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Representative Functions, Variational Convergence and Almost Convexity

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Abstract
We develop a new epi-convergence based on the use of *bounded* convergent nets on the product topology of the strong topology on the primal space and weak star topology on the dual space of a general real Banach space. We study the propagation of the associated variational convergences through conjugation of convex functions defined on this product space. These results are then applied to the problem of construction of a bigger-conjugate representative function for the recession operator associated with a maximal monotone operator on this real Banach space. This is then used to study the relationship between the recession operator of a maximal monotone operator and the normal–cone operator associated with the closed, convex hull of the domain of that monotone operator. This allows us to show that the strong closure of the domain of any maximal monotone operator is convex in a general real Banach space.

Keywords Maximal monotone operators · Representative functions · Almost convexity · Recession operators

Mathematics Subject Classification 47H05 · 46N10 · 47H04 · 49J53

Introduction

Monotone operators have attracted the attention of researchers for many decades due to their important place in the theory of functional analysis and optimisation [2, 26, 30] and [3]. In [18, 21] Martínez-Legaz, Svaiter and Penot pioneered the use of monotone operator theory using representative functions, a tool which has come to be indispensable for the study of this topic. Representative functions are proper convex functions on $X \times X^*$ that characterise their associated monotone operator as the set of points of coincidence with the duality pairing. Indeed for any representative function this contact set is always a monotone set and so each representative function represents a given monotone operator. The notion of a representative function was introduced by Fitzpatrick in [14] where he gives an explicit

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formula for the minimal such function which is now called the *Fitzpatrick representative function* which may be used to represent any maximal monotone operator.

When one constructs a monotone operator from other maximal monotone operators the question arises as to how to obtain a representative function for the resultant operator, based on the knowledge of the Fitzpatrick function of the constituent maximal monotone operators. There have been many studies that have covered this issue from the point-of-view of certain binary operations of convex analysis [2, 30, 33] (and other references contained therein) but we wish, here, to single out constructions where we have a family of monotone operators and take a set-valued limit in the construction [15, 16, 19, 20, 22–24, 27]. The content of the last list of publications is much more varied in the nature of the convergences that are used in these constructions. Moreover, the spaces on which these constructions work are often affected by the convergence used, and the desired interaction of these convergences with conjugation (a necessary tool in the use of representative functions and convex analysis to study monotonicity). This has often resulted at least in one of two restrictive assumptions. The convergence is defined via a strong metric characterization (and possibly other additional assumptions on the family) or the space is assumed to be reflexive where conjugation is characterised in a fashion consistent with imposition of strong topologies on the primal space $X \times X^*$ and weak topologies on the dual space $(X \times X^*)^{*\dagger} = X \times X^*$ (the conjugate transpose operation).

When X is reflexive, we conveniently find that the closures of a convex set with respect to all combinations of weak and strong product topologies that are possible for a convex set $C \subseteq X \times X^*$, all coincide due to the coincidence of the weak and weak* topologies and coincidence of the strong and weak continuous linear forms. Outside of a reflexive space this happy situation is completely missing and we must face the issue of compatibility of convergences and duality (and hence conjugacy) directly. This endeavour constitutes part of the study undertaken in this paper.

In this paper we also continue the study the recession operator [5, 13, 25] associated with any monotone operator T. We will focus on a construction of a bigger–conjugate representative function for the recession operator of a maximal monotone operator T.

Definition 1 Let $T : X \rightrightarrows X^*$ be an operator. The recession operator, $\operatorname{rec} T : X \rightrightarrows X^*$ (see [5] for a sequential version)

$$(\operatorname{rec} T)(z) := \{z^* \in X^* \mid \exists t_\alpha \to 0^+, \left(z_\alpha, z_\alpha^*\right) \in T \text{ such that } \left(z_\alpha, t_\alpha z_\alpha^*\right) \to^{s \times bd\mathrm{w}^*} \left(z, z^*\right)\},$$

where $z_\alpha \to^s z$ denote strong (norm) convergence in X along the net and $t_\alpha z_\alpha^* \to^{\mathrm{w}^*} z^*$ denotes weak* convergence in the dual space X^* and $\left(z_\alpha, t_\alpha z_\alpha^*\right) \to^{s \times bd\mathrm{w}^*} (z, z^*)$ denotes this joint convergence with norm bounded nets.

One of the reason for interest in this operator is that it provides a natural connection between the domain of the original maximal monotone operator T and its strong closure in that it is shown in [13, Lemma 11] that $\operatorname{dom}(\operatorname{rec} T) = \overline{\operatorname{dom}} T$.

In our study we single out a particular subclass of representative functions for study, those whose conjugate–transpose are pointwise larger that the original representative function. This class of *bigger–conjugate representative functions* interact with closure operations in interesting ways [13, Theorem 8]. In this paper we show how one may construct bigger–conjugate representative functions from the Fitzpatrick function of the original maximal monotone operator T that represents the recession monotone operator $\operatorname{rec} T$, in the sense that the set-of-contact with the duality product (the set which it represents) contains the graph of $\operatorname{rec} T$. Indeed the monotone operator it represents is $N_{\overline{\operatorname{co}} \operatorname{dom} T}$, where $\overline{\operatorname{co}} \operatorname{dom} T$ denotes the (strong) convex closure of the domain of T. Indeed whenever $\operatorname{rec} T$ is maximal this result implies $\overline{\operatorname{co}} \operatorname{dom} T = \overline{\operatorname{dom}} T$. Convexity of the closure of the domain, for a monotone operator, is here referred to as the ***almost convexity property.*** We note that the maximality of rec T for maximal operators T has already been shown to be true in reflexive spaces in [11], providing a another proof of the almost–convexity property in this context. The almost–convexity problem is important for two reasons: 1) its resolution would be helpful as a tool to aid studies of the sum theorem (where domain assumptions can be essential); 2) a counterexample to almost–convexity for a maximal monotone operator is also a counterexample to the sum theorem (it is well known that if the sum theorem holds in a given Banach space, then on this space maximal monotone operators possess the almost–convexity property). We provide necessary and sufficient condition for almost–convexity similar to the necessary conditions used in [30] and the related works of [6] and these conditions also generalise the recent result of [32] (which only applied to bounded domains). We go on to establish the almost–convexity for maximal monotone operators on any real Banach space, establishing a long-held conjecture.

The main obstacle to the program revolves around the study of the variational convergence of convex functions on $X \times X^*$ with respect to an epi-convergence based on bounded $s \times \mathrm{w}^*$-convergent nets. This being a convergence that is not (in general) induced by a topology, we need to undertake a study of its interaction with conjugation (based on $X \times X^*$ endowed with the $s \times \mathrm{w}^*$ topology paired with $X^* \times X$, endowed with the $\mathrm{w}^* \times s$ topology). This requires the building of a theory of epi-convergence based on this new convergence notion. The primary target is the development of some result on the propagation of these variational convergences through conjugation. This allows us to demonstrate that the representative function we construct is indeed bigger–conjugate and also allows us to obtain an explicit formula for it.

The paper is organised as follows: basic definitions are given in Sect. 1, in Sect. 2 we begin our discussion of the "closure operation" induced by the convergence of bounded strong×weak* convergent nets on $X \times X^*$. In Sect. 3 we define the variational $s \times bd\mathrm{w}^*$-convergence concept and consider the problem of characterising the convergence for monotone families of convex functions. Section 4 is devoted to the study of the propagation of the conjugation operation through $s \times bd\mathrm{w}^*$-epiconvergence of variational convergent families. In Sect. 5 we summarise some tools we use from monotone operator theory. In Sect. 6 we carry the construction of the bigger–conjugate representative function discussed above. Finally in Sects. 7 and 8 we use these tools to study the almost convexity property for maximal monotone operators. In the final Sect. 8 we provide a proof of almost–convexity.

1 Preliminaries

We denote by X a real Banach space and X^* is its topological dual, paired via the duality product $\langle x, x^* \rangle : X \times X^* \to \mathbb{R}$. In this and the papers [11, 13] all topological closures of set in $X \times X^*$ are with respect to the $s \times \mathrm{w}^*$ topology so as to respect the basic duality relationships for conjugation on $X \times X^*$ paired with $X^* \times X$, with the latter endowed with the $\mathrm{w}^* \times s$ topology. The interior of a set $C \subseteq X$ is denoted by $\operatorname{int} C$ and its (strong) closure by $\overline{C}$. The convex hull of a set $T \subseteq X \times X^*$ (which is often identified with the graph of the associated operator $T : X \rightrightarrows X^*$ taking $x \in X$ to $T(x) \subseteq X^*$) will be denoted by $\operatorname{co} T$ and the convex $(s \times \mathrm{w}^*)$-closure by $\overline{\operatorname{co}}\, T$. The complement of set T (in the ambient space) will be denoted T^c. The indicator function $\delta_C(x)$ of a set $C \subseteq X$ takes the value 0 for $x \in C$ and $+\infty$ otherwise. Denote by $\dagger : (x^*, x) \leftrightarrow (x, x^*)$ the transpose operator. We denote both the $s \times \mathrm{w}^*$ (resp. $\mathrm{w}^* \times s$)- closed ball of radius $K > 0$ by

$$\overline{B_K}(0) := \left\{ \left(x, x^*\right) \text{ (resp. } \left(x^*, x\right)) \mid \max\{\|x\|, \left\|x^*\right\|_*\} \le K \right\} \subseteq X \times X^* \text{ (resp.} \subseteq X^* \times X),$$

and by $B_K(0)$ the corresponding open ball (this is to avoid cumbersome notation like $\overline{B_K}^{\dagger}(0) \subseteq X^* \times X$). By $PC(X \times X^*)$ we denote the proper convex functions $f : X \times X^* \to \mathbb{R}_{+\infty} := \mathbb{R} \cup \{+\infty\}$. Denote by $\Gamma(X \times X^*)$ the set of all ($\mathrm{w}^* \times s$)-lower–semicontinuous, proper convex functions. When going from sets in $X \times X^*$ to ones in $X \times X^* \times \mathbb{R}$ (i.e. epi f) we may use the norm $\|(x, x^*, \alpha)\| = \max\{\|(x, x^*)\|, |\alpha|\}$ i.e. when dealing with epi-graphs we will also use the box norm to extend to the extra single dimension. All closures can be interpreted according to its context, strong in X and w^* in X^*, which closure or ball will be clear from the context. Noting that convergent weak* *sequences* are necessarily bounded, a slight generalisation on this type of limit involves bounded $s \times \mathrm{w}^*$–convergent nets. This is a *convergence* notion and is not directly associated with a topological convergence (differing in nature from the *bounded–weak* topology [17]* denoted by bw*).

We can embed a convex set $C \subseteq X \times X^*$ into the space $X^{**} \times X^*$ in the usual isometric sense that one usually considers $X \subseteq X^{**}$. When we do this with epigraphs of functions $f : X \times X^* \to \mathbb{R}_{+\infty}$: we denote the resulting function by $\widehat{f} : X^{**} \times X^* \to \mathbb{R}_{+\infty}$. We also denote the conjugation with respect to pairing $\sigma_{\boldsymbol{w}}(X \times X^*)$ with $\sigma_{\boldsymbol{w}^*}\left((X \times X^*)^*\right)$ by $f \mapsto f^{\star}$. Note that $\widehat{f}^{\star}$ acts on the space $(X \times X^*)^* = X^* \times X^{**}$. Denote the transpose operator $\dagger : (x^*, x) \leftrightarrow (x, x^*)$ and the transpose conjugate of f by

$$f^{*\dagger}(x, x^*) := f^*(x^*, x) = \sup_{(z,z^*) \in X \times X^*} \left\{\langle (x, x^*), (z, z^*) \rangle - f(z, z^*)\right\} \tag{1}$$

The conjugate $f^{*\dagger}$ in the sense of (1) and the traditional conjugate $\widehat{f}^{\star\dagger}$ are compatible in the sense that

$$(\widehat{f}^{\star\dagger})|_{X \times X^*} \equiv f^{*\dagger} \quad \text{on } X \times X^* \quad \text{i.e. epi}\, \widehat{f}^{\star\dagger} \cap \left(X \times X^* \times \mathbb{R}\right) = \text{epi}\, f^{*\dagger}. \tag{2}$$

A *representative function* for a monotone operator T is a convex function $f : X \times X^* \to \mathbb{R}_{+\infty}$ with $f \geq \langle \cdot, \cdot \rangle$ and $T \subseteq \{(x, x^*) \mid f(x, x^*) = \langle x, x^* \rangle\} := M_f$. The interest in representative functions stems from that fact that M_f is always monotone. Martínez-Legaz and Svaiter [18] also introduced the monotone polar for a monotone set $T \subseteq X \times X^*$, by $T^{\mu} := \{(x, x^*) \in X \times X^* \mid \langle x - y, x^* - y^* \rangle \geq 0, \forall (y, y^*) \in T\}$. In [18] it is noted that: $T \subseteq T^{\mu}$ means T is monotone; with T maximal if and only if $T^{\mu} = T$. Related notions are of *pre-maximal monotonicity*, $T^{\mu\mu} = T^{\mu}$ (i.e. T^{μ} is maximal and T has a unique maximal extension) and of *monotonic closure*, $T^{\mu\mu} = T$.

The class of *bigger–conjugate* representative functions for T is defined as

$$bR(T) := \left\{f \in PC(X \times X^*) \mid f^{*\dagger} \geq f \geq \langle \cdot, \cdot \rangle,\ T \subseteq M_f\right\}.$$

The interest in the $f \in bR(T)$ stems from the fact that $M_f = M_{f^{*\dagger}}$, with their assured maximality as monotone sets when X is reflexive [10]. In [13] it is shown that in a general real Banach space, representable monotone extensions of T, which are given by M_f for $f \in bR(T)$, are *maximal–like* in that they are monotonically closed i.e. $M_f^{\mu\mu} = M_f$.

Denoting the restriction of a function $F : X^* \times X^{**} \to \mathbb{R}_{\infty}$ by $\widehat{F} : X^* \times X \to \mathbb{R}_{\infty}$ given by $\widehat{F}(x^*, x) = F(x^*, x^{**})$ when $\hat{x} = x^{**}$ then the Fitzpatrick function

$$F_T(x, x^*) = \left[(\widehat{(\langle \cdot, \cdot \rangle + \delta_T)^{\star}})\right]^{\dagger}(x, x^*) = \sup_{(z,z^*) \in T} \left\{\langle (x, x^*), (z, z^*) \rangle - \langle z, z^* \rangle\right\}$$

is a representative function for T, when T is maximal monotone [14]. As the Fitzpatrick function is defined via a conjugate-transpose restricted to $X \times X^*$ it is (by definition) $s \times \mathrm{w}^*$-closed (and hence strongly closed as well). We note that almost all duality theorems for conjugation (i.e. Fenchel duality) are based on the duality pairing that gives rise to the conjugate $f \mapsto f^{\widehat{\star}}$ which unfortunately will not be available when using the transpose conjugate (1).

2 A Closure Operation for Convex Subsets in $X \times X^*$

When X is not reflexive it is well known that $(x, x^*) \mapsto \langle x, x^* \rangle$ is not continuous under any topology $s \times \tau$ compatible with duality (i.e. $(X^*, \tau)^* = X$) unless X is at least reflexive, [33]. Indeed to be continuous with respect to $\tau = \text{bw}^*$ we need X finite dimensional [33]. This is why we are interested in w*-convergence of bounded nets as it is well known that the duality product is continuous with respect to this convergence [13].

In the following we study the closure for convex sets $C \subseteq X \times X^*$ with respect to bounded $s \times \text{w}^*$ convergent nets since in the subsequent analysis we need to construct a bigger-conjugate representative function $h \in bR\,(\text{rec}\, T)$. The main difficulty is actually showing our construction indeed has a bigger conjugate. This necessitates the introduction of the following closure operation in order to develop an appropriate duality theory. The product topology $s \times \text{w}^*$ is problematic to study directly as one member is sequentially determined and the other is not. We will discuss in this section a novel way of characterising this product topology that allows a classical approach to its study.

Definition 2 For $C \subseteq X \times X^*$ denote

$$\overline{C}^{s \times bd\text{w}^*} := \bigcup_{K>0} \overline{C \cap \overline{B_K}(0)}^{s \times \text{w}^*}. \tag{3}$$

Clearly $C \subseteq \overline{C}^{s \times bd\text{w}^*} \subseteq \overline{C}^{s \times \text{w}^*}$ and it is immediate that when C is $s \times \text{w}^*$-closed, have $C = \overline{C}^{s \times bd\text{w}^*}$. Note that $\overline{C}^{s \times bd\text{w}^*}$ consists of all $s \times$ w* accumulation points of bounded nets from C, motivating the notation.

This closure must be strictly stronger than the $s \times \text{w}^*$ closure (indeed for the case $C = \{0\} \times C^*$, this closure corresponds to that associated with the classical bounded–weak*-topology on X^* which is itself also strictly stronger than the weak* topology). We say C is $s \times bd\text{w}^*$-closed iff $C \supseteq \overline{C}^{s \times bd\text{w}^*}$ (and so $C = \overline{C}^{s \times bd\text{w}^*}$).

Lemma 3 *Let* $C \subseteq X \times X^*$.

1. *Then* $\overline{C \cap \overline{B_K}(0)}^{s \times \text{w}^*} \subseteq \overline{C}^{s \times bd\text{w}^*} \cap \overline{B_K}(0)$ *for all* $K > 0$.
2. *If* C *is convex so is* $\overline{C}^{s \times bd\text{w}^*}$.
3. *If* $\{C_i\}_{i \in I}$ *are* $s \times bd\text{w}^*$*-closed then so is* $\cap_{i \in I} C_i$.

Proof The first inclusion follows from definitions. Since $\overline{C}^{s \times bd\text{w}^*}$ is a union of the nested convex sets $\left\{ \overline{C \cap \overline{B_K}(0)}^{s \times \text{w}^*} \right\}_{K>0}$ we have $\overline{C}^{s \times bd\text{w}^*}$ convex. For the last conclusion, since $C_i \supseteq \overline{C}_i^{s \times bd\text{w}^*}$,

$$\bigcap_{i \in I} C_i \supseteq \bigcap_{i \in I} \overline{C}_i^{s \times bd\text{w}^*} = \bigcap_{i \in I} \bigcup_{K>0} \overline{C_i \cap \overline{B_K}(0)}^{s \times \text{w}^*}$$

$$\supseteq \bigcup_{K>0} \overline{\left(\bigcap_{i \in I} C_i \right) \cap \overline{B_K}(0)}^{s \times \text{w}^*} = \overline{\bigcap_{i \in I} C_i}^{s \times bd\text{w}^*}.$$

□

We will now assume C is convex to obtain a much stronger result. We will need to appeal to the following variant of a separation theorem. Denote $Y := X^* \times X$ and consider the canonical embedding of a subset $C \subseteq X \times X^*$ into $Y^* = (X^* \times X)^* = X^{**} \times X^*$ by $J_Y(C) \cap (X^{**} \times X^*)$ (later we will simply write either $\widehat{C}$ or $C \subseteq X^{**} \times X^*$, accepting the abuse of notation). We impose the $\boldsymbol{w}^*$-topology on the dual space $Y^* := (X^* \times X)^* = X^{**} \times X^*$.

Proposition 4 *Suppose $C, D \subseteq X \times X^*$ are convex. Denote the $\boldsymbol{w}^*$-closure of $J_{X\times X^*}(C) \cap (X^{**} \times X^*)$ (and $J_{X\times X^*}(D) \cap (X^{**} \times X^*)$) as a subset of Y^* by $\overline{C}^{\boldsymbol{w}^*}$ (and $\overline{D}^{\boldsymbol{w}^*}$).*

1. *For any convex set C,*

$$\overline{C}^{\boldsymbol{w}^*}|_{X\times X^*} = \overline{C}^{s\times \mathrm{w}^*}. \tag{4}$$

2. *Suppose that $\overline{C}^{\boldsymbol{w}^*} \cap \overline{D}^{\boldsymbol{w}^*} = \emptyset$ with D bounded. Then there exists $(z^*, z) \in Y$ $(= X^* \times X)$ such that*

$$\delta_C^*\left(z^*, z\right) \le \alpha < \delta_D^*\left(z^*, z\right). \tag{5}$$

Conversely if C, D can be strictly separated by a $s \times \mathrm{w}^*$-continuous hyperplane, in the sense of (5), then $\overline{C}^{\boldsymbol{w}^*} \cap \overline{D}^{\boldsymbol{w}^*} = \emptyset$.

Proof Part 1: With $Y = X^* \times X$, then from the embedding of Y into Y^*, we may view C and D as subsets of $Y^* = X^{**} \times X^*$. Now $\overline{C}^{s\times \mathrm{w}^*}$ is the intersection of enclosing half-spaces in $X \times X^*$ formed from $(x^*, x) \in X^* \times X$, and $\overline{C}^{\boldsymbol{w}^*}$ is the intersection of those in $X^{**} \times X^*$ formed from $(x^*, x) \in X^* \times X = Y$. Hence

$$\overline{C}^{s\times \mathrm{w}^*} = \overline{C}^{\boldsymbol{w}^*} \cap \left(X \times X^*\right) = \overline{C}^{\boldsymbol{w}^*}|_{X\times X^*},$$

which is (4).

Part 2: Now suppose $\overline{C}^{\boldsymbol{w}^*} \cap \overline{D}^{\boldsymbol{w}^*} = \emptyset$. We then have $\overline{D}^{\boldsymbol{w}^*}$ compact in Y^* and $\overline{C}^{\boldsymbol{w}^*}$ closed. We now invoke Theorem 1.1.5 of [34] to obtain $(z^*, z) \in X^* \times X$ satisfying

$$\langle\left(z^*, z\right), \left(v^{**}, v^*\right)\rangle \le \alpha_1 < \alpha_2 \le \langle\left(z^*, z\right), \left(u^{**}, u^*\right)\rangle$$

for all $(v^{**}, v^*) \in \overline{C}^{\boldsymbol{w}^*}$ and $(u^{**}, u^*) \in \overline{D}^{\boldsymbol{w}^*}$. As $C \subseteq \overline{C}^{s\times \mathrm{w}^*} \subseteq \overline{C}^{\boldsymbol{w}^*}$ (and similarly for D) we have (5). Thus when $\overline{C}^{\boldsymbol{w}^*} \cap \overline{D}^{\boldsymbol{w}^*} = \emptyset$ we can strictly separate then with a $s \times \mathrm{w}^*$-continuous hyperplane.

On the other hand whenever this possible, we have $(z^*, z) \in Y := X^* \times X$ which is the pre-dual of $Y^* := (X^* \times X)^* = X^{**} \times X^*$ and α_1, α_2 such that for all $(v\ , v^*) \in C$ and $(u\ , u^*) \in D$, we have

$$\langle\left(z^*, z\right), \left(v\ , v^*\right)\rangle \le \alpha_1 < \alpha_2 \le \langle\left(z^*, z\right), \left(u\ , u^*\right)\rangle.$$

Hence for $H_\alpha := \{(y^{**}, y^*) \mid \langle(z^*, z), (y^{**}\ , y^*)\rangle \le \alpha\}$ we have

$$\overline{C}^{\boldsymbol{w}^*} \subseteq H_{\alpha_1} \text{ and } \overline{D}^{\boldsymbol{w}^*} \subseteq H_{\alpha_2}^c$$

and so $\overline{C}^{\boldsymbol{w}^*} \cap \overline{D}^{\boldsymbol{w}^*} \subseteq H_{\alpha_1} \cap H_{\alpha_2}^c = \emptyset$. □

Corollary 5 *Suppose $C \subseteq X \times X^*$ is convex. Denote $Y = X^* \times X$ and the $\boldsymbol{w}^*$-closure of $J_X(C) \cap (X^{**} \times X^*)$ as a subset of $Y^* := (X^* \times X)^*$ by $\overline{C}^{\boldsymbol{w}^*}$. Then C is $s \times \mathrm{w}^*$-closed iff there exists a w^*-closed convex set $D \subseteq Y^*$ such that $C = D|_{X \times X^*}$. In particular C is $s \times \mathrm{w}^*$-closed iff $C = \overline{C}^{\boldsymbol{w}^*} \cap (X \times X^*)$.*

Proof We have C is $s \times \mathrm{w}^*$-closed iff $C = \overline{C}^{s \times \mathrm{w}^*} = \overline{C}^{\boldsymbol{w}^*} \cap (X \times X^*)$ and so for $D := \overline{C}^{\boldsymbol{w}^*}$ we have $C = D|_{X \times X^*}$. On the other hand when $C = D|_{X \times X^*}$ for $D = \overline{D}^{\boldsymbol{w}^*}$ then $C = \overline{D}^{\boldsymbol{w}^*} \cap (X \times X^*)$ with this set $s \times \mathrm{w}^*$-closed. □

Remark 6 Note, that by Corollary 5, a convex set A is $s \times \mathrm{w}^*$-closed iff there exists a $\boldsymbol{w}^*$-closed set B in $Y^* = X^{**} \times X^*$ with $A = B \cap (X \times X^*)$ from which it then follows that $A = \overline{B \cap (X \times X^*)}^{\boldsymbol{w}^*} \cap (X \times X^*)$.

3 Variational Limits

The recession operator $\operatorname{rec} T$ is defined as a Kuratowski-Painlevé limit of a family of sets $\{tT\}_{t>0}$ with respect to the $s \times bd\mathrm{w}^*$-convergence. This leads us to study such limits in more detail. A problem that arises often in defining variational limits, in the context of convergences that are *not defined by a topology*, is that of devising a consistent set of fundamental definitions (this also arises with sequential convergences based on weak topologies). We must do this in our context, and so pursue the framework of a Kuratowski/Painlevé-type convergence (see [1, Sect. 5.2]), based on *bounded* convergent $s \times \mathrm{w}^*$ nets. Denote by $(s \times \mathrm{w}^*)\text{-}\limsup_{t \to +\infty} A_t$ and $(s \times \mathrm{w}^*)\text{-}\liminf_{t \to +\infty} A_t$ the usual Kuratowski–Painlevé convergence with respect to the product topology of strong with weak*. Denote

$$\mathcal{N}\left(x, x^*\right) := \left\{U \times W \mid x \in U \in \mathcal{N}(x) \text{ and } x^* \in W \in \mathcal{N}\left(x^*\right)\right\}$$

where $\mathcal{N}(x)$ is the strong neighbourhood basis at x (in X) and $\mathcal{N}(x^*)$ is a weak* neighbourhood basis at x^* (in X^*).

Definition 7 Let $\{A_t\}_{t>0}$ be a family of subsets of $X \times X^*$. Then, we define

$$bds\mathrm{w}^*\text{-}\limsup_{t \to +\infty} A_t := \Big\{\left(x, x^*\right) \mid \exists \text{ net } \left(x_\alpha, x_\alpha^*, t_\alpha\right) \to^{s \times \mathrm{w}^*} \left(x, x^*, +\infty\right) \\ \text{with } \left(x_\alpha, x_\alpha^*\right) \in A_{t_\alpha} \text{ and } \left\{x_\alpha^*\right\} \text{ bounded}\Big\}, \quad (6)$$

$$bds\mathrm{w}^*\text{-}\liminf_{t \to +\infty} A_t := \big\{\left(x, x^*\right) \mid \exists K > 0 : \\ \forall V \in \mathcal{N}\left(x, x^*\right)\ (\exists t_V > 0)(\forall t > t_V)\ (A_t \cap \overline{B_K}(0) \cap V \neq \emptyset\big\}. \quad (7)$$

We always have $bds\mathrm{w}^*\text{-}\limsup_{t \to +\infty} A_t \supseteq bds\mathrm{w}^*\text{-}\liminf_{t \to +\infty} A_t$. The set $bds\mathrm{w}^*\text{-}\limsup_{t \to +\infty} A_t$ *may not* be $s \times \mathrm{w}^*$-closed.

