systems.

2.2. *Generally Valid, Subtly Circular*

Dedekind describes elementary arithmetic as being "based on the formation of ordinal and cardinal numbers" and continues, "the successive progress from one member of the series of the absolute whole numbers to the next is the first and simplest operation of arithmetic; all other operations rest on it".[10] Addition, multiplication, and exponentiation are obtained by iterating "the first and simplest operation," addition, and multiplication, respectively, and then joining these iterations into single acts. For the further development of arithmetic these definitions of the basic operations are insufficient as they are restricted to the very small domain of the positive integers. The demand that one should be able to carry out the inverse operations of subtraction, division etc. without any restrictions leads to the creation of "the negative, fractional, irrational and finally also the so-called imaginary numbers". Indeed, Dedekind views this last demand as another formulation of the demand "to create anew by each of these operations the whole given number domain".

Having expanded the domain of numbers by means of the inverse operations, a crucial question arises, namely, how to extend the definitions of the fundamental operations so that they are applicable to the newly created numbers. Here Dedekind joins the above general reflections and considers in detail the extension of multiplication from the natural numbers to all integers. The extension of the definition of multiplication is non-arbitrary, Dedekind asserts, if one follows his principle of the general validity of laws as the source for deriving "the meaning of the operations for the new number domains". This source cannot be exploited without a *subtle circularity* for addition itself: the new numbers are generated by the unrestricted inverse of a restricted operation, which is then extended to this generated broader domain! In spite of Dedekind's protestation – that the definition of the extended operation "involves an a priori complete arbitrariness" (see next paragraph) – he appeals to the very character of that generation. The intricate dependency can be observed most clearly also in Dedekind's considerations for the extension of multiplication from the natural numbers to all integers.

As noted earlier, Dedekind defines multiplication for the natural numbers as joining the iteration of addition into one single act, and it is of course assumed now that addition and subtraction are already available for all integers. Prima facie, the definition of multiplication via iteration makes sense only if the multiplicator is positive; the multiplicator is the number which indicates how often one has to iterate the addition of the multiplicand. The multiplicand can be positive or negative. Dedekind asserts:

> A special definition is therefore needed in order to admit negative multiplicators as well, and to liberate in this way the operation from the initial restriction; but such a definition involves an *a priori* complete arbitrariness, and it would only later be determined whether this arbitrarily chosen definition would bring any real advantage to arithmetic; and even if this succeeded, one could only call it a lucky guess, a happy coincidence – the sort of thing a scientific method ought to avoid.[11]

What considerations might provide grounds for a principled definition of the extended operation of multiplication? – "One has to investigate", Dedekind demands, "which laws govern the product, if the multiplicator is successively subjected to the same changes by which the series of negative numbers is generated from the series of the absolute whole numbers in the first place". The broader domain is obtained, of course, by the unrestricted inversion of addition, i.e., by considering $(m - n)$ for arbitrary natural numbers m and n. Dedekind observes that $a \times (m+1) = a \times m + a$, which yields the "addition theorem for the multiplicator" $a \times (m+n) = a \times m + a \times n$. From this follows the "subtraction theorem" $a \times (m - n) = a \times m - a \times n$, but only as long as the minuend m is greater than the subtrahend n. Taking this law as valid also for the case that the difference representing the multiplicator is negative, one obtains the definition of multiplication for the generated new numbers.[12] Thus, Dedekind concludes, "It is no longer an accident that the general law for multiplication is in both cases exactly the same". Dedekind obtains in a similar way the generalized definition of exponentiation for rational numbers.

2.3. *Imaginary and Real Problems*

The extension of the basic operations to the real and imaginary numbers is only alluded to. Dedekind claims, "These advances [obtained by creating the new classes of numbers] are so immense that it is difficult to decide which of the many paths that are opened up here one should follow first". So much is clear, however, that the operations of arithmetic have to be extended to these new classes and that no extension is possible along the lines sketched above without grasping the "generation" of the real and imaginary numbers. Here, "at least with the treatment of the imaginary numbers", the main difficulties for the systematic development of arithmetic begin. Dedekind ends the discussion of the number systems in a very surprising way:

> However, one might well hope that a truly solid edifice of arithmetic will be attained by persistently applying the principle not to permit ourselves any arbitrariness, but always to be led on by the discovered laws. Everybody knows that until now, an unobjectionable theory of the imaginary numbers, not to mention those newly invented by Hamilton, does not exist, or at any rate has not been published yet.[13]

Four years later, in the fall of 1858, Dedekind lectured on the infinitesimal calculus at the "Eidgenössisches Polytechnikum" in Zürich. He reports in *1872* that he was motivated – by the "overwhelming feeling of dissatisfaction" with the need to appeal to geometric evidences when discussing certain limit considerations – to search for "a purely arithmetic and completely rigorous foundation of the principles of infinitesimal analysis". He found it in his examination of continuity and the resulting definition of real numbers as, or rather through, cuts of rationals.

Dedekind discussed the solution with his friend Heinrich Durège at the time and presented the material to the "Wissenschaftlicher Verein" in Braunschweig on 11 January 1864, but also in some of his lectures on the differential and integral calculus.[14] Already in 1870 he had the intention of publishing his theory of continuity according to a letter from his friend Adolf Dauber.[15] We have the extended draft *1871/1872* of the essay *Stetigkeit und irrationale Zahlen*; that was seemingly written in late 1871 and early 1872. We should notice that by this time Dedekind had isolated the concept of a *field* (Körper) in Supplement X of Dirichlet (1871). This concept plays a significant role in *1872*; the novel and careful definition of the arithmetic operations on rational cuts via their definition on rationals will be discussed in Section 4.2. This fits marvelously with Dedekind's evolving view of natural numbers and their extensions to integers and rationals around 1872.

Assuming that the difficulty mentioned explicitly in the *Habilitationsrede* (to obtain an "irreproachable theory of imaginary numbers") has been resolved[16] and that the definition of real numbers in terms of cuts answers Dedekind's concerns for a rigorous foundation of analysis, two questions are clearly implicit in the above and remain open for Dedekind in 1872: (i) What are (the principles for) natural numbers? and (ii) How are the integers and rational numbers obtained, or how are they created, starting with the natural numbers? In the *Habilitationsrede* Dedekind takes for granted that the new mathematical objects (the negative and fractional numbers) have been obtained already from the natural numbers; the central issue is there, how to extend the basic arithmetic operations to the wider number systems. Question (ii) is addressed in a sequence of manuscripts contained in Cod. Ms. Dedekind III, 4, and it seems that the issues were settled to Dedekind's satisfaction before the essay on continuity and irrational numbers was completed. Question (i) was not settled at that time; on the contrary, Dedekind struggled with it intermittently over the next six years. The intense work is reflected in the manuscript *1872/1878*; it served as the very first draft for the 1888 essay on the nature and meaning of numbers and is published as Appendix LVI in Dugac (1976) with the title *Gedanken über die Zahlen*. We will analyze that work in Section 5.1, whereas the next two sections are devoted to turning the puzzle of manuscripts on extensions into an informative mosaic that answers question (ii).

3. EXTENDING DOMAINS

In this section two important steps are described and analyzed: (i) the successor operation is separated clearly from the other arithmetic operations as the one that generates the domain of natural numbers (above, and even in *1872*, *all* the arithmetic operations are on a par); (ii) basic domains of integers and rationals are characterized axiomatically (guaranteeing invertibility of addition and multiplication, but also providing without subtle circularity the basis for extending the arithmetic operations). The axiomatic "analysis" is complemented by a "synthesis" in Section 4.1: Dedekind gives an explicit definition of appropriate domains that form models of those axioms.[17] These considerations foreshadow the broad methodological moves in *1888*; namely, an axiomatic characterization of simply infinite systems and the explicit definition of a model. To obtain an appropriate axiomatic analysis of the natural numbers as a simply infinite system will take significantly more work. That is obtained in the manuscript *1872/1878*, at the end of which we find the first formulation of the Dedekind–Peano axioms. But one step at a time!

3.1. *Analyzing Naïvely*

The manuscript *1872/1878* has the subtitle *Attempt to analyze the number concept from the naïve point of view* (Versuch einer Analyse des Zahlbegriffs vom naiven Standpuncte aus). Is it in the "naïve" approach to the topic that Dedekind sees, as he does in *1888*, a certain similarity between his view and that of Helmholtz and Kronecker? What did Dedekind have in mind, when calling his approach naïve? An answer to the second question seems to be given in his letter to Keferstein by the remark addressing the rhetorical question, "How did my essay come into being?"

> Surely not all at once, rather it is a synthesis constructed after protracted labor, which is based on a preceding analysis of the sequence of natural numbers as it presents itself, in experience so to speak, to our consideration.[18]

A thoroughgoing analysis of the data of ordinary mathematical experience, free from philosophical preconceptions, is fundamental for Dedekind. Such an analysis, as Dedekind demanded already in *1854*, should lead to notions that reflect the nature of the subject and prove their efficacy in its development. The independence from traditional philosophical preconceptions is brought out clearly, when

Dedekind at the very beginning of *1872/1878* writes that the notions he uses for the foundation of the number concept "remain necessary for arithmetic even when the notion of cardinal number is assumed as immediately evident ('inner intuition')".[19]

Recall that in *1854* elementary arithmetic begins with the formation of ordinal and cardinal numbers. Dedekind views the "successive progress from one member of the sequence of positive integers to the next" as "the first and simplest operation of arithmetic" on which all other operations rest. Addition is obtained by joining iterations of this "first and simplest operation" into a single act; in completely parallel ways one obtains multiplication from addition and exponentiation from multiplication. This standpoint concerning the character of natural numbers is hardly changed, when Dedekind expresses his views in Section 1 of *1872*. There he uses *chain*, the central notion of *1888*, not yet in the precise sense of the later work, but rather as a fitting informal notion to capture the structural character of the domain that has been obtained by successively generating its objects through the "simplest arithmetical act:"

I regard the whole of arithmetic as a necessary, or at least natural, consequence of the simplest arithmetical act, that of counting, and counting itself is nothing other than the successive creation of the infinite series of positive integers in which each individual is defined by the one immediately preceding; the simplest act is to pass from an already-created individual to its successor that is to be newly created. The chain of these numbers already forms in itself an exceedingly useful instrument for the human mind; it presents an inexhaustible wealth of remarkable laws, which one obtains by introducing the four fundamental operations of arithmetic.[20]

One should notice that Dedekind speaks of counting as "nothing other" than the *successive creation* of the individual positive integers.

An elementary and restricted development of arithmetic is given in the contemporaneous manuscript *Arithmetische Grundlagen*; this manuscript is found in three distinct versions in Dedekind's Nachlass (Cod. Ms. Dedekind III, 4, II).[21] The development uses only the definition principle by recursion and the proof principle by induction. The first version starts out in the following way:

§1

Act of creation 1; $1+1=2$; $2+1=3$; $3+1=4$ ... numbers (ordinal).

§2

Definition of addition by $a+(b+1)=(a+b)+1$. After this, consequences are – according to the nature of the subject – always to be deduced by complete induction.[22]

This is only slightly modified in the second version that reads:

§1

Creation of the numbers: 1; $1+1=2$; $2+1=3$; $3+1=4$ ... from each number *a* the *following* number $a+1$ is formed by the act $+1$. Therefore, everything by complete induction.

§2

Definition of addition: $a+(b+1)=(a+b)+1$.[23]

In both versions elementary arithmetic is then briefly and very thoroughly developed. Dedekind establishes (in different ways) associativity and commutativity of addition and multiplication and ends with a proof of the distributive law $a\times(b+c) = a\times b + a\times c$. In the second version, he remarks on the margin that this law can be obtained much more directly from the definition of multiplication and the associativity of addition. Such a more direct argument is indeed presented in the third version.

Most remarkable about the third version of *Arithmetische Grundlagen* is the fact that Dedekind separates the generating "successor operation" from addition, i.e., the sequence of numbers is now indicated by 1, $\varphi(1)=2$, $\varphi(2)=3$, $\varphi(3)=4$, ..., and the recursive definition of addition is given by the two equations $a+\varphi(\mathrm{b}) = \varphi(a+b)$ and $a+1 = \varphi(a)$ instead of just by the single equation $a+(b+1) = (a+b)+1$. This notational change to the unary successor operation indicates the beginning of a quite dramatic conceptual shift that finds its systematic expression in the manuscript *1872/1878* and provides one solid reason for thinking that *Arithmetische Grundlagen* was completed in (early) 1872.

