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BV Sweeping Process Involving Prox-Regular Sets and a Composed Perturbation

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Abstract

In this paper we study the existence and properties of solutions for a discontinuous sweeping process involving prox-regular sets in a Hilbert spaces. The variation of the moving set is controlled by a positive Radon measure and the perturbation is the sum of two multivalued mappings. The values of the first one are closed, bounded, not necessarily convex sets. It is measurable in the time variable, Lipschitz continuous in the phase variable, and it satisfies a conventional growth condition. The values of the second one are closed, convex, not necessarily bounded sets. We assume that this mapping has a closed with respect to the phase variable graph.

Other assumptions concern the intersection of the second mapping with the multivalued mapping defined by the growth conditions. We suppose that this intersection has a measurable selector and it possesses some compactness properties.

We prove the existence of right-continuous solutions of bounded variation for our inclusion. If the values of the first inclusion are closed convex sets, then the solution set is a closed subset of the space of right-continuous functions of bounded variation with sup-norm. If, in addition, the values of the moving sets are compact sets, then the solution set is compact in the space of right-continuous functions of bounded variation endowed with the topology of uniform convergence on an interval.

The proofs are based on the author's theorem on continuous with respect to a parameter selectors passing through fixed points of contraction multivalued maps with closed, nonconvex, decomposable values depending on the parameter and some compactness criteria (an analog of the Arzelà–Ascoli theorem) for sets in the space of right-continuous functions of bounded variation with sup-norm. The classical Ky Fan fixed point theorem is also used. The results that we obtain are new.

Keywords Sweeping process · Prox-regular sets · Nonconvex-valued perturbation

Mathematics Subject Classification 28B20 · 34G25 · 47J35 · 34A60 · 47H10 · 49J52 · 49J53

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1 Introduction

Let H be a separable Hilbert space with the norm $\|\cdot\|$, metric $d(\cdot,\cdot)$, inner product $\langle\cdot,\cdot\rangle$ and zero element Θ. We introduce the following notation: $R^+ = [0,+\infty)$, $T = [0,a] \subset R^+$, $a > 0$, $C : T \rightrightarrows H$ is a multivalued mapping with closed r-prox regular values [1], $U, V : T \times H \rightrightarrows H$ is a multivalued mapping with closed values.

By $L^1(T,H)$ we denote the space of Lebesgue integrable functions from T to H. The symbol ω-$L^1(T,H)$ denotes the space $L^1(T,H)$ endowed with the weak topology.

The space of all right continuous functions of bounded variation $x : T \to H$ with the norm

$$\|x(\cdot)\|_{BV} = \sup\{\|x(t)\|;\ t \in T\}$$

is denoted by $BV_+(T,H)$.

By λ we denote the Lebesgue measure. In what follows, without explicitly mentioning this, we assume that the inequality

$$|d(y,C(t)) - d(y,C(s))| \le \mu(]s,t]),\ y \in H, \tag{1.1}$$

$s \le t,\ s,t \in T$, holds, where μ is a positive Radon measure on T satisfying the inequality

$$\sup_{s\in]0,a]} \mu(\{s\}) < r/2. \tag{1.2}$$

Consider the measurable sweeping process

$$-dx \in \mathcal{N}(C(t);x(t)) + F(t,x(t)), \tag{1.3}$$

$$F(t,x) = U(t,x) + V(t,x), \tag{1.4}$$

$$x(0) = x_0 \in C(0),$$

where $\mathcal{N}(C(t);x)$ is the proximal normal cone [2] to the a $C(t)$ at a point $x \in C(t)$ and $U, V : T \times H \rightrightarrows H$ are multivalued mappings with closed values.

Definition 1.1 By a solution to the inclusions