Remark 8 These notions have an obvious extension to subsets of $X \times X^* \times \mathbb{R}$ (to include epigraphs of functions on $X \times X^*$) where $s \times w^*$ then stands for $s \times w^* \times \tau_{\mathbb{R}}$, the product with the standard topology on the reals.

Remark 9 We note that when $(x, x^*) \in bds\text{w}^*\text{-}\liminf_{t\to+\infty} A_t$ and $U \times W \in \mathcal{N}(x, x^*)$ then there exists $K > 0$ such that $\{t \in \mathbb{R}_+ \mid [[U \times W] \cap \overline{B_K}(0)] \cap A_t \neq \emptyset\}$ is residual. Moreover $(x, x^*) \in bds\text{w}^*\text{-}\limsup_{t\to+\infty} A_t$ then we have $\{t \in \mathbb{R}_+ \mid [[U \times W] \cap \overline{B_K}(0)] \cap A_t \neq \emptyset\}$ contains a cofinal subset.

Remark 10 Note that if we define $\frac{1}{t}T := \{(x, \frac{1}{t}x^*) \mid (x, x^*) \in T\}$ then

$$bds\text{w}^*\text{-}\limsup_{t\to+\infty} \frac{1}{t}T = \operatorname{rec} T \tag{8}$$

once again identifying $\operatorname{rec} T$ with its graph and $\operatorname{rec} T(x)$ with the image of the associated multi-function.

Definition 11 We say that a family $\{f_t\}_{t>0}$ of proper functions $s \times \text{w}^*$**- boundedly converges** at (x, x^*) as $t \to \infty$ iff both of the following coincide:

$$\left(bd\text{-}e\text{-}\liminf_{t\to+\infty} f_t\right)(x, x^*) := \inf\{\alpha \mid (x, x^*, \alpha) \in bds\text{w}^*\text{-}\limsup_{t\to+\infty} (\operatorname{epi} f_t)\} \quad \text{and}$$

$$\left(bd\text{-}e\text{-}\limsup_{t\to+\infty} f_t\right)(x, x^*) := \inf\{\alpha \mid (x, x^*, \alpha) \in bds\text{w}^*\text{-}\liminf_{t\to+\infty} (\operatorname{epi} f_t)\}.$$

When these coincide for all (x, x^*) then we denote the associated function f by:

$$f = bd\text{-}e\text{-}\lim_{t\to+\infty} f_t.$$

We also note in passing that order is preserved when applying these limits, in that $f_t \leq g_t$ implies $bd\text{-}e\text{-}\liminf_{t\to+\infty} f_t \leq bd\text{-}e\text{-}\liminf_{t\to+\infty} g_t$ etc.

This definition is used to take into account the possibility that the infimal value may not actually be in the limit of epigraphs. When $\{f_t\}_{t>0}$ is monotonically non-decreasing this problem does not occur and we can then simply identify $\operatorname{epi}(bd\text{-}e\text{-}\lim_{t\to+\infty} f_t) = bds\text{w}^*\text{-}\lim_{t\to+\infty} (\operatorname{epi} f_t)$ (see later for details).

Clearly

$$bd\text{-}e\text{-}\liminf_{t\to+\infty} f_t \leq bd\text{-}e\text{-}\limsup_{t\to+\infty} f_t.$$

We may move from a limit with $t \to +\infty$ to one with $\tau \to 0^+$ (as is later done in Sect. 6) via the simple transformation $\tau = \frac{1}{t+\alpha}$ for any $\alpha > 0$ and so for now we focus on limits with $t \to +\infty$. Characterisation similar to those of other epi-limits of functions can be made, but in this case the attainment of the infimum in the following is not assured. We say an epi-limit is attained if there exists a net attaining the infimum in (9). In the case when X is reflexive then we find that weak* and weak topologies coincide and so for convex functions this convergence is characterised sequentially (using Mazur characterisation of weak vs strong closures of convex sets). Hence a diagonalisation argument may be used to assert attainment as weakly convergent sequences are bounded.

Lemma 12 *Let* $\{f_t\}_{t>0}$ *be a family of proper functions. Then*

$$\left(bd\text{-}e\text{-}\liminf_{t\to+\infty} f_t\right)(x, x^*) = \inf_{\substack{\{(x_\alpha, x_\alpha^*) \to^{s\times bd\text{w}^*} (x, x^*)\} \\ \{t_\alpha \to +\infty\}}} \liminf_{\alpha} f_{t_\alpha}(x_\alpha, x_\alpha^*). \tag{9}$$

Proof We have

$$\gamma := bd\text{-}e\text{-}\liminf_{t\to+\infty} f_t\left(x,x^*\right)$$
$$= \inf\left\{\beta \mid \exists \text{ net } \left(x_\alpha, x_\alpha^*, t_\alpha\right) \to^{s\times \mathrm{w}^*} \left(x, x^*, +\infty\right) \text{ with } \beta = \liminf_\alpha f_{t_\alpha}\left(x_\alpha, x_\alpha^*\right)\right\}$$

and so for any $\varepsilon > 0$ we have the existence of $\left(x_\alpha, x_\alpha^*, t_\alpha\right) \to^{s\times \mathrm{w}^*} (x, x^*, +\infty)$ such that $\left(x_{t_\alpha}, x_{t_\alpha}^*, \gamma_\alpha\right) \in \operatorname{epi} f_{t_\alpha}$ with $\gamma_\alpha \to \hat{\gamma} \le \gamma + \frac{\varepsilon}{2}$. That is, for any $\varepsilon > 0$

$$\left(bd\text{-}e\text{-}\liminf_{t\to+\infty} f_t\right)\left(x,x^*\right) + \varepsilon \ge \inf_{\left(x_\alpha,x_\alpha^*\right)\to^{s\times bd\mathrm{w}^*}(x,x^*)} \inf_{\{t_\alpha\to+\infty\}} \liminf_\alpha f_{t_\alpha}\left(x_\alpha, x_\alpha^*\right)$$

$$\text{or} \quad \left(bd\text{-}e\text{-}\liminf_{t\to+\infty} f_t\right)\left(x,x^*\right) \ge \inf_{\left(x_\alpha,x_\alpha^*\right)\to^{s\times bd\mathrm{w}^*}(x,x^*)} \inf_{\{t_\alpha\to+\infty\}} \liminf_\alpha f_{t_\alpha}\left(x_\alpha, x_\alpha^*\right).$$

Take an arbitrary $t_\alpha \to +\infty$ and suppose $\left(x_\alpha, x_\alpha^*\right) \to^{s\times bd\mathrm{w}^*} (x, x^*)$. Place $\gamma' := \liminf_\alpha f_{t_\alpha}\left(x_\alpha, x_\alpha^*\right)$. Then for $\gamma'_{t_\alpha} := f_{t_\alpha}\left(x_\alpha, x_\alpha^*\right)$ we have $\left(x_\alpha, x_\alpha^*, \gamma'_{t_\alpha}\right) \in \operatorname{epi} f_{t_\alpha}$ and $\liminf_\alpha \gamma'_{t_\alpha} = \gamma'$. Then $\left(x, x^*, \gamma'\right) \in bd\mathrm{w}^*\text{-}\limsup_{t\to+\infty} \operatorname{epi} f_t$. Hence

$$\liminf_\alpha f_{t_\alpha}\left(x_\alpha, x_\alpha^*\right) = \gamma' \ge bd\text{-}e\text{-}\liminf_{t\to+\infty} f_t\left(x,x^*\right).$$

As this holds for all $t_\alpha \to +\infty$ and $\left(x_\alpha, x_\alpha^*\right) \to^{s\times bd\mathrm{w}^*} (x, x^*)$ we have

$$\inf_{\left\{\left(x_t,x_t^*\right)\to^{s\times bd\mathrm{w}^*}(x,x^*)\right\}} \inf_{\{t_\alpha\to+\infty\}} \liminf_\alpha f_{t_\alpha}\left(x_{t_\alpha}, x_{t_a}^*\right) \ge \left(bd\text{-}e\text{-}\liminf_{t\to+\infty} f_t\right)\left(x,x^*\right). \qquad \square$$

Note that for any $K > 0$ we have $(s \times \mathrm{w}^*)\text{-}\limsup_{t\to+\infty}\left(A_t \cap \overline{B_K}(0)\right) = bds\mathrm{w}^*\text{-}\limsup_{t\to+\infty}\left(A_t \cap \overline{B_K}(0)\right)$ (and similarly for the limit infimum). We may now give a characterisation of these limits, paralleling that given for the Kuratowski–Painlevé limit of variational analysis. Note that the final union means that the limiting sets are not necessarily $s \times \mathrm{w}^*$-closed.

Proposition 13 *Consider a family $\{A_t\}_{t>0}$ of subsets of $X \times X^*$. Then*

$$bds\mathrm{w}^*\text{-}\limsup_{t\to+\infty} A_t = \bigcup_{K>0}\bigcap_{\eta>0}\overline{\left[\left(\bigcup_{t\ge\eta} A_t\right)\cap \overline{B_K}(0)\right]}^{s\times \mathrm{w}^*} \quad \text{and} \tag{10}$$

$$bds\mathrm{w}^*\text{-}\liminf_{t\to+\infty} A_t = \bigcup_{K>0}\bigcap_{\substack{I\subseteq\mathbb{R}_+\\ \text{cofinal}}}\overline{\left[\left(\bigcup_{t\in I} A_t\right)\cap \overline{B_K}(0)\right]}^{s\times \mathrm{w}^*}. \tag{11}$$

Moreover we have $bds\mathrm{w}^*\text{-}\limsup_{t\to+\infty} A_t = A$ if $(s \times \mathrm{w}^*)\text{-}\limsup_{t\to+\infty}\left(A_t \cap \overline{B}_K(0)\right) = A \cap \overline{B}_K(0)$ for all sufficiently large $K > 0$, and $bds\mathrm{w}^*\text{-}\liminf_{t\to+\infty} A_t = A$ if $(s \times \mathrm{w}^*)\text{-}\liminf_{t\to+\infty} A_t \cap \overline{B}_K(0) = A \cap \overline{B}_K(0)$ for all sufficiently large $K > 0$. When all $\{A_t\}_{t>0}$ are convex then so is $bds\mathrm{w}^*\text{-}\liminf_{t\to+\infty} A_t$.

Proof See the Appendix for proof. $\square$

We can interpret such limits for subsets $\{A_t\}_{t>0}$ of $Y = X^* \times X$ by embedding $\hat{A}_t := J_{X\times X^*}(A_t) \cap (X^{**} \times X^*) \subseteq Y^*$ where $Y = X^* \times X$.

Definition 14 Suppose $\{A_t\}_{t>0}$ are a family of subsets of $Y := X^* \times X$. Embedding $\hat{A}_t := J_{X\times X^*}(A_t) \cap (X^{**} \times X^*) \subseteq Y^*$ denote:

$$bd\left(w \times \mathrm{w}^*\right)\text{-}\limsup_{t\to+\infty} A_t := \left[\bigcup_{K>0}\bigcap_{\eta>0}\overline{\left[\bigcup_{t\geq\eta}\hat{A}_t \cap \overline{B_K}(0)\right]}^{\boldsymbol{w}^*}\right] \cap \left(X \times X^*\right)$$

$$= \left[\bigcup_{K>0} \boldsymbol{w}^*\text{-}\limsup_{t\to+\infty}\left[\hat{A}_t \cap \overline{B_K}(0)\right]\right] \cap \left(X \times X^*\right) \quad \text{and}$$

$$bd\left(w \times \mathrm{w}^*\right)\text{-}\liminf_{t\to+\infty} A_t := \left[\bigcup_{K>0}\bigcap_{\substack{I\subseteq\mathbb{R}_+\\ \text{cofinal}}}\overline{\left[\bigcup_{t\in I}\hat{A}_t \cap \overline{B_K}(0)\right]}^{\boldsymbol{w}^*}\right] \cap \left(X \times X^*\right)$$

$$= \left[\bigcup_{K>0} \boldsymbol{w}^*\text{-}\liminf_{t\to+\infty}\left[\hat{A}_t \cap \overline{B_K}(0)\right]\right] \cap \left(X \times X^*\right).$$

Remark 15 Note that the limit–infimum and –supremum on the right-hand-side are in the Kuratowski–Painlevé sense.

Definition 16 For $g : X \times X^* \to \mathbb{R}_{+\infty}$ form $\widehat{g} : X^{**} \times X^* \to \mathbb{R}_{+\infty}$ by $\widehat{g}(x, x^*) = g(x, x^*)$ if $(x, x^*) \in X \times X^*$ and $+\infty$ otherwise. Denote, for $(x^{**}, x^*) \in X^{**} \times X^*$,

$$\left(bd\boldsymbol{w}^*\text{-}e\text{-}\liminf_{t\to+\infty}\widehat{g_t}\right)\left(x^{**}, x^*\right)$$

$$:= \inf\left\{\gamma \;\middle|\; \left(x^{**}, x^*, \gamma\right) \in \left[\bigcup_{K>0} \boldsymbol{w}^*\text{-}\limsup_{t\to+\infty}\left[\operatorname{epi}\widehat{g_t} \cap \overline{B_K}(0)\right]\right]\right\} \quad \text{and}$$

$$bd\left(w \times \mathrm{w}^*\right)\text{-}e\text{-}\liminf_{t\to+\infty} f_t^{*\dagger} := \left(bd\boldsymbol{w}^*\text{-}e\text{-}\liminf_{t\to+\infty}\widehat{f_t{}^{*\dagger}}\right)|_{X\times X^*}.$$

Note that $\bigcup_{K>0} \boldsymbol{w}^*\text{-}\limsup_{t\to+\infty}\left[\operatorname{epi}\widehat{g_t} \cap \overline{B_K}(0)\right]$ consists of the limits of all selections of bounded $\boldsymbol{w}^*$-convergent subnets. The set $\operatorname{epi}(bd\boldsymbol{w}^*\text{-}e\text{-}\liminf_{t\to+\infty}\widehat{g_t})$ has the vertical recession direction we associate with an epi-graph but the set may not be closed. Note that $\widehat{f_t^\star}$ acts on the space $(X \times X^*)^* = X^* \times X^{**}$. Note the conjugate in our prior sense $f^{*\dagger}$ and that $f_t^{\widehat{\star}\dagger}$ which passes from $Y := X \times X^*$ to $Y^* = X^{**} \times X^*$ are compatible in the sense that

$$(f_t{}^{\widehat{\star}\dagger})|_{X\times X^*} \equiv f_t{}^{*\dagger} \quad \text{on } X \times X^* \quad \text{i.e. } \operatorname{epi} f_t{}^{\widehat{\star}\dagger} \cap \left(X \times X^* \times \mathbb{R}\right) = \operatorname{epi} f_t{}^{*\dagger}. \tag{12}$$

In general we only have

$$\left(bd\boldsymbol{w}^*\text{-}e\text{-}\limsup_{t\to+\infty}\operatorname{epi} f_t{}^{\widehat{\star}\dagger}\right)\Big|_{X\times X^*} = bd\left(w \times \mathrm{w}^*\right)\text{-}\limsup_{t\to+\infty}\operatorname{epi} f_t{}^{*\dagger}$$

due the potential failure of the limit set to be convex. Note, that by Corollary 5, for convex sets A_t are $s \times \mathrm{w}^*$-closed iff there exists a $\boldsymbol{w}^*$-closed set B_t in $X^{**} \times X^*$ with $A_t = B_t \cap (X \times X^*)$ from which it then follows that $A_t = \overline{B_t \cap (X \times X^*)}^{\boldsymbol{w}^*} \cap (X \times X^*)$.

The following may be proved along similar lines to that in Lemma 12 and so the proof is omitted.

Lemma 17 *Let $\{f_t\}_{t>0}$ be a family of proper functions on $X \times X^*$. Then*

$$\left(bd\boldsymbol{w}^*\text{-}e\text{-}\liminf_{t\to+\infty} \widehat{f_t}\right)\left(x^{**}, x^*\right) = \inf_{\substack{\left\{\left(x_\alpha, x_\alpha^*\right)\to^{bd\boldsymbol{w}^*}\left(x^{**}, x^*\right)\right\} \\ \{t_\alpha\to+\infty\}}} \liminf_{\alpha} \widehat{f_{t_\alpha}}\left(x_\alpha, x_\alpha^*\right). \tag{13}$$

Let us now consider monotonic limit of families of functions.

Proposition 18 *Let $\{f_t\}_{t>0}$ be a family of $[-\infty, +\infty]$-valued functions.*

1. *When $\{f_t\}_{t>0}$ are convex then so is $f := bd$ -e-$\limsup_{t\to+\infty} f_t$.*
2. *Assume for each (x, x^*) that $t \mapsto f_t(x, x^*)$ is monotonically non-decreasing (as $t \to \infty$). We have*

$$\bigcap_{t>0} \overline{\text{epi } f_t}^{s\times bd\mathrm{w}^*} = \bigcap_{t>0}\bigcup_{K>0} \overline{\text{epi } f_t \cap \overline{B_K}(0)}^{s\times\mathrm{w}^*} \supseteq bds\mathrm{w}^*\text{-}\limsup_{t\to+\infty} \text{epi } f_t \tag{14}$$
$$= \bigcup_{K>0}\bigcap_{t>0} \overline{\text{epi } f_t \cap \overline{B_K}(0)}^{s\times\mathrm{w}^*} = bds\mathrm{w}^*\text{-}\liminf_{t\to+\infty} \text{epi } f_t \supseteq \bigcap_{t>0} \overline{\text{epi } f_t}^{s\times bd\mathrm{w}^*}$$

   and so bd-e-$\lim_{t\to+\infty}$ epi f_t exists.
3. *When all f_t are $s \times bd\mathrm{w}^*$ (resp. $s \times \mathrm{w}^*$)-closed then $f := bd$-e-$\lim_{t\to\infty} f_t$ is also $s \times bd\mathrm{w}^*$ (resp. $s \times \mathrm{w}^*$)-closed and coincides with the pointwise limit (i.e. epi $f = \bigcap_{t>0}$ epi f_t), and $(s \times \mathrm{w}^*)$-$\lim_{t\to+\infty}\left[\text{epi } f_t \cap \overline{B_K}(0)\right] = \text{epi } f \cap \overline{B_K}(0)$ for each $K > 0$.*

Proof 1) Convexity of $f := bd$-e-$\limsup_{t\to+\infty} f_t$ follows immediately from the last assertion of Proposition 13.

2) Assume $t \mapsto f_t(x, x^*)$ is monotonically non-decreasing. As epi $f_t \subseteq$ epi f_τ for all $t \geq \tau$, it follows that $\bigcup_{t\geq\tau}$ epi $f_t =$ epi f_τ so

$$bds\mathrm{w}^*\text{-}\limsup_{t\to+\infty} \text{epi } f_t = \bigcup_{K>0}\bigcap_{t>0} \overline{\text{epi } f_t \cap \overline{B_K}(0)}^{s\times\mathrm{w}^*} \subseteq \overline{\text{epi } f_\tau}^{s\times bd\mathrm{w}^*} \quad \text{for all } \tau > 0 \tag{15}$$

$$\text{so} \qquad bds\mathrm{w}^*\text{-}\limsup_{t\to+\infty} \text{epi } f_t \subseteq \bigcap_{\tau>0} \overline{\text{epi } f_\tau}^{s\times bd\mathrm{w}^*} = \bigcap_{t>0}\bigcup_{K>0} \overline{\text{epi } f_t \cap \overline{B_K}(0)}^{s\times\mathrm{w}^*}.$$

To show equality of the limits we need to demonstrate that $bds\mathrm{w}^*$-$\liminf_{t\to+\infty}$ epi $f_t \supseteq$ $bds\mathrm{w}^*$-$\limsup_{t\to+\infty}$ epi f_t. Let $(x, x^*) \in bd\mathrm{w}^*$-$\limsup_{t\to+\infty}$ epi f_t. Then there exists $K >$ 0 and $\{t_\alpha\}$ with $t_\alpha \to +\infty$ and $\left\{\left(x_\alpha, x_\alpha^*\right)\right\}$ with $\left(x_\alpha, x_\alpha^*\right) \to (x, x^*)$ for which $\left(x_\alpha, x_\alpha^*\right) \in$ epi $f_{t_\alpha} \cap \overline{B}_K(0)$ for all α. Let W be a neighbourhood of (x, x^*). Then, there exists $\bar{\alpha}$ such that for all $\alpha \geq \bar{\alpha}$ we have $\left(x_\alpha, x_\alpha^*\right) \in$ epi $f_{t_\alpha} \cap \overline{B}_K(0) \cap W$. Let $\tau \geq t_{\bar{\alpha}}$. Then (since $t_\alpha \to +\infty$) there exists $t_{\alpha'} \geq \tau$ with $\alpha' \geq \bar{\alpha}$, so epi $f_\tau \cap \overline{B}_K(0) \cap W \supseteq$ epi $f_{t_{\alpha'}} \cap \overline{B}_K(0) \cap W$. Therefore $(x, x^*) \in bd\mathrm{w}^*$-$\liminf_{t\to+\infty}$ epi f_t.

3) When each epi f_η is $s \times bd\mathrm{w}^*$ (or $s \times \mathrm{w}^*$)-closed, by Lemma 3 part 3 we have $\bigcap_{t>0}$ epi f_t is $s \times bd\mathrm{w}^*$ (or $s \times \mathrm{w}^*$)-closed, and so, by (14) the limit exists, as epi $f =$

$\bigcap_{\eta>0} \operatorname{epi} f_\eta$, which is $s \times bd\mathrm{w}^*$ (or $s \times \mathrm{w}^*$)-closed. Moreover, as

$$\operatorname{epi} f \cap \overline{B_K}(0) = \bigcap_{\eta>0} \overline{\operatorname{epi} f_\eta \cap \overline{B_K}(0)}^{s\times \mathrm{w}^*} = \bigcup_{H>0} \left[\bigcap_{\eta>0} \overline{\bigcup_{t\geq\eta} \operatorname{epi} f_t \cap \overline{B_K}(0)}^{s\times \mathrm{w}^*} \right] \cap \overline{B}_H(0)$$

we have $(s \times \mathrm{w}^*)$-$\limsup_{t\to+\infty} \operatorname{epi} f_t \cap \overline{B_K}(0) = \operatorname{epi} f \cap \overline{B_K}(0)$. A similar calculation can be used to show $(s \times \mathrm{w}^*)$-$\liminf_{t\to\infty} \operatorname{epi} f_t \cap \overline{B_K}(0) = \operatorname{epi} f \cap \overline{B_K}(0)$. □

We briefly discuss monotonically non-increasing families.

Proposition 19 *Assume for each $(x, x^*) \in X \times X^*$ we have $t \mapsto g_t(x, x^*)$ is monotonically non-increasing (as $t \to \infty$), where the family $\{g_t\}_{t\in\mathbb{R}_+}$ are $[-\infty, +\infty]$-valued functions.*

1. *Then*

$$bds\mathrm{w}^*\text{-}\liminf_{t\to\infty} \operatorname{epi} g_t = bds\mathrm{w}^*\text{-}\limsup_{t\to+\infty} \operatorname{epi} g_t = \overline{\left(\bigcup_{t\geq 0} \operatorname{epi} g_t\right)}^{s\times bd\mathrm{w}^*},$$

   and so $g := bd$-e-$\lim_{t\to\infty} g_t$ exists.
2. *If each g_t is convex, then $g := bd$-e-$\lim_{t\to+\infty} g_t$ is convex.*

Proof As $\{g_t\}_{t>0}$ is monotonically non-increasing, which implies $\operatorname{epi} g_t \subseteq \operatorname{epi} g_\tau$ for $t \leq \tau$, we have $\left(\bigcup_{t\in I} \operatorname{epi} g_t\right) = \left(\bigcup_{t\geq 0} \operatorname{epi} g_t\right)$ for any residual or cofinal set $I \subseteq \mathbb{R}_+$. Let $K > 0$. Then for $\operatorname{epi} g := bd\mathrm{w}^*$-$\limsup_{t\to+\infty} \operatorname{epi} g_t$, by Proposition 13, for all $\tau > 0$,

$$\begin{aligned}
\operatorname{epi} g &= \bigcup_{H>0} \bigcap_{\eta>0} \overline{\left(\bigcup_{t\geq\eta} \operatorname{epi} g_t\right) \cap \overline{B_H}(0)}^{s\times \mathrm{w}^*} = \bigcup_{H>0} \overline{\left(\bigcup_{t\geq 0} \operatorname{epi} g_t\right) \cap \overline{B_H}(0)}^{s\times \mathrm{w}^*} \\
&= \bigcup_{H>0} \overline{\left(\bigcup_{t\geq\tau} \operatorname{epi} g_t\right) \cap \overline{B_H}(0)}^{s\times \mathrm{w}^*} \\
&= bds\mathrm{w}^*\text{-}\lim_{t\to\infty} \operatorname{epi} g_t = \overline{\left(\bigcup_{t\geq 0} \operatorname{epi} g_t\right)}^{s\times bd\mathrm{w}^*}
\end{aligned}$$

When $\{g_t\}_{t>0}$ is a family of convex functions then $\bigcup_{t\geq 0} \operatorname{epi} g_t$ is convex (being a monotonically non-decreasing nested set of convex sets) and hence $\overline{\left(\bigcup_{t\geq 0} \operatorname{epi} g_t\right)}^{s\times bd\mathrm{w}^*}$ is convex by Lemma 3. □

Utilising these observations about monotonic limits we find that the limit–infimum lends itself to another useful interpretation.