3.2. *Creating in Circles*

The third version makes also quite clear that Dedekind is trying to use these foundations for constructing the extended number systems, here, of all integers. Dedekind defines subtraction by $a-b=c$, in case $a=b+c$; this is taken, implicitly, as the motivation for considering an extension of the positive integers that contains 0 (zero) and the negative numbers 1*, 2*, 3*, etc. The successor operation is suitably extended by setting, in particular, $0+1=1$, $1^*+1=0$, $2^*+1=1^*$, $3^*+1=2^*$, etc. Having defined the predecessor operation $b=a-1$, in case $b+1=a$, he considers $1-1$, $(1-1)-1$, etc. as the *new numbers*.

Together with the systematic development up to the distributive law (central for restricting the possible extensions of multiplication in *1854* and called there the "addition theorem for the multiplicator") this sets the stage for a development along the lines suggested in his *Habilitationsrede*. Indeed, it sets the stage in a much more refined way, but it leaves in place the subtle circularity we diagnosed in *1854*; it is now directly visible through the juxtaposition of the non-positive numbers 0, 1*, 2*, etc. and the new numbers $1-1$, $(1-1)-1$, $((1-1)-1)-1$, etc. Thus, Dedekind assumes here a domain containing also zero and the negative numbers in order to define the extended successor operation. That allows him, in turn, to define the general predecessor operation and to describe the desired extension of the system of natural numbers by the *new numbers*.[24] But of what objects does the first extension really consist? What are the negative numbers? (Dedekind's answers to these questions are discussed fully in Section 4.)

There is no indication on the manuscript itself as to when *Arithmetische Grundlagen* was written. We conjecture, for three reasons, that it was completed in early 1872. The first reason is simply the fact that the beginnings of the various versions are in accord with the informal description in *1872*. The second reason was mentioned already, when we looked at the third version and noticed an important and rather unique overlap with *1872/1878*. Finally, the third reason is provided by the systematic context of creating the system of rational numbers on these arithmetic foundations. In *1871/1872* Dedekind emphasizes that the rational numbers are *a free creation*. He also claims that the "instrument mathematicians have constructed by creating the rational numbers" has to be refined by the creation of the irrational numbers in a purely arithmetic way.

> Just as negative and fractional rational numbers are formed by a free creation, and just as the laws of operating with these numbers are reduced to the laws of operating with positive integers (at least it *should* be done in this way), in the same way the irrational numbers must also be defined by means of the rational numbers.[25]

This long sentence is repeated almost verbatim in the publication *1872*. Here it is (and we urge readers to notice the italized replacement for the parenthetical remark in the above quotation):

Just as negative and fractional rational numbers are formed by a free creation, and just as the laws of operating with these numbers *must and can* be reduced to the laws of operating with positive integers, in the same way the irrational numbers must also be completely defined by means of the rational numbers alone.[26]

What is the mathematical substance that allows us to understand the shift from *should* to *must and can*?

We conjecture that the material contained in Cod. Ms. Dedekind III, 4 provides the answer: having established proper *arithmetic foundations*, Dedekind convinces himself in detail that the system of rational numbers can be created, and that the laws for calculating with these numbers can be reduced to those for calculating with the positive whole numbers. This is done, however, in a completely novel axiomatic way.

3.3. *Analyzing Axiomatically*

Dedekind's *Nachlass* contains several manuscripts dealing with the extension of the natural numbers to the integers and rational numbers. Particular ways of extending the number concept are pursued in the following manuscripts: (i) Cod. Ms. Dedekind III, 4, I, pp. 1–4, entitled *Die Schöpfung der Null und der negativen ganzen Zahlen*, (ii) Cod. Ms. Dedekind III, 4, I, pp. 5–7, without title, but we will refer to it as *Ganze und rationale Zahlen*, and (iii) Cod. Ms. Dedekind III, 2, I, entitled *Die Erweiterung des Zahlbegriffs auf Grund der Reihe der natürlichen Zahlen*. The first two manuscripts, we conjecture, were written in 1872.[27] The third one was written after 1888, as it refers explicitly to the essay *1888*; it gives an altogether modern approach. In this subsection we give a detailed account of the first manuscript.

Our description in Section 3.2 of how to generate the integers from the natural numbers is based on remarks in the third version of *Arithmetische Grundlagen*. The generation proceeds essentially in two steps, the creation of the negative numbers motivated by the demand of the general invertibility of addition and the creation of the *new numbers* by the generalized predecessor operation. In *Die Schöpfung der Null und der negativen ganzen Zahlen* a beautifully detailed presentation of the first step of those considerations is given. That is *one* way of describing it; more accurately, however, Dedekind separates cleanly the discussion of the general invertibility of addition, extension of operations and the permanence of laws from the generation of mathematical objects satisfying those laws.

The first manuscript formulates at the outset basic facts regarding the series of natural numbers N: (1) N is closed under addition; addition is (2) commutative and (3) associative; (4) if $a > b$, then there exists one and only one natural number c, such that $b + c = a$, whereas in the opposite case, when $a \leq b$, no such number c exists. Dedekind notes that the fourth condition states a certain *irregularity* and raises the crucial question, whether it is possible to extend the sequence N to a system M (by the addition of elements or numbers to be newly generated) in such a way that M satisfies conditions (1)–(3) and also (4′), i.e., for any two elements a and b from M, there exists exactly one element c, such that $b + c = a$. And he asks, how rich must the *smallest* such system M be.

In the following *Investigation*, which is also called *Analysis*, Dedekind assumes the existence of such a system M. He reasons that M must contain a unique element 0 (called zero), such that $a + 0 = a$; furthermore, for every element a in N there must be a new element a^* in M, such that $a + a^* = 0$. Thus, *any* system M satisfying (1)–(4′) must contain in addition to the elements of N the new element zero and all the different new elements a^*. Dedekind considers now the system P consisting of just N together with these new elements and shows that P has already the *completeness* expressed by conditions (1)–(4′); P is obviously the smallest such system, as it must be contained in any complete system M. The investigation is carried out in exemplary mathematical clarity, but it assumes quite explicitly the existence of a suitable M. This methodologically crucial issue is presumably addressed in the second, and unfortunately incomplete, section of the manuscript that is entitled *Synthesis*. Here is the full text of that section:

> From the sequence N of natural numbers a is to be created a system P, which contains in addition to the elements a also an element 0 and for each a a corresponding element a^*, with the stipulation that all these elements in P are *different* from each other (easy to formulate more precisely; on the *possibility* of such a creation, see farther below).[28]

There is no "farther below" and thus no discussion of the *possibility of such a creation*. The manuscript ends abruptly on page 4 with the remark just quoted. The folder contains, however, additional material that was written at a later date (as argued above), but its substance was undoubtedly clear to Dedekind in 1872 and can be understood as realizing such a creation.

4. CREATING MODELS

The systematic considerations are continued in *Ganze und Rationale Zahlen*. This manuscript has two main parts: the first deals with the *extension of the domain* of all natural numbers to that of all integers; the second is concerned with the "transition from the domain G of all whole numbers to the field R of all rational numbers". The first part consists of three handwritten pages together with a few *Zettel* filled with detailed calculations concerning integers; the second part sketches very briefly similar considerations for the rationals on just one page. We describe the first part in detail, despite the fact that the steps are routine for a modern reader: Dedekind has finally found a way out of the subtle circularity involved in his earlier considerations of the various number systems (and their creation from the natural numbers).

4.1. *Pairs as Numbers*

Dedekind starts out with the domain N of all natural numbers together with the operations of addition and multiplication. Both operations satisfy the commutative and associative laws, and the distributive law connects them. The domain G of *all* whole numbers is then formed from N, as Dedekind puts it, *by extension*: "Any two numbers m, n in N generate a number (m, n) in G". Dedekind defines two pairs of numbers (m, n) and (m', n') as *identical* when $m+n'=m'+n$ and verifies that this relation is symmetric and transitive. As it is obviously also reflexive, it is an equivalence relation. Then he defines *addition* on pairs by letting the sum of (m, n) and (m', n') be identical to the pair $(m+m', n+n')$. Having checked that the defined addition yields identical results when applied to identical pairs, he verifies easily the associative and commutative laws. *Multiplication* for pairs (m, n) and (m', n') is given by $(mm'+nn', mn'+m'n)$ and is treated in a completely parallel way: uniqueness is checked (that is actually a quite lengthy argument and spills over onto the *Zettel*) and laws are verified; the final step is the verification of the distributive law.

This is the central part of constructing the integers as pairs of natural numbers that represent positive and negative numbers, but of course also zero. It is reminiscent of the very early considerations in *1854*, when Dedekind extends subtraction from the natural numbers to the integers and, in essence, uses differences between natural numbers to *represent* negative numbers. Thinking of the pairs (m, n) as differences $m-n$ and using the ordinary calculation rules, the operations are obtained in a direct way and obey the standard laws. A parallel construction is sketched in the second part of this manuscript to obtain the rationals R from the integers G: for pairs (m, n) and (m', n') – where m, n, m' and n' are in G, but n and n' are different from zero – "identity" is defined by $mn' = m'n$; this is again an equivalence relation. Thinking of pairs (m, n) as fractions m/n, addition and multiplication are defined via the ordinary calculation rules as $(mn' + nm', nn')$, respectively (mm', nn'). The various laws can be verified. It is also clear, though Dedekind does not prove it, that the inverted operations can be performed without any restriction.

We emphasize that this manuscript is in very rough form and indicates only the bare minimum of the needed considerations. But even so, it does provide a quite novel way in which to ensure the permanence of laws. Dedekind does not create – out of thin air – new individual elements: he rather obtains by pairing natural numbers, respectively integers, new systems of genuine mathematical objects. The arithmetic operations are then defined in terms of the operations on natural numbers, respectively integers. These systems satisfy the laws or axiomatic conditions for integers and rationals, i.e., Dedekind exhibits models for these laws. In fact, the models presented are exactly the ones that are still being employed today: except that in a modern exposition one would deal with equivalence classes of pairs.

That is done very beautifully in the final and later manuscript concerned with the extension of the number systems, *Die Erweiterung des Zahlbegriffs auf Grund der Reihe der natürlichen Zahlen*. It should be noticed clearly, however, that Dedekind could have taken this last step in 1872. There was no ideological reason for avoiding infinite mathematical objects; indeed, he had used such objects in the ideal-theoretic investigations of Supplement X for the second edition of Dirichlet's *Zahlentheorie* of 1871, but also in the 1872 essay on continuity and irrational numbers. Yet there is one question that is left open: The rational numbers, "are" they these specific infinite objects? – A pertinent answer can be extracted from *1872*, as Dedekind's essay answers an analogous question for the reals. In the introduction to *1888*, Dedekind situates his treatment of the natural numbers in the general context of providing, as he puts it, a completely clear picture of the science of numbers. He refers to the example of the real numbers presented in *1872* and remarks that the other classes of numbers can be treated easily in a *quite similar fashion*. It is this observation, made also very clearly in Dedekind's letter to Weber dated 24 January 1888, that allows us to use the methodological considerations concerning the reals for our present context of the rational numbers. What has Dedekind to say about the question, what the reals really "are"? (The reader should look also at the related considerations in Section 6.2 and note 62.)

4.2. *Systems as Numbers*

In his considerations of this very question Dedekind heeds, first of all, his own later warning in a letter to Lipschitz of 27 July 1876 that "nothing is more dangerous in mathematics than to make existence assumptions without sufficient proof". This refers to the definition of the system of real numbers. Recall that the system of reals is to allow us to pursue all phenomena of the geometric line in a purely arithmetic way. Thus, it has to be defined by means of rational numbers and (the laws for) the arithmetic operations have to be reduced to (those for) the operations on rational numbers. The construction has to be done in such a way that the resulting system has the same kind of continuity or completeness as the geometric line. We will emphasize, on the one hand, the considerations involved in extending the system of rationals to that of the reals and bring out, on the other hand, the new answer to the question that parallels the above for rationals: "Are" the constructed objects, i.e., the cuts, *really* the real numbers? (The central issues are discussed in almost identical ways in *1871/1872* and *1872*.)