Lemma 20 *Suppose $\{f_t\}_{t>0}$ is a family of extended-real-valued functions. Then*

$$bd\text{-}e\text{-}\liminf_{t\to\infty} f_t = bd\text{-}e\text{-}\lim_{\eta\to\infty}\left(\inf_{t\geq\eta} f_t\right) \geq bd\text{-}e\text{-}\lim_{\eta\to\infty}\left(\overline{\inf_{t\geq\eta} f_t}^{s\times bd\mathrm{w}^*}\right).$$

Proof Note that $g_\eta := \overline{\inf_{t\geq\eta} f_t}^{s\times bd\mathrm{w}^*}$ is a monotonically non-decreasing family of $s \times bd\mathrm{w}^*$-closed functions and so by Proposition 18 we have existence of the epi-limits $bd\text{-}e\text{-}\lim_{t\to\infty} g_t = \sup_\eta g_\eta$ and $bd\text{ -}e\text{-}\lim_{\eta\to\infty}\left(\inf_{t\geq\eta} f_t\right)$. By Proposition 13 with $A_t :=$ epi f_t

$$
\begin{aligned}
&bds\mathrm{w}^*\text{-}\limsup_{t\to+\infty} \operatorname{epi} f_t \\
&= \bigcup_{K>0}\bigcap_{\eta>0}\overline{\left[\left(\bigcup_{t\geq\eta}\operatorname{epi} f_t\right)\cap \overline{B_K}(0)\right]}^{s\times\mathrm{w}^*} = \bigcup_{K>0}\bigcap_{\eta>0}\overline{\left(\operatorname{epi}\left(\inf_{t\geq\eta} f_t\right)\right)\cap \overline{B_K}(0)}^{s\times\mathrm{w}^*} \\
&= bds\mathrm{w}^*\text{-}\lim_{\eta\to+\infty}\operatorname{epi}\left(\inf_{t\geq\eta} f_t\right) \subseteq \bigcup_{K>0}\bigcap_{\eta>0}\left(\operatorname{epi} g_\eta\right)\cap\overline{B_K}(0) = \bigcap_{\eta>0}\operatorname{epi} g_\eta \\
&= \operatorname{epi}\left(\sup_\eta g_\eta\right) = \operatorname{epi}\left(bd\text{-}e\text{-}\lim_{\eta\to\infty} g_t\right) = \operatorname{epi}\left[bd\text{-}e\text{-}\lim_{\eta\to\infty}\left(\overline{\inf_{t\geq\eta} f_t}^{s\times\mathrm{w}^*}\right)\right]. \qquad \square
\end{aligned}
$$

4 The $s \times bd\mathrm{w}^*$-Convergence and Conjugation

We will need to understand how the limits used in construction of a representative function for the monotone operator rec T, interact with conjugacy, in order to show that such a function is bigger-conjugate. In this section we explore the interaction of conjugation with this new convergence notion.

The following has been observed for almost all viable epi-limits (see [21, Lemma 1]). We provide a proof for our context.

Proposition 21 *Let $\{f_t\}_{t>0}$ be a family of functions. Then*

$$
\left(bd\text{-}e\text{-}\liminf_{t\to\infty} f_t^*\right)^* \leq bd\text{-}e\text{-}\limsup_{t\to\infty} f_t. \tag{16}
$$

If $\{f_t\}_{t>0}$ are proper convex, so is $bd\text{-}e$ -$\limsup_{t\to\infty} f_t$.

Proof By Proposition 18, the convexity of f_t yields same for $f := bd\text{-}e\text{-}\limsup_{t\to\infty} f_t$. Let $H := bd\text{-}e$ -$\liminf_{t\to+\infty} f_t^*$ and we will show $f \geq H^*$. If $f(x, x^*) = +\infty$ or $H \equiv +\infty$ (so $H^* \equiv -\infty$) there is nothing to prove. We may now assume $f(x, x^*) < +\infty$ and H is not identically $+\infty$. Let $+\infty > \gamma > f(x, x^*)$. As $(x, x^*) \in bds\mathrm{w}^*\text{-}\liminf_{t\to+\infty} \operatorname{epi} f_t$, then for some $K > 0$, we have for all $U \times W \in \mathcal{N}(x, x^*)$ there exists $t_V > 0$ such that for $t > t_V$ we have $(\operatorname{epi} f_t) \cap [U \times W] \cap \overline{B_K}(0) \neq \emptyset$, and so there exists $\left(x_t, x_t^*\right) \in \operatorname{epi} f_t \cap \overline{B_K}(0) \cap [U \times W]$ so $f_t\left(x_t, x_t^*\right) \leq \gamma_t$ with $\left(x_t, x_t^*\right) \in U \times W$ and $\left\|\left(x_t, x_t^*\right)\right\| \leq K$. Now for $+\infty > \beta > H(y, y^*)$ we have $(y, y^*) \in bds\mathrm{w}^*\text{-}\limsup_{t\to+\infty} \operatorname{epi} f_t^*$ so there exists $\left(y_{t_\alpha}, y_{t_\alpha}^*, \beta_{t_\alpha}\right) \to (y, y^*, \beta)$ as $t \to \infty$ with $\left\|y_{t_\alpha}^*\right\| \leq K' < +\infty$ bounded and $+\infty > \beta_{t_\alpha} \geq f_{t_\alpha}^*\left(y_{t_\alpha}^*, y_{t_\alpha}\right)$. We know that $f_{t_\alpha}^*\left(y_{t_\alpha}^*, y_{t_\alpha}\right) > -\infty$ since the presumption that $f_{t_\alpha}^*\left(y_{t_\alpha}^*, y_{t_\alpha}\right) = -\infty$ would imply $f_{t_\alpha} \geq f_{t_\alpha}^{**} \equiv +\infty$ and we know that $f(x, x^*) < +\infty$. Similarly if it was the case that $f_{t_\alpha}\left(x_{t_\alpha}, x_{t_\alpha}^*\right) = -\infty$ then $f_{t_\alpha}^* \equiv +\infty$, implying $\beta_{t_\alpha} = +\infty$, counter to construction. By the Fenchel inequality for all α we have,

$$
\gamma_{t_\alpha} + \beta_{t_\alpha} \geq f_{t_\alpha}\left(x_{t_\alpha}, x_{t_\alpha}^*\right) + f_{t_\alpha}^*\left(y_{t_\alpha}^*, y_{t_\alpha}\right) \geq \langle\left(x_{t_\alpha}, x_{t_\alpha}^*\right), \left(y_{t_\alpha}^*, y_{t_\alpha}\right)\rangle.
$$

For each $\varepsilon > 0$ we may take $W \in \mathcal{N}(x^*)$ such that $\sup_{u^* \in W} \langle y, u^* - x^* \rangle \leq \varepsilon$ where $B_\varepsilon(x) \times W \times I_\varepsilon(\gamma) \in \mathcal{N}(x, x^*, \gamma)$ has $(x, x^*) \in B_\varepsilon(x) \times W \in \mathcal{N}(x, x^*)$ and $\gamma_{t_\alpha} \leq \gamma + \varepsilon$ with $\left(x_{t_\alpha}, x^*_{t_\alpha}\right) \in B_\varepsilon(x) \times W$. In this case we observe that $\limsup_\alpha \langle \left(x_{t_\alpha}, x^*_{t_\alpha}\right), \left(y^*_{t_\alpha}, y_{t_\alpha}\right) \rangle \leq \gamma + \beta + \varepsilon$. Now

$$
\begin{aligned}
\langle \left(x_{t_\alpha}, x^*_{t_\alpha}\right), \left(y^*_{t_\alpha}, y_{t_\alpha}\right) \rangle &= \langle \left(x, x^*\right), \left(y, y^*\right) \rangle + \langle x, y^*_{t_\alpha} - y^* \rangle + \langle y, x^*_{t_\alpha} - x^* \rangle \\
&\quad + \langle x_{t_\alpha} - x, y^*_{t_\alpha} \rangle + \langle y_{t_\alpha} - y, x^*_{t_\alpha} \rangle \\
&\geq \langle \left(x, x^*\right), \left(y, y^*\right) \rangle + \langle x, y^*_{t_\alpha} - y^* \rangle + \langle y, x^*_{t_\alpha} - x^* \rangle \\
&\quad - K' \left\| x_{t_\alpha} - x \right\| + K \left\| y_{t_\alpha} - y \right\|.
\end{aligned}
$$

Since $y^*_{t_\alpha} \to^{w^*} y^*$, $x^*_{t_\alpha} \to^{w^*} x^*$, $x_{t_\alpha} \to^{s} x$ it follows that

$$
\begin{aligned}
\gamma + \beta + \varepsilon &\geq \langle \left(x, x^*\right), \left(y, y^*\right) \rangle - \varepsilon \left(K + K'\right) + \limsup_\alpha \langle y, x^*_{t_\alpha} - x^* \rangle \\
&\geq \langle \left(x, x^*\right), \left(y, y^*\right) \rangle - \varepsilon \left(K + K' + 1\right).
\end{aligned}
$$

As $\varepsilon > 0$ was arbitrary we have $\gamma + \beta \geq \langle (x, x^*), (y^*, y) \rangle$ and so $\gamma \geq \langle (x, x^*), (y^*, y) \rangle - \beta$. As $\beta > H(y^*, y)$ and $\gamma > f(x, x^*)$ are arbitrary we get (for any (y^*, y) with $H(y^*, y) < +\infty$)

$$
f\left(x, x^*\right) \geq \langle \left(x, x^*\right), \left(y^*, y\right) \rangle - H\left(y^*, y\right), \quad \text{for all } \left(y, y^*\right) \text{ so } f(x, x^*) \geq H^*\left(x, x^*\right).
$$

Then $bd\text{-}e\text{-}\limsup_{t \to \infty} f_t = f \geq H^* = \left(bd\text{-}e\text{-}\liminf_{t \to \infty} f^*_t\right)^*$. □

We may leverage this result for non-decreasing nets to obtain a continuity result for conjugates.

Proposition 22 *Let $\{f_t\}_{t>0}$ be a family of functions with $t \mapsto f_t(x, x^*)$ monotonically non-decreasing (as $t \to \infty$). Then*

$$
\left(bd\text{-}e\text{-}\lim_{t \to \infty} f^{**}_t\right)^* = \overline{bd\text{-}e\text{-}\lim_{t \to \infty} f^*_t}^{w^* \times s}. \tag{17}
$$

When all f_t are $s \times w^*$-closed, convex with $f_t > -\infty$ then $bd\text{-}e\text{-}\lim_{t \to \infty} f_t$ exists and is also $s \times w^*$-closed and convex with

$$
bd\text{-}e\text{-}\lim_{t \to \infty} f_t = \left(bd\text{-}e\text{-}\lim_{t \to \infty} f^*_t\right)^* \quad \textit{and} \quad \left(bd\text{-}e\text{-}\lim_{t \to \infty} f_t\right)^* = \overline{bd\text{-}e\text{-}\lim_{t \to \infty} f^*_t}^{w^* \times s}. \tag{18}
$$

Proof Let $g_t := f^{*\dagger}_t$ then $\{g_t\}_{t>0}$ is monotonically non-increasing. Hence by Proposition 19 we have $g_t \geq g := bd\text{-}e\text{-}\lim_{t \to \infty} g_t$ for all $t > 0$ and so $g^*_t \leq g^*$, implying $bd\text{-}e\text{-}\lim_{t \to \infty} g^*_t \leq \overline{g^*}^{bd w^* \times s} = g^*$ and hence

$$
\left(bd\text{-}e\text{-}\lim_{t \to \infty} f^{**}_t\right)^{*\dagger} = \left(bd\text{-}e\text{-}\lim_{t \to \infty} g^*_t\right)^* \geq g^{**} = \overline{bd\text{-}e\text{-}\lim_{t \to \infty} f^{*\dagger}_t}^{s \times w^*}.
$$

Now apply Lemma 21 to g_t to get

$$
\left(bd\text{-}e\text{-}\liminf_{t \to \infty} g^*_t\right)^* \leq bd\text{-}e\text{-}\limsup_{t \to \infty} g_t \quad \text{or}
$$

$$\left(bd\text{-}e\text{-}\lim_{t\to\infty} f_t^{**}\right)^{*\dagger} \leq bd\text{-}e\text{-}\lim_{t\to\infty} f_t^{*\dagger},$$

$$\text{implying} \quad \left(bd\text{-}e\text{-}\lim_{t\to\infty} f_t^{**}\right)^{*} \leq \overline{bd\text{-}e\text{-}\liminf_{t\to\infty} f_t^{*}}^{\mathrm{w}^*\times s} \leq \left(bd\text{-}e\text{-}\lim_{t\to\infty} f_t^{**}\right)^{*}.$$

For the case when f_t is convex and $s \times \mathrm{w}^*$–closed with $f_t > -\infty$, then $f_t^{**} = f_t$, so $(bd\text{-}e\text{-}\lim_{t\to\infty} f_t)^* = \overline{bd\text{-}e\text{-}\lim_{t\to\infty} f_t^*}^{s\times\mathrm{w}^*}$ and in particular, due to Proposition 18 then $bd\text{-}e\text{-}\lim_{t\to\infty} f_t = \sup_t f_t > -\infty$ is closed and convex so $bd\text{-}e\text{-}\lim_{t\to\infty} f_t = (bd\text{-}e\text{-}\lim_{t\to\infty} f_t)^{**}$ along with $\left(\overline{bd\text{-}e\text{-}\lim_{t\to\infty} f_t^*}^{s\times\mathrm{w}^*}\right)^* = \left(bd\text{-}e\text{-}\lim_{t\to\infty} f_t^*\right)^*$, via the fact that our conjugation is based on the pairing of $X \times X^*$ with $X^* \times X$ and associated $s \times \mathrm{w}^*$-continuous linear functions on $X \times X^*$ (so the $s \times \mathrm{w}^*$-closure does not affect the value of the conjugate). □

Again we may utilise the symmetry that is present in the pairing of $\sigma_{s\times\mathrm{w}^*}(X \times X^*)$ with $\sigma_{\mathrm{w}^*\times s}(X^* \times X)$. A result that uses the passage of the conjugate $* : \sigma_{s\times\mathrm{w}^*}(X \times X^*) \to \sigma_{\mathrm{w}^*\times s}(X^* \times X)$ is also applicable to inverse $* : \sigma_{\mathrm{w}^*\times s}(X^* \times X) \to \sigma_{s\times\mathrm{w}^*}(X \times X^*)$ (using the transpose operator).

Proposition 23 *Let $\{f_t\}_{t>0}$ be a family of convex functions, $f_t \not\equiv +\infty$ with $t \mapsto f_t(x, x^*)$ monotonically non-increasing (as $t \to \infty$). Then $bd\text{-}e\text{-}\lim_{t\to\infty} f_t^*$ is $(\mathrm{w}^* \times s)$–closed and*

$$bd\text{-}e\text{-}\lim_{t\to\infty} f_t^* = \overline{bd\text{-}e\text{-}\lim_{t\to\infty} f_t^*}^{\mathrm{w}^*\times s} = \left(bd\text{-}e\text{-}\lim_{t\to\infty} f_t\right)^*. \tag{19}$$

Proof If $\{f_t\}_{t>0}$ is monotonically non-increasing then for $g_t := f_t^*$ we have $\{g_t\}_{t>0}$ is monotonically non-decreasing and closed. Hence applying Proposition 22 we have (noting that either $g_t > -\infty$ or $f_t \geq f_t^{**} = g_t^* \equiv +\infty$)

$$\begin{aligned} bd\text{-}e\text{-}\lim_{t\to\infty} f_t^* = bd\text{-}e\text{-}\lim_{t\to\infty} g_t &= \left(bd\text{-}e\text{-}\limsup_{t\to\infty} g_t^*\right)^* \\ &= \left(bd\text{-}e\text{-}\limsup_{t\to\infty} f_t^{**}\right)^* = \left(bd\text{-}e\text{-}\limsup_{t\to\infty} \overline{f_t}^{s\times\mathrm{w}^*}\right)^*. \end{aligned} \tag{20}$$

In the following, by Proposition 19 we have the first equality, with the inequality following from $f_t \geq \inf_{t>0} f_t$ so

$$bd\text{-}e\text{-}\limsup_{t\to\infty} \overline{f_t}^{s\times\mathrm{w}^*} = \overline{\inf_{t>0} \overline{f_t}^{s\times\mathrm{w}^*}}^{s\times bd\mathrm{w}^*} \geq \overline{\inf_{t>0} f_t}^{s\times\mathrm{w}^*} = \overline{\left(\overline{\inf_{t>0} f_t}^{s\times bd\mathrm{w}^*}\right)}^{s\times\mathrm{w}^*} \tag{21}$$

Thus, as the conjugate is not affected by the $s \times \mathrm{w}^*$-closure, and by Proposition 19 we have $bd\text{-}e\text{-}\lim_{t\to\infty} f_t = \overline{\inf_{t>0} f_t}^{s\times bd\mathrm{w}^*}$, it follows from (20)-(21) that:

$$bd\text{-}e\text{-}\lim_{t\to\infty} f_t^* \leq \left(\overline{\left(\overline{\inf_{t>0} f_t}^{s\times bd\mathrm{w}^*}\right)}^{s\times\mathrm{w}^*}\right)^* = \left(\overline{\inf_{t>0} f_t}^{s\times bd\mathrm{w}^*}\right)^* = \left(bd\text{-}e\text{-}\lim_{t\to\infty} f_t\right)^*. \tag{22}$$

By Proposition 21 we have $\left(bd\text{-}e\text{-}\lim_{t\to\infty} f_t^*\right)^* \leq bd\text{-}e\text{ -}\lim_{t\to\infty} f_t$ so

$$bd\text{-}e\text{-}\lim_{t\to\infty} f_t^* \geq \overline{bd\text{-}e\text{-}\lim_{t\to\infty} f_t^*}^{\mathrm{w}^*\times s} = \left(bd\text{-}e\text{-}\lim_{t\to\infty} f_t^*\right)^{**} \geq \left(bd\text{-}e\text{-}\lim_{t\to\infty} f_t\right)^*. \tag{23}$$

Combining (22) and (23) we get (19). □

We note the following for later use when further investigating the continuity of conjugation in relation to these variational limits.

Lemma 24 *Let* $\{f_t\}_{t>0}$ *be a family of* $s \times bd\mathrm{w}^*$*-(resp.* $s \times \mathrm{w}^*$*-)closed functions on* $X \times X^*$ *for which* $(\inf_t f_t)(x, x^*) > -\infty$ *for all* (x, x^*)*. Recalling that* $\widehat{f_t}$ *denotes the embedding of* f_t *within the conjugate space* $(X \times X^*)^* = X^* \times X^{**}$*, (by setting* $\widehat{f_t} = f_t$ *on the subspace* $X \times X^*$ *and* $+\infty$ *otherwise) with the associated conjugation* $\widehat{\star} : X \times X^* \to (X \times X^*)^* = X^* \times X^{**}$*. Then on* $X \times X^*$

$$\overline{\left(bd\boldsymbol{w}^*\text{-}e\text{-}\liminf_{t\to+\infty} \widehat{f_t}\right)^{\widehat{\star}\dagger}}^{s\times bd\mathrm{w}^*} = \left(bd\text{-}e\text{-}\liminf_{t\to\infty} f_t\right)^{*\dagger}. \tag{24}$$

Proof We have, using the $\boldsymbol{w}^*$-continuity of $(x^{**}, x^*) \mapsto \langle (x^{**}, x^*), (y^*, y)\rangle = \langle x^{**}, y^*\rangle + \langle x^*, y\rangle$, the Fenchel inequality $\langle (x_\alpha, x_\alpha^*), (y^*, y)\rangle - f_{t_\alpha}(x_\alpha, x_\alpha^*) \le f_t^*(y^*, y)$ and Lemma 17 that

$$\begin{aligned}
&\left(bd\boldsymbol{w}^*\text{-}e\text{-}\liminf_{t\to+\infty} \widehat{f_t}\right)^{\widehat{\star}\dagger}(y, y^*) \\
&= \sup_{(x^*, x^{**})} \left\{ \langle (x^{**}, x^*), (y^*, y)\rangle - \inf_{\substack{\{(x_\alpha, x_\alpha^*) \to^{bd\boldsymbol{w}^*} (x^{**}, x^*)\} \\ \{t_\alpha \to +\infty\}}} \liminf_\alpha \widehat{f_t}(x_t, x_t^*) \right\} \\
&= \sup_{\substack{\{(x_\alpha, x_\alpha^*) \to^{bd\boldsymbol{w}^*} (x^{**}, x^*) \in X^{**}\times X^*\} \\ \{t_\alpha \to +\infty\}}} \limsup_\alpha [\{\langle (x_\alpha, x_\alpha^*), (y^*, y)\rangle - f_{t_\alpha}((x_\alpha, x_\alpha^*))\}] \\
&\le \sup_{\{t_\alpha \to +\infty\}} \limsup_\alpha f_{t_\alpha}^{*\dagger}(y, y^*) = \lim_{\eta\to+\infty} \sup_{t\ge\eta} f_t^{*\dagger}(y, y^*) \\
&= \lim_{\eta\to+\infty} \left(\inf_{t\ge\eta} f_t\right)^{*\dagger}(y, y^*).
\end{aligned}$$

That is, we have a monotonically nondecreasing family $\{g_\eta = \inf_{t\ge\eta} f_t\}$ where

$$\left(bd\boldsymbol{w}^*\text{-}e\text{-}\liminf_{t\to+\infty} \widehat{f_t}\right)^{\widehat{\star}\dagger}|_{X\times X^*} \le \lim_{\eta\to+\infty} (g_\eta)^{*\dagger}. \tag{25}$$

As $\{g_\eta^*\}$ is closed and monotonically non-increasing, Proposition 19 implies

$$bd\text{-}e\text{-}\lim_{\eta\to\infty} (g_\eta)^{*\dagger} = \overline{\inf_\eta (g_\eta)^{*\dagger}}^{s\times bd\mathrm{w}^*} = \overline{\lim_{\eta\to+\infty} (g_\eta)^{*\dagger}}^{s\times bd\mathrm{w}^*}.$$

Using this observation and Proposition 22 (18), we have

$$\begin{aligned}
\overline{\lim_{\eta\to+\infty} (g_\eta)^{*\dagger}}^{s\times bd\mathrm{w}^*} &= bd\text{-}e\text{-}\lim_{\eta\to\infty} (g_\eta)^{*\dagger} = \overline{bd\text{-}e\text{-}\lim_{\eta\to\infty} (g_\eta)^{*\dagger}}^{s\times\mathrm{w}^*} \\
&= \left(bd\text{-}e\text{-}\lim_{\eta\to\infty} g_\eta\right)^{*\dagger}.
\end{aligned}$$

Combining this with (25) and Lemma 20 we have

$$\overline{\left(bd\boldsymbol{w}^{*}\text{-}e\text{-}\liminf_{t\to+\infty}\widehat{f_t}\right)^{\widehat{\star}\dagger}}^{s\times bd\mathrm{w}^*} \leq \overline{\lim_{\eta\to+\infty}\left(g_\eta\right)^{*\dagger}}^{s\times bd\mathrm{w}^*} = \left(bd\text{-}e\text{-}\lim_{\eta\to\infty} g_\eta\right)^{*\dagger}$$
$$= \left(bd\text{-}e\text{-}\lim_{\eta\to\infty}\left(\inf_{t\geq\eta} f_t\right)\right)^{*\dagger} = \left(bd\text{-}e\text{-}\liminf_{t\to\infty} f_t\right)^{*\dagger}.$$

Hence

$$\overline{\left(bd\boldsymbol{w}^{*}\text{-}e\text{-}\liminf_{t\to+\infty}\widehat{f_t}\right)^{\widehat{\star}\dagger}}^{s\times bd\mathrm{w}^*} \leq \left(bd\text{-}e\text{-}\liminf_{t\to\infty} f_t\right)^{*}.$$

The reverse inequality is immediate from $\operatorname{epi}\left(bd\boldsymbol{w}^{*}\text{-}e\text{-}\liminf_{t\to+\infty}\widehat{f_t}\right) \supseteq \operatorname{epi}\left(bd\text{-}e\text{-}\widehat{\liminf}_{t\to\infty} f_t\right)$ as then, when restricting to $X^*\times X$, we have

$$\left(bd\boldsymbol{w}^{*}\text{-}e\text{-}\liminf_{t\to+\infty}\widehat{f_t}\right)^{\widehat{\star}} \geq \left(bd\text{-}e\text{-}\widehat{\liminf_{t\to\infty}} f_t\right)^{\widehat{\star}} = \left(bd\text{-}e\text{-}\liminf_{t\to\infty} f_t\right)^{*}.$$

The last inequality is preserved on taking the $(bd\mathrm{w}^*\times s)$-closure as the conjugate $(bd\text{-}e\text{-}\lim_{t\to\infty} f_t)^*$ is actually $(\mathrm{w}^*\times s)$-closed. This gives the reverse inequality. □

We will strengthen our conjugation results by adapting Theorem 2 of [21]. We first show the following result.