Cuts are partitions (A_1, A_2) of the system of rationals with the property that all a_1 in A_1 are less than all a_2 in A_2; they are viewed extensionally: $(A_1, A_2) = (B_1, B_2)$ if and only if A_1 and A_2 have the same members as B_1 and B_2, respectively. If A_2 contains a smallest element a', then the cut (A_1, A_2) is said to have been *engendered* by a'; the fact that not all cuts are engendered by rationals constitutes the *incompleteness* or *discontinuity* of the domain of rationals.[29] Dedekind continues, in the section entitled *Creation of irrational numbers*:

> Thus, whenever we have a cut (A_1, A_2) produced by no rational number, we *create* a new number, an *irrational* number α, which we regard as completely defined by this cut (A_1, A_2); we shall say that the number α corresponds to this cut, or that it produces this cut. From now on, therefore, to every definite cut there corresponds a definite rational or irrational number, and we regard two numbers as *different* or *unequal* if and only if they correspond to essentially different cuts.[30]

The system of real numbers consists thus of all rational numbers (corresponding of course to the cuts engendered by them) together with these newly *created* irrational ones or, to put it in other words, the system of rationals has been extended by these irrational numbers. The crucial point is this: reals are not identified with cuts, but rather "correspond" to cuts; the latter are for Dedekind genuine mathematical objects, and the relations between reals and operations on them are defined in terms of the corresponding cuts.

The ordering between two reals α and β corresponding to the cuts (A_1, A_2) and (B_1, B_2) is defined as follows: $\alpha < \beta$ if and only if $A_1 \subset B_1$ (if, for any rational cut, the rational that engenders the cut is always, say, in the right part of the cut). Addition and multiplication of reals is defined in terms of the corresponding operations for the rationals. Consider two reals α and β that correspond to the cuts (A_1, A_2) and (B_1, B_2); the sum $\alpha+\beta$ corresponds to the cut (C_1, C_2), where C_1 consists of all c that are smaller than a_1+b_1 for some a_1 in A_1 and b_1 in B_1, and C_2 consists of the remaining rational numbers. Multiplication can be defined in a similar way, and it is not difficult to verify the arithmetic laws for a field. Dedekind verifies also the order laws and proves that the system of reals is continuous. The system of reals or, more directly, the system of all cuts has been recognized as a complete ordered field.

It should be noticed that Dedekind uses "creation" here with a different sense than in the early discussions: for one, not individual mathematical objects are created, but rather systems thereof; in addition, the elements of those systems correspond to the elements of an already established system. (In the case of the reals, they correspond to rational cuts.) Perhaps to emphasize this new sense, Dedekind speaks in both *1871/1872* and *1872* of *free* creation.

4.3. *Free Creation*

Dedekind had excellent reasons for not identifying the real numbers with cuts of rationals. He articulated them very clearly in his early correspondence with Lipschitz already in 1876 and, as we mentioned above, in his letter to Weber dated 24 January 1888. The correspondence with Lipschitz was partially stimulated by the preparation of Dedekind's essay *Sur la théorie des nombres entiers algébriques*, published in 1877 in the Bulletin des sciences mathématiques.

Lipschitz had actually suggested that Dedekind be invited to report on his work in algebraic number theory.[31] The resulting attempt by Dedekind to present his work (essentially contained in Supplement X of the second edition of Dirichlet's *Zahlentheorie*) in a new and possibly more accessible way contains in the *Introduction* a long methodological note; it is attached to remarks about Kummer's ideal numbers and his own ideals. In that note he points to *1872* as making even more evident – for the case of introducing the irrational numbers and defining the arithmetic operations on them – the "legitimacy, or rather necessity, of such demands, which must always be imposed with the introduction or creation of new arithmetic elements". He refers here to the demands concerning the precise definition of new mathematical objects in terms of already existing ones and the general definition of operations on them in terms of the given ones. In contemporary language, the structures of pairs and cuts provide models of the axioms for integers, rationals, and reals; the particular elements of these structures are not identified with the respective numbers, but the latter are specifically obtained by an *abstracting* free creation.

If we think of the genetic method as underlying the construction of mathematical objects, systems of which are models of appropriate axiom systems, we can see very clearly how it complements in Dedekind's hands an axiomatic approach. However, an arithmetization of analysis that satisfies Dedekind's methodological demands for creating the irrational numbers has not been achieved yet: for that it is essential to characterize the very basis of the construction, the natural numbers. First steps beyond *Arithmetische Grundlagen* are taken in the manuscript *1872/1878* for *1888* that was written, modified, and extended between 1872 and 1878. At the end of this period Dedekind must have thought about publishing a booklet with the very title of *1888*, as Heinrich Weber writes in a letter of 13 November 1878:[32] "I am awaiting your book *Was sind und was sollen die Zahlen* with great anticipation". In the Introduction to *1888* on page IV, the earlier manuscript is said to contain "all essential basic thoughts of my present essay". Dedekind mentions as the main points the "sharp distinction between the finite and the infinite", the concept of cardinal, the justification of proof by induction and definition by recursion.

The emphasis in the draft is, however, almost exclusively on the proof principle; there are some very brief, almost cryptic hints concerning definition by recursion. From a modern perspective there is so much more to the final essay; for one, the detailed metamathematical considerations and Dedekind's reflections based on them. In the letter to Keferstein they are properly emphasized, and we will discuss them in Section 6, in particular, the existence and uniqueness, up to isomorphism, of simply infinite systems. That will be the background for discussing the free creation of numbers with a more systematically founded perspective.

5. CHAIN OF A SYSTEM

Weber's "great anticipation" was more than justified already in 1878, as Dedekind's reflections had led him to a novel conceptualization of natural numbers within, what he viewed as, a *logical* framework using the fundamental concepts of *system* and *mapping*. Indeed, in the manuscript *1872/1878* Dedekind writes:

> If one accurately tracks what we are doing when we count a set or a number of things, one is necessarily led to the concept of correspondence or mapping.
> The concepts of system, of mapping, which shall be introduced in the following in order to ground the concept of number, cardinal number, remain indispensable for arithmetic even if one wants to assume the concept of cardinal number as being immediately evident ("inner intuition").[33]

This is the basis for the *radical break* with the considerations in *1854* and the description of the positive integers in *1872*, a break that was hinted at by the notational change from the creative act $+1$ to the successor operation φ in the third version of *Arithmetische Grundlagen*. However, the facts one is forced to accept from an informal analysis of number using these new conceptual tools "are still far from being adequate for completely characterizing the nature of the number sequence N;"[34] for that the general notion of the *chain of a system A* is introduced. The specialization of A to the system $\{1\}$ leads to the "complete" characterization of N as a simply infinite system.

5.1. *Mappings (Between 2 and 3)*

We do not mean to discuss mappings between the integers two and three, but rather emphasize the significance of the notion of mapping that emerged in Dedekind's work between the publication of the second and third edition of Dirichlet's *Zahlentheorie* in 1871 and 1879, respectively. This was a fruitful and important period in Dedekind's work on algebraic number theory: he published the essay *Sur la théorie des nombres entiers algébriques* and worked intermittently, but strenuously, on a proper formulation of his *Gedanken über Zahlen*. The broad considerations, which were central for the mathematical and the foundational work, are highlighted in the *announcement* of the third edition and in a footnote to that very work. Indeed, Dedekind refers back to these considerations in (a note to §161 of) the fourth edition of 1894 indicating very clearly, how important those reflections were for him:

> It is stated already in the third edition of the present work (1879, footnote on p. 470) that the entire science of numbers is also based on this intellectual ability to compare a thing a with a thing a', or to relate a to a', or to let a correspond to a', without which no thinking at all is possible. The development of this thought has meanwhile been published in my essay "Was sind und was sollen die Zahlen?" (Braunschweig, 1888); ..."[35]

This remark is attached to a discussion of the general notion of mapping. The evolution of that notion in Dedekind's work is one of the foci of our second paper, but the material from the manuscript *Gedanken über Zahlen* reveals already crucial aspects of this development and its significance.

The manuscript contains three distinct layers.[36] In its initial attempt to characterize natural numbers via *chains*, the first layer uses the notions *mappable*, *corresponding*, and *image*, which match *1871* (Section I of §159 in Supplement X) as well as *1872* in terminology and outlook. In its second attempt, calling a chain now a *group* (sic), the manuscript introduces for the first time in Dedekind's writings the term *mapping* (Abbildung). Dedekind distinguishes without any explanation between *injective* (deutliche) and *non-injective* (undeutliche) mappings. The second layer is the longest and most intricate one, and it alone discusses finite cardinals. The third layer is close to the eventual presentation of this material in *1888* and takes mappings officially as objects of study; it matches the remarks and note in *1879* mentioned above.

5.2. *Thoughts on Numbers*

Let us indicate briefly the *common* arithmetic content. In each layer Dedekind considers a system S and a (n arbitrary) mapping φ from S to S.[37] If φ is injective, the system S is called *infinite* just in case there is a proper subset U of S, such that the system $\varphi(S)$ of images is a subset of U. The other notions are defined relative to S and φ. A subset K of S is called a *chain* if and only if it is closed under φ. A subset B of S is called *dependent on A* if and only if B is a subset of any chain that contains A, and (A) is the system of all things dependent on A. Finally, Dedekind establishes as the central claim that (A) *is a chain*. As a justification for induction one can easily show that, given two subsets A and K of S,

$$\text{If } A \subseteq K \quad \text{and } \varphi(K) \subseteq K, \quad \text{then } (A) \subseteq K.$$

Assume $A \subseteq K$ and $\varphi(K) \subseteq K$, consider an arbitrary a in (A), and distinguish two cases. In the first case a is in A, then – by the assumption $A \subseteq K$ – a is in K. In the second case $\{a\}$ is dependent on A, but not in A, i.e., contained in any chain that contains A. But K is such a chain; thus $\{a\}$ is a subset of K, and a is an element of K. This sequence of steps anticipates that in *1888*, except for the definition of (A) via the dependency relation.

The second layer defines the dependency relation just for elements and calls the system (a) of all elements dependent on a the *sequence of a*. For injective mappings and infinite systems S Dedekind establishes as a theorem that (1) *is an infinite system*. Every element of (1) is called a *number*; proof by induction is justified as above, and the issue of definition by recursion is raised here, briefly. Dedekind notes on the margin:

> The proof of the correctness of the method of proof from n to $n+1$ is correct; in contrast, the proof (completeness) of the definition of concepts by the method from n to $n+1$ is not yet sufficient at this point; the existence (consistent) of the concept remains in doubt. This will become possible only by *injectivity*, by the consideration of the system $[n]$!!!!!! Foundation.[38]

This is a pregnant remark and, together with theorems established on pp. 300–304, points ahead to central issues in *1888*. To support that claim, we have to explain first of all the notation $[n]$. Informally, $[n]$ is the system of all numbers less than or equal to n, for any n in (1); systematically, $[n]$ is defined as the system of numbers not contained in (n'), and it is shown to be finite. (In *1888* the systems $[n]$ are denoted by Z_n.) Dedekind formulates as a theorem that *a system B is infinite, if every system* $[n]$ *can be mapped injectively into B*. He remarks on the margin, "To prove this is circuitous, but possible". (Umständlich, aber möglich zu beweisen.) This is, of course, the central and deep fact used to establish in §14 of *1888* that Dedekind's definition of infinite is equivalent to the standard one.[39] That proof requires definition by recursion and a form of the axiom of choice:[40] to secure generally the existence of a mapping satisfying recursion equations the systems $[n]$ are invoked and the *injectivity* of the mapping φ is needed (Remark 130 of *1888*). All of this seems to be hinted at in the remark quoted above; it is a dramatic step for gaining a proper perspective.

The third layer is a very polished version of the considerations leading up to Theorem 31 that states, (A) *is a chain*. But this time there is a most interesting and important note next to the statement of the theorem: "(A) is the 'smallest' chain that contains the system A". The layer ends with brief remarks on the "direct treatment of the system Z of natural (i.e., whole positive rational) numbers". We quote those in full and mention that Dedekind wrote next to the sentence just quoted "better N than Z:"

Characteristic of the system Z. There is an injective mapping from Z – if T is a part of Z, then the image of T is denoted by T'– which has the following property.