Proposition 25 *Assume that* $\{f_t\}_{t\in\mathbb{R}_+}$ *is a family of* $(s\times\mathrm{w}^*)$*–closed, proper, convex functions. Assume in addition that for any cofinal subset* $I_1\subseteq\mathbb{R}_+$ *there exists a cofinal subset* I_2 *of* I_1 *and a bounded net* $\left\{\left(x_\alpha^*, x_\alpha\right)\right\}_{t_\alpha\in I_2}$ *such that* $\left\{f_{t_\alpha}^*\left(x_\alpha^*, x_\alpha\right)\right\}_{t_\alpha\in I_2}$ *is bounded above. Denote* $f := bd\text{-}e\text{-}\limsup_{t\to\infty} f_t$. *Then, on* $X\times X^*$ *we have*

$$\overline{f}^{s\times bd\mathrm{w}^*} \leq \left(bd\boldsymbol{w}\text{-}e\text{-}\liminf_{t\to+\infty}\widehat{f_t^{\star}}\right)^{\widehat{\star}}. \tag{26}$$

Furthermore, if $\inf_t f_t^*(x^*,x) > -\infty$ *for all* $(x^*,x)\in X^*\times X$ *then on* $X\times X^*$:

$$\overline{bd\text{-}e\text{-}\limsup_{t\to\infty} f_t}^{s\times bd\mathrm{w}^*} = \left(bd\text{-}e\text{-}\liminf_{t\to\infty} f_t^*\right)^{*}. \tag{27}$$

Proof Given $(\bar{x},\bar{x}^*)\in X\times X^*$ and $\bar{\gamma} < \overline{f}^{s\times bd\mathrm{w}^*}(\bar{x},\bar{x}^*)$ we note that $(\bar{x},\bar{x}^*,\bar{\gamma}) \notin \overline{bds\mathrm{w}^*\text{-}\liminf_{t\to\infty}(\operatorname{epi} f_t)}^{s\times bd\mathrm{w}^*}$, so for any $K > \max\{|\bar{\gamma}|, \|(\bar{x},\bar{x}^*)\|\} \geq 0$ we can find a cofinal subset I_1 of $\mathbb{R}_+$, a norm ball $U\ni\bar{x}$, a weak* neighbourhood $W\ni\bar{x}^*$ and $\rho>0$ for which

$$\begin{aligned}
&[[(U\times W)]\times[\bar{\gamma}+(-\rho,\rho)]]\cap\operatorname{epi} f_{t_\alpha}\cap\overline{B}_K(0)\\
&=\left(\left[[(U\times W)\cap\overline{B}_K(0)]\times[\bar{\gamma}+[-\rho,\rho]]\cap[-K,K]\right]\right)\cap\operatorname{epi} f_{t_\alpha}\\
&=\emptyset
\end{aligned}$$

for each $t_\alpha \in I_1$. After adjusting $\rho > 0$ accordingly we may claim

$$\left[[\left(\{\bar{x}\} \times \overline{W \cap B_K(0)}^{w^*} \right)] \times [\bar{\gamma} + [-\rho, \rho]] \right] \cap \text{epi } f_{t_\alpha}$$
$$\subseteq ([(U \times W)] \times [\bar{\gamma} + [-\rho, \rho]]) \cap \text{epi } f_{t_\alpha} \cap \overline{B}_K(0) = \emptyset.$$

The Hahn-Banach separation theorem holds (for the separation of the $(s \times w^*)$-closed convex set epi f_{t_α} and the $(s \times w^*)$-compact convex set $\{\bar{x}\} \times \overline{W \cap B_K(0)}^{w^*}$) within a locally convex linear topological space [34, Theorem 1.1.3] and $X \times X^*$ is made so by endowing (as usual) X with the strong topology and X^* with the weak* topology. Hence we have $\left(y_\alpha^*, y_\alpha, -\lambda_\alpha\right) \in X^* \times X \times \mathbb{R}$ of unit norm (the $\lambda_\alpha \geq 0$ implied by the epigraphical recession direction in epi f_{t_α}) for which

$$\langle \left(y_\alpha^*, y_\alpha\right), \left(x, x^*\right) \rangle - \lambda_\alpha \gamma \leq \langle \left(y_\alpha^*, y_\alpha\right), \left(z, z^*\right) \rangle - \lambda_\alpha r \quad \text{for all } \left(x, x^*, \gamma\right) \in \text{epi } f_{t_\alpha}$$
$$\text{and } \left(z, z^*, r\right) \in [\left(\{\bar{x}\} \times W \cap \overline{B_K}(0)\right)] \times [\bar{\gamma} + (-\rho, \rho)].$$

Using the fact this holds for all $r \in \bar{\gamma} + (-\rho, \rho)$ we have

$$\langle \left(y_\alpha^*, y_\alpha\right), \left(x, x^*\right) \rangle - \lambda_\alpha \gamma \leq \langle \left(y_\alpha^*, y_\alpha\right), \left(\bar{x}, \bar{x}^*\right) \rangle - \lambda_\alpha (\bar{\gamma} - \rho)$$
$$\text{for all } \left(x, x^*, \gamma\right) \in \text{epi } f_{t_\alpha}. \tag{28}$$

Now let I_2 be cofinal in I_1 and take a bounded $\left\{\left(x_\alpha^*, x_\alpha\right)\right\}_{t_\alpha \in I_2}$ be such that $f^*\left(x_\alpha^*, x_\alpha\right)$ is bounded above i.e. $f_{t_\alpha}^*\left(x_\alpha^*, x_\alpha\right) < b$ and $\left\|\left(x_\alpha^*, x_\alpha\right)\right\| \leq H$ (some H). Then for each $t_\alpha \in I_2$ we have

$$\langle \left(x_\alpha^*, x_\alpha\right), \left(x, x^*\right) \rangle - \gamma \leq b \quad \text{for all } \left(\left(x, x^*\right), \gamma\right) \in \text{epi } f_{t_\alpha}. \tag{29}$$

Taking (28) along $I_2 \subseteq I_1$ we may multiply (28) by $q > 0$ and add to (29) to get

$$\begin{aligned}
&\langle \left(x_\alpha^*, x_\alpha\right) + q\left(y_\alpha^*, y_\alpha\right), \left(x, x^*\right) \rangle - (1 + \lambda_\alpha q)\ \gamma \\
&\leq \langle \left(x_\alpha^*, x_\alpha\right) + q\left(y_\alpha^*, y_\alpha\right), \left(\bar{x}, \bar{x}^*\right) \rangle - \langle \left(x_\alpha^*, x_\alpha\right), \left(\bar{x}, \bar{x}^*\right) \rangle + b - \lambda_\alpha q \bar{\gamma} - \lambda_\alpha q \rho \\
&\leq \langle \left(x_\alpha^*, x_\alpha\right) + q\left(y_\alpha^*, y_\alpha\right), \left(\bar{x}, \bar{x}^*\right) \rangle \\
&\quad - (1 + \lambda_\alpha q)\bar{\gamma} + \bar{\gamma} + \|x_\alpha\| \left\|\bar{x}^*\right\| + \left\|x_\alpha^*\right\| \|\bar{x}\| + b - \lambda_\alpha q \rho \\
&\leq \langle \left(x_\alpha^*, x_\alpha\right) + q\left(y_\alpha^*, y_\alpha\right), \left(\bar{x}, \bar{x}^*\right) \rangle - (1 + \lambda_\alpha q)\bar{\gamma} \\
&\quad + \left[\bar{\gamma} + \left(\left\|\bar{x}^*\right\| + \|\bar{x}\|\right) H + b - \lambda_\alpha q \rho\right].
\end{aligned} \tag{30}$$

Now choose $\bar{q} > 0$ sufficiently large so that $[\bar{\gamma} + [\|\bar{x}^*\| + \|\bar{x}\|] H + b - q\rho] \leq 0$ for all $q \geq \bar{q}$. Indeed we may take

$$\bar{q} = \frac{1}{\rho}\left\{\bar{\gamma} + \left[\left\|\bar{x}^*\right\| + \|\bar{x}\|\right] H + b\right\}.$$

Let $q = 2\bar{q}$ and set $\left(z_\alpha^*, z_\alpha\right) := (1 + 2\lambda_\alpha q)^{-1}\left[\left(x_\alpha^*, x_\alpha\right) + 2q\left(y_\alpha^*, y_\alpha\right)\right]$; on division of (30) by $(1 + 2\lambda_\alpha q)$ we obtain

$$\langle \left(z_\alpha^*, z_\alpha\right), \left(x, x^*\right) \rangle - \gamma \leq \langle \left(z_\alpha^*, z_\alpha\right), \left(\bar{x}, \bar{x}^*\right) \rangle - \bar{\gamma} \quad \text{for all } \left(\left(x, x^*\right), \gamma\right) \in \text{epi } f_{t_\alpha}.$$

Thus $f^*_{t_\alpha}\left(z^*_\alpha, z_\alpha\right) \leq \gamma_\alpha := \langle\left(z^*_\alpha, z_\alpha\right), (\bar{x}, \bar{x}^*)\rangle - \bar{\gamma}$ for $t_\alpha \in I_2$. As $\left\{\left(x^*_\alpha, x_\alpha\right)\right\}_{t_\alpha \in I_2}$ and $\left\{\left(y^*_\alpha, y_\alpha\right)\right\}_{t_\alpha \in I_2}$ are bounded, as $q = 2\overline{q}$ we have $\left\{\left(z^*_\alpha, z_\alpha, \gamma_\alpha\right)\right\}_{t_\alpha \in I_2}$ bounded for all α (in norm by, say, $\overline{K} = (\|(\bar{x}, \bar{x}^*)\| + \bar{\gamma})(H + 2\bar{q}))$ and so may extract a $\boldsymbol{w}^*$-convergent (in $X^* \times X^{**} = (X \times X^*)^*$) subnet $\left\{\left(z^*_\alpha, z_\alpha\right)\right\}_{t_\alpha \in I_3 \subseteq I_2}$ with

$$\left(z^*_\alpha, z_\alpha\right) := (1 + \lambda_\alpha q)^{-1}\left[\left(x^*_\alpha, x_\alpha\right) + q\left(y^*_\alpha, y_\alpha\right)\right] \to^{\boldsymbol{w}^*}_{\alpha \in I_3} \left(y^*, y^{**}\right) \in \overline{B_{\bar{K}}}(0).$$

Also as $\gamma_\alpha \to \langle (y^*, y^{**}), (\bar{x}, \bar{x}^*)\rangle - \bar{\gamma}$ $(\leq \overline{K})$ we have

$$\begin{aligned}\left(y^*, y^{**}, \langle\left(y^*, y^{**}\right), \left(\bar{x}, \bar{x}^*\right)\rangle - \bar{\gamma}\right) &\in \boldsymbol{w}^*\text{-}\limsup_{t \to +\infty}\left(\operatorname{epi}\widehat{f_t^\star} \cap \overline{B_{\overline{K}}}(0)\right) \\ &\subseteq \bigcup_{K>0} \boldsymbol{w}^*\text{-}\limsup_{t \to +\infty}\left(\operatorname{epi}\widehat{f_t^\star} \cap \overline{B_K}(0)\right).\end{aligned}$$

That is, $\left(b\boldsymbol{w}^*\text{-}e\text{-}\liminf_{t\to+\infty}\widehat{f_t^\star}\right)(y^*, y^{**}) \leq \langle (y^*, y^{**}), (\bar{x}, \bar{x}^*)\rangle - \bar{\gamma}$, which implies

$$\bar{\gamma} \leq \left(bd\boldsymbol{w}^*\text{-}e\text{-}\liminf_{t\to+\infty}\widehat{f_t^\star}\right)^{\widehat{\star}}\left(\bar{x}, \bar{x}^*\right),$$

and so $\overline{f}^{s \times bd\mathrm{w}^*}(\bar{x}, \bar{x}^*) \leq \left(b\boldsymbol{w}^*\text{-}e\text{-}\liminf_{t\to+\infty}\widehat{f_t^\star}\right)^{\widehat{\star}}(\bar{x}, \bar{x}^*)$. As we have this inequality for arbitrary $(\bar{x}, \bar{x}^*)$ we get (26). Using (24), it follows that on $X \times X^*$, have $\overline{f}^{s \times bd\mathrm{w}^*} \leq \overline{\left(bd\boldsymbol{w}^*\text{-}e\text{-}\liminf_{t\to+\infty}\widehat{f_t^{\star\dagger}}\right)^{\widehat{\star}}}^{s \times bd\mathrm{w}^*} = \left(bd\text{-}e\text{-}\liminf_{t\to\infty} f_t^{*\dagger}\right)^{*\dagger}$. To get (27), using (16) we have $f \geq \left(bd\text{-}e\text{-}\liminf_{t\to\infty} f_t^{*\dagger}\right)^{*\dagger}$ with the latter $(s \times \mathrm{w}^*)$-closed and hence also for $s \times bd\mathrm{w}^*$. Taking a $s \times bd\mathrm{w}^*$-closure we have the reverse inequality, giving the results. □

Again we may utilise the symmetry that is present in the pairing of $\sigma_{s \times \mathrm{w}^*}(X \times X^*)$ with $\sigma_{\mathrm{w}^* \times s}(X^* \times X)$.

Corollary 26 *Assume that* $\{f_t\}_{t \in \mathbb{R}_+}$ *is a family of* $(s \times \mathrm{w}^*)$*-closed, proper, convex functions for which* $(\inf_t f_t)(x, x^*) > -\infty$ *for all* (x, x^*)*, and that for any cofinal subset* I_1 *of* $\mathbb{R}_+$ *there is a cofinal* $I_2 \subseteq I_1$ *and a bounded net* $\left\{\left(x_\alpha, x^*_\alpha\right)\right\}_{t_\alpha \in I_2}$ *such that* $\left\{f_{t_\alpha}\left(x_\alpha, x^*_\alpha\right)\right\}_{t_\alpha \in I_2}$ *is bounded above. Then we have*

$$\overline{bd\text{-}e\text{-}\limsup_{t\to\infty} f_t^*}^{s \times bd\mathrm{w}^*} = \left(bd\text{-}e\text{-}\liminf_{t\to\infty} f_t\right)^*. \tag{31}$$

Proof We apply (27) to the family of convex functions $\left\{f_t^{*\dagger}\right\}_{t>0}$ that are well defined on $X \times X^*$. We then we need to assume that for any cofinal subset I_2 of I_1 there is a bounded net $\left\{\left(x_\alpha, x^*_\alpha\right)\right\}_{t_\alpha \in I_2}$ such that $\left\{\left(f^*_{t_\alpha}\right)^*\left(x_\alpha, x^*_\alpha\right)\right\}_{t_\alpha \in I_2}$ is bounded above. As $\left(f^*_{t_\alpha}\right)^* = f_{t_\alpha}$ this corresponds to our stated assumption. The result now follows. □

5 Some Tools from Monotone Operator Theory

When $h \in PC(X \times X^*)$ may not be representative (i.e. h need not majorise the duality product on $X \times X^*$), we form $M_h^{\leq} := \{(x, x^*) \in X \times X^* \mid h(x, x^*) \leq \langle x, x^*\rangle\}$, and note that when, instead, $h \geq \langle\cdot,\cdot\rangle$ (i.e. h is representative) it is well known that we have $M_h^{\leq} = M_h$, a monotone set. Denote $R(T) := \{h \in PC(X \times X^*) \mid h \geq \langle\cdot,\cdot\rangle \text{ and } T \subseteq M_h\}$. When $F_T = \langle\cdot,\cdot\rangle - \inf_{(y,y^*)\in T}\langle\cdot - y, \cdot - y^*\rangle$ is not representative we may study the set $T^\mu = M_{F_T}^{\leq}$. One always has $T^{\mu\mu\mu} = T^\mu$ and if T is monotone then $T \subseteq T^\mu$. In [18] $T \mapsto T^\mu$ is shown to be a polarity and as a consequence $A \subseteq B$ implies $A^\mu \supseteq B^\mu$, $T \subseteq T^{\mu\mu}$ and $(A \cup B)^\mu = A^\mu \cap B^\mu$ (for any sets $A, B \subseteq X \times X^*$). From definitions it is clear that T^μ always has w^*-closed convex images. If T is monotone but not maximal, the Fitzpatrick function F_T may not be representative. On the other hand the Penot/Svaiter function $P_T := F_T^{*\dagger} \geq \langle\cdot,\cdot\rangle$, does indeed represent T in that $T \subseteq M_{P_T}$. Following [18] we say T is representable when there exists $h \in R(T)$ with $T = M_h$.

Recall $bR(T) := \left\{h \in PC(X \times X^*) \mid P_T \geq h^{*\dagger} \geq h \geq \langle\cdot,\cdot\rangle\right\}$ are the bigger–conjugate representative functions with $T \subseteq M_h \subseteq T^\mu$. It is known, [18, 30] that when $h \in R(T)$ is closed or if $h \in bR(T)$, then $h \in [F_T, P_T]$ where $[F_T, P_T] = \{g \in PC(X, X^*) \mid F_T \leq g \leq P_T\}$, in the pointwise partial order [12, Lemma 2.1]. Note that if $h \in bR(T)$ then we have $T \subseteq M_h$ and $h \in bR(M_h)$ (see [13, Lemma 3]). When $h \in bR(T)$ then $h \geq F_{M_h}$, a detailed proof of which may be found in [12, Lemma 2.6B] or [13, Lemma 3].

Proposition 27 ([13, Theorem 8]) ***(Monotonic Closure Theorem)*** *Let $h \in bR(T)$. Then M_h is monotonically closed, i.e. $M_h^{\mu\mu} = M_h$.*

Moreover this set is unique in the following sense.

Lemma 28 [12, *Lemma* 2.11], [30] *Let $T : X \rightrightarrows X^*$ be a monotone operator, let $k, h \in bR(T)$ for which $h \leq k$. Then $M_k = M_h \supseteq T$.*

We note that in [31, Theorem 11.2] are examples of representable operators that are not monotonically closed and so the use of bigger–conjugate representative functions plays an important role.

Remark 29 As noted in [13] if $F_{M_h} \geq \langle\cdot,\cdot\rangle$, then $\overline{h}^{s\times w^*} \in bR(T)$ which indicates that we only really need to consider $(s \times w^*)$–closed representative functions when seeking to characterise maximality but this not mandatory and may be difficult to enforce in some constructions. Of course when $h \in bR(T)$ and $h \geq \overline{h}^{s\times \mathrm{w}^*} \in bR(T)$ then $M_{\overline{h}^{s\times \mathrm{w}^*}} = M_h$.

We will summarise some results we require that appear in [13]. First note that in [7] it is shown that *not* every maximal monotone operator has a $(s \times w^*)$-closed graph (even for the case of a subdifferential of a convex function). As we do not *a priori* assume any closure property for elements of $R(T)$ or $bR(T)$ we study an appropriate closure.

Lemma 30 ([13, Lemma 5]) *Suppose T is monotone, and $\left\{(x_\alpha, x_\alpha^*)\right\}_\alpha \subseteq T$ is a **bounded** net converging in the $(s \times \mathrm{w}^*)$–topology to (x, x^*). Then $(x, x^*) \in T^{\mu\mu}$ i.e. $T \subseteq \overline{T}^{s\times bd\mathrm{w}^*} \subseteq T^{\mu\mu}$. In particular when T is monotonically closed (as is the case when $h \in bR(T)$ and $T = M_h$) we have $T (= T^{\mu\mu}) = \overline{T}^{s\times bd\mathrm{w}^*}$.*

Proof Let $(y, y^*) \in T^\mu$. Then as $\left\{(x_\alpha, x_\alpha^*)\right\}_\alpha \subseteq T$ is bounded, there is $K > 0$ such that $\left\|x_\alpha^*\right\| \leq K$ for all α so that

$$\begin{aligned} 0 &\leq \langle y - x_\alpha, y^* - x_\alpha^*\rangle \\ &= \langle y - x, y^* - x_\alpha^*\rangle + \langle x - x_\alpha, y^* - x_\alpha^*\rangle \\ &\leq \langle y - x, y^* - x_\alpha^*\rangle + \left[\left\|y^*\right\| + K\right]\|x - x_\alpha\|. \end{aligned} \tag{32}$$

As $x_\alpha \to x$ in the norm topology and $x_\alpha^* \to x^*$ in the weak* topology we have, on taking the limit,

$$0 \le \langle y - x, y^* - x^* \rangle \quad \text{for all } \left(y, y^*\right) \in T^\mu.$$

Hence $(x, x^*) \in T^{\mu\mu}$ and so $T \subseteq \overline{T}^{s\times bdw^*} \subseteq T^{\mu\mu}$. □

This closure is particularly well suited to the study of representative functions. Denote by $\overline{h}^{s\times bdw^*}$ the convex function with epigraph $\overline{\operatorname{epi} h}^{s\times bdw^*}$.

Lemma 31 *Let $h \in R(T)$.*

1. *Then $\overline{h}^{s\times bdw^*} \in R(T)$ and $M_{\overline{h}^{s\times bdw^*}} \supseteq \overline{M_h}^{s\times bdw^*} \supseteq M_h \supseteq T$, and so when $\overline{h}^{s\times bdw^*} = h$ we have $M_h = \overline{M_h}^{s\times bdw^*}$.*
2. *When $h \in bR(T)$ then $\overline{h}^{s\times bdw^*} \in bR(T)$ and $M_{\overline{h}^{s\times bdw^*}} = \overline{M_h}^{s\times bdw^*} = (M_h)^{\mu\mu} = M_h \supseteq T$.*

Proof Part 1) First we note that $\overline{h}^{s\times bdw^*}$ majorises the duality product. Indeed, as $h \ge \langle \cdot, \cdot \rangle$ we have, from (32) (with $(y, y^*) = (x, x^*)$), that there is $\left(x_\alpha, x_\alpha^*\right) \to^{s\times bdw^*} (x, x^*)$ for which

$$\overline{h}^{s\times bdw^*}\left(x, x^*\right) = \lim_\alpha h\left(x_\alpha, x_\alpha^*\right) \ge \lim_\alpha \langle x_\alpha, x_\alpha^* \rangle = \langle x, x^* \rangle,$$

giving the inequality for the closure.

On the other hand, if $(x, x^*) \in \overline{M_h}^{s\times bdw^*}$, and $\left(x_\alpha, x_\alpha^*\right) \to^{s\times bdw^*} (x, x^*)$ with $\left(x_\alpha, x_\alpha^*\right) \in M_h$ we have

$$\langle x, x^* \rangle = \lim_\alpha h\left(x_\alpha, x_\alpha^*\right) \ge \overline{h}^{s\times bdw^*}\left(x, x^*\right) \ge \langle x, x^* \rangle$$

and so $(x, x^*) \in M_{\overline{h}^{s\times bdw^*}}$. It follows that $M_h \subseteq \overline{M_h}^{s\times bdw^*} \subseteq M_{\overline{h}^{s\times bdw^*}}$.

Part 2) When $h \in bR(T)$ then $h \le h^{*\dagger}$, so from the first part, $\langle \cdot, \cdot \rangle \le \overline{h}^{s\times bdw^*} \le h \le h^{*\dagger} \le \left(\overline{h}^{s\times bdw^*}\right)^{*\dagger}$ and so $\overline{h}^{s\times bdw^*} \in bR(T)$. This implies, using [30], that $M_{\overline{h}^{s\times bdw^*}} = M_h$ $(= \overline{M_h}^{s\times bdw^*})$. □

We make note of the following for later use. Recall Definition 1 for the recession operator $\operatorname{rec} M$.

Proposition 32 ([13, Proposition 5]) *Suppose $T : X \rightrightarrows X^*$ is a monotone operator. If $x, y \in \overline{\operatorname{dom}} T$ then for any $\lambda \in [0, 1]$ we have $(\lambda x + (1 - \lambda) y, 0) \in (\operatorname{rec} T)^{\mu\mu}$ and so*

$$\operatorname{co} \overline{\operatorname{dom}} T \subseteq P_X \left[(\operatorname{rec} T)^{\mu\mu}\right]. \tag{33}$$

Moreover when $\operatorname{rec} T$ is monotonically closed then $\overline{\operatorname{dom}} T$ is convex.

Remark 33 When X is reflexive it is well known that for maximal monotone T the strong closure of the domain $\overline{\operatorname{dom}} T$ is convex. When X is reflexive and T is maximal monotone, then it is shown in [13] that $(\operatorname{rec} T)(x) = N_{\overline{\operatorname{co}} \operatorname{dom} T}(x)$ for all $x \in \overline{\operatorname{dom}} T$. Consequently that $\operatorname{rec} T = N_{\overline{\operatorname{co}} \operatorname{dom} T}$ on $\overline{\operatorname{dom}} T$ and hence $(\operatorname{rec} T)^{\mu\mu} = (N_{\overline{\operatorname{co}} \operatorname{dom} T})^{\mu\mu} = N_{\overline{\operatorname{co}} \operatorname{dom} T}$ being maximally monotone.

6 A Bigger-Conjugate Representative Function for rec T

The objective of the section is to construct a concrete representative $h \in bR\,(\operatorname{rec} T)$ using a variational limit. In doing so we further connect to this operator, the normal cone operator associated with the convex closure of the domain of T. The construction we use is the following.

Definition 34 Let X be an arbitrary Banach space. Suppose $T : X \rightrightarrows X^*$ is maximal monotone and denote its Fitzpatrick function by F_T. Then we form

$$F_T 0_2^+ := \overline{bd\text{-}e\text{-}\liminf_{t \to_+ 0} F_{tT}}^{s \times bd\mathrm{w}^*}.$$

where $(tT)(x) := tT(x)$.

We begin by taking $(z, z^*) \in T$ and translating this point to $(0, 0)$ noting that as

$$F_{T-(z,z^*)}\left(x, x^*\right) = F_T\left(x + z, x^* + z^*\right) - \left(\langle x + z, x^* + z^*\rangle - \langle x, x^*\rangle\right), \tag{34}$$

any analysis based on $s \times \mathrm{w}^*$ convergent *bounded nets* will be unaffected by such a translation, which is the case for the arguments to follow. Thus without loss of generality we assume $(0, 0) \in T$, so that F_T, $F_T 0^+ \geq 0$ on $X \times X^*$.

We will build up the properties of this function in a series of Lemmas and Propositions.

Lemma 35 *Let X be an arbitrary real Banach space. Suppose $T : X \rightrightarrows X^*$ is monotone, with Fitzpatrick function F_T. Suppose also that $(0, 0) \in T$. Then*:

1. *On $X \times X^*$,*

$$F_T 0_2^+ \geq 0.$$

Moreover, denoting $g_t(x, x^*) := F_{tT}\ (x, x^*) = tF_T\left(x, \frac{x^*}{t}\right)$ then for all $(y, y^*) \in X \times X^*$,

$$g_t^{*\dagger}\left(y, y^*\right) = tP_T\left(y, \frac{y^*}{t}\right) = P_{tT}\left(y, y^*\right) \geq F_{tT}\left(y, y^*\right) = tF_T\left(y, \frac{y^*}{t}\right) = g_t\left(y, y^*\right).$$

2. *We have*

$$\begin{aligned} &\operatorname{Pr}_X \operatorname{dom} F_T 0_2^+ \subseteq \overline{\operatorname{Pr}_X \operatorname{dom} F_T}, \qquad \textit{and, when } T \textit{ maximal},\\ &\operatorname{Pr}_X \operatorname{dom} F_T 0_2^+ \subseteq \overline{\operatorname{co}} \operatorname{dom} T. \end{aligned} \tag{35}$$

Furthermore when T is maximal, then $F_T 0_2^+ \geq \langle \cdot, \cdot \rangle$ on $X \times X^*$, with also $(F_T 0_2^+)(x, x^*) = \langle x, x^* \rangle$ for all $(x, x^*) \in \operatorname{rec} T$, so $F_T 0_2^+ \in R\,(\operatorname{rec} T)$ (i.e. $\operatorname{rec} T \subseteq M_{F_T 0_2^+}$).