I Z' is a part of Z.
II There is a number (i.e., a thing contained in Z), which is not contained in Z'. This number shall be called "one" and is denoted by 1.
III A number chain (i.e., each part T of Z, whose image T' is a part of T') that contains the number 1 is identical with Z.[41]

This "characteristic" corresponds perfectly to the axiomatic conditions for a simply infinite system in *1888*, i.e., we have here the very first formulation of the so-called Peano Axioms.[42]

5.3. *Axioms for Numbers*

In the systematic analysis of *1888* we use Dedekind's letter to Keferstein, but also his official reply *1890** to Keferstein's review of *1888*. Dedekind makes his methodological considerations much more explicit in these documents than in the essay itself. Indeed, in the letter Dedekind poses these motivating questions:

What are the mutually independent fundamental properties of the sequence N, that is, those properties that are not derivable from one another but from which all others follow? And how should we divest these properties of their specifically arithmetic character so that they are subsumed under more general notions and under activities of the understanding *without* which no thinking is possible at all but *with* which a foundation is provided for the reliability and completeness of proofs and for the formulation of consistent definitions of concepts?[43]

When one poses the problem in this way, Dedekind continues, then one is forced to accept the following facts: the number sequence N is a system of elements or individuals, called *numbers*; the relation between these elements is given by a mapping φ from N to N; φ must be *similar* (ähnlich, this term replaces "deutlich" used in the earlier discussion); the image of N under φ is a proper part of N, and 1 is the only element not in the image. The central methodological problem, Dedekind emphasizes, is the precise characterization of just those individuals that are obtained by iterated application of φ to 1; this is to be achieved in general logical terms, not presupposing arithmetic notions. Before addressing this central problem, Dedekind introduces as above, relative to a system S and an arbitrary mapping φ from S to S, the general concept of a chain. Then he defines directly, using the insight gained in the third layer of *1872/1878*, the *chain* A_0 *of a system* A as the intersection of all chains containing A. A_0 obviously contains A as a subset, is closed under the operation φ, and is minimal among the chains that contain A, i.e., if $A \subseteq K$ and $\varphi(K) \subseteq K$, then $A_0 \subseteq K$. These properties characterize A_0 uniquely. From the minimality of A_0 it is easy to prove a general induction principle in the form:

$$(*) \qquad \text{if } A \subseteq \Sigma \quad \text{and } \varphi(A_0 \cap \Sigma) \subseteq \Sigma, \quad \text{then } A_0 \subseteq \Sigma;$$

Σ denotes the extension of any property E pertaining to the elements of S.

After this preparatory step Dedekind specializes the consideration to the chain N of the system $\{1\}$ for the similar mapping φ, i.e., the *simply infinite system* (N, φ, 1). The essence of this system is given by the axiomatic conditions α, β, γ, and δ of *Erklärung 71* in corresponding order: $\varphi(N) \subseteq N, N = 1_0, 1 \notin \varphi(N)$, and φ is a similar mapping. Condition β expresses in Dedekind's notation that N is the chain $\{1\}_0$ of the system $\{1\}$; it is the basis for the usual induction principle for natural numbers formulated now as follows:

$$(**) \qquad \text{If } \{1\} \subseteq \Sigma \text{ and } \varphi(N \cap \Sigma) \subseteq \Sigma, \text{ then } N \subseteq \Sigma$$

The considerations leading to $(**)$ are completely parallel to those for $(*)$ above. Indeed, reordering conditions α, β, γ, and δ, reformulating them a little, and using $(**)$ as the induction principle yields:

$$1 \in N,$$
$$(\forall n \in N)\ \varphi(n) \in N,$$
$$(\forall n, m \in N)(\varphi(n) = \varphi(m) \Rightarrow n = m),$$
$$(\forall n \in N)\ \varphi(n) \neq 1, \text{and}$$
$$(1 \in \Sigma\ \&\ (\forall n \in N)(n \in \Sigma \Rightarrow \varphi(n) \in \Sigma)) \Rightarrow (\forall n \in N)\ n \in \Sigma.$$

These statements make explicit the principles underlying Dedekind's earlier "characteristic of the system Z" and are mere notational variants of the five axioms for the positive integers formulated in Peano's *1889*.

Hilbert's axiomatization for N in his *1905* also uses these axioms, clearly extracted from Dedekind's characterization of simply infinite systems.[44] Hilbert's syntactic consistency proof in that paper was to guarantee the existence of the "smallest infinite". Thus, his proof was to serve the dual purpose of Dedekind's argument for the existence of a simply infinite system. Already in his *1900a* and *1900b* Hilbert intended to insure the existence of a set, here the set of real numbers, by a "direct" proof of the consistency of an appropriate axiomatic theory. The theory was formulated in the style of Dedekind: one considers a system of objects satisfying certain axiomatic conditions, and the systematic development of the theory makes use of these conditions only. In contrast to Dedekind, Hilbert called a theory consistent if it does not allow to establish in finitely many steps a contradiction; note that this is only a quasi-syntactic specification of consistency, as the steps that are allowed in proofs were not made explicit. We turn our attention now to Dedekind's way of thinking about, and addressing, the issue of existence and consistency; this is the last component in the assembly of a Dedekindian perspective on numbers and the nature of mathematics.

6. ABSTRACT TYPE

The number sequence N is characterized completely as the abstract type of a simply infinite system, Dedekind writes to Keferstein; how is this to be understood? The answer to the question will evolve through a sequence of detailed metamathematical, reflective steps concerning simply infinite systems. The steps are guided by the systematic insights gained in the earlier investigations; thus, Dedekind is concerned with the "possibility of the creation of a simply infinite system" – to use the language of the early axiomatic analysis of number systems reported in Section 3; cf. in particular Section 3.3. We first discuss the existence proof for simply infinite systems and then complement the literal uniqueness of the chain of the system $\{1\}$ by the completely new sense of uniqueness "up to isomorphism". Finally, we describe Dedekind's view of the science of numbers or arithmetic that is based on the metamathematical work.

6.1. *Logical Existence*

A simply infinite system is defined as a triple $(N, \varphi, 1)$ or, in contemporary model-theoretic terminology, as a structure that satisfies the conditions α, β, γ, and δ from Dedekind's *Erklärung* 71. We saw that these conditions correspond to the so-called Peano Axioms. Given the earlier concerns, it is perfectly natural for Dedekind to ask: "Does such a system exist at all in our realm of thoughts?"[45] The affirmative answer to this question is given by a *logical existence proof*, and Dedekind explains to Keferstein that without such a proof "it would remain always doubtful, whether the concept of such a system does not perhaps contain internal contradictions".[46] In his official response *1890** to Keferstein's review article Dedekind asserts more strongly, "as long as such a proof has not been given one may fear that the above definition of the system N contains an internal contradiction, whereby the certainty of arithmetic would be lost".[47] That is the reason, he emphasizes in his letter, why the proofs for theorems 66 and 72 of his essay are necessary.

The crucial considerations are presented in the proof of theorem 66. Theorem 72 just states that every infinite system contains a simply infinite one as a part, and *that* assertion can be established straightforwardly. To establish theorem 66, i.e., the claim that there is an infinite system, Dedekind formulates and proves the claim for a specific system, namely, for his *Gedankenwelt*. Dedekind's *Gedankenwelt* is defined as "the totality S of all things that can be an object of my thinking". For an arbitrary element s of S, the thought s' that "s can be an object of my thinking" is itself an element of S. The operation φ that leads from s to s' is injective, and the set of images S' is a proper part of S, as Dedekind's own self, for example, is in S but not in S'. Thus, S together with φ is indeed an infinite system. The terminology of "thing" and "system" was already introduced at the very beginning of the *1872/1878* manuscript and not only in *1888*: "A *thing*", it says there, "is any object of our thinking; ..." and "A *system* ... S of things is determined, if one can judge of any thing, whether or not it belongs to the system". Dedekind notes there also that such a system of things is treated as a new thing when contrasted with the other things. These remarks can be found in Dugac (1976, 293); we mention them here to make perfectly clear that the use of these notions in the argument above does not locate it close to the actual writing of *1888*.

In *1890** Dedekind reproduces the proof of theorem 66, asserts that he considers it not only as *correct*, but as *rigorously correct* (streng richtig), and explicates it in an informative way without, as he claims, adding anything new. The explication consists in expanding the specification of φ by a parenthetical remark. Instead of considering "the thought s' that ...", Dedekind considers here "the thought s' (expressible in the form of a sentence or judgment) that ...". This seems to indicate directly that Dedekind's thoughts are not to be viewed as psychological ideas. There is also indirect evidence: Frege asserts in his manuscript *Logik* that he uses the word "Gedanke" in an unusual way and remarks that "Dedekind's usage agrees with mine".[48] Such a Fregean understanding is reinforced, when Dedekind continues his explication by claiming that the thought s' can be an object of his thinking. After all, "I may think, e.g., of this thought s', that it is obvious, that it has a subject and a predicate, etc". (ich darf z.B. von diesem Gedanken s' denken, dass er selbstverständlich ist, dass er ein Subjekt und ein Prädikat besitzt u.s.w.). Consequently, the thought s' is an element of S.

In *1888*, Dedekind writes in the footnote to theorem 66, "A similar consideration is found in §13 of Bolzano's *Paradoxien des Unendlichen* (Leipzig 1851)". The similarity of their considerations is particularly striking, when we compare Bolzano's argument with Dedekind's in the explicated form pertaining to thoughts that are expressible in the form of sentences. Bolzano establishes that "the set of sentences and truths in themselves" (die Menge der Sätze und Wahrheiten an sich) is an infinite multiplicity. This is achieved by considering first any truth T whatsoever and then using the construction principle *the proposition A is true* to step from any true proposition A to a distinct new and true proposition. Bolzano concludes that *this* set of all propositions constructed from T "enjoys a multiplicity surpassing every individual integer" and is therefore infinite, according to his definition.[49] (The characterization of "this" set, or of similarly constructed ones, as the chain of $\{T\}$ was for Dedekind according to his letter to Keferstein, "... one of the most difficult points of my analysis and its mastery required lengthy reflection". (p. 100))

Excursion. The need to prove the existence of an infinite system is not even discussed in *1872/1878*. A proof is given in the manuscript from 1887 that precedes the final writing of *1888*, and this seems to be the first appearance in Dedekind's manuscripts or published writings. Through a letter from Cantor to Dedekind dated 7 October 1882 we know that the former sent with his letter also a copy of Bolzano's booklet to Dedekind.[50] These three facts are taken as evidence, for example by Dugac, that Dedekind adapted Bolzano's considerations concerning the objective existence of the infinite. (Cf. Dugac (1976, 81 and 88), but also Sinaceur (1974, 254), Belna (1996, 37, 38 and 54ff) and Ferreirós (1999, 243–246); on p. 243 Ferreirós takes it for granted that Dedekind knew Bolzano's proof when giving his own and "transformed it to suit his different philosophical ideas and his strict definition of infinity".)

We first describe in (almost tedious) detail, where the claim and proof for the existence of an infinite system occur first, namely, in the fourth section of *1887* that is entitled "The finite and infinite". It starts out with a definition.

40. Definition. S is called an infinite system, if there is an injective mapping from S, such that the image of S is a proper part of S; in the opposite case S is called a finite system.[51]

This is followed by the remark that "all hitherto known definitions of the finite and the infinite are completely useless, to be rejected by all means".[52] Next comes a proposition, numbered 41, which states that the union S of the singleton $\{a\}$ and T is finite, if T is finite.[53] As in other manuscripts of Dedekind's, the pages of *1887* are vertically divided in half. The main text is written on one half, whereas the other half is reserved for later additions. On this particular page a number of important additions have been made. Already its first line indicates that the manuscript is still being reorganized in significant ways: Dedekind refers to remarks on a separate page and writes that the "first two propositions of §7 belong here". This is followed by three propositions, numbered 40^x, 40^{xx}, and 40^{xxx}, the last of which claims: "There are infinite systems". Dedekind adds parenthetically, "Remarks on separate page", and mentions there that the following proposition can be added immediately to the *fundamental definition 40*:

Proposition: There are infinite systems; the system S of all those things s (this word understood in the sense given in the introduction) that *can be* objects of my thinking, is infinite (my realm of thoughts).[54]

The proposition is established by a proof of roughly the same character as that given in the sources we discussed already.[55]

In the preface to the second edition of *1888*, Dedekind emphasizes that Cantor and Bolzano had also recognized the property he uses as the definition of an infinite system. However,

... neither of these authors made the attempt to use this property as the definition of the infinite and to establish upon this foundation with rigorous logic the science of numbers. But this is precisely the content of my difficult labor, which in all its essentials I had completed several years before the publication of Cantor's memoir [i.e., Cantor 1878] and at a time when the work of Bolzano was completely unknown to me, even by name.[56]

Whether and how Dedekind was influenced by Bolzano's work in formulating his proof of theorem 66 remains a topic of speculation. The known facts, as we recounted them, allow a different interpretation than that given by Dugac and Ferreirós: the manuscript of 1887 is so different from *1872/1878* that one might conjecture with good reason that Dedekind had other intermediate manuscripts or, at least, additional notes to bridge this remarkable conceptual and mathematical gap. The issue of providing "models" for axioms had been pressing already at that time, as we pointed out in Section 4. Given Dedekind's own remarks concerning the connection with Bolzano's and Cantor's work (indicated in the above quote), we *speculate* that he must have completed one such intermediate manuscript *no later than* 1878. Dedekind's own remarks, quoted above at the end of Section 4, about these early considerations do not shed decisive light on the issue at hand: in his response to Weber's inquiry concerning the status of *Was sind und was sollen die Zahlen?* from 1878 he gives a description that fits the available material in *1872/1878*, and views it as a "rough draft;" in the Introduction to *1888* the earlier material is said to include also the justification of definition by recursion (that is barely hinted at in the folder of *1872/1878*); finally, in his remark concerning Cantor and Bolzano in the preface to the second edition of *1888* we just quoted, he claims to have completed the work "in all its essentials" *several years* before the appearance of Cantor's 1878 paper. *End of Excursion*.

Dedekind himself just points out a "similarity" between Bolzano's and his own considerations. We point to a central dissimilarity and, without further elaboration, to the fact that Dedekind's formulations are dramatically more rigorous.[57] Bolzano bases his considerations concerning the *objective existence* of the infinite implicitly on the existence of the species of integers and explicitly on the existence of the set of sentences and truths in themselves, whereas Dedekind uses only one universal system, his *Gedankenwelt*; a simply infinite system and the natural numbers are obtained from it.

6.2. *Mathematical Uniqueness*

How then are natural numbers obtained in Dedekind's case? Any infinite system whatsoever has as a part a simply infinite one that is unique as a minimal chain (of a chosen element 1), as we observed above. To insist on minimality has the metamathematical reason emphasized by Dedekind in both *1890* and *1890**, namely, that it excludes "intruders;" these intruders are, in modern terminology, non-standard elements. The minimality captures the informal, motivating idea that every element of the chain is obtained by the finite iteration of the operation φ applied to 1.[58] This is also the basis for establishing that simply infinite systems are unique in a novel sense.

Given the analysis via minimal chains, it is most direct to use the general concept of a mapping and to conceive of a bijection ψ between two arbitrary simply infinite systems based on operations φ and θ, respectively. ψ would map the first element of one system to the first element of the other; in addition, the bijection would satisfy the recursion equation $\psi(\varphi(n)) = \theta(\psi(n))$. It is *one thing* to graphically draw such a connection, but *quite another thing* (i) to have the appropriate mathematical (or logical) notions to capture the essence of the situation and (ii) to prove the unique existence of such a structure-preserving mapping. The *one thing* is undoubtedly in everybody's mind, certainly Bolzano's and also Kronecker's, for example, in §1 of his *1887* entitled *Definition des Zahlbegriffs*. The *other thing* is what Dedekind does in §9 of *1888!*

Dedekind isolates the crucial feature in theorem 126, *Satz der Definition durch Induktion*: let (N, φ, 1) be a simply infinite system, let θ be an arbitrary mapping from a system Ω to itself, and let ω be an element of Ω; then there is exactly one mapping ψ from N to Ω that satisfies the conditions

I $\psi(N) \subseteq \Omega$,

II $\psi(1) = \omega$,

III $\psi(\varphi(n)) = \theta(\psi(n))$.[59]

The justification requires subtle metamathematical considerations; i.e., a proof by induction of the existence of approximations to the intended mapping for initial segments of N. The basic idea was used later in axiomatic set theory and extended to transfinite recursion; Gödel used it within formal arithmetic.[60] In the context of his investigation, Dedekind draws two conclusions with the help of theorem 126: on the one hand, all simply infinite systems are similar (theorem 132), and on the other hand, any system that is similar to a simply infinite one is itself simply infinite (theorem 133).

These results, together with some observations in remark 134 to which we will return below, "justify completely" the explication of the concept of number Dedekind provided already in *Erklärung* 73:

If in the consideration of a simply infinite system N ordered by a mapping φ we entirely neglect the special character of the elements, simply retaining their distinguishability and taking into account only the relations in which they are placed to one another by the ordering mapping φ, then these elements are called *natural numbers* or *ordinal numbers* or simply *numbers*, and the base-element 1 is called the *base-number* of the *number-series N*. With reference to this freeing of the elements from every other content (abstraction) we are justified in calling the numbers a free creation of the human mind.[61]

In the earlier manuscript *1887* one finds, after an almost identical remark, a more expanded and explicit formulation concerning the result of the abstraction; Dedekind writes there:

By this abstraction, the originally given elements n of N are turned into new elements $\mathcal{n}$, namely into numbers (and N itself is consequently also turned into a new abstract system $\mathcal{N}$). Thus, one is justified in saying that the numbers owe their existence to an act of free creation of the mind. For our mode of expression, however, it is more convenient to speak of the numbers as of the original elements of the system N and to disregard the transition from N to $\mathcal{N}$, which itself is an injective mapping. Thereby, as one can convince oneself using the theorems regarding definition by recursion, nothing essential is changed, nor is anything obtained surreptitiously in illegitimate ways.[62]

This is *Dedekind abstraction* in its clearest and most direct formulation (and obviously follows the spirit of the remarks concerning the free creation of the real numbers in *1872*). Though these newly created objects are indeed the numbers, there is nevertheless no need to insist on treating *them* as the subject of the science of numbers. After having established as proposition 106 the similarity of all simply infinite systems, Dedekind concludes his deliberations in *1887* under the heading "Creation of the pure natural numbers:"

It follows from the above, that the laws regarding the relations between the numbers are completely independent from the choice of that simply infinite system N, which we called the number sequence, and that they are also independent from the mapping of N that orders N as a simple sequence.[63]

How does that claim follow "from the above?" – In remark 134 of *1888* Dedekind gives a rough argument that can be interpreted as showing that "categoricity" implies "elementary equivalence". The arguments for theorems 132 and 133 make use of the canonical bijection that transforms elements of one simply infinite system into corresponding elements of the other and that can even be claimed to transform, as Dedekind does, the successor operation of one system into that of the other. Thus, if one considers only propositions in which the particular character of the elements is neglected and only notions are used that arise from the successor function in one system, then these propositions have quite general validity for any other simply infinite system.[64] This gives finally the complete justification of the above remarks and allows the proper characterization of the "object of the *science of numbers* or *arithmetic*" as presented in the second half of the long note 73:

> The relations or laws which are derived entirely from the conditions α, β, γ, δ in 71, and therefore, are always the same in all ordered simply infinite systems, whatever names may happen to be given to the individual elements (compare 134), form the next [in *Ewald*, one finds "first" here] object of the *science of numbers* or *arithmetic*.[65]

We begin to explore the meaning of this characterization next.

6.3. *The Science of Numbers*

From the very start, two broad and intimately connected issues were of paramount importance for Dedekind's foundational reflections, namely, finding fundamental concepts and principles, but also using them for the systematic development of a subject. As the reader may recall, in *1854* Dedekind views introducing a concept as formulating an hypothesis concerning the inner nature of a science. In *1888* introducing the concept "simply infinite system" is more than formulating an hypothesis concerning the essential character of number theory: it rather emerges from a deep insight into a capacity of the human mind – "without which no thinking is possible" and with which we have the essential basis for erecting "the entire science of numbers". (That is forcefully expressed in the Preface to the first edition of *1888*, pp. III–IV.) The principle of proof by induction and that of definition by recursion (or induction) can be obtained, and these principles allow the unique characterization of the number sequence – a most significant theoretical insight.

Both principles are also indispensable for a systematic development: proof by induction is the pervasive form of argumentation in number theory, and definition by recursion yields the standard operations like addition, multiplication, and exponentiation. As to the definition principle Dedekind emphasizes in his letter to Keferstein the need "to formulate the *definitions* of operations on numbers consistently for *all* numbers n". This is one aspect of Dedekind's general concern to provide the conceptual tools for the development of arithmetic and to establish their efficacy in recognizing general truths. Let us recall the parallel considerations in *1872*. There, the notion of continuity allows the characterization of the real numbers, the operations on reals are defined via operations on the rationals, and their basic properties are verified. Then Dedekind develops a fundamental part of analysis and establishes, in particular, that the principle of continuity implies the theorem that bounded increasing sequences have a limit. This theorem is actually shown to imply the principle of continuity; thus, we have here – as far as we know – the very first theorem of "reverse mathematics". This equivalence has for Dedekind significant methodological impact; in his letter to Lipschitz dated 27 July 1876 he emphasizes this equivalence, when making the point that his definition of irrational numbers has not created any number, "which was not already grasped more or less clearly in the mind of every mathematician".[66]

The reflections concerning induction and recursion have consequently two fundamental goals: to serve as the methodological frame for Dedekind's answer to the question *Was sind die Zahlen?* and to provide the systematic tools for developing number theory. We saw how they are used to justify the abstractionist move, when the natural numbers were viewed as a "free creation of the human mind;" we also discussed the basic role of induction and recursion in the development of number theory. However, we did not address the question, "What is number theory?" – For Dedekind, as is clearly stated in the second half of note 73, number theory is the "theory" of the sequence N, which is completely characterized as the abstract type of a simply infinite system. Thus, we are facing two central methodological questions, which emerge from the general discussion in remark 134 and note 73:

(i) How are the concepts characterized that arise out of the successor operation? (Begriffe, die aus der Anordnung φ entspringen, remark 134);
(ii) How are the laws obtained that are derived exclusively from the conditions for simply infinite systems? (Gesetze, welche ganz allein aus den Bedingungen α, β, γ, δ in 71 abgeleitet werden, note 73)?

The answer to both of these questions is, "By logic!" i.e., logic is to specify principles of concept formation (or of definition) and principles of proof; these "logical" principles are not explicitly formulated. A satisfactory analysis has to be sufficiently restricted in order to ensure, what Dedekind argued for, namely, that any proposition using only "those" concepts and having been inferred by only "those" proof principles is valid for any simply infinite system. Dedekind had an appropriate abstract understanding of "theory" already in the 1870s; that is clear not only from his contemporaneous mathematical work, but also from more general methodological discussions, for example, in the letters to Lipschitz dated 10 June and 27 July 1876. (We quoted from the second letter already earlier.) In the second letter he claims, for us most interestingly, that the continuity of space is by no means inseparably connected to Euclid's geometry. He proposes to establish the claim by an analysis of the whole system of Euclidean geometry making clear that the continuity principle is not being used. In a parenthetical addition he remarks on a sure, infallible method for such an analysis. This method consists in replacing all "terms of art" (Kunstausdrücke) by arbitrary newly invented and until now meaningless words; he continues:

> ... the system must not collapse [by such a replacement], if it has been constructed correctly, and I claim for example, that my theory of the real numbers withstands this test.[67]

These remarks precede Hilbert's pronouncement on "Tische", "Stühle", and "Bierseidel" by quite a few years! In her brief note to *1872*, Emmy Noether points to the letters Dedekind sent in 1876 to Lipschitz and asserts that they not only contain pertinent remarks on the essay itself but also express, in her view, an "axiomatic standpoint" (axiomatische Auffassung).[68] Dedekind's above remarks are the most explicit and concise expression of this standpoint.