3. *We have $\left(F_T 0_2^+\right)^{*\dagger} \geq F_T 0_2^+$, and so if $F_T 0_2^+$ is convex and T maximal, then $F_T 0_2^+ \in bR\,(\operatorname{rec} T)$. Furthermore,* ***if*** *$F_T 0_2^+ = \overline{bd\text{-}e\text{-}\lim_{t \to_+ 0} F_{tT}}^{s \times bd\mathrm{w}^*}$ (i.e.* ***exists as an epi-limit****) then also $F_T 0_2^+$ is convex with $F_T 0_2^+ \geq F_{\operatorname{rec} T}$ and so*

$$\left(F_{\operatorname{rec} T}\right)^{*\dagger} = P_{\operatorname{rec} T} \geq \left(F_T 0_2^+\right)^{*\dagger} \quad \textit{on} \quad X \times X^*.$$

Proof Part 1: As $(0,0) \in T$ it is immediate that for all (x,x^*) we have $F_T(x,x^*) \geq \langle x,0\rangle + \langle x^*,0\rangle - \langle 0,0\rangle = 0$, so clearly $F_T 0_2^+ \geq 0$. Next note that

$$t F_T\left(x,\frac{x^*}{t}\right) = \langle x,x^*\rangle \quad \text{iff} \quad F_T\left(x,\frac{x^*}{t}\right) = \langle x,\frac{x^*}{t}\rangle \quad \text{iff} \quad x^* \in tT(x).$$

Indeed

$$\begin{aligned} F_{tT}\left(x,x^*\right) &= \sup_{(y,ty^*)\in tT}\left(\langle x,ty^*\rangle + \langle y,x^*\rangle - t\langle y,y^*\rangle\right) \\ &= t\sup_{(y,y^*)\in T}\left(\langle x,y^*\rangle + \langle y,\frac{x^*}{t}\rangle - \langle y,y^*\rangle\right) = tF_T\left(x,\frac{x^*}{t}\right) = g_t\left(x,x^*\right). \end{aligned}$$

Hence $g_t^{*\dagger} = F_{tT}^{*\dagger} = P_{tT}$. Furthermore, we have (as $t>0$) and $P_T := F_T^{*\dagger}$ that

$$\begin{aligned} g_t^{*\dagger}\left(y,y^*\right) &= \sup_{(x,x^*)}\left\{\langle\left(x,x^*\right),\left(y,y^*\right)\rangle - tF_T\left(x,\frac{x^*}{t}\right)\right\} \\ &= t\sup_{(x,x^*)}\left\{\left\langle\left(x,\frac{x^*}{t}\right),\left(y,\frac{y^*}{t}\right)\right\rangle - F_T\left(x,\frac{x^*}{t}\right)\right\} \\ &= t\sup_{(x,x^*)}\left\{\left\langle\left(x,x^*\right),\left(y,\frac{y^*}{t}\right)\right\rangle - F_T\left(x,x^*\right)\right\} \\ &= tP_T\left(y,\frac{y^*}{t}\right) \geq tF_T\left(y,\frac{y^*}{t}\right) = g_t\left(y,y^*\right). \end{aligned} \tag{36}$$

Part 2: Clearly we always have $\Pr_X \operatorname{dom} F_T 0_2^+ \subseteq \overline{\Pr_X \operatorname{dom} F_T}$ while $\overline{\operatorname{co}} \operatorname{dom} T = \overline{\Pr_X \operatorname{dom} F_T}$, when T is maximal [6, Theorem 3.6] which gives (35). Now assume T is maximal monotone. Take $(x,x^*) \in \operatorname{rec} T$ so there exist nets $t_\alpha \to 0$ and $x_\alpha^* \in t_\alpha T(x_\alpha)$, with $\left\{\left(x_\alpha,x_\alpha^*\right)\right\}$ bounded and $\left(x_\alpha,x_\alpha^*\right) \to^{s\times w^*} (x,x^*)$. That $x_\alpha^* \in t_\alpha T(x_\alpha)$, implies (for all α) that $F_T\left(x_\alpha,\frac{x_\alpha^*}{t_\alpha}\right) = \langle x_\alpha,\frac{x_\alpha^*}{t_\alpha}\rangle$ or $t_\alpha F_T\left(x_\alpha,\frac{x_\alpha^*}{t_\alpha}\right) = \langle x_\alpha,x_\alpha^*\rangle$. Thus

$$\liminf_\alpha t_\alpha F_T\left(x_\alpha,\frac{x_\alpha^*}{t_\alpha}\right) = \lim_\alpha \langle x_\alpha,x_\alpha^*\rangle = \langle x,x^*\rangle. \tag{37}$$

Furthermore for arbitrary $(x,x^*) \in X \times X^*$, and any net $\left(w_\alpha,w_\alpha^*\right) \to (x,x^*)$ with $\left\{w_\alpha^*\right\}$ bounded and $t_\alpha \to_+ 0$, we have

$$\liminf_\alpha t_\alpha F_T\left(w_\alpha,\frac{w_\alpha^*}{t_\alpha}\right) \geq \liminf_\alpha t_\alpha\langle w_\alpha,\frac{w_\alpha^*}{t_\alpha}\rangle = \lim_\alpha\langle w_\alpha,w_\alpha^*\rangle = \langle x,x^*\rangle.$$

Hence

$$bd\mathrm{w}^*\text{-}\limsup_{t\to+0}(\operatorname{epi} g_t) \subseteq \operatorname{epi}\langle\cdot,\cdot\rangle \quad \text{or} \quad bd\text{-}e\text{-}\liminf_{t\to+0} g_t \geq \langle\cdot,\cdot\rangle, \quad \text{giving} \tag{38}$$

$$F_T 0_2^+ \geq \overline{bd\text{-}e\text{-}\liminf_{t\to+0} g_t}^{s\times bd\mathrm{w}^*} \geq \langle\cdot,\cdot\rangle, \quad \text{on } X\times X^*,$$

the last inequality following from the $(s \times bd\mathrm{w}^*)$-continuity of the duality product. Moreover by (37) and Lemma 12 we have

$$F_T 0_2^+ \left(x.x^*\right) = \overline{bd\text{-}e\text{-}\liminf_{t \to +0} g_t}^{s \times bd\mathrm{w}^*} \left(x, x^*\right) \leq \liminf_{\alpha} F_{t_\alpha T} \left(x_\alpha, x_\alpha^*\right)$$
$$= \liminf_{\alpha} t_\alpha F_T \left(x_\alpha, \frac{x_\alpha^*}{t_\alpha}\right) = \langle x, x^* \rangle,$$

for any $(x, x^*) \in \operatorname{rec} T$. Thus $F_T 0_2^+ = \langle \cdot, \cdot \rangle$ on $\operatorname{rec} T$.

Part 3: We now show that $h := F_T 0_2^+$ satisfies $h \leq h^{*\dagger}$. In order to apply Proposition 25 to $g_t = F_{tT}$ we need to supply a bounded family $\left\{\left(x_\tau, x_\tau^*\right)\right\}_{\tau > 0}$ with $\left\{g_{1/\tau}^{*\dagger}\left(x_\tau, x_\tau^*\right)\right\}$ bounded (then given a cofinal subnet I_1 we have same holding on $I_2 = I_1$). To this end, note that as $(0, 0) \in T$ then $P_T(0, 0) = \langle 0, 0 \rangle = 0$ where P_T is strongly–closed convex on $X \times X^*$. Let $\left(x_t, x_t^*\right) \in B_t(0, 0) \cap \operatorname{dom} P_T$ be such that $\lim_{t \to 0} P_T\left(x_t, x_t^*\right) = \liminf_{(y, y^*) \to (0,0)} P_T\left(y, y^*\right) = 0$. Then for $t \in (0, 1]$, we have $\left\{\left(x_t, t x_t^*\right)\right\}_{t \in (0,1]}$ a bounded net, for which $g_t^{*\dagger}\left(x_t, t x_t^*\right) = t P_T\left(x_t, \frac{t x_t^*}{t}\right) = t P_T\left(x_t, x_t^*\right) \to_{t \to 0} 0$. Using the convention $\tau = \frac{1}{t}$ we have provided the desired bounded family $\left\{\left(x_\tau, x_\tau^*\right)\right\}_{\tau > 0}$ with $\left\{g_{1/\tau}^{*\dagger}\left(x_\tau, x_\tau^*\right)\right\}$ bounded. We note that the family $\left\{g_t^{*\dagger} := P_{tT}\right\}_{t > 0}$ consists of positive functions. Thus, Proposition 25 yields

$$\overline{bd\text{-}e\text{-}\limsup_{\tau \to +\infty} g_{1/\tau}}^{s \times bd\mathrm{w}^*} = \left(bd\text{-}e\text{-}\liminf_{\tau \to +\infty} g_{1/\tau}^*\right)^*, \tag{39}$$

and by (36) we have $g_{1/\tau} \leq g_{1/\tau}^{*\dagger}$. Using these facts we obtain

$$h = F_T 0_2^+ = \overline{bd\text{-}e\text{-}\liminf_{t \to +0} F_{tT}}^{s \times bd\mathrm{w}^*} \leq \overline{bd\text{-}e\text{-}\limsup_{\tau \to +\infty} g_{1/\tau}}^{s \times bd\mathrm{w}^*} = \left(bd\text{-}e\text{-}\liminf_{\tau \to +\infty} g_{1/\tau}^{*\dagger}\right)^{*\dagger}$$
$$\leq \left(bd\text{-}e\text{-}\liminf_{\tau \to +\infty} g_{1/\tau}\right)^{*\dagger} = \left(\overline{bd\text{-}e\text{-}\liminf_{t \to +0} g_t}^{s \times bd\mathrm{w}^*}\right)^{*\dagger} = h^{*\dagger}.$$

Thus if $F_T 0_2^+$ is convex, then it is bigger–conjugate convex. In particular this holds when $F_T 0_2^+$ exists as an epi-limit. Assume now this epi–limit condition on $F_T 0_2^+$. Consider the Fitzpatrick function for $\operatorname{rec} T$ i.e.

$$F_{\operatorname{rec} T}\left(x, x^*\right) = \left(\langle \cdot, \cdot \rangle + \delta_{\operatorname{rec} T}\right)^* \left(x, x^*\right)$$
$$= \left(\langle \cdot, \cdot \rangle + \delta_{bd\mathrm{w}^*\text{-}\limsup_{t \to +0} tT}\right)^* \left(x, x^*\right)$$
$$= \left(bd\text{-}e\text{-}\liminf_{t \to +0} \left(\langle \cdot, \cdot \rangle + \delta_{tT}\right)\right)^* \left(x, x^*\right)$$
$$\leq \left(bd\text{-}e\text{-}\liminf_{t \to +0} \overline{\operatorname{co}}^{s \times \mathrm{w}^*} \left(\langle \cdot, \cdot \rangle + \delta_{tT}\right)\right)^* \left(x, x^*\right)$$
$$= \overline{bd\text{-}e\text{-}\limsup_{t \to +0} \left(\langle \cdot, \cdot \rangle + \delta_{tT}\right)^*}^{s \times bd\mathrm{w}^*} \left(x, x^*\right) = \left(F_T 0_2^+\right)\left(x, x^*\right),$$

where the third equality follows easily from definitions, the inequality from the order reversal of conjugation and the following equality from an application of Proposition 25. For Proposition 25 to be applicable we observed that from the above $g_t^{*\dagger} = P_{tT} = \overline{\text{co}}^{s\times \text{w}^*}(\langle \cdot, \cdot\rangle + \delta_{tT}) \geq 0$ has already been shown to satisfy the necessary boundedness preconditions for the application of Proposition 25.

Finally we note that we have already shown that when T is maximal, $(F_T 0_2^+)(x, x^*) = F_{\text{rec}\,T}(x, x^*) = \langle x, x^*\rangle$ for $(x, x^*) \in \text{rec}\, T$ always i.e. $M_{F_T 0_2^+} \supseteq \text{rec}\, T$. □

We recall the following results from [28] and [11, Corollary 23].

Remark 36 If $C \subseteq X \times X^*$ is closed convex, then the usual concept of recession corresponds to the "asymptotic cone" 0^+C defined by: $(y, y^*) \in 0^+C$ iff for some $c \in C$ (and hence for any $c \in C$) $c + \mathbb{R}_+(y, y^*) \subseteq C$ or $C + t(y, y^*) \subseteq C$ for all $t > 0$. When C is not closed, 0^+C may itself not be closed. When C is not closed, 0^+C can be defined by a limiting process. In [28] we have $0^+C = \limsup_{\lambda\downarrow 0} \lambda C := \bigcap_{\varepsilon>0}\left[\overline{\bigcup_{0<\lambda<\varepsilon} \lambda C}\right]$ (where the limit supremum is taken with respect to an appropriate topology associated with the duality pairing).

Lemma 37 ([11, *Lemma* 22]) *Let* $f \in \Gamma(X \times X^*)$. *The polar of the convex cone generated by* dom f^* *i.e.*

$$\left(\text{cone}\,\text{dom}\, f^*\right)^\circ := \left\{\left(x, x^*\right) \mid \delta^*_{\text{dom}\, f^*}\left(x^*, x\right) \leq 0\right\}$$

is the same as the asymptotic cone $0^+[f \leq \alpha]$ *for any* $\alpha > \inf f$.

The transpose operator can take a polar set (which resides in the dual $X^* \times X$) back into the primal space $X \times X^*$. This permits statements which otherwise would not be possible in general Banach space theory.

Corollary 38 ([11, *Corollary* 23]) *Suppose* $C \subseteq X \times X^*$ *is a* $s \times \text{w}^*$*-closed convex set. Then*

$$\left(\text{dom}\,\delta_C^*\right)^\circ = \left\{\left(w, w^*\right) \mid \delta^*_{\text{dom}\,\delta_C^*}\left(w^*, w\right) \leq 0\right\} = 0^+C.$$

Thus for $h \in bR(T)$ *we have*

$$\overline{\text{dom}}^{s\times \text{w}^*}\delta^*_{\overline{\text{co}}\, M_h} = \left(0^+\overline{\text{co}}\, M_h\right)^\circ \quad \textit{and} \quad \overline{\text{dom}}^{s\times \text{w}^*}\delta^*_{\text{dom}\, F_{M_h}} = \left(0^+\overline{\text{dom}\, F_{M_h}}^{s\times \text{w}^*}\right)^\circ. \tag{40}$$

Recall that the *asymptotic function* $f0^+$ of a proper, $(s \times \text{w}^*)$-closed, convex function f defined on $X \times X^*$ is given by

$$\begin{aligned}\left(f0^+\right)\left(x, x^*\right) &:= \lim_{\tau\to\infty} \frac{1}{\tau}\left[f\left(\left(y, y^*\right) + \tau\left(x, x^*\right)\right) - f\left(y, y^*\right)\right] \\ &= \sup_{\tau>0} \frac{1}{\tau}\left(f\left(\left(y, y^*\right) + \tau\left(x, x^*\right)\right) - f\left(y, y^*\right)\right),\end{aligned}$$

for any (or all) $(y, y^*) \in \text{dom}\, f$, noting that ***the above expression is independent of the choice of*** $(y, y^*) \in \text{dom}\, f$.

Remark 39 Recall from [28, Theorem 3B] that $(f0^+)(x, x^*) \leq \mu$ equivalent to the following:

1. $f((y,y^*)+\tau(x,x^*)) \le f(y,y^*)+\tau\mu$ for some (and equivalently for all) $(y,y^*) \in \operatorname{dom} f$ and $\tau \ge 0$.
2. $f((y,y^*)+(x,x^*)) - f(y,y^*) \le \mu$ for all $(y,y^*) \in \operatorname{dom} f$.
3. There exists a net (x_i, x_i^*) and $t_i > 0$ such that $\lim_i t_i = 0$ and $\lim_i (x_i, x_i^*) = (x, x^*)$ with

$$\lim_i t_i f\left(\frac{1}{t_i}(x_i, x_i^*)\right) \le \mu.$$

4. $\langle (x,x^*),(w,w^*)\rangle \le \mu$ for all $(w,w^*) \in \operatorname{dom} f^{*\dagger}$.

Remark 40 We note that $0^+\overline{\operatorname{dom} F_{M_h}}^{s\times w^*}$ is closed while $0^+ \operatorname{dom} F_{M_h}$ may not be.

Proposition 41 ([11, *Proposition* 26]) *Suppose* $h \in bR(T)$. *Then*

$$F_{M_h}0^+ = \delta^{*\dagger}_{\overline{\text{co}}^{s\times w^*} M_h} \tag{41}$$

Furthermore, for any $\alpha > \inf \mathcal{F}_{M_h}$,

$$0^+[F_{M_h} \le \alpha] \subseteq \operatorname{dom} \delta^{*\dagger}_{\overline{\text{co}}^{s\times w^*} M_h} \subseteq 0^+ \operatorname{dom} F_{M_h}. \tag{42}$$

We have seen that the problem of constructing a member of $bR(\operatorname{rec} T)$ for any monotone operator T, has been reduced to the problem of showing the existence of a particular epi-limit. We will now consider the recession function associated with the Fitzpatrick function.

Proposition 42 *Let* X *be an arbitrary real Banach space. Suppose* $T : X \rightrightarrows X^*$ *is monotone and* $(0,0) \in T$. *Denote its Fitzpatrick function by* F_T. *For* $(x,x^*),(y,y^*) \in X \times X^*$, *define*:

$$f_\tau((y,y^*),(x,x^*)) := \begin{cases} \frac{1}{\tau}\left[F_T((y,y^*)+\tau(x,x^*)) - F_T(y,y^*)\right] & \text{if } (y,y^*) \in \operatorname{dom} F_T \\ +\infty & \text{otherwise} \end{cases}. \tag{43}$$

Then $\tau \mapsto f_\tau$ *is monotonically nondecreasing as* $\tau \to \infty$, *and convex in* (x,x^*). *Also the pointwise limit of the family* $\tau \mapsto f_\tau((y,y^*),(\cdot,\cdot))$ *is independent of the choice of* $(y,y^*) \in \operatorname{dom} F_T$, *and exists as a convex function coinciding with* F_T0^+ *on* $X \times X^*$, *with* $\operatorname{dom} F_T0^+ \subseteq \operatorname{dom} F_T$. *Moreover for any* $(x,x^*) \in \operatorname{dom} F_T0_2^+ \subseteq \overline{\operatorname{dom} F_T}^{s\times bdw^*}$ *we have*

$$\left(F_T0_2^+\right)(x,x^*) = \overline{bd\text{-}e\text{-}\lim_{\tau\to+\infty} f_\tau((x,x^*),\cdot)}^{s\times bdw^*}(0,x^*) \quad \textit{with the epi-limit existing,}$$

$$\textit{with also,} \quad \left(F_T0_2^+\right)(x,x^*) = \overline{(F_T0^+(0,\cdot))}^{bdw^*}(x^*) + \delta_{\overline{\Pr_X \operatorname{dom} F_T}}(x)$$

for all $(x,x^*) \in X \times X^*$.

Furthermore, when T *is maximal*

$$\left(F_T0_2^+\right)(x,x^*) = \overline{(F_T0^+(0,\cdot))}^{bdw^*}(x^*) + \delta_{\overline{\text{co}}\operatorname{dom} T}(x) \quad \textit{for all } (x,x^*) \in X \times X^*. \tag{44}$$

(*In particular, when* T *maximal, then* $F_T0_2^+$ *is convex.*)

Proof Due to the convexity of F_T, [29, Theorems 23.1 or 8.5] we have $\tau \mapsto f_\tau$ (pointwise) monotonically nondecreasing (on $\operatorname{dom} F_T$) as $\tau \to \infty$, and infinite when $(y,y^*) \notin$

$\operatorname{dom} F_T$. So, $\tau \mapsto f_\tau$ is pointwise monotonically nondecreasing as $\tau \to \infty$. Moreover, $\sup_{\tau>0} f_\tau((y,y^*),(x,x^*)) \le \mu$ is equivalent to

$$\left(\left(y,y^*\right), F_T\left(y,y^*\right)\right) + \tau\left(\left(x,x^*\right),\mu\right) \in \operatorname{epi} F_T \quad \text{for all } \tau > 0$$
$$\text{or} \qquad \left(\left(x,x^*\right),\mu\right) \in 0^+ \operatorname{epi} F_T,$$

using the fact that $\operatorname{epi} F_T$ is $(s \times \mathrm{w}^*)$-closed. This means the bound of μ holds for all choices of $(y,y^*) \in \operatorname{dom} F_T$ (including $(0,0) \in \operatorname{dom} F_T$, and $(y,y^*) = (x,x^*)$, when $(x,x^*) \in \operatorname{dom} F_T$). Note that there exists $\mu < +\infty$ with $((x,x^*),\mu) \in 0^+ \operatorname{epi} F_T$ iff $(F_T 0^+)(x,x^*) < +\infty$ iff $(x,x^*) \in \operatorname{dom} F_T 0^+$. As $(0,0) \in \operatorname{dom} F_T$ and $F_T(0,0) = 0$ we have $((x,x^*),\mu) \in 0^+ \operatorname{epi} F_T$ with $\mu < +\infty$ iff $\tau((x,x^*),\mu) \in \operatorname{epi} F_T$ for all $\tau > 0$ and hence $(x,x^*) \in \operatorname{dom} F_T$ follows (take $\tau = 1$). Thus

$$\operatorname{dom} F_T 0^+ \subseteq \operatorname{dom} F_T. \tag{45}$$

Moreover it then follows that $((x,x^*),\mu) \in 0^+ \operatorname{epi} F_T$ iff $((x,x^*), F_T(x,x^*)) + \tau((x,x^*),\mu) \in \operatorname{epi} F_T$ for all $\tau > 0$ iff $f_\tau((x,x^*),(x,x^*)) \le \mu$ for all $\tau > 0$ iff $(F_T 0^+)(x,x^*) = \sup_\tau f_\tau((x,x^*),(x,x^*)) \le \mu$.

Define $h : X \times X^* \to \mathbb{R}_{+\infty}$ as follows:

$$h\left(x,x^*\right) := \sup_\tau f_\tau\left(\left(x,y^*\right),\left(0,x^*\right)\right) = \left(F_T 0^+\right)\left(0,x^*\right), \tag{46}$$

for any y^* such that $(x,y^*) \in \operatorname{dom} F_T$, if such y^* exists; with $h(x,x^*) := +\infty$ otherwise. Such a y^* exists iff $x \in \operatorname{Pr}_X \operatorname{dom} F_T$. Utilising these observations we note $h(x,x^*) = \sup_\tau f_\tau((x,x^*),(0,x^*))$ is clearly finite if $(x,x^*) \in \operatorname{dom} F_T 0^+ \subseteq \operatorname{dom} F_T$. In summary:

$$h\left(x,x^*\right) = \begin{cases} \left(F_T 0^+\right)\left(0,x^*\right) & \text{if } x \in \operatorname{Pr}_X \operatorname{dom} F_T \\ +\infty & \text{otherwise} \end{cases}. \tag{47}$$

Note that (47) implies $h \ge 0$, and $h(x,0) = 0$ for $x \in \operatorname{Pr}_X \operatorname{dom} F_T$, together with

$$\begin{aligned} &\operatorname{dom}\left(F_T 0^+(0,\cdot)\right) \cap \left(\operatorname{Pr}_X \operatorname{dom} F_T \times X^*\right) \\ &\qquad \subseteq \operatorname{dom} h \cap \left(\operatorname{Pr}_X \operatorname{dom} F_T \times X^*\right) = \operatorname{dom} h. \end{aligned} \tag{48}$$

By definitions, on $X \times X^*$,

$$h\left(x,x^*\right) \ge \left(F_T 0^+\right)\left(0,x^*\right).$$

Thus $\operatorname{dom} h \subseteq \operatorname{dom}\left(F_T 0^+(0,\cdot)\right)$ and utilising (48)

$$\operatorname{dom}\left(F_T 0^+(0,\cdot)\right) \cap \left(\operatorname{Pr}_X \operatorname{dom} F_T \times X^*\right) = \operatorname{dom} h. \tag{49}$$

Denote by $\overline{h}$ the function with the epigraph $\overline{\operatorname{epi} h}^{s \times bd\mathrm{w}^*}$ and similarly for $\overline{(F_T 0^+(0,\cdot))}^{s \times bd\mathrm{w}^*} = \overline{(F_T 0^+(0,\cdot))}^{bd\mathrm{w}^*}$. Using the identity (47) and inclusion (49), on taking lower closures we have

$$\begin{aligned} \overline{h}\left(x,x^*\right) &= \overline{(F_T 0^+(0,\cdot)) + \delta_{\operatorname{Pr}_X \operatorname{dom} F_T \times X^*}(\cdot)}^{s \times bd\mathrm{w}^*}\left(x,x^*\right) \\ &= \overline{(F_T 0^+(0,\cdot))}^{bd\mathrm{w}^*}\left(x^*\right) + \delta_{\overline{\operatorname{Pr}_X \operatorname{dom} F_T}}(x). \end{aligned} \tag{50}$$

Now we address the relationship of these quantities to $F_T 0_2^+$. By Proposition 18, part 3, and the $(s \times bd\mathrm{w}^*)$-closedness of $f_\tau((y, y^*), (\cdot, \cdot)))$ (for any fixed $(y, y^*) \in \operatorname{dom} F_T$) we have bd-e-$\lim_{\tau \to +\infty} f_\tau\left((y, y^*), \cdot\right) = F_T 0^+$ on $X \times X^*$ existing as an epi-limit, and is also a $(s \times bd\mathrm{w}^*)$-closed convex function. Consequently the epi-limit is independent of the choice of $(y, y^*) \in \operatorname{dom} F_T$.

Taking $(y, y^*) = (0, 0) \in \operatorname{dom} F_T$ then $f_\tau\left((0, 0), \cdot\right) = \frac{1}{\tau} F_T(\tau \cdot)$ and for all $(x, x^*) \in X \times X^*$

$$\begin{aligned}(F_T 0^+)\left(x, x^*\right) &= \sup_\tau f_\tau\left((0, 0), \left(x, x^*\right)\right) \\ &= \left(bd\text{-}e\text{-}\lim_{t \to +0} t F_T\left(\frac{(\cdot, \cdot)}{t}\right)\right)\left(x, x^*\right).\end{aligned}$$

Let $(x, x^*) \in X \times X^*$. For any $\epsilon > 0$, suppose the net $\left(x_\alpha, x_\alpha^*\right) \to_{s \times bd\mathrm{w}^*} (x, x^*)$ is chosen so that $\lim_\alpha t_\alpha F_T\left(x_\alpha, \frac{x_\alpha^*}{t_\alpha}\right) \le \left(bd\text{-}e\text{-}\liminf_{t \to +0} F_{tT}\right)(x, x^*) + \epsilon$ whenever the latter is finite. Then $\left(t_\alpha x_\alpha, x_\alpha^*\right) \to_{t_\alpha \to 0} (0, x^*)$ so that

$$\begin{aligned}(F_T 0^+)\left(0, x^*\right) &= \sup_{\tau > 0} f_\tau((0, 0), (0, x^*)) \le \liminf_\alpha t_\alpha F_T\left(\frac{1}{t_\alpha}\left(t_\alpha x_\alpha, x_\alpha^*\right)\right) \\ &= \liminf_\alpha t_\alpha F_T\left(x_\alpha, \frac{x_\alpha^*}{t_\alpha}\right) \le \left(bd\text{-}e\text{-}\liminf_{t \to +0} F_{tT}\right)\left(x, x^*\right) + \epsilon.\end{aligned}$$

As $\epsilon > 0$ is arbitrary we conclude, for all $(x, x^*) \in X \times X^*$,

$$(F_T 0^+)\left(0, x^*\right) = \sup_{\tau > 0} f_\tau((0, 0), (0, x^*)) \le (bd\text{-}e\text{-}\liminf_{t \to +0} F_{tT})(x, x^*), \tag{51}$$

and so

$$\overline{(F_T 0^+(0, \cdot))}^{bd\mathrm{w}^*}\left(x^*\right) \le \overline{bd\text{-}e\text{-}\liminf_{t \to +0} F_{tT}}^{s \times bd\mathrm{w}^*}\left(x, x^*\right) = (F_T 0_2^+)\left(x, x^*\right). \tag{52}$$

On the other hand, $F_T(0, 0) = 0$ so

$$\begin{aligned}(F_T 0^+)\left(0, x^*\right) &= \sup_{\tau > 0} \frac{1}{\tau}\left[F_T\left((0, 0) + \tau\left(0, x^*\right)\right) - F_T(0, 0)\right] = \sup_{\tau > 0} f_\tau((0, 0), (0, x^*)) \\ &= \sup_{\tau > 0} \frac{1}{\tau} F_T\left(\tau\left(0, x^*\right)\right) = \lim_{t \to +0} F_{tT}\left(0, x^*\right) \\ &\ge \inf_{\substack{\left\{(x_\alpha, x_\alpha^*) \to^{s \times bd\mathrm{w}^*} (0, x^*)\right\} \\ \{t_\alpha \to +\infty\}}} \liminf_\alpha F_{t_\alpha T}\left(\left(x_\alpha, x_\alpha^*\right)\right) \\ &= \left(bd\text{-}e\text{-}\liminf_{t \to +0} F_{tT}\right)\left(0, x^*\right) \ge (F_T 0_2^+)\left(0, x^*\right).\end{aligned}$$

Combining this with (51) we have

$$(F_T 0^+)\left(0, x^*\right) = \sup_{\tau > 0} f_\tau((0, 0), (0, x^*)) = (bd\text{-}e\text{-}\liminf_{t \to +0} F_{tT})(0, x^*) \tag{53}$$

and, combining instead with (52), gives $\overline{(F_T 0^+(0, \cdot))}^{bd\mathrm{w}^*}(x^*) = (F_T 0_2^+)(0, x^*)$, for all $x^* \in X^*$.