7. CONCLUDING REMARKS

Let us return to the narrower historical context sketched in our Introduction and elaborated in Section 3. Kronecker and Helmholtz share, so we claim there, the "naïve" starting-point with Dedekind, but it is only Dedekind who builds a conceptual framework, in which he can express sharply the (naïve) analysis, carry out fruitful metamathematical investigations, and provide the tools for a systematic treatment of number theory. The underlying distinctive methodological themes and their evolution will be at the center of a second essay entitled *Dedekind's structuralism: mappings and models*. We will argue in particular for what was already indicated above: Dedekind's *Stetigkeit und irrationale Zahlen* is a significant stepping-stone in this development.

The essay is commonly viewed as providing the final step in a *genetic* presentation of the reals via cuts. However, from the perspective of *1888* and the (unpublished) work that contains all its central notions already before 1878, *1872* can be seen as containing a thoroughly axiomatic characterization of the reals as a complete ordered field together with a semantic consistency proof for these axioms; that was observed already in Sieg (1990, 264–265). Dedekind's investigation of the correspondence between the geometric line and the system of all cuts contains the crucial elements of a proof of the categoricity of the axioms. What is missing at this stage of his foundational reflections in *1872* is the *general concept of mapping*. In our second essay we will discuss the emergence of this notion in Dedekind's mathematical and foundational work, as well as the detailed connections with (what Hilbert and Bernays later called) *existential axiomatics* and the *reductive structuralism* that arose from Hilbert's Program, properly understood.

There remains a great deal of important historico-analytical work that can and should be done, in spite of Ferreirós's marvelous book. The latter has opened a larger vista for Dedekind's work and is wonderfully informative in so many different and detailed ways. However, it seems to us to be deeply conflicted about, indeed, sometimes to misjudge, the general character of the foundational essays and manuscripts. Let us mention three important aspects. First of all, there is *no program* of a constructionist sort in *1854* that is then being pursued in Dedekind's later essays, as claimed on pp. 217–218. Secondly, there is *no conflict*, and consequently no choice has to be made, between a genetic and an axiomatic approach for Dedekind. That conflict is frequently emphasized. It underlies the long meta-discussion (on pp. 119–124) where the question is raised, why authors around 1870, including of course Dedekind, pursued the genetic and not the axiomatic approach. In that meta-discussion Ferreirós seeks reasons "for the limitations of thought in a period", but only reveals the limitations of our contemporary perspective. Finally, there is *no supersession* of Dedekind's "deductive method" (described on pp. 246–248) by the axiomatic method of Hilbert's, but the former is rather the very root of the latter. Hilbert's first axiomatic formulations in *Über den Zahlbegriff* and *Grundlagen der Geometrie* are patterned after Dedekind's. Indeed, Hilbert is a logicist in Dedekind's spirit at that point, and it is no accident that, as late as 1917/1918, he was attracted by attempts to provide a logicist foundation of mathematics.[69]

We mention three broad and temporally distinct directions for such historico-analytic work, namely, (i) a thorough-going exploration of the early mathematical and philosophical context of Dedekind's work, in particular, the impact of Gauss, Dirichlet, Riemann, and Herbart, (ii) a detailed examination of the deep interaction between Dedekind's foundational and mathematical work, in particular, the work on algebraic number theory in the 1870s, and (iii) a thorough investigation of Dedekind's influence on Hilbert's mathematical work, in particular, on the *Zahlbericht*.

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NOTES

[*] Herbart, as quoted in Scholz (1982, 437).

[1] In Hilbert (1931, 487): "Im Jahre 1888 machte ich als junger Privatdozent von Königsberg aus eine Rundreise an die deutschen Universitäten. Auf meiner ersten Station, in Berlin, hörte ich in allen mathematischen Kreisen bei jung und alt von der damals eben erschienenen Arbeit Dedekinds *Was sind und was sollen die Zahlen?* sprechen – meist in gegnerischem Sinne. Die Abhandlung ist neben der Untersuchung von Frege der wichtigste erste tiefgreifende Versuch einer Begründung der elementaren Zahlenlehre". On this trip Hilbert visited also Paul du Bois-Reymond who told Hilbert "die dedekindsche Arbeit 'Was sollen Zahlen' sei ihm grässlich" (in Hilbert's report, *Cod.* Ms. 741, 1/5 and also mentioned in Dugac (1976, 203)).

In the preface to the second edition of his *1888*, Dedekind reports: "Die vorliegende Schrift hat bald nach ihrem Erscheinen neben günstigen auch ungünstige Beurteilungen gefunden, ja es sind ihr arge Fehler vorgeworfen. Ich habe mich von der Richtigkeit dieser Vorwürfe nicht überzeugen können und lasse jetzt die seit kurzem vergriffene Schrift, zu deren öffentlicher Verteidigung es mir an Zeit fehlt, ohne jede Änderung wieder abdrucken, indem ich nur folgende Bemerkungen dem ersten Vorwort hinzufüge". The preface was written in August 1893.

[2] Frobenius refers to Dedekind's investigations in Supplement XI of Dirichlet (1894). The letter is found in Dugac (1976, 269). Here is the German text: "Hoffentlich gehen Sie vielfach die Wege von Dedekind, vermeiden aber die gar zu abstrakten Winkel, die er jetzt so gern aufsucht. Seine neueste Auflage enthält so viele Schönheiten, der §173 ist hochgenial, aber seine Permutationen sind zu körperlos, und es ist doch auch unnöthig, die Abstraktion so weit zu treiben".

[3] Ewald (1996, 791). "Aber ich weiß sehr wohl, daß gar mancher in den schattenhaften Gestalten, die ich ihm vorführe, seine Zahlen, die ihn als treue und vertraute Freunde durch das ganze Leben begleitet haben, kaum wiedererkennen mag; er wird durch die lange, der Beschaffenheit unseres Treppenverstandes entsprechende Reihe von einfachen Schlüssen, durch die nüchterne Zergliederung der Gedankenreihen, auf denen die Gesetze der Zahlen beruhen, abgeschreckt und ungeduldig darüber werden, Beweise für Wahrheiten verfolgen zu sollen, die ihm nach seiner vermeintlichen inneren Anschauung von vornherein einleuchtend und gewiß erscheinen". Dedekind expressed such sentiments also in a letter to Klein written on 6 April 1888; the letter is contained in Appendix XXV of Dugac (1976, 188, 189). Even a quite positive review like that by Meyer remarks: "Der Verfasser sieht bei seinen Darlegungen von specifischen mathematischen Kenntnissen völlig ab, er wendet sich demgemäss an jeden Gebildeten. [...] Für unsere Vorstellung allerdings sinken die gemeinhin Zahlen genannten Dinge vermöge der erwähnten Abstractionen zu blossen Schatten herab, dafür sind sie aber auch aller subjectiven Willkür entzogen, und, strengen rein logischen Regeln unterworfen, bieten sie für den Arithmetiker völligen Ersatz für jene populären Zahlen". (Cf. Dugac (1976) on Meyer, pp. 93, 176.)

[4] These manuscripts are analyzed in Section 5; their dating is Dedekind's own.

[5] These observations are made in the first note to the preface of the first edition: "Das Erscheinen dieser Abhandlungen [i.e., the essays by Helmholtz and Kronecker] ist die Veranlassung, die mich bewogen hat, nun auch mit meiner, in mancher Beziehung ähnlichen, aber durch ihre Begründung doch wesentlich verschiedenen Auffassung hervorzutreten, die ich mir seit vielen Jahren und ohne jede Beeinflussung von irgendwelcher Seite gebildet habe".

[6] *As to Ferreirós*, we refer to the discussion, *Dedekind's deductive method*, in section 5.3 of his book *1999*, where it is claimed on p. 247, that "... Dedekind's deductive method seems rather strange, and could be even called anti-axiomatic... The underlying elementary logic [in Dedekind *1888*] – although transparently employed – is not made explicit, and above all arithmetic is understood as requiring no axiom. All of this places Dedekind's contribution in a peculiar historical position, as an intermediate step that would quickly be abandoned (or, if you wish, superseded)". That perspective is taken then in Section 6.3 to judge Dedekind's influence on Hilbert and his school with the central question formulated on p. 246: "... why Dedekind's strong deductivism did not lead to an axiomatic approach". *As to Corry*, we refer to the discussion of Dedekind's and Hilbert's influence "on the rise of the structural approach to algebra" in his book *1996* as expressed, for example, on pp. 170–171; though both Dedekind and Hilbert introduced "a kind of axiomatic analysis when dealing with their algebraic entities" and in their algebraic works "displayed and promoted" structural features, they did so "independently of any adoption of the modern axiomatic approach". For Dedekind, in particular, Corry notes on p. 129 in a section entitled "Dedekind and the structural image of algebra": "Although Dedekind used many of the concepts that were later to become

the hard core of structural algebra, these concepts play very different roles in those of his works in which they appear. Therefore, they cannot be identified with the notion of an algebraic structure". Finally, *as to McCarty*, he argues that the solutions to the "mysteries" in Dedekind's thought are "to be found in the doctrines of Kant's Transcendental Dialectic". (p. 70) These points of difference are taken up also below.

[7] Dedekind (1854, 428). The German text: "Diese Vorlesung hat nicht etwa ... die Einführung einer bestimmten Klasse neuer Funktionen in die Mathematik, sondern vielmehr allgemein die Art und Weise zum Gegenstande, wie in der fortschreitenden Entwicklung dieser Wissenschaft (i.e., der Mathematik) neue Funktionen, oder, wie man ebensowohl sagen kann, neue *Operationen* zu der Kette der bisherigen hinzugefügt werden".

[8] Dedekind (1888, VI). The German text: "... die größten und fruchtbarsten Fortschritte in der Mathematik und anderen Wissenschaften sind vorzugsweise durch die Schöpfung und Einführung neuer Begriffe gemacht, nachdem die häufige Wiederkehr zusammengesetzter Erscheinungen, welche von den alten Begriffen nur mühselig beherrscht werden, dazu gedrängt hat".

[9] Ewald (1996, 756) [4]. The German text – Dedekind (1854, 430) – is as follows: "So zeigt sich wohl, daß die aus irgendeinem Motive eingeführten Begriffe, weil sie anfangs zu beschränkt oder zu weit gefaßt waren, einer Abänderung bedürfen, um ihre Wirksamkeit, ihre Tragweite auf ein größeres Gebiet erstrecken zu können. Dieses Drehen und Wenden der Definitionen, den aufgefundenen Gesetzen oder Wahrheiten zuliebe, in denen sie eine Rolle spielen, bildet die größte Kunst des Systematikers". In the Introduction to the second edition of Dirichlet (1863) he emphasized this general aspect for the particular mathematical work. He presented in the tenth supplement his general theory of ideals in order, as he put it, "to cast, from a higher standpoint, a new light on the main subject of the whole book". In German, "Endlich habe ich in dieses Supplement eine allgemeine Theorie der Ideale aufgenommen, um auf den Hauptgegenstand des ganzen Buches von einem höheren Standpunkte aus ein neues Licht zu werfen;" he continues, "hierbei habe ich mich freilich auf die Darstellung der Grundlagen beschränken müssen, doch hoffe ich, daß das Streben nach charakteristischen Grundbegriffen, welches in anderen Teilen der Mathematik mit so schönem Erfolg gekrönt ist, mir nicht ganz mißglückt sein möge". (Dedekind 1932, 396, 397)

[10] Ewald (1996, 757ff). The German text, Dedekind (1854, 430–431), is as follows: "Die Elementararithmetik geht aus von der Bildung der Ordinal- und Kardinalzahlen; der sukzessive Fortschritt von einem Gliede der Reihe der absoluten ganzen Zahlen zu dem nächstfolgenden ist die erste und einfachste Operation der Arithmetik; auf ihr fußen alle andern. Faßt man die mehrere Male hintereinander wiederholte Ausführung dieser Elementaroperation in einem einzigen Akt zusammen, so gelangt man zum Begriff der Addition. Aus diesem bildet sich auf ähnliche Weise der der Multiplikation, aus diesem der der Potenzierung".