Recall that $\overline{h}$ denotes the closure $\overline{h}^{s\times bd\mathrm{w}^*}$. Then for each $(x,x^*)\in X\times X^*$ there exists a bounded net $\left(x_\alpha,x_\alpha^*\right)\to^{s\times bd\mathrm{w}^*}(x,x^*)$ such that

$$\overline{h}\left(x,x^*\right)=\lim_\alpha h\left(x_\alpha,x_\alpha^*\right).$$

Suppose $\overline{h}\,(x,x^*)<+\infty$ (noting that we have $\overline{h}\geq 0$). Let $\left(x_\alpha,x_\alpha^*\right)\to^{s\times bd\mathrm{w}^*}(x,x^*)$ be any fixed bounded net with $\lim_\alpha h\left(x_\alpha,x_\alpha^*\right)=\overline{h}\,(x,x^*)$. Then, for $\left(x_\alpha,y_\alpha^*\right)\in\operatorname{dom}F_T$, as $-\infty<\widehat{f_\tau}\left(x_\alpha,x_\alpha^*\right):=f_\tau\left(\left(x_\alpha,y_\alpha^*\right),\left(0,x_\alpha^*\right)\right)\leq h\left(x_\alpha,x_\alpha^*\right)$, we have for any $\varepsilon>0$, and $\tau_\alpha\to+\infty$ that eventually,

$$\widehat{f_{\tau_\alpha}}\left(x_\alpha,x_\alpha^*\right)-\varepsilon\leq\overline{h}\left(x,x^*\right).$$

Due to (43), with $\tau\ =\frac{1}{t}$, in the expression for $\widehat{f_\tau}\,(w,w^*):=f_\tau\left((w,y^*),(0,w^*)\right)\ \ (w\in\Pr_X\operatorname{dom}F_T)$ we have, recalling (for any y^* for which $(w,y^*)\in\operatorname{dom}F_T$),

$$\begin{aligned} g_t\left(w,w^*\right)&=\frac{1}{\tau}F_T\left(w,y^*+\tau w^*\right),\quad\text{that}\\ g_t\left(w,w^*\right)&=\widehat{f_\tau}\left(w,w^*\right)+tF_T\left(w,y^*\right).\end{aligned}\tag{54}$$

As we have chosen a fixed bounded net $\left(x_\alpha,x_\alpha^*\right)\to^{s\times\mathrm{w}^*}(x,x^*)$ so *for any* $t_\alpha\downarrow 0^+$ and associated $\tau_\alpha\to\infty$ we will have $-\infty<\limsup_\alpha\widehat{f_{\tau_\alpha}}\left(x_\alpha,x_\alpha^*\right)\leq\limsup_\alpha h\left(x_\alpha,x_\alpha^*\right)\leq\overline{h}\,(x,x^*)<+\infty$. Thus we eventually have $x_\alpha\in\Pr_X\operatorname{dom}F_T$ and for any $\left(x_\alpha,y_\alpha^*\right)\in\operatorname{dom}F_T$ with $\frac{y_\alpha^*}{\tau_\alpha}\to 0$ (which we achieve via our selection of t_α) we have a bounded net $\left(x_\alpha,x_\alpha^*+\frac{y_\alpha^*}{\tau_\alpha}\right)\to_\alpha^{s\times bd\mathrm{w}^*}(x,x^*)$ with

$$\left(F_T0_2^+\right)\left(x,x^*\right)\leq\ \liminf_\alpha g_{t_\alpha}\left(x_\alpha,x_\alpha^*\right)=\liminf_\alpha\frac{1}{\tau_\alpha}F_T\left(x_\alpha,\tau_\alpha\left(x_\alpha^*+\frac{y_\alpha^*}{\tau_\alpha}\right)\right),$$

noting that by Lemma 35, $\Pr_X\operatorname{dom}F_T0_2^+\subseteq\overline{\Pr_X\operatorname{dom}F_T}$. Then from

$$\begin{aligned} g_{t_\alpha}\left(x_\alpha,x_\alpha^*\right)-\widehat{f_{\tau_\alpha}}\left(x_\alpha,x_\alpha^*\right)&=t_\alpha F_T\left(x_\alpha,y_\alpha^*\right)\quad\text{we get}\\ \liminf_\alpha g_{t_\alpha}\left(x_\alpha,x_\alpha^*\right)-\limsup_\alpha\widehat{f_{\tau_\alpha}}\left(x_\alpha,x_\alpha^*\right)&\leq\liminf_\alpha\left[g_{t_\alpha}\left(x_\alpha,x_\alpha^*\right)-\ \widehat{f_{\tau_\alpha}}\left(x_\alpha,x_\alpha^*\right)\right]\\ &=\liminf_\alpha t_\alpha F_T\left(x_\alpha,y_\alpha^*\right)\\ \text{and so}\qquad\left(F_T0_2^+\right)\left(x,x^*\right)&\leq\liminf_\alpha g_{t_\alpha}\left(x_\alpha,x_\alpha^*\right)\\ &\leq\limsup_\alpha\widehat{f_{\tau_\alpha}}\left(x_\alpha,x_\alpha^*\right)+\liminf_\alpha t_\alpha F_T\left(x_\alpha,y_\alpha^*\right)\\ &\leq\limsup_\alpha h\left(x_\alpha,x_\alpha^*\right)+\liminf_\alpha t_\alpha F_T\left(x_\alpha,y_\alpha^*\right)\\ &\leq\overline{h}\left(x,x^*\right)+\liminf_\alpha t_\alpha F_T\left(x_\alpha,y_\alpha^*\right),\end{aligned}\tag{55}$$

where the fourth inequality follows from the fact that $\widehat{f_{\tau_\alpha}}\leq h$. Note that $F_T\left(x_\alpha,y_\alpha^*\right)$ is finite for all α. Since this inequality holds for all $t_\alpha\to 0$, (with $t_\alpha y_\alpha^*=\frac{y_\alpha^*}{\tau_\alpha}\to 0$) we select $t_\alpha:=\min\left\{\frac{\epsilon_\alpha}{|F_T(x_\alpha,y_\alpha^*)|+1},\frac{\epsilon_\alpha}{\|y_\alpha^*\|}\right\}$ where $\epsilon_\alpha\to 0$ so that $t_\alpha\to 0$. Then $0\leq\liminf_\alpha t_\alpha F_T\left(x_\alpha,y_\alpha^*\right)\leq$

$\epsilon_\alpha \to 0$ and $\frac{y_\alpha^*}{\tau_\alpha} = t_\alpha y_\alpha^* \to 0$. Thus

$$(F_T 0_2^+)\left(x, x^*\right) \le (bd\text{-}e\text{-}\liminf_{t\to+0} F_{tT})(x, x^*) \le \liminf_{\alpha} g_{t_\alpha}(x_\alpha, x_\alpha^*)$$
$$\le \limsup_{\alpha} f_{\tau_\alpha}(x_\alpha, x_\alpha^*), (0, x_\alpha^*)) \le \overline{h}\left(x, x^*\right).$$

In particular we have established

$$(bd\text{-}e\text{-}\liminf_{t\to+0} F_{tT})(x, x^*) \le \overline{h}(x, x^*) \le h(x, x^*) \quad \text{for all } (x, x^*) \in X \times X^*, \tag{56}$$

and so $F_T 0_2^+ \le \overline{h}$ on $X \times X^*$. For the reverse inequality, we note, for $(x, x^*) \in \operatorname{dom} F_T 0_2^+$, that from (35), have $x \in \overline{\operatorname{Pr}_X \operatorname{dom} F_T}$ so that using (50) and (52), it follows that

$$\overline{h}\left(x, x^*\right) = \overline{(F_T 0^+(0, \cdot))}^{bd\mathrm{w}^*}\left(x^*\right) + \delta_{\overline{\operatorname{Pr}_X \operatorname{dom} F_T}}(x) \le (F_T 0_2^+)\left(x, x^*\right), \tag{57}$$

forcing equality on $X \times X^*$ and verifying the result.

Note that when maximal, the relation (44) obtains from use of [6, Thm 3.6]. □

Corollary 43 *Let X be an arbitrary real Banach space. Suppose $T : X \rightrightarrows X^*$ is maximal monotone and $(0, 0) \in T$. Denote its Fitzpatrick function by F_T. Then there exists a convex function h on $X \times X^*$ for which*: (*i*)

$$h = bd\text{-}e\text{-}\liminf_{t\to+0} F_{tT} \quad \textit{on} \quad \operatorname{Pr}_X \operatorname{dom} F_T \times X^*; \tag{58}$$

(*ii*) $\overline{h}^{s\times bd\mathrm{w}^*} = F_T 0_2^+$; (*iii*) $\operatorname{dom} h = \operatorname{Pr}_X \operatorname{dom} F_T \times \operatorname{dom} F_T 0^+(0, \cdot)$ *where* $\operatorname{dom} F_T 0^+(0, \cdot)$ *is a convex cone in X^*. Furthermore for any $x \in \operatorname{Pr}_X \operatorname{dom} F_T$ we have the epi-limit* (58) *attained in that there exists y^* such that $(x, y^*) \in \operatorname{dom} F_T$ and*

$$h(x, x^*) = \lim_{t\to 0} F_{tT}(x, x^* + t y^*).$$

Proof We have already defined a convex function via (46) which we will show satisfies the assertions of this Corollary. First we note that from (56) it follows that $h(x, x^*) \ge (bd\text{-}e\text{-}\liminf_{t\to+0} F_{tT})(x, x^*)$. Then, from the representation (54) of g_t, the fact that when $x \in \operatorname{Pr}_X \operatorname{dom} F_T$ there exists $y^* \in X^*$ such that $(x, y^*) \in \operatorname{dom} F_T$ and $F_T \ge 0$ we have: (recalling the functions $\widehat{f}$ from the preceding proof)

$$(bd\text{-}e\text{-}\liminf_{t\to+0} F_{tT})(x, x^*) = (bd\text{-}e\text{-}\liminf_{t\to+0} g_t)(x, x^*)$$
$$= bd\text{-}e\text{-}\liminf_{t\to+0}(\widehat{f}_{\frac{1}{t}}\left(x, x^*\right) + t F_T\left(x, y^*\right))$$
$$\ge bd\text{-}e\text{-}\lim_{\tau\to+\infty} \widehat{f}_\tau\left(x, x^*\right) = h(x, x^*)$$

where we have used Proposition 18 part 3 and the pointwise monotonicity of $\tau \mapsto \widehat{f}_\tau(\cdot)$ to obtain that last equality. This implies, for $x \in \operatorname{Pr}_X \operatorname{dom} F_T$ that we have equality:

$$h(x, x^*) = (bd\text{-}e\text{-}\liminf_{t\to+0} F_{tT})(x, x^*) = \sup_{\tau>0} f_\tau((x, y^*), (0, x^*)) = (F_T 0^+)\left(0, x^*\right), \tag{59}$$

where we have used (53) and the independence of the recession function on the choice of base-point $(x, y^*) \in \operatorname{dom} F_T$.

As $\sup_{\tau>0} f_\tau((x, y^*), (0, x^*)) = \lim_{t\to 0}(t F_T(x, \frac{x^*+ty^*}{t}) - t F_T(x, y^*)) = \lim_{t\to 0} F_{tT}(x, x^* + ty^*)$ (on placing $\tau = \frac{1}{t}$) it is evident that the associated epi-limit is actually attained when $x \in \operatorname{Pr}_X \operatorname{dom} F_T$ i.e. $(bd\text{-}e\text{-}\liminf_{t\to_+ 0} F_{tT})(x, x^*) = \lim_{t\to 0} F_{tT}(x, x^* + ty^*)$. From (49) and (50) it is evident that $\operatorname{dom} h = \operatorname{Pr}_X \operatorname{dom} F_T \times \operatorname{dom} F_T 0^+(0, \cdot)$. Finally we note that positive homogeneity implies $\operatorname{dom} F_T 0^+(0, \cdot)$ is a cone in X^* and the convexity of $x^* \mapsto F_T(0, x^*)$ implies that this (fixed) cone is convex i.e. the domain of h is rectangular. □

The next result augments that of [25, Lemma 2] which deals with a parallel characterisation of the representative function of the range recession operator in a reflexive space.

Corollary 44 *Let $g \in bR(T)$ and assume $(0, 0) \in M_g$. Then we have the following.*

1. *When M_g is maximal, then $(F_{M_g} 0_2^+)(x, x^*) = \delta^*_{\overline{\operatorname{co}} \operatorname{dom} M_g}(x^*) + \delta_{\overline{\operatorname{co}} \operatorname{dom} M_g}(x)$ for all $(x, x^*) \in X \times X^*$.*
2. *When $\overline{\operatorname{co}} \operatorname{dom} M_g = \overline{\operatorname{Pr}_X \operatorname{dom} F_{M_g}}$ (which is true when M_g is maximal, [6]) we have*

$$M_{F_{M_g} 0_2^+} = N_{\overline{\operatorname{co}} \operatorname{dom} M_g},$$

   which is maximal.
3. *When a monotone operator $T \ni (0, 0)$ is maximal, then $M_{F_T 0_2^+} = N_{\overline{\operatorname{co}} \operatorname{dom} T}$ is also maximal and $F_T 0_2^+ = \delta^*_{\overline{\operatorname{co}} \operatorname{dom} T} + \delta_{\overline{\operatorname{co}} \operatorname{dom} T} \geq \langle \cdot, \cdot \rangle$.*

Proof We use Proposition 41 and 42 to note that

$$(F_{M_g} 0^+)(0, \cdot) = \delta^{*\dagger}_{\overline{\operatorname{co}} M_g}\left(0, x^*\right) = \delta^*_{\overline{\operatorname{co}} \operatorname{dom} M_g}\left(x^*\right)$$

which is w*-lower–semicontinuous. Moreover, $\operatorname{co} \operatorname{dom} M_g \subseteq P_X \operatorname{dom} P_{M_g} \subseteq P_X \operatorname{dom} F_{M_g}$ with $\overline{\operatorname{co}} \operatorname{dom} M_g = \overline{\operatorname{Pr}_X \operatorname{dom} F_{M_g}}$ (when M_g is maximal) so $\overline{(F_{M_g} 0^+(0, \cdot))}^{bd\mathrm{w}^*}(x^*) = \delta^*_{\overline{\operatorname{co}} \operatorname{dom} M_g}(x^*)$ and

$$\begin{aligned}
(F_{M_g} 0_2^+)\left(x, x^*\right) &= \overline{(F_{M_g} 0^+(0, \cdot))}^{bd\mathrm{w}^*}\left(x^*\right) + \delta_{\overline{\operatorname{Pr}_X \operatorname{dom} F_{M_g}}}(x) \\
&= \delta^*_{\overline{\operatorname{co}} \operatorname{dom} M_g}\left(x^*\right) + \delta_{\overline{\operatorname{Pr}_X \operatorname{dom} F_{M_g}}}(x) \\
&= \delta^*_{\overline{\operatorname{co}} \operatorname{dom} M_g}\left(x^*\right) + \delta_{\overline{\operatorname{co}} \operatorname{dom} M_g}(x) \text{ when } M_g \text{ is maximal.}
\end{aligned}$$

Thus whenever $\overline{\operatorname{co}} \operatorname{dom} M_g = \overline{\operatorname{Pr}_X \operatorname{dom} F_{M_g}}$ we have

$$\begin{aligned}
M^{\leq}_{F_{M_g} 0_2^+} &= \left\{ \left(x, x^*\right) \,\middle|\, \langle x, x^* \rangle = \delta^*_{\overline{\operatorname{co}} \operatorname{dom} M_g}\left(x^*\right) + \delta_{\overline{\operatorname{Pr}_X \operatorname{dom} F_{M_g}}}(x) \right\} \\
&= \left\{\left(x, x^*\right) \mid x \in \overline{\operatorname{co}} \operatorname{dom} M_g \text{ and } \langle y - x, x^* \rangle \leq 0 \text{ for all } y \in \overline{\operatorname{co}} \operatorname{dom} M_g\right\} \\
&= N_{\overline{\operatorname{co}} \operatorname{dom} M_g}. \qquad (60)
\end{aligned}$$

When T is maximal, so $g := F_T \in bR(T)$, then $M_g = T$ is established, so by Part 2, $M_{F_T 0_2^+} = N_{\overline{\operatorname{co}} \operatorname{dom} T}$ which is maximal. □

Recall that we call a monotone operator T of type "Brøndsted-Rockafellar" (BR) [4], if, whenever $(x, x^*) \in X \times X^*$ and $\alpha, \beta > 0$ satisfy

$$\inf_{(y,y^*)\in T} \langle x - y, x^* - y^* \rangle \geq -\alpha\beta\,,$$

then there exists $(w, w^*) \in T$ with $\|x - w\| \leq \alpha$ and $\|x^* - w^*\| \leq \beta$.

Remark 45 Note that when $(x, x^*) \in \operatorname{dom} F_T$ we have $+\infty > F_T(x, x^*) = \langle x, x^* \rangle - \inf_{(y,y^*)\in T} \langle x - y, x^* - y^* \rangle$ so

$$\inf_{(y,y^*)\in T} \langle x - y, x^* - y^* \rangle = \langle x, x^* \rangle - F_T\left(x, x^*\right) \geq -b > -\infty$$

and if T is of BR-type then hence for any $\alpha, \beta > 0$ such that $\alpha\beta = b$ we have existence of $(w, w^*) \in T$ with $\|x - w\| \leq \alpha$ and $\|x^* - w^*\| \leq \beta$. Hence (using the $\alpha = \frac{1}{n}$ and $\beta = bn$) we have existence of $w_n \in \operatorname{dom} T$ such that $w_n \to x$ and hence $x \in \overline{\operatorname{dom}}\, T$. That is, $\operatorname{Pr}_X \operatorname{dom} F_T \subseteq \overline{\operatorname{dom} T}$ $(\subseteq \overline{\operatorname{Pr}_X \operatorname{dom} F_T})$. Hence $\overline{\operatorname{dom} T} = \overline{\operatorname{Pr}_X \operatorname{dom} F_T}$ is convex.

The following obtains by a similar argument as given in [13, Proposition 13].

Lemma 46 *Suppose* $T : X \rightrightarrows X^*$ *is a monotone operator of type* (*BR*) *where* X *is an arbitrary real Banach space. Then for all* $x \in \overline{\operatorname{dom} T}$ $(= \overline{\operatorname{co}} \operatorname{dom} T$ *from remark* 45) *we have* $N_{\overline{\operatorname{dom} T}}(x) = (\operatorname{rec} T)(x) = \operatorname{rec}(T^\mu)(x)$. *Thus* $\operatorname{rec} T = N_{\overline{\operatorname{dom} T}}$ *and hence* $\operatorname{rec} T$ *is maximal.*

Proof See appendix for proof. □

7 Conditions for the Almost–Convexity Property

It is conjectured that the domain of a maximal monotone operator T has the *almost convex* property.

Definition 47 We say that the **almost convex property** (ACP) holds for a monotone operator T iff we have $\overline{\operatorname{dom} T}$ convex.

This property has been studied by a number of authors and good summary of the best results to date may be found in [6]. Also see [32] for some recent insights. Note that by definition $\operatorname{dom}(\operatorname{rec} T) \subseteq \overline{\operatorname{dom} T}$—indeed, in [13] equality is shown by demonstrating that $x \in \overline{\operatorname{dom} T}$ implies $(x, 0) \in \operatorname{rec} T$. Thus maximality of $\operatorname{rec} T$ immediately delivers the almost–convexity property. We note that the maximality of $\operatorname{rec} T$ has already been shown to be true in reflexive spaces in [11], providing another proof of the almost convexity property in reflexive spaces. In non-reflexive spaces we know that all monotone operators that are subdifferentials and those with non-empty interiors in their domains also possess the almost convexity property.

Proposition 48 *Let* X *be an arbitrary real Banach space and* $T : X \rightrightarrows X^*$ *be monotone. Suppose* $g \in bR(T)$ *is such that* $\overline{\operatorname{co}} \operatorname{dom} M_g = \overline{\operatorname{Pr}_X \operatorname{dom} F_{M_g}}$ (*which is true when* M_g *is maximal* [6]). *Then* $\operatorname{rec} M_g$ *is a "pre-maximal monotone operator", in the sense that*

$$\left(\operatorname{rec} M_g\right)^\mu = M^{\leq}_{F_{\operatorname{rec} M_g}} = M_{F_{M_g} 0_2^+} = N_{\overline{\operatorname{co}} \operatorname{dom} M_g} \quad \textit{iff} \quad \operatorname{dom}\left(\operatorname{rec} M_g\right)^\mu \subseteq \overline{\operatorname{Pr}_X \operatorname{dom} F_{M_g}}.$$

Thus when T *and* $\operatorname{rec} T$ *are maximal, we have* $\operatorname{rec} T = N_{\overline{\operatorname{co}} \operatorname{dom} T}$.

Proof First we note that by [13, Lemma 11], $\operatorname{dom}\left(\operatorname{rec} M_g\right)=\overline{\operatorname{dom}} M_g$ so we have $0 \in \operatorname{rec} M_g\,(x)$ for all $x \in \overline{\operatorname{dom}} M_g$ and

$$\begin{aligned} F_{\operatorname{rec} M_g}(x, x^*) &= \sup_{(y,y^*)\in \operatorname{rec} M_g} \langle x, y^*\rangle + \langle y, x^*\rangle - \langle y, y^*\rangle \\ &\geq \sup_{y\in\overline{\operatorname{dom}} M_g} \langle x, 0\rangle + \langle y, x^*\rangle - \langle y, 0\rangle = \delta^*_{\overline{\operatorname{co}}\operatorname{dom} M_g}\left(x^*\right) \end{aligned}$$

and by Lemma 35 part 3, $F_{\operatorname{rec} M_g} \leq F_{M_g} 0_2^+$ and so we have from Corollary 44 part 2 that

$$M_{F_{M_g}0_2^+} = N_{\overline{\operatorname{co}}\operatorname{dom} M_g} \subseteq \left(\operatorname{rec} M_g\right)^{\mu} = M^{\leq}_{F_{\operatorname{rec} M_g}} \subseteq \left\{\left(x, x^*\right) \mid \delta^*_{\overline{\operatorname{co}}\operatorname{dom} M_g}\left(x^*\right) \leq \langle x^*, x\rangle\right\}.$$

So, we have $\operatorname{dom}\left(\operatorname{rec} M_g\right)^{\mu} \subseteq \overline{\operatorname{Pr}_X \operatorname{dom} F_{M_g}} = \overline{\operatorname{co}}\operatorname{dom} M_g$ iff

$$\left(\operatorname{rec} M_g\right)^{\mu} = M^{\leq}_{F_{\operatorname{rec} M_g}} \subseteq \left\{\left(x, x^*\right) \mid \delta^*_{\overline{\operatorname{co}}\operatorname{dom} M_g}\left(x^*\right) \leq \langle x^*, x\rangle,\, x \in \overline{\operatorname{co}}\operatorname{dom} M_g\right\} = N_{\overline{\operatorname{co}}\operatorname{dom} M_g}.$$

This implies $\left(\operatorname{rec} M_g\right)^{\mu}$ is maximal (confirming pre-maximality). Furthermore when $\left(\operatorname{rec} M_g\right)^{\mu} = N_{\overline{\operatorname{co}}\operatorname{dom} M_g}$ we have $\operatorname{dom}\left(\operatorname{rec} M_g\right)^{\mu} = \overline{\operatorname{co}}\operatorname{dom} M_g = \overline{\operatorname{Pr}_X \operatorname{dom} F_{M_g}}$.

When T and $\operatorname{rec} T$ are maximal we may choose $g = F_T$ and get $\operatorname{dom}\left(\operatorname{rec} M_g\right)^{\mu} = \operatorname{dom}\left(\operatorname{rec} T^{\mu}\right)^{\mu} = \operatorname{dom}\operatorname{rec} T \subseteq \overline{\operatorname{Pr}_X \operatorname{dom} F_{M_g}}$ and so $\left(\operatorname{rec} M_g\right)^{\mu} = \operatorname{rec} T = N_{\overline{\operatorname{co}}\operatorname{dom} T}$. □

We now focus on studying when $F_M 0_2^+$ is a bigger–conjugate representative function for $\operatorname{rec} T$ with the domain $\overline{\operatorname{dom}} T$. In the following the condition (61) has a striking similarity to that used in [30] and related works [6].

Proposition 49 *Let X be an arbitrary Banach space. Suppose $T : X \rightrightarrows X^*$ is maximal monotone with Fitzpatrick function F_T. Denote $g := F_T 0_2^+$. Suppose also that*

$$\sup_{(y,y^*)\in T} \langle x - y, y^*\rangle = +\infty \quad \textit{for all } x \notin \overline{\operatorname{dom}} T. \tag{61}$$

Then $\operatorname{rec} T \subseteq M_g$ and $\operatorname{dom}\left(\operatorname{rec} T\right)^{\mu\mu} \subseteq \operatorname{dom} M_g \subseteq \overline{\operatorname{dom}} T$.

Proof Via the maximality of T the Fitzpatrick function F_T represents T in that $T = M_{F_T}$. We note that by Lemma 35, $g \in bR\,(\operatorname{rec} T)$ and $M_g = N_{\overline{\operatorname{co}}\operatorname{dom} T}$ for $g = F_T 0_2^+$. We will make use of the following inequality: for any $(x, x^*) \in X \times X^*$ and $(y, y^*) \in T$ and $\tau > 0$ we have from the convexity of $F_T \in bR\,(T)$ that

$$\begin{aligned} \frac{1}{\tau+1} F_T\left(x, (1+\tau)\, x^*\right) + \frac{\tau}{1+\tau} F_T\left(y, y^*\right) &\geq F_T\left(\frac{1}{\tau+1} x + \frac{\tau}{1+\tau} y,\, x^* + \frac{\tau}{1+\tau} y^*\right) \\ &\geq \langle \frac{1}{\tau+1} x + \frac{\tau}{1+\tau} y,\, x^* + \frac{\tau}{1+\tau} y^*\rangle \end{aligned}$$

and when $(y, y^*) \in T$ we have $F_T\,(y, y^*) = \langle y, y^*\rangle$ and there follows:

$$\begin{aligned} \frac{1}{\tau+1} F_T\left(x, (1+\tau)\, x^*\right) &\geq \left(\frac{1}{\tau+1}\right)^2 \langle x + \tau y, (1+\tau)\, x^* + \tau y^*\rangle - \frac{\tau}{1+\tau}\langle y, y^*\rangle \\ &= \langle x, x^*\rangle - \frac{\tau}{(\tau+1)^2}\langle x - y, x^* - y^*\rangle - \left(\frac{\tau}{1+\tau}\right)^2 \langle x - y, x^*\rangle \end{aligned}$$

$$= \langle x, x^* \rangle + \frac{\tau (\tau - 1)}{(\tau + 1)^2} \langle x - y, x^* \rangle + \frac{\tau}{(\tau + 1)^2} \langle x - y, y^* \rangle. \quad (62)$$

We next show that $\operatorname{dom} M_g \subseteq \overline{\operatorname{dom}} T$. We assume $x \notin \overline{\operatorname{dom}} T$. Thus there exists $\delta > 0$ such that $B_\delta(x) \cap \overline{\operatorname{dom}} T = \emptyset$. We may also assume that $x \in \overline{\operatorname{co}} \operatorname{dom} T$, otherwise by (44) we have $\left(F_T 0_2^+\right)(x, x^*) = +\infty > \langle x, x^* \rangle$ and $x \notin M_g$ is deduced immediately.