[11] Ewald (1996, 758) [8]. The German text – Dedekind (1854, 431–432) – is as follows: "Es bedarf daher einer besonderen Definition, um auch negative Multiplikatoren zuzulassen, und auf diese Weise die Operation von der anfänglichen Beschränkung zu befreien; eine solche involviert aber eine a priori vollständige Willkürlichkeit, und es würde sich erst später entscheiden, ob denn die so beliebig gewählte Definition der Arithmetik einen wesentlichen Nutzen brächte; und glückte es auch, so könnte man

dies doch immer nur ein zufälliges Erraten, ein glückliches Zutreffen nennen, von welchem eine wissenschaftliche Methode sich frei halten soll".

[12] Notice a most interesting feature: a negative number can be taken to be represented, for the purpose of defining the extended operation, by a pair of positive ones. If the multiplicand *a* were to be represented in the same way, this approach is technically very close to the later one discussed in Section 4.1.

[13] Ewald (1996, 759) [9]. The German text – Dedekind (1854, 434) – is as follows: "Indessen ist wohl zu hoffen, daß man durch beharrliche Anwendung des Grundsatzes, sich auch hier keine Willkürlichkeit zu erlauben, sondern immer durch die gefundenen Gesetze selbst sich weiterleiten zu lassen, zu einem wirklich festen Gebäude der Arithmetik gelangen wird. Bis jetzt ist bekanntlich eine vorwurfsfreie Theorie der imaginären, geschweige denn der neuerdings von Hamilton erdachten Zahlen entweder nicht vorhanden, oder doch wenigstens noch nicht publiziert". To see why Gauss's geometric interpretation of complex numbers did not satisfy Dedekind's purely arithmetic ambitions, it is instructive to read Gauss's defense of the use of complex numbers in his *1831*, in particular, pp. 310–331.

[14] See, for example, the outline for such a course in the winter semester of 1862/1863, published in Dugac (1976) as Appendix IV.

[15] Dauber asks Dedekind in a letter of 20 June 1871, whether Dedekind had come closer to realizing his plans for publishing his theory of continuity, and remarks that Dedekind had written him about such plans a year earlier. The letter is part of Appendix XXVI in Dugac (1976); the remark can be found on p. 192.

[16] Ferreirós reports on p. 220 of his *1999* that Dedekind borrowed in 1857 Hamilton's *Lectures on Quaternions* from the Göttingen Library. Hamilton gives in the *Preface* to his book the definition of complex numbers as pairs of reals. Pairs are viewed there as genuine mathematical objects for which operations can be defined appropriately; see Hamilton (1853, 381–385). "Thus", Ferreirós concludes convincingly, "Dedekind could regard the problem of complex numbers as satisfactorily solved, ..." Indeed, Dedekind uses Hamilton's way later; cf. Section 4.1.

[17] Dedekind uses the concepts *analysis* and *synthesis* here. Dedekind's use illuminates, but is also illuminated by, that of the ancient geometers, in particular, as formulated by Pappus; see, Beaney (2003).

[18] Van Heijenoort (1967, 99). The German text is: "Gewiss nicht in einem Zuge, sondern sie ist eine nach langer Arbeit aufgebaute Synthesis, die sich auf eine vorausgehende Analyse der Reihe der natürlichen Zahlen stützt, so wie diese sich, gewissermassen erfahrungsmässig, unserer Betrachtung darbietet".

[19] Dugac (1976, 293). The German text is included in footnote 33 .

[20] Ewald (1996, 768). The German text – Dedekind (1872, 5–6) – is as follows: "Ich sehe die ganze Arithmetik als eine notwendige oder wenigstens natürliche Folge des einfachsten arithmetischen Aktes, des Zählens, an, und das Zählen selbst ist nichts anderes als die sukzessive Schöpfung der unendlichen Reihe der positiven ganzen Zahlen, in welcher jedes Individuum durch das unmittelbar vorhergehende definiert ist; der einfachste Akt ist der Übergang von einem schon erschaffenen Individuum zu dem darauffolgenden neu zu erschaffenden. Die Kette dieser Zahlen bildet an sich schon ein überaus nützliches Hilfsmittel für den menschlichen Geist, und sie bietet einen unerschöpflichen Reichtum an merkwürdigen Gesetzen dar, zu welchen man durch die Einführung der vier arithmetischen Grundoperationen gelangt".

[21] Ferreirós discusses *Arithmetische Grundlagen* on p. 218 and, more extensively, on pp. 222–224. Our perspectives are different on the dating of the manuscript and on the "rational reconstruction" of the mathematical content and context. Our reasons for differing are presented with the detailed discussion of the manuscript below. On one crucial issue we do agree with Ferreirós, namely, that the introduction of the successor operation in (what we take to be) the third version of the manuscript is of utmost significance and a central result of the informal analysis.

[22] "§1 Schöpfungsakt 1; $1+1=2$; $2+1=3$; $3+1=4$ ... Zahlen (Ordinal). §2 Erklärung der Addition durch $a+(b+1)=(a+b)+1$. Hiernach Folgerungen, der Natur der Sache nach [,] immer durch die vollständige Induktion abzuleiten".

[23] "§1 *Erschaffung der Zahlen*: 1; $1+1=2$; $2+1=3$; $3+1=4$ . . . aus jeder Zahl a wird durch den Act $+1$ die *folgende* Zahl $a+1$ gebildet. – Deshalb Alles durch vollständige Induction. §2 Erklärung der Addition: $a+(b+1)=(a+b)+1$".

[24] That way of proceeding was not uncommon at the time; indeed, Heine pursues a similar route in his *Elemente der Functionenlehre*. Though Heine's is a natural way of proceeding, Dedekind must have found it (and his own approach) quite unsatisfactory at this juncture. Heine answers the general question "*What are numbers?*" not by a conceptual definition, but rather by taking a purely formal standpoint (acerbically criticized by Frege): "In the definition [of numbers] I adopt the purely formal standpoint, *by calling certain tangible marks numbers*, such that the existence of these numbers is not in question". Dedekind received Heine's paper, when working on the draft of *1872*. Heine describes his way of introducing the negative numbers on pp. 173–174 of his essay. As to possible precedents of Dedekind's way of proceeding cf. Ferreirós (1999, 219), note 1.

[25] In Dugac (1976, 205). "So wie die negativen und gebrochenen rationalen Zahlen durch eine freie Schöpfung hergestellt, und wie die Gesetze der Rechnungen mit diesen Zahlen auf die Gesetze der Rechnungen mit ganzen positiven Zahlen zurückgeführt werden (so *sollte* es wenigstens geschehen), ebenso müssen auch die irrationalen Zahlen durch die rationalen Zahlen definiert werden". The emphasis of "sollte" is Dedekind's.

[26] Ewald (1996, 771). The German text is: "So wie die negativen und gebrochenen rationalen Zahlen durch eine freie Schöpfung hergestellt, und wie die Gesetze der Rechnungen mit diesen Zahlen auf die Gesetze der Rechnungen mit ganzen positiven Zahlen zurückgeführt werden *müssen und können*, ebenso hat man dahin zu streben, daß auch die irrationalen Zahlen durch die rationalen Zahlen allein vollständig definiert werden".

[27] To be more precise, we conjecture that the first manuscript was written in 1872, whereas the second one was written much later, but that its essential content goes back to 1872. (The evidence for the conjecture that the second manuscript was written later is quite direct: one part of the detailed calculations is written on the back of a receipt for a journal subscription – from 1907.)

[28] The German text on p. 4 of Cod. Ms. Dedekind III, 4, I, is this: "Man erschaffe aus der Reihe N der natürlichen Zahlen a ein System P, welches außer den Elementen a noch ein Element 0, und zu jedem a ein entsprechendes Element a^* enthält, mit der Festlegung, daß alle diese Elemente in P von einander *verschieden* sind (leicht genauer auszudrücken; über die *Möglichkeit* einer solchen Schöpfung weiter unten)".

[29] There is a simple issue of whether the partition (A', B') that is exactly like (A, B) except that b' is no longer the smallest element of B but the largest element in A' should also be a cut or not; Dedekind discusses these matters in *1871/1872* on p. 11, i.e., on p. 207 in Dugac (1976). For his own presentation, he decides, to consider such cuts as not *essentially different*.

[30] Ewald (1996, 773). "Jedesmal nun, wenn ein Schnitt (A_1, A_2) vorliegt, welcher durch keine rationale Zahl hervorgebracht wird, so *erschaffen* wir eine neue, eine *irrationale* Zahl α, welche wir als durch diesen Schnitt (A_1, A_2) vollständig definiert ansehen; wir werden sagen, daß die Zahl α diesem Schnitt entspricht, oder daß sie diesen Schnitt hervorbringt. Es entspricht also von jetzt ab jedem bestimmten Schnitt eine und nur eine rationale oder irrationale Zahl und wir sehen zwei Zahlen stets und nur dann als *verschieden* oder *ungleich* an, wenn sie wesentlich verschiedenen Schnitten entsprechen".

[31] Cf. the letter from Lipschitz to Dedekind dated 11 March 1876; in Lipschitz (1986, 47–48).

[32] The letter is found in Appendix L of Dugac (1976, 272). In German the remark is: "Deinem Buch *Was sind und was sollen die Zahlen* sehe ich mit grosser Spannung entgegen". Dedekind responded on 19 November 1878, saying: "Du fragst auch nach meiner Untersuchung über den Uranfang der Arithmetik: "Was sind und was sollen die Zahlen?" Sie ruht und ich zweifle, ob ich sie je publiciren werde; sie ist auch nur in rohem Entwurfe aufgeschrieben, mit dem Motto: "Was beweisbar ist, soll in der Wissenschaft nicht ohne Beweis geglaubt werden". Die Hauptsache ist die Unterscheidung des Zählbaren vom Unzählbaren, und der Begriff der Anzahl, und die Begründung der sog. vollständigen Induction". Dedekind (1932, 486).

[33] From *1872/1878*, printed in Dugac (1976, 293): "Verfolgt man genau, was wir beim Abzählen der Menge oder Anzahl von Dingen thun, so wird man nothwendig auf den Begriff der Correspondenz oder Abbildung geführt.

Die Begriffe des Systems, der Abbildung, welche im Folgenden eingeführt werden, um den Begriff der Zahl, der Anzahl zu begründen, bleiben auch dann für die Arithmetik unentbehrlich, selbst wenn man den Begriff der Anzahl als unmittelbar evident ("innere Anschauung") voraussetzen wollte".

[34] Dedekind in his letter to Keferstein, in van Heijenoort (1967, 100). The German text is: "Aber ich habe in meiner Entgegnung … gezeigt, daß diese Tatsachen noch lange nicht ausreichen, um das Wesen der Zahlenreihe N vollständig zu erfassen".

[35] "Schon in der dritten Auflage dieses Werkes (1879, Anmerkung auf S. 470) ist ausgesprochen, daß auf dieser Fähigkeit des Geistes, ein Ding a mit einem Ding a' zu vergleichen, oder a auf a' zu beziehen, oder dem a ein a' entsprechen zu lassen, ohne welche überhaupt kein Denken möglich ist, auch die gesamte Wissenschaft der Zahlen beruht. Die Durchführung dieses Gedankens ist seitdem veröffentlicht in meiner Schrift "Was sind und was sollen die Zahlen?" (Braunschweig 1888); …"

[36] The first layer extends in Dugac (1978) from p. 293 to p. 297, the second from p. 297 to p. 304, and the third from p. 304 to p. 309. The order of the layers reflects, quite clearly, the temporal evolution of Dedekind's ideas, with only one exception: much of the material in the right-hand columns on pp. 293–294 must have been added later. In particular, we conjecture that the remarks quoted above from p. 293 of the manuscript (at the very beginning of this part of our paper) are from a later date; they fit systematically best with the beginning of the third layer. The material

on p. 294 uses notations that are introduced and explained only on p. 308, respectively on p. 301.

[37] It should be emphasized that the (in our view, original part of the) first layer does not have the explicit notation φ for a mapping; §159 of *1871* does, but only for substitutions, i.e., isomorphisms between fields.

[38] Dugac (1976, 300): "Der Beweis der Richtigkeit der Beweismethode von n auf $n+1$ ist richtig; dagegen ist der Beweis (Vollständigkeit) der Begriffserklärung durch die Methode von n auf $n+1$ an dieser Stelle noch nicht genügend; die Existenz (widerspruchsfrei) des Begriffs bleibt zweifelhaft. Dies wird erst möglich durch die *Deutlichkeit*, durch die Betrachtung des Systems [n]!!!!!! Fundament".