Taking the supremum over all $(y, y^*) \in T$ in (62) we obtain

$$\begin{aligned} g_{\frac{1}{\tau+1}}\left(x, x^*\right) &= \frac{1}{\tau+1} F_T\left(x, (1+\tau) x^*\right) \\ &\geq \langle x, x^* \rangle + \frac{\tau (\tau - 1)}{(\tau + 1)^2} \sup_{y \in \operatorname{dom} T} \langle x - y, x^* \rangle + \frac{\tau}{(\tau + 1)^2} \sup_{(y, y^*) \in T} \langle x - y, y^* \rangle = +\infty. \end{aligned}$$

As this holds also for all $B_\delta(x) \times X^*$ we have $bd\text{-}e\text{-}\liminf_{t \to +0} g_t \geq +\infty$ on $B_\delta(x) \times X^*$ for some $\delta > 0$. Hence

$$\left(F_T 0_2^+\right)\left(x, x^*\right) = \left(\overline{bd\text{-}e\text{-}\liminf_{t \to +0} g_t}^{s \times bdw^*}\right)\left(x, x^*\right) = +\infty > \langle x, x^* \rangle \text{ for all } x^* \in X^*.$$

Hence $\left(x, x^*\right) \notin \operatorname{dom} M_{F_T 0_2^+}$ and so $\operatorname{dom} M_{F_T 0_2^+} \subseteq \overline{\operatorname{dom}} T$. As $\operatorname{dom} M_g \subseteq \overline{\operatorname{dom}} T$ it immediately follows, since $(\operatorname{rec} T) \subseteq M_g$ by Lemma 35, that we have $(\operatorname{rec} T)^{\mu\mu} \subseteq \left(M_g\right)^{\mu\mu} = M_g$ (due to Proposition 27) so $\operatorname{dom} (\operatorname{rec} T)^{\mu\mu} \subseteq \operatorname{dom} M_g \subseteq \overline{\operatorname{dom}} T$. □

We will link the almost–convexity to the recession operator and also generalise the recent result of [32].

Theorem 50 *Let X be an arbitrary Banach space. Suppose $T : X \rightrightarrows X^*$ is maximal monotone with Fitzpatrick function F_T and denote $g := F_T 0_2^+$. When $\operatorname{dom} M_g \subseteq \overline{\operatorname{dom}} T$ (as is the case under the assumptions of Proposition 49) then it follows that T is almost-convex. Moreover the condition (61) of Proposition 49 is necessary and sufficient for almost-convexity of T.*

Proof By Proposition 42, g is convex, so Lemma 35 gives $g \in bR(\operatorname{rec} T)$ with $\operatorname{rec} T \subseteq M_g$. Since $M_g = (M_g)^{\mu\mu}$ (thanks to Proposition 27), then Proposition 32 yields

$$\operatorname{co} \overline{\operatorname{dom}} T \subseteq P_X\left[(\operatorname{rec} T)^{\mu\mu}\right] = \operatorname{dom} (\operatorname{rec} T)^{\mu\mu} \subseteq \operatorname{dom} M_g \subseteq \overline{\operatorname{dom}} T\,,$$

whence $\overline{\operatorname{dom}} T = \operatorname{co} \overline{\operatorname{dom}} T$, implying convexity of the closure of the domain. Thus (61) of Proposition 49 is sufficient for almost-convexity of T.

For the converse, suppose $\operatorname{dom} T$ has convex closure. Then by Corollary 44 (part 3) we have $(F_T 0_2^+)(x, x^*) = \delta^*_{\overline{\operatorname{dom}} T}(x^*) + \delta_{\overline{\operatorname{dom}} T}(x)$ and so $x \notin \overline{\operatorname{dom}} T$ iff $(F_T 0_2^+)(x, x^*) = +\infty$. For any $t_\beta \to 0^+$, $\left(x_\beta, x_\beta^*\right) \to^{s \times bdw^*} (x, x^*)$ with $x_\beta^* \in (\operatorname{dom} T)^\perp$ we have from the identity $g_{t_\beta} = F_{t_\beta T}$, the definition of the Fitzpatrick function $F_{t_\beta T}$, (and direct calculation) that

$$\liminf_\beta g_{t_\beta}\left(x_\beta, x_\beta^*\right) = \liminf_\beta \sup_{(y, y^*) \in T} t_\beta \langle x_\beta - y, y^* \rangle \geq (F_T 0_2^+)\left(x, x^*\right) = +\infty.$$

Hence $\sup_{(y, y^*) \in T} \langle x_\beta - y, y^* \rangle = +\infty$ eventually. In particular we have $\sup_{(y, y^*) \in T} \langle x - y, y^* \rangle = +\infty$ for $x \notin \overline{\operatorname{dom}} T$. □

We have $g = F_{M_g} 0_2^+ \in bR\,(\operatorname{rec} T)$ but do not yet know in general that $\overline{\operatorname{rec} M_g}^{s\times bd\mathrm{w}^*} = (M_g)^{\mu\mu}$.

Lemma 51 *Suppose X be an arbitrary real Banach space and $T : X \rightrightarrows X^*$ is a maximal monotone operator. Then $F_T 0_2^+ \in bR\,(\operatorname{rec} T)$ with*

$$\overline{\operatorname{rec} T}^{s\times bd\mathrm{w}^*} = \overline{bds\mathrm{w}^*\text{-}\limsup_{t\to +0} (tT)^{\mu}}^{s\times bd\mathrm{w}^*} \subseteq M_{F_T 0_2^+} = N_{\overline{\mathrm{co}}\,\mathrm{dom}\,T}.$$

Proof The first equality holds since maximality of T implies $(tT)^{\mu} = tT$, so we may apply (8) to obtain $\operatorname{rec} T = bds\mathrm{w}^*\text{-}\limsup_{t\to+0} (tT)^{\mu}$. We next investigate

$$\begin{aligned}
\operatorname{rec} T = bds\mathrm{w}^*\text{-}\limsup_{t\to +0} (tT)^{\mu} &= \Big\{ (x, x^*) \;\Big|\; \exists\ (x_\alpha, x_\alpha^*, t_\alpha) \to^{s\times bd\mathrm{w}^*} (x, x^*, 0) \\
&\qquad \text{with } (x_\alpha, x_\alpha^*) \in M_{F_{t_\alpha T}} = (t_\alpha T)^{\mu} \Big\} \\
&= \Big\{ (x, x^*) \mid \exists\ (x_\alpha, x_\alpha^*, t_\alpha) \to^{s\times bd\mathrm{w}^*} (x, x^*, 0) \text{ with } F_{t_\alpha T}(x_\alpha, x_\alpha^*) \le \langle x_\alpha, x_\alpha^* \rangle \Big\} \\
&= \Big\{ (x, x^*) \mid \exists\ (x_\alpha, x_\alpha^*, t_\alpha, \beta_\alpha) \to^{s\times bd\mathrm{w}^*} (x, x^*, 0, \beta) \\
&\qquad \text{with } (x_\alpha, x_\alpha^*, \beta_\alpha) \in \operatorname{epi} F_{t_\alpha T} \text{ and } \beta_\alpha \le \langle x_\alpha, x_\alpha^* \rangle \Big\} \\
&\subseteq \Big\{ (x, x^*) \;\Big|\; \exists \beta :\ (x, x^*, \beta) \in bds\mathrm{w}^*\text{-}\limsup_{t\to 0} \operatorname{epi} F_{tT} \text{ with } \beta \le \langle x, x^* \rangle \Big\}.
\end{aligned}$$

Absorbing the extra dimension into the closure of the epi-graph, applying Lemma 31 part 2 (since $F_T 0_2^+ \in bR\,(\operatorname{rec} T)$, by Proposition 42)

$$\begin{aligned}
\overline{\operatorname{rec} T}^{s\times bd\mathrm{w}^*} &\subseteq \Big\{ (x, x^*) \;\Big|\; \exists \beta : (x, x^*, \beta) \in \overline{bds\mathrm{w}^*\text{-}\limsup_{t\to 0} \operatorname{epi} F_{tT}}^{s\times bd\mathrm{w}^*} \text{ with } \beta \le \langle x, x^* \rangle \Big\} \\
&= \Big\{ (x, x^*) \;\Big|\; \exists \beta :\ (x, x^*, \beta) \in \operatorname{epi} \Big(\overline{bd\text{-}e\text{-}\liminf_{t\to +0} F_{tT}}^{s\times bd\mathrm{w}^*} \Big) \text{ and } \beta \le \langle x, x^* \rangle \Big\} \\
&= \big\{ (x, x^*) \mid \exists \beta :\ (x, x^*, \beta) \in \operatorname{epi} F_T 0_2^+ \text{ and } \beta \le \langle x, x^* \rangle \big\} \\
&= \big\{ (x, x^*) \mid (F_T 0_2^+)(x, x^*) \le \langle x, x^* \rangle \big\} = M_{F_T 0_2^+} = N_{\overline{\mathrm{co}}\,\mathrm{dom}\,T}.
\end{aligned}$$

□

Definition 52 Suppose X be an arbitrary real Banach space and $T : X \rightrightarrows X^*$ is a maximal monotone operator. We define the ε-maximal enlargement of T as

$$\begin{aligned}
T^{\mu[\varepsilon]} &:= \big\{ (x, x^*) \mid F_T(x, x^*) \le \langle x, x^* \rangle + \varepsilon \big\} \\
&\equiv \big\{ (x, x^*) \mid \langle x - y, x^* - y^* \rangle \ge -\varepsilon \text{ for all } (y, y^*) \in T \big\}.
\end{aligned}$$

Note that this set is clearly $(s \times bd\mathrm{w}^*)$-closed (but not necessarily $(s \times \mathrm{w}^*)$-closed). Note that as $tT = \{(x, tx^*) \mid (x, x^*) \in T\}$ then if follows that $t\left(T^{\mu[\varepsilon]}\right) = (tT)^{\mu[t\varepsilon]}$.

Proposition 53 *Suppose X be an arbitrary real Banach space and $T : X \rightrightarrows X^*$ is a maximal monotone operator. Then for* $\operatorname{rec}(T^{\mu_0}) := bds\mathrm{w}^*\text{-}\limsup_{\substack{t\to+0\\ \varepsilon\to^+0}} (tT)^{\mu[\varepsilon]}$ *we have*

$$\overline{\operatorname{rec}(T^{\mu_0})}^{s\times bd\mathrm{w}^*} = M_{F_T 0_2^+} = N_{\overline{\mathrm{co}}\,\mathrm{dom}\,t} \supseteq \overline{\operatorname{rec} T}^{s\times bd\mathrm{w}^*}.$$

Proof Consider $\varepsilon > 0$ and

$$\left\{ (x, x^*) \,\middle|\, \exists \beta : (x, x^*, \beta) \in \overline{bds\text{w}^*\text{-}\limsup_{t \to 0} \operatorname{epi} F_{tT}}^{s \times bd\text{w}^*}, \text{ with } \beta \leq \langle x, x^* \rangle \right\}$$

$$= \left\{ (x, x^*) \,\middle|\, \exists \left(y_\gamma, y_\gamma^*, t_\gamma, \beta_\gamma\right) \to^{s \times bd\text{w}^*} \left(x_\alpha, x_\alpha^*, 0, \beta_\alpha\right) \text{ and } \left(x_\alpha, x_\alpha^*, \beta_\alpha\right) \to_\alpha^{s \times bd\text{w}^*} \left(x, x^*, \beta\right) \right.$$
$$\left. \varepsilon_\gamma \to^+ 0, \text{ with } \left(y_\gamma, y_\gamma^*, \beta_\gamma\right) \in \operatorname{epi} F_{t_\gamma T} \text{ and } \beta_\gamma - \varepsilon_\gamma \leq \langle y_\gamma, y_\gamma^* \rangle \right\}$$

$$= \left\{ (x, x^*) \,\middle|\, \exists \left(y_\gamma, y_\gamma^*, t_\gamma, \varepsilon_\gamma\right) \to^{s \times bd\text{w}^*} \left(x_\alpha, x_\alpha^*, 0, 0\right) \right.$$
$$\left. \text{with } F_{t_\gamma T}\left(y_\gamma, y_\gamma^*\right) \leq \langle y_\gamma, y_\gamma^* \rangle + \varepsilon_\gamma, \left(x_\alpha, x_\alpha^*\right) \to_\alpha^{s \times bd\text{w}^*} \left(x, x^*\right) \right\}$$

$$= \left\{ (x, x^*) \,\middle|\, \exists \left(y_\gamma, y_\gamma^*, t_\gamma\right) \to^{s \times bd\text{w}^*} \left(x_\alpha, x_\alpha^*, 0\right), \varepsilon_\gamma \to^+ 0, \right.$$
$$\left. \text{with } \left(y_\gamma, y_\gamma^*\right) \in \left(t_\gamma T\right)^{\mu_{\varepsilon_\gamma}}, \left(x_\alpha, x_\alpha^*\right) \to_\alpha^{s \times bd\text{w}^*} \left(x, x^*\right) \right\}$$

$$= \overline{bds\text{w}^*\text{-}\limsup_{\substack{t \to +0 \\ \varepsilon \to^+ 0}} (tT)^{\mu[\varepsilon]}}^{s \times bd\text{w}^*}.$$

Hence

$$\overline{\operatorname{rec}\left(T^{\mu_0}\right)}^{s \times bd\text{w}^*}$$
$$= \left\{ (x, x^*) \,\middle|\, \left(F_T 0_2^+\right)(x, x^*) = \left(\overline{bd\text{-}e\text{-}\liminf_{t \to +0} F_{tT}}^{s \times bd\text{w}^*} \right)(x, x^*) \leq \langle x, x^* \rangle \right\}$$
$$= \left\{ (x, x^*) \mid \left(F_T 0_2^+\right)(x, x^*) = \langle x, x^* \rangle \right\} = M_{F_T 0_2^+} = N_{\overline{\operatorname{co}} \operatorname{dom} T}. \qquad \square$$

Again the condition (63) is known to hold in reflexive space [9, Corollary 3.8].

Theorem 54 *Suppose X is an arbitrary real Banach space and $T : X \rightrightarrows X^*$ is a maximal monotone operator. Then* $\operatorname{dom} \operatorname{rec}\left(T^{\mu_0}\right) \subseteq \overline{\operatorname{dom}} T$ *is a sufficient condition for the domain of T to be almost convex.*

Proof We have this condition implying

$$\operatorname{dom} \overline{\operatorname{rec}\left(T^{\mu_0}\right)}^{s \times bd\text{w}^*} = \overline{\operatorname{dom}} \operatorname{rec} T, \tag{63}$$

which is a sufficient condition for the domain of T to be almost convex since by Proposition 53 we have $\operatorname{dom} \overline{\operatorname{rec} T}^{s \times bd\text{w}^*} \supseteq \operatorname{dom} N_{\overline{\operatorname{co}} \operatorname{dom} T} = \overline{\operatorname{co}} \operatorname{dom} T$. Lemma 51 provides the reverse inclusion. It is clear that $\operatorname{dom} \overline{\operatorname{rec} T}^{s \times bd\text{w}^*} = \overline{\operatorname{dom}} T$ so $\overline{\operatorname{dom}} T = \overline{\operatorname{co}} \operatorname{dom} T$. $\square$

Once again we note that it has already been shown in [8, Corollary 5.3.16] that in a reflexive Banach space one has $d\left(T^{\mu_\varepsilon}, T\right) \leq \sqrt{2\varepsilon}$ and $\operatorname{dom} T^{\mu_\varepsilon} \subseteq \overline{\operatorname{dom}} T$ for any maximal monotone operator. Consequently condition (63) may be verified in a reflexive space as $t\left(T^{\mu_\varepsilon}\right) = (tT)^{\mu_{\varepsilon t}}$ where tT is maximal when T is maximal.

8 Almost Convexity

We will need to utilise the duality mapping which can be viewed as the subdifferential of convex functions:

$$\mathcal{J}_X(x) = \partial\left(\frac{1}{2}\|\cdot\|^2\right)(x) \subseteq X^* \quad \text{and} \quad \mathcal{J}_{X^*}\left(x^*\right) = \partial\left(\frac{1}{2}\|\cdot\|_*^2\right)\left(x^*\right) \subseteq X^{**}.$$

When X is reflexive we may assume $x^* \mapsto \frac{1}{2}\|x^*\|_*^2$ is Fréchet differential and then $\mathcal{J}_{X^*}(x^*) \subseteq X$. In the context of non-reflexive spaces it is unclear whether either $\mathcal{J}_X$ (resp. $\mathcal{J}_{X^*}$) are onto X^* (resp. X). We will need to consider the (approximate) minimisers for the problem for a maximal monotone operator T: given $(z, z^*) \in T^{\mu[\varepsilon/2]}$ solve for the minimum (for $\lambda > 0$) of:

$$G_\lambda\left(y, y^*\right) := F_T\left(y, y^*\right) - \langle y, y^*\rangle + \frac{1}{2\lambda}\left\|\left(z, z^*\right) - \left(y, y^*\right)\right\|^2. \tag{64}$$

Note that $\frac{\varepsilon}{2} \geq F_T(z, z^*) - \langle z, z^*\rangle = G_\lambda(z, z^*) \geq \inf_{X\times X^*} G_\lambda$. Note also that any point satisfying $\frac{\varepsilon}{2} \geq G_\lambda(y, y^*) \geq 0$ satisfies $(y, y^*) \in T^{\mu[\varepsilon/2]}$. When X is reflexive we may seek a solution of $\inf_{(y,y^*)\in X\times X^*} G_\lambda(y, y^*)$ by considering the optimality condition for the problem (note that in the Fréchet sense $D_{(u,u^*)}\langle u, u^*\rangle(\cdot) = \langle(u^*, u), \cdot\rangle$):

$$(0,0) \in \partial F_T\left(y, y^*\right) - \left(y^*, y\right) + \frac{1}{\lambda}\left(-\mathcal{J}_{X\times X^*}\right)\left(\left(z, z^*\right) - \left(y, y^*\right)\right).$$

To ensure a solution we need the onto property, so in absence of this we will focus on approximate minimisers and variational principles. This is partly motivated by the following observation.

Proposition 55 *Suppose X is an arbitrary real Banach space and $T: X \rightrightarrows X^*$ is a monotone operator. Then $\{(x, x^*) \mid (x^*, x) \in \partial_\varepsilon F_T(x, x^*)\} \subseteq T^{\mu[\varepsilon/2]}$.*

When M is maximal we have $T^{\mu[\varepsilon/2]} = \{(x, x^*) \mid (x^*, x) \in \partial_\varepsilon F_T(x, x^*)\}$, and hence

$$(0,0) \in \partial_\varepsilon F_T\left(z, z^*\right) - \left(z^*, z\right) + \frac{1}{\lambda}\left(-\mathcal{J}_{X\times X^*}\right)\left(\left(z, z^*\right) - \left(z, z^*\right)\right)$$

for any $(z, z^*) \in T^{\mu[\varepsilon/2]}$.

Proof See the Appendix. □

Let us return to the problem (64). The following is a consequence of the Ekeland variational principle.

Proposition 56 *Suppose X is an arbitrary real Banach space, $T: X \rightrightarrows X^*$ a maximal monotone operator. Then for any $(z, z^*) \in X \times X^*$, there exists a constant $\Lambda = \Lambda(z, z^*) > 0$ such that for all $0 < \lambda < \Lambda$:*

$$F_T\left(z, z^*\right) - \langle z, z^*\rangle \geq \frac{\lambda}{2(1+\lambda)^2}[d\left(\left(z, z^*\right), T\right)]^2. \tag{65}$$

Moreover for any (z, z^*) we have:

$$(F_T 0_2^+)(z, z^*) - \langle z, z^*\rangle \geq \frac{1}{2}[d\left(z, \overline{\operatorname{dom} T}\right)]^2. \tag{66}$$

Proof If $(z, z^*) \in T$ or $(z, z^*) \notin \operatorname{dom} F_T$, we may take $\Lambda(z, z^*) = 1$, as a trivial case. Hereafter, consider $(z, z^*) \in (\operatorname{dom} F_T) \backslash T$. Place $\varepsilon := F_T(z, z^*) - \langle z, z^* \rangle$, so $\varepsilon > 0$.

As $(z, z^*) \notin T$ there exist $(x, x^*) \in T$ and $\delta > 0$ such that $\langle x - z, x^* - z^* \rangle = -2\delta < 0$ (with also $\frac{\varepsilon}{2} \geq G_\lambda(z, z^*) \geq 2\delta > 0$ for all $\lambda > 0$). Because T is maximal we have $\overline{T}^{s \times s^*} = T$ and so we may assume $\|(z, z^*) - (x, x^*)\| = \max\{\|x - z\|, \|x^* - z^*\|\} = N\bar{\delta} > 0$ where $\bar{\delta}, N > 0$ are chosen so that $B_{\bar{\delta}}(z, z^*) \cap T = \emptyset$ and $\langle x - u, x^* - u^* \rangle \leq -\delta < 0$ for all $(u, u^*) \in B_{\bar{\delta}}(z, z^*)$. Note that since $(x, x^*) \in T$ we have $(x^*, x) \in \partial F_T(x, x^*)$ so for all $(u, u^*) \in X \times X^*$,

$$\begin{aligned} G_\lambda\left(u, u^*\right) &\geq F_M\left(x, x^*\right) + \langle\left(x, x^*\right), \left(u, u^*\right) - \left(x, x^*\right)\rangle - \langle u, u^* \rangle \\ &\quad + \frac{1}{2\lambda}\left\|\left(z, z^*\right) - \left(u, u^*\right)\right\|^2 \\ &= \langle\left(x, x^*\right), \left(u, u^*\right)\rangle - \langle x, x^* \rangle - \langle u, u^* \rangle + \frac{1}{2\lambda}\left\|\left(z, z^*\right) - \left(u, u^*\right)\right\|^2 \\ &= -\langle x - u, x^* - u^* \rangle + \frac{1}{2\lambda}\left\|\left(z, z^*\right) - \left(u, u^*\right)\right\|^2 =: g_\lambda\left(u, u^*\right). \end{aligned}$$

Then $g_\lambda(z, z^*) = -\langle x - z, x^* - z^* \rangle = 2\delta$ for all $\lambda > 0$. Now for $(u, u^*) \in B_{\bar{\delta}}(z, z^*)$ we have $g_\lambda(u, u^*) \geq \delta + \frac{1}{2\lambda}\|(z, z^*) - (u, u^*)\|^2 \geq \delta > 2\delta - \frac{\varepsilon}{2} \geq g_\lambda(z, z^*) - \varepsilon$ for any $\lambda > 0$. Moreover, for all $(u, u^*) \in X \times X^*$, we have

$$\begin{aligned} g_\lambda\left(u, u^*\right) &\geq -\|x - z\|\left\|x^* - z^*\right\| - \max\left\{\|x - z\|, \left\|x^* - z^*\right\|\right\}\left[\|z - u\| + \left\|z^* - u^*\right\|\right] \\ &\quad - \|z - u\|\left\|z^* - u^*\right\| + \frac{1}{2\lambda}\|z - u\|^2 + \frac{1}{2\lambda}\left\|z^* - u^*\right\|^2 \\ \text{i.e.} \quad & g_\lambda\left(u, u^*\right) + \|x - z\|\left\|x^* - z^*\right\| \geq -N\bar{\delta}[\alpha + \beta] - \alpha\beta + \frac{1}{2\lambda}\left[\alpha^2 + \beta^2\right] \\ &= -N\bar{\delta}[\alpha + \beta] + \left(\frac{1}{2\lambda} - \frac{1}{2}\right)\left[\alpha^2 + \beta^2\right] + (\alpha - \beta)^2 \\ &\geq -N\bar{\delta}[\alpha + \beta] + \frac{1}{2\lambda}(1 - \lambda)\left[\alpha^2 + \beta^2\right] \end{aligned} \tag{67}$$

where $N\bar{\delta} = \max\{\|x - z\|, \|x^* - z^*\|\}$ and $\alpha := \|z - u\|$, $\beta := \|z^* - u^*\|$. Next, note for all $(u, u^*) \notin B_{\bar{\delta}}(z, z^*)$ that $\max\{\alpha, \beta\} \geq \bar{\delta}$. As the right hand side of (67) tends to infinity as $\lambda \to 0$, we have the existence of $\lambda_1 > 0$ for which $0 < \lambda \leq \lambda_1$ implies for all $(u, u^*) \notin B_{\bar{\delta}}(z, z^*)$ that $g_\lambda(u, u^*) \geq 2\delta$, and so $g_\lambda(u, u^*) \geq 2\delta = g_\lambda(z, z^*) \geq g_\lambda(z, z^*) - \varepsilon$ for $0 < \lambda \leq \lambda_1$. Thus for $0 < \lambda \leq \lambda_1$ we have $g_\lambda(u, u^*) \geq g_\lambda(z, z^*) - \varepsilon$ for all $(u, u^*) \in X \times X^*$.

As $(u, u^*) \mapsto g_\lambda(u, u^*) = -\langle x - u, x^* - u^* \rangle + \frac{1}{2\lambda}\|(z, z^*) - (u, u^*)\|^2$ is jointly lower–semicontinuous with respect to the strong (norm) topologies, and bounded below, we are able to apply the Ekeland variational principle for any fixed $0 < \lambda \leq \lambda_1$. Thus for any $\eta > 0$ there exists a point (y, y^*) such that i) $g_\lambda(y, y^*) + \frac{\eta}{2}\|(y, y^*) - (z, z^*)\| \leq g_\lambda(z, z^*)$; ii) $\|(y, y^*) - (z, z^*)\| \leq \frac{\varepsilon}{\eta}$; iii) $g_\lambda(u, u^*) + \frac{\eta}{2}\|(y, y^*) - (u, u^*)\| > g_\lambda(y, y^*)$ for all $(u, u^*) \neq (y, y^*)$ i.e. (y, y^*) is a strict minimiser. Suppose $(y, y^*) \notin B_{\bar{\delta}}(z, z^*)$ then $g_\lambda(y, y^*) \geq 2\delta$ and so i) implies $2\delta + \frac{\eta}{2}\bar{\delta} \leq g_\lambda(y, y^*) + \frac{\eta}{2}\|(y, y^*) - (z, z^*)\| \leq g_\lambda(z, z^*) = 2\delta$, a contradiction. Thus $(y, y^*) \in B_{\bar{\delta}}(z, z^*)$ and $-\langle x - y, x^* - y^* \rangle \geq \delta > 0$ and hence i) implies (for any fixed

$0 < \lambda \leq \lambda_1$ and any given $\eta > 0$)

$$\begin{aligned} F_T\left(z,z^*\right) - \langle z,z^*\rangle = G\left(z,z^*\right) &\geq g_\lambda\left(z,z^*\right) \geq g_\lambda\left(y,y^*\right) \\ &= -\langle x-y,x^*-y^*\rangle + \frac{1}{2\lambda}\left\|\left(z,z^*\right)-\left(y,y^*\right)\right\|^2 \\ &\geq \frac{1}{2\lambda}\left\|\left(z,z^*\right)-\left(y,y^*\right)\right\|^2. \end{aligned}$$

Now we consider the necessary optimality condition at (y, y^*) for $(u,u^*) \mapsto g_\lambda(u,u^*) + \frac{\eta}{2}\|(u,u^*)-(y,y^*)\|$ i.e.

$$\left(u,u^*\right) \mapsto -\langle x-u,x^*-u^*\rangle + \frac{1}{2\lambda}\left\|\left(z,z^*\right)-\left(u,u^*\right)\right\|^2 + \frac{\eta}{2}\left\|\left(u,u^*\right)-\left(y,y^*\right)\right\|.$$

Applying the Clarke subdifferential and its calculus, noting that $D_{(u,u^*)}\langle u,u^*\rangle(\cdot) = \langle (u^*,u),\cdot\rangle$ in the Fréchet sense and that both $\frac{1}{2}\|\cdot\|^2$ and $\|\cdot\|$ are finite, convex and hence also regular, we obtain

$$(0,0) \in \left(x^*,x\right) - \left(y^*,y\right) + \frac{1}{\lambda}(-\mathcal{J})_{X\times X^*}\left(\left(z,z^*\right)-\left(y,y^*\right)\right) + \frac{\eta}{2}\overline{B}_1(0)$$

$$\text{and so } \left(z,z^*\right)-\left(x,x^*\right) \in \left(z,z^*\right)-\left(y,y^*\right) + \frac{1}{\lambda}(-\mathcal{J})_{X\times X^*}\left(\left(z,z^*\right)-\left(y,y^*\right)\right) + \frac{\eta}{2}\overline{B}_1(0).$$

That is

$$d\left(\left(z,z^*\right),T\right) \leq \left\|\left(z,z^*\right)-\left(x,x^*\right)\right\| \leq \left(1+\frac{1}{\lambda}\right)\left\|\left(z,z^*\right)-\left(y,y^*\right)\right\| + \frac{\eta}{2}.$$

Hence for a given $0 < \lambda \leq \lambda_1$ and any given $\eta > 0$

$$F_T\left(z,z^*\right) - \langle z,z^*\rangle \geq \frac{1}{2\lambda}\left(1+\frac{1}{\lambda}\right)^{-2}\left[d\left(\left(z,z^*\right),T\right)-\frac{\eta}{2}\right]^2.$$

As η is arbitrary we have $F_T(z,z^*) - \langle z,z^*\rangle \geq \frac{1}{2}\frac{\lambda}{(1+\lambda)^2}d^2((z,z^*),T)$ as long as $0 < \lambda \leq \lambda_1$ and so we may take $\Lambda = \lambda_1$, and we have (65) following.