[39] Such a standard definition is given, for example, in Bolzano's *Paradoxien des Unendlichen*, Sections 8–9.

[40] That is now well-known and was first established in Tarski (1924); additional details are found in Belna (1996) on p. 41. As a matter of historical record, Zermelo remarked already on the use of the axiom of choice in Dedekind's proof on p. 188 of Zermelo (1908).

[41] "*Charakteristik des Systems Z*. Es giebt eine deutliche Abbildung von Z - ist T ein Theil von Z, so soll das Bild von T mit T' bezeichnet werden -, welche folgende Eigenschaft besitzt.

I. Z' ist Theil von Z.
II. Es giebt eine Zahl (d.h. ein in Z enthaltenes Ding), welche nicht in Z' enthalten ist. Diese Zahl soll "Eins" heissen und mit 1 bezeichnet werden.
III. Eine Zahlkette (d.h. jeder Theil T von Z, dessen Bild T' ein Theil von T ist), welche die Zahl 1 enthält, ist identisch mit Z".

[42] Peano mentions in the Introduction of his *1889*: "In this paper I have used the research of others". In particular, he states later in the paragraph that begins with the sentence just quoted, "Also quite useful to me was the recent work by Dedekind, *Was sind und was sollen die Zahlen* (Braunschweig 1888), in which questions pertaining to the foundations of numbers are acutely examined". (p. 103). Belna (1996), on p. 60, refers to a text from 1891, in which "Peano recognizes that his axioms 'are due to Dedekind' and drawn from #71 of the latter's book". Stein remarks in his *2000a* that "Giuseppe Peano directly borrowed his axioms for arithmetic" from Dedekind's characterization of the system of natural numbers as a simply infinite system. Peirce made priority claims at a number of occasions; they are discussed very well, and accorded their proper place, in Belna (1996) on pp. 57–59. It is quite clear from the above discussion that Dedekind gives an analysis of natural numbers in *1872/1878* that culminates in their axiomatic characterization. However, the further claim – as found in Belna (1996) on p. 58 and Stein (2000a) – that there is no essential difference (except by the absence of the theorem concerning the existence of infinite systems) between the *1872/1878* manuscript and *1888* is not correct; for example, none of the metamathematical results and broader conceptual reflections discussed in Section 6 are contained in *1872/1878*.

[43] Van Heijenoort (1967, 99–100), except for a correction in the very last sentence, where "formulation of consistent definitions of concepts" replaces "construction of consistent notions and definitions". The German text, also reprinted in Sinaceur (1974, 272), is as follows: "Welches sind die von einander unabhängigen Grundeigenschaften dieser Reihe N, d.h. diejenigen Eigenschaften, welche sich nicht aus

einander ableiten lassen, aus denen aber alle anderen folgen? Und wie muss man diese Eigenschaften ihres spezifisch arithmetischen Characters entkleiden, der Art, dass sie sich allgemeinen Begriffen und solchen Tätigkeiten des Verstandes unterordnen, *ohne* welche überhaupt kein Denken möglich ist, *mit* welchen aber auch die Grundlage gegeben ist für die Sicherheit und Vollständigkeit der Beweise, wie für die Bildung widerspuchsfreier Begriffs-Erklärungen?"

[44] For details, see Sieg (2002, 366–371). Hilbert does not formulate the induction principle; he just claims that it *can* be formulated in a way that is suitable for his investigations.

[45] The German text is: "Existiert überhaupt ein solches System in unserer Gedankenwelt?" – In van Heijenoort (1967) "Gedankenwelt" is misleadingly translated as "realm of ideas".

[46] The German text is: "Ohne den logischen Existenz-Beweis würde es immer zweifelhaft bleiben, ob nicht der Begriff eines solchen Systems vielleicht innere Widersprüche enthält"

[47] Sinaceur (1974, 266). The German text is: "... so lange ein solcher Beweis nicht geliefert ist, darf man befürchten, dass die obige Definition des Systems N einen inneren Widerspruch enthält, womit dann die Gewissheit der Arithmetik hinfällig würde".

[48] Quoted from p. 138 of Frege (1969); cf. *ibid.* pp. 147–148, where Frege analyzes Dedekind's proof, approvingly. – McCarty asserts in his *1995* that Section 66 distinguishes itself "as the most blatantly psychologistic". To support this claim in note 5, p. 93, and also to bolster his contention of a strong connection between Kant and Dedekind on p. 71, McCarty relies on the mistranslation of "Gedankenwelt" as "realm of ideas" in van Heijenoort (1967). McCarty writes on p. 71: "... we will find the mathematical objects of Dedekind among the pure ideas of Kant. Dedekind did, after all, write to Keferstein that he must locate the infinite system of natural numbers 'in the realm of our ideas'."

[49] Bolzano (1851, 258). Bolzano's definition, given on p. 254, is as follows: "... I propose the name *infinite multitude* for one so constituted that every single finite multitude represents only a part of it". Note that Bolzano uses finite multitude and whole number synonymously; see also note 57.

[50] Dugac (1976, 256). Cantor characterizes Bolzano's booklet as "a peculiar little work" (ein merkwürdiges Werkchen) of which he happened to have a second copy.

[51] Dedekind (1887). "40. Erklärung: S heißt ein unendliches System, wenn es eine derartige deutliche Abbildung von S gibt, daß das Bild von S ein echter Teil von S ist; im entgegengesetzten Fall heißt S ein endliches System". (The underlining is Dedekind's.)

[52] Dedekind (1887). "Anmerkung: alle bisher bekannten Definitionen des Endlichen und Unendlichen sind gänzlich unbrauchbar, durchaus zu verwerfen".

[53] Dedekind (1887). "41. Satz: Ist $S = M(a,T)$, wo a ein Element von S, und T ein endliches System bedeutet, so ist auch S ein endliches System". M is the union operation. This is essentially Theorem 70 in *1888*.

[54] Dedekind (1887). "*Satz*. Es giebt unendliche Systeme; das System S aller derjenigen Dinge s (dieses Wort in dem in der Einleitung angegebenen Sinne verstanden), welche Gegenstand meines Denkens *sein können*, ist unendlich (meine Gedankenwelt)".

[55] More will be said on the details of the reorganization in Section 5 of *1887* at another occasion

[56] Ewald (1996, 796). The German text – in Dedekind (1888, IX–X) – is as follows: "... keiner der genannten Schriftsteller hat den Versuch gemacht, diese Eigenschaft zur Definition des Unendlichen zu erheben und auf dieser Grundlage die Wissenschaft von den Zahlen streng logisch aufzubauen, und gerade hierin besteht der Inhalt meiner mühsamen Arbeit, die ich in allem Wesentlichen schon mehrere Jahre vor dem Erscheinen der Abhandlung von G. Cantor [i.e., Cantor 1878] und zu einer Zeit vollendet hatte, als mir das Werk von Bolzano selbst dem Namen nach gänzlich unbekannt war".

[57] We discuss how Dedekind obtains natural numbers below. Compare that approach to Bolzano's quick step in §8, where – after describing the formation of series that start from a particular individual of a species A and proceed by adjoining a fresh individual from that species – says: "Such multitudes I call *finite* [endlich] or *countable* [zählbar], or quite boldly: *numbers*; and more specifically: *whole* numbers – under which the first term shall also be comprised". Here and in §13, where Bolzano establishes the existence of an infinite set, the proper general ("logical") characterization of the set of objects that are obtained from an initial one via some successor operation is completely missing in Bolzano.

[58] Cf. Sinaceur (1974, 268)

[59] Dedekind points out on p. 27 of *1888* what is obvious, namely, that condition I is a consequence of II and III; he only includes it on account of greater clarity (*Deutlichkeit*).

[60] In contrast to induction, the recursion principle is not correct for arbitrary chains; that is discussed in *Bemerkung* 130 of *1888*.

[61] Ewald (1996, 809). The German text is: "Wenn man bei der Betrachtung eines einfach unendlichen, durch eine Abbildung φ geordneten Systems N von der besonderen Beschaffenheit der Elemente gänzlich absieht, lediglich ihre Unterscheidbarkeit festhält und nur die Beziehungen auffaßt, in die sie durch die ordnende Abbildung φ zueinander gesetzt sind, so heißen diese Elemente *natürliche Zahlen* oder *Ordinalzahlen* oder auch schlechthin *Zahlen*, und das Grundelement 1 heißt die *Grundzahl* der *Zahlenreihe* N. In Rücksicht auf diese Befreiung der Elemente von jedem anderen Inhalt (Abstraktion) kann man die Zahlen mit Recht eine freie Schöpfung des menschlichen Geistes nennen". – The resonance with the remarks concerning the real numbers in *1872* is not accidental, as Dedekind makes quite clear in his letter to Weber of 24 January 1888; the letter is found in Dedekind (1932, 488–490).

[62] The German text is found at the beginning of Section 5 of *1887*: "Da durch diese Abstraction die ursprünglich vorliegenden Elemente n von N (und folglich auch N selbst in ein neues abstraktes System $\mathcal{N}$) in neue Elemente $\mathcal{n}$, nämlich in Zahlen umgewandelt sind, so kann man mit Recht sagen, daß die Zahlen ihr Dasein einem freien Schöpfungsacte des Geistes verdanken. Für die Ausdrucksweise ist es aber bequemer, von den Zahlen wie von den ursprünglichen Elementen des Systems N zu sprechen, und den Übergang von N zu $\mathcal{N}$, welcher selbst eine deutliche Abbildung ist, außer Acht zu lassen, wodurch, wie man sich mit Hilfe der Sätze über Definition durch Recursion ... überzeugt, nichts Wesentliches geändert, auch Nichts auf unerlaubte Weise erschlichen wird".

[63] This is the full text of section 107 in *1887*; the German is: "Aus dem Vorhergehenden ergiebt sich, daß die Gesetze über die Beziehungen zwischen den Zahlen gänzlich unabhängig von der Wahl desjenigen einfach unendlichen Systems *N* sind, welches wir die Zahlenreihe genannt haben, sowie auch unabhängig von der Abbildung von *N*, durch welche *N* als einfache Reihe geordnet ist".
[64] This is nothing but a paraphrase of Dedekind's considerations in #134 of *1888*.
[65] Ewald (1996, 809). The German text from Dedekind (1888) is: "Die Beziehungen oder Gesetze, welche ganz allein aus den Bedingungen α, β, γ, δ in 71 abgeleitet werden und deshalb in allen geordneten einfach unendlichen Systemen immer dieselben sind, wie auch die den einzelnen Elementen zufällig gegebenen Namen lauten mögen (vgl. 134), bilden den nächsten Gegenstand der *Wissenschaft von den Zahlen* oder der *Arithmetik*".
[66] The full German text in Dedekind (1932, 475), is as follows: "Ebenso wenig habe ich gemeint, durch meine Definition der irrationalen Zahlen irgend eine Zahl erschaffen zu haben, die nicht vorher schon in dem Geiste eines jeden Mathematikers mehr oder weniger deutlich aufgefaßt war; dies geht aus meiner ausdrücklichen Erklärung (S. 10 und 30) hervor, daß die durch meine Definition der irrationalen Zahlen erreichte Vollständigkeit oder Stetigkeit (*A*) des reellen Zahlengebietes wesentlich äquivalent ist mit dem von allen Mathematikern anerkannten und benutzten Satze (*B*): 'Wächst eine Größe beständig, aber nicht über alle Grenzen, so nähert sie sich einem Grenzwerth' ".
[67] The German text of the whole passage, on p. 479 of Dedekind (1932), is: "... eine untrügliche Methode einer solchen Analyse besteht für mich darin, alle Kunstausdrücke durch beliebige neu erfundene (bisher sinnlose) Worte zu ersetzen, das Gebäude darf, wenn es richtig konstruiert ist, dadurch nicht einstürzen, und ich behaupte z.B., daß meine Theorie der reellen Zahlen diese Probe aushält."
[68] In Dedekind (1932, 334).
[69] See Sieg (1999) and Mancosu (1999).

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