Proof of (66): As done earlier (without loss of generality) we will translate the graph of T so that we may assume $(0,0) \in T$. By Corollary 43 we have a convex function h for which $F_T 0_2^+ = \overline{h}^{s\times bdw^*}$ with $\operatorname{dom} h = \operatorname{Pr}_X \operatorname{dom} F_T \times \operatorname{dom} F_T 0^+(0,\cdot)$ and that for $(z,z^*) \in \operatorname{dom} h$ there exists a net attaining the epi-limit defining h i.e. there exists y^* such that $h(z,z^*) = \lim_{\tau\to+0} F_{\tau T}(z,z^*+\tau y^*)$. From the positive homogeneity of h we have additionally that for any $\lambda' > 0$:

$$\begin{aligned} h(x,x^*) &= \frac{1}{\lambda'}h(z,\lambda' z^*) = \lim_{t\to+0}\frac{1}{\lambda'}F_{tT}(z,\lambda' z^*+ty^*) \\ &= \lim_{t\to+0}\left(\frac{t}{\lambda'}\right)F_T(z,y^*+\left(\frac{\lambda'}{t}\right)z^*) = \lim_{\tau\to+0}\tau F_T(z,y^*+\frac{1}{\tau'}z^*) \end{aligned}$$

and so we may take a convergent *sequence* $\tau_n \to 0$ and place $t_n\left(\lambda'\right) = \lambda'\tau_n$ and get

$$h(x,x^*) = \lim_n \frac{1}{\lambda'}F_{t_n(\lambda')T}(z,\lambda' z^*+t_n\left(\lambda'\right)y^*).$$

Note that when $0 < \lambda < \lambda'$ we have $t_n(\lambda) < t_n(\lambda')$. Now for each n, there exists a $\Lambda_n \to 0$ such that (65) holds for the maximal monotone operator $t_n(\lambda')T$ and the point $(z, \lambda'^* + t_n(\lambda'^*)$ for $0 < \lambda < \Lambda_n$. In the following we will use the fact that for each n we have $\sup_{0<\lambda'<\Gamma'}(\cdot) \geq \inf_{0<\lambda'<\Lambda_n}(\cdot) \geq \inf_{\substack{0<\lambda'<\Lambda_n \\ n\geq\Gamma}}(\cdot)$ irrespective of the relative sizes of Λ_n and Γ'. Indeed suppose $\Gamma' \geq \Lambda_n$ then $\sup_{0<\lambda'<\Gamma'}(\cdot) \geq \sup_{0<\lambda'<\Lambda_n}(\cdot) \geq \inf_{0<\lambda'<\Lambda_n}(\cdot)$. On the other hand when $\Gamma' < \Lambda_n$ we have $\sup_{0<\lambda'<\Gamma'}(\cdot) \geq \inf_{0<\lambda'<\Gamma'}(\cdot) \geq \inf_{0<\lambda'<\Lambda_n}(\cdot)$ again.

Then as $F_T(0,0) = 0$ implies $h(z,0) = 0$ for all z, we have, for any $z \in \operatorname{co}\operatorname{dom} T$ (using the fact that a limit can be written as a limit supremum) that:

$$
\begin{aligned}
&(h(z,\cdot))'_{x^*}(0; z^*) \\
&= \inf_{\lambda'\geq 0} \frac{1}{\lambda'}(h(z, 0+\lambda'^*) - h(z,0)) = \lim_{\lambda'\to 0} \frac{1}{\lambda'}(h(z,0+\lambda'^*) - h(z,0)) \\
&= \lim_{\lambda'\to 0} \lim_{n} \frac{1}{\lambda'} F_{t_n(\lambda')T}(z, \lambda' z^* + t_n\left(\lambda'\right) y^*) \\
&= \inf_{\Gamma'} \sup_{0<\lambda'<\Gamma'} \sup_{\Gamma} \inf_{n\geq\Gamma} F_{t_n(\lambda')T}(z, \lambda' z^* + t_n\left(\lambda'\right) y^*) \\
&= \inf_{\Gamma'} \sup_{\Gamma} \sup_{0<\lambda'<\Gamma'} \inf_{n\geq\Gamma} \frac{1}{\lambda} F_{t_n(\lambda')T}(z, \lambda' z^* + t_n\left(\lambda'\right) y^*) \\
&\geq \sup_{\Gamma} \inf_{\substack{0<\lambda'<\Lambda_n \\ n\geq\Gamma}} \inf_{n\geq\Gamma} \frac{1}{\lambda'} F_{t_n(\lambda')T}(z, \lambda' z^* + t_n\left(\lambda'\right) y^*) \\
&\geq \sup_{\Gamma} \inf_{n\geq\Gamma} \inf_{0<\lambda<\Lambda_n} \frac{1}{\lambda'} F_{t_n(\lambda')T}(z, \lambda' z^* + t_n\left(\lambda'\right) y^*) \quad (\text{and as } t_n\left(\lambda'\right) = \lambda'\tau_n) \\
&\geq \liminf_{n} \inf_{0<\lambda'\leq\Lambda_n} \left(\langle z, z^* + \tau_n y^* \rangle + \frac{1}{2}(\frac{1}{1+\lambda'})^2 d^2((z, \lambda' z^* + t_n\left(\lambda'\right) y^*), t_n(\lambda')T) \right) \\
&\geq \langle z, z^* \rangle + \liminf_{n} \frac{1}{2}(\frac{1}{1+\lambda_n})^2 \inf_{0<\lambda\leq\Lambda_n} d^2((z, \lambda' z^* + t_n\left(\lambda'\right) y^*), t(\lambda)_n T) \\
&\geq \langle z, z^* \rangle + \frac{1}{2} \liminf_{n} d^2(z, \Pr_X \left[\bigcup_{0<\lambda\leq\Lambda_n} t_n(\lambda)T) \right]) \geq \langle z, z^* \rangle + \frac{1}{2} d^2(z, \overline{\operatorname{dom}} T),
\end{aligned} \tag{68}
$$

where we have used the fact that for all $t > 0$ we have $\Pr_X(tT) \subseteq \overline{\operatorname{dom}} T$.

As the $\lambda \mapsto \frac{1}{\lambda}(h(z, 0+\lambda z^*) - h(z,0))$ non-decreasing we have (as $h(z,0) = 0$)

$$
h(z, z^*) = h(z, 0+z^*) - h(z,0) \geq (h(z,\cdot))'_{x^*}(0; z^*)
$$

from which (66) follows from (68) when $z \in \overline{\operatorname{co}}\operatorname{dom} T = \overline{\Pr_X \operatorname{dom} F_T}$, since $F_T 0_2^+ = \overline{h}^{s\times bd w^*}$. When $z \notin \overline{\operatorname{co}}\operatorname{dom} T$ then by Proposition 42, the left hand side of (66) is $+\infty$. □

We may now provide a proof of the conjectured almost–convexity for general monotone operators.

Theorem 57 *Suppose X is an arbitrary real Banach space and $T : X \rightrightarrows X^*$ is a maximal monotone operator. Then T is almost–convex, in that $\overline{\operatorname{dom}}\, T = \overline{\operatorname{co}}\operatorname{dom} T$, which is convex.*

Proof Suppose $z \in \overline{\text{co}}\,\text{dom}\,T$. Using (66) we find that these exists $(z, z^*) \in M_{F_T 0_2^+} = N_{\overline{\text{co}}\,\text{dom}\,T}$ and we have

$$0 = (F_T 0_2^+)(z, z^*) - \langle z, z^* \rangle \geq \frac{1}{2}[d(z, \overline{\text{dom}\,T})]^2.$$

Hence $\overline{\text{co}}\,\text{dom}\,T \subseteq \overline{\text{dom}\,T}$. □

Appendix

Proof of Proposition 13 (Proof of (10)): When $(x, x^*) \in bds\text{w}^*\text{-}\limsup_{t\to+\infty} A_t$ then there exists $I = \{t_\alpha\}$ cofinal in $\mathbb{R}_+$ and $\left(x_{t_\alpha}, x_{t_\alpha}^*\right) \to (x, x^*)$ as $t_\alpha \to +\infty$ with $\left(x_{t_\alpha}, x_{t_\alpha}^*\right) \in A_{t_\alpha}$ and $\left\|x_{t_\alpha}^*\right\| \leq K$ (for some fixed bound $K > 0$). Let $\eta \in \mathbb{R}_+$ and U a norm–open ball around x and W a weak* neighbourhood of x^*. There exists $t_\alpha \in I$ with $t_\alpha \geq \eta$ so that $\left[[U \times W] \cap \overline{B_K}(0)\right] \cap A_{t_\alpha} \neq \emptyset$ and so $\left[[U \times W] \cap \overline{B_K}(0)\right] \cap A_t \neq \emptyset$ for some $t \geq \eta$. Thus $(x, x^*) \in \overline{\left[\left(\bigcup_{t\geq\eta} A_{t_\alpha}\right) \cap \overline{B_K}(0)\right]}^{s\times\text{w}^*}$ from which the inclusion

$$bds\text{w}^*\text{-}\limsup_{t\to+\infty} A_t \subseteq \bigcup_{K>0}\bigcap_{\eta>0} \overline{\left[\left(\bigcup_{t\geq\eta} A_t\right) \cap \overline{B_K}(0)\right]}^{s\times\text{w}^*}$$

follows. For the inverse inclusion, suppose $(x, x^*) \notin bds\text{w}^*\text{-}\limsup_{t\to+\infty} A_t$. We *claim* that (for each $K > 0$) there exists U a norm–open ball around x and W a weak* neighbourhood of x^* such that $\left\{t \in \mathbb{R}_+ \mid \left[[U \times W] \cap \overline{B_K}(0)\right] \cap A_t \neq \emptyset\right\}$ fails to be cofinal:

(To prove this claim, presume otherwise. Then for some $K > 0$ and any neighbourhood $U \times W$, we have $\left\{t \in \mathbb{R}_+ \mid \left[[U \times W] \cap \overline{B_K}(0)\right] \cap A_t \neq \emptyset\right\}$ cofinal. Then

$$(U \times W)_\eta := \bigcup_{\substack{t\geq\eta \\ t\in I}} \left\{\left[[U \times W\,] \cap \overline{B_K}(0)\right] \cap A_t\right\} \neq \emptyset,$$

since I is cofinal. Then we note that $\mathcal{F}_I := \{(U \times W)_\eta \neq \emptyset \mid U \times W \text{ nbhd of } (x, x^*), \eta > 0\}$ is a filterbase in that for any pair $(U \times W)_{\eta_i} \in \mathcal{F}$ for $i = 1, 2$ we have for all $\eta \geq \max\{\eta_1, \eta_2\}$ that

$$\begin{aligned}(U_1 \times W_1)_{\eta_1} \cap (U_2 \times W_2)_{\eta_2} &\supseteq \bigcup_{\substack{t\geq\eta \\ t\in T}} \left\{\left[[(U_1 \cap U_2) \times (W_1 \cap W_2)\,] \cap \overline{B_K}(0)\right] \cap A_t\right\} \\ &= ((U_1 \cap U_2) \times (W_1 \cap W_2))_\eta\,.\end{aligned}$$

As we have $U \times W \supseteq (U \times W)_\eta \in \mathcal{F}_I$ it follows that $\mathcal{F}_I \to (x, x^*)$. As we may construct a net from a filter basis (via selection $\left(x_{t_\alpha}, x_{t_\alpha}^*\right) \in (U \times W)_{t_\alpha}$, for $t_\alpha \in I$ where $\alpha \equiv U \times W \in \mathcal{N}(x, x^*)$ partially ordered via set-inclusion). Indeed for any $U \times W \ni (x, x^*)$ and $\eta > 0$ there exists $t_\alpha \in I$, $t_\alpha \geq \eta$ such that $\left(x_{t_\alpha}, x_{t_\alpha}^*\right) \in \left[[U \times W\,] \cap \overline{B_K}(0)\right] \cap A_{t_\alpha}$. Thus there exists a net $\left(x_{t_\alpha}, x_{t_\alpha}^*\right) \in A_{t_\alpha} \cap \overline{B_K}(0)$, with $t_\alpha \in I$ $(t_\alpha \to \infty)$, converges to (x, x^*). Hence $(x, x^*) \in bds\text{w}^*\text{-}\limsup_{t\to+\infty} A_t$, counter to assumption. This establishes the claim.)

Hence the complement

$$\left\{t \in \mathbb{R}_+ \mid \left[[U \times W] \cap \overline{B_K}(0)\right] \cap A_t = \emptyset\right\}$$

contains a residual set, implying $(x,x^*) \notin \overline{\left[\left(\bigcup_{t\geq\eta} A_t\right) \cap \overline{B_K}(0)\right]}^{s\times w^*}$, for some $\eta > 0$, irrespective of the size of $K > 0$.

(Proof of (11)): Consider $(x,x^*) \in bds\text{w}^*\text{-}\liminf_{t\to+\infty} A_t$ first. Let $I \subseteq \mathbb{R}_+$ be a cofinal set and let $V \in \mathcal{N}(x,x^*)$. Then $V \cap \cup_{t\in T} A_t \cap \overline{B_K}(0) \supseteq V \cap A_{t'_V} \cap \overline{B_K}(0) \neq \emptyset$ for some $t'_V > t_V$ with $t'_V \in I$. Then $(x,x^*) \in \overline{\bigcup_{t\in T} A_t \cap \overline{B_K}(0)}^{s\times w^*}$. As I was arbitrary, $(x,x^*) \in \bigcap_{\substack{I\subseteq\mathbb{R}_+\\ \text{cofinal}}} \overline{\left[\left(\bigcup_{t\in T} A_t\right) \cap \overline{B_K}(0)\right]}^{s\times w^*}$ with the existence of $K > 0$ providing the union.

Conversely, suppose $(x,x^*) \notin bds\text{w}^*\text{-}\liminf_{t\to+\infty} A_t$. Then we claim that for each $K > 0$ there exists $U \times W \in \mathcal{N}(x,x^*)$ such that $\left\{t \in \mathbb{R}_+ \mid \left[[U\times W] \cap \overline{B_K}(0)\right] \cap A_t \neq \emptyset\right\}$ is not residual. (Indeed, if this claim is false, then $\exists K > 0$: $\forall U \times W \in \mathcal{N}(x,x^*)$, $\left\{t \in \mathbb{R}_+ \mid \left[[U\times W] \cap \overline{B_K}(0)\right] \cap A_t \neq \emptyset\right\}$ is residual so there exists t_V with $[[U\times W] \cap \overline{B_K}(0)] \cap A_t \neq \emptyset$ for all $t > t_V$ which means that $(x,x^*) \in bds\text{w}^*\text{-}\liminf_{t\to+\infty} A_t$ counter to assumption.) Hence $I := \left\{t \in \mathbb{R}_+ \mid \left[[U\times W] \cap \overline{B_K}(0)\right] \cap A_t = \emptyset\right\}$ is cofinal, so

$$(x,x^*) \notin \overline{\left[\left(\bigcup_{t\in I} A_t\right) \cap \overline{B_K}(0)\right]}^{s\times w^*} \supseteq \bigcap_{\substack{I\subseteq\mathbb{R}_+\\ \text{cofinal}}} \overline{\left[\left(\bigcup_{t\in I} A_t\right) \cap \overline{B_K}(0)\right]}^{s\times w^*}.$$

As $K > 0$ was arbitrary we have $(x,x^*) \notin \bigcup_{K>0} \bigcap_{\substack{I\subseteq\mathbb{R}_+\\ \text{cofinal}}} \overline{\left[\left(\bigcup_{t_\alpha\in I} A_{t_\alpha}\right) \cap \overline{B_K}(0)\right]}^{s\times w^*}$. This establishes (11).

When $(s\times w^*)\text{-}\limsup_{t\to+\infty}\left(A_t \cap \overline{B}_K(0)\right) = A \cap \overline{B}_K(0)$ for all $K > 0$ sufficiently large, then taking the union across $K > 0$ in the last identity

$$A = \bigcup_{K>0}\left[A \cap \overline{B}_K(0)\right] = \bigcup_{K>0}\bigcap_{\eta>0} \overline{\left[\bigcup_{t\geq\eta} A_t \cap \overline{B_K}(0)\right]}^{s\times w^*} = bds\text{w}^*\text{-}\limsup_{t\to+\infty} A_t.$$

A similar argument covers the other case.

Finally assume all $\{A_t\}_{t>0}$ are convex. Let $(x,x^*), (y,y^*) \in bds\text{w}^*\text{-}\liminf_{t\to+\infty} A_t$ and $W \in \mathcal{N}(0,0)$ such that $W + W' \subseteq V \in \mathcal{N}(0,0)$. Then there exist $K', K'' > 0$ such that for $t > t^1_W$ there exists $\left(x_t, x_t^*\right) \in A_t \cap \overline{B}_{K'}(0) \cap [(x,x^*) + W]$ and for $t > t^2_W$ we have $\left(y_t, y_t^*\right) \in A_t \cap \overline{B}_{K''}(0) \cap [(y,y^*) + W]$ so for $\lambda \in [0,1]$, $t > t_V := \max\left\{t^1_W, t^2_W\right\}$ and $K := \max\{K', K''\}$

$$\begin{aligned}
&\lambda\left(x_t, x_t^*\right) + (1-\lambda)\left(y_t, y_t^*\right)\\
&\in A_t \cap \overline{B}_K(0) \cap \left[\lambda\left[\left(x,x^*\right) + W\right] + (1-\lambda)\left[\left(y,y^*\right) + W\right]\right]\\
&\quad = A_t \cap \overline{B}_K(0) \cap \left[\lambda\left(x,x^*\right) + (1-\lambda)\left(y,y^*\right) + \lambda W + (1-\lambda) W\right]\\
&\quad \subseteq A_t \cap \overline{B}_K(0) \cap \left[\lambda\left(x,x^*\right) + (1-\lambda)\left(y,y^*\right) + V\right]
\end{aligned}$$

and so $A_t \cap \overline{B}_K(0) \cap [\lambda(x,x^*) + (1-\lambda)(y,y^*) + V] \neq \emptyset$ for $t > t_V$. So $\lambda(x,x^*) + (1-\lambda)(y,y^*) \in bds\text{w}^*\text{-}\liminf_{t\to+\infty} A_t$. □

Proof of Lemma 46 We show that $N_{\overline{\text{co}}\,\text{dom}\,T}(x) \subseteq (\text{rec}\,T)(x) = \text{rec}\,(T^\mu)(x)$ for any $x \in \overline{\text{dom}\,T}$ and as a consequence of the inclusion $(\text{rec}\,T^\mu)(x) \subseteq N_{\overline{\text{co}}\,\text{dom}\,T}(x)$ of [13, Proposition 13] gives the desired equality. Take arbitrary $(x,x^*) \in \text{rec}\,(T^\mu)$, $(y,y^*) \in T$ and

$z^* \in N_{\overline{\text{co}}\,\text{dom}\,T}(x)$. Then there exists $t_\alpha \downarrow 0$ and $x_\alpha^* \in T^\mu(x_\alpha)$ with $x_\alpha \to x$ and a bounded net $t_\alpha x_\alpha^* \to x^*$. Consider

$$\langle y - x_\alpha, y^* - \left(x_\alpha^* + \frac{1}{t_\alpha} z^*\right)\rangle = \langle y - x_\alpha, y^* - x_\alpha^*\rangle - \frac{1}{t_\alpha}\langle y - x_\alpha, z^*\rangle$$

$$= \frac{1}{t_\alpha}\langle x - y, z^*\rangle - \frac{1}{t_\alpha}\langle x - x_\alpha, z^*\rangle \geq -\frac{1}{t_\alpha}\|x - x_\alpha\|\left\|z^*\right\| := -\varepsilon_\alpha.$$

As $(y, y^*) \in T$ was arbitrary we have

$$\left(x_\alpha, x_\alpha^* + \frac{1}{t_\alpha} z^*\right) \in \left\{(w, w^*) \mid \langle w - y, w^* - y^*\rangle \geq -\varepsilon_\alpha, \forall \left(y, y^*\right) \in T\right\}.$$

We may now apply the (BR) property to assert that for any $\eta > 0$ there exists $\left(w_\alpha, w_\alpha^*\right) \in T$ such that $\|w_\alpha - x_\alpha\| \leq \frac{\varepsilon_\alpha}{\eta}$ and $\left\|w_\alpha^* - \left(x_\alpha^* + \frac{1}{t_\alpha} z^*\right)\right\| \leq \eta$. Consequently we have

$$\begin{aligned} \|w_\alpha - x\| &\leq \|w_\alpha - x_\alpha\| + \|x - x_\alpha\| \\ &\leq \frac{1}{\eta t_\alpha}\|x - x_\alpha\|\left\|z^*\right\| + \|x - x_\alpha\| = \|x - x_\alpha\|\left[\frac{\|z^*\|}{\eta t_\alpha} + 1\right] \\ \text{and} \quad & \left\|t_\alpha w_\alpha^* - \left(t_\alpha x_\alpha^* + z^*\right)\right\| \leq \eta t_\alpha. \end{aligned} \tag{69}$$

Using the freedom we have on the choice of η for each given α, we take $\eta = \eta_\alpha := \frac{\|x - x_\alpha\|^{\frac{1}{2}}}{t_\alpha} > 0$; then we find that $\frac{\|x - x_\alpha\|}{\eta_\alpha t_\alpha} = \|x - x_\alpha\|^{\frac{1}{2}} \to 0$ and $\eta_\alpha t_\alpha = \|x - x_\alpha\|^{\frac{1}{2}} \to 0$. As we have a bounded net $t_\alpha x_\alpha^* \to^{w^*} x^*$ it follows for (69) that $\left\{t_\alpha w_\alpha^*\right\}_\alpha$ is bounded with $t_\alpha w_\alpha^* \to^{w^*} x^* + z^*$ and as $\left(w_\alpha, w_\alpha^*\right) \in T$ with $w_\alpha \to x$ we have $x^* + z^* \in (\text{rec}\,T)(x)$. As $x^* = 0 \in \text{rec}\,(T^\mu)(x)$ it follows $z^* \in (\text{rec}\,T)(x)$ and so that $N_{\overline{\text{co}}\,\text{dom}\,T}(x) \subseteq (\text{rec}\,T)(x)$. Moreover as $z^* = 0 \in N_{\overline{\text{co}}\,\text{dom}\,T}(x)$ we also have $\text{rec}\,(T^\mu)(x) \subseteq (\text{rec}\,T)(x)$, forcing equality. As $\overline{\text{dom}}\,T$ is convex we have $N_{\overline{\text{dom}\,T}}(x) = (\text{rec}\,T)(x)$ for all $x \in \overline{\text{dom}}T = \overline{\text{co}}\,\text{dom}\,T$ and hence $N_{\overline{\text{dom}\,T}} = \text{rec}\,T$. □

Proof of Proposition 55 When $(x, x^*) \in \partial_\varepsilon F_T(x, x^*)$ then $F_T(y, y^*) - F_T(x, x^*) \geq \langle (x, x^*), (y, y^*) - (x, x^*)\rangle - \varepsilon$ for all (y, y^*) so

$$\begin{aligned} \langle\left(x, x^*\right), \left(x, x^*\right)\rangle - F_T\left(x, x^*\right) &\geq \sup_{(y, y^*)}\left[\langle\left(x, x^*\right), \left(y, y^*\right)\rangle - F_T\left(y, y^*\right)\right] \\ &= F_T^{*\dagger}\left(x, x^*\right) - \varepsilon \\ &= P_T\left(y, y^*\right) - \varepsilon \geq F_T\left(x, x^*\right) - \varepsilon \end{aligned}$$

$$\text{and hence} \qquad 2\langle x, x^*\rangle - 2F_T\left(x, x^*\right) \geq -\varepsilon \qquad \text{or} \qquad \frac{\varepsilon}{2} + \langle x, x^*\rangle \geq F_T\left(x, x^*\right).$$

Thus $(x, x^*) \in T^{\mu[\varepsilon/2]}$. On the other hand when T is maximal $F_T(y, y^*) \geq \langle y, y^*\rangle$ for all (y, y^*) and we have $(x, x^*) \in T^{\mu[\varepsilon/2]}$, implying

$$\begin{aligned} \langle x, x^*\rangle - \langle\left(x, x^*\right), \left(y, y^*\right)\rangle + \langle y, y^*\rangle = \langle x - y, x^* - y^*\rangle \geq -\frac{\varepsilon}{2} \\ \text{so} \qquad F_T\left(y, y^*\right) - F_T\left(x, x^*\right) \geq \langle y, y^*\rangle - F_T\left(x, x^*\right) \end{aligned}$$

$$\geq \langle \left(x, x^*\right), \left(y, y^*\right) \rangle - \langle x, x^* \rangle - \frac{\varepsilon}{2} - F_T \left(x, x^*\right)$$
$$\geq \langle \left(x, x^*\right), \left(y, y^*\right) \rangle - 2\langle x, x^* \rangle - \varepsilon$$

using $F_T(x, x^*) \leq \langle x, x^* \rangle + \frac{\varepsilon}{2}$ (as $(x, x^*) \in T^{\mu[\varepsilon/2]}$) in the last inequality. This implies

$$F_T\left(y, y^*\right) - F_T\left(x, x^*\right) \geq \langle \left(x, x^*\right), \left(y, y^*\right) - \left(x, x^*\right) \rangle - \varepsilon \quad \text{for all } \left(y, y^*\right)$$

so $(x, x^*) \in \partial_\varepsilon F_T(x, x^*)$. □

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Declarations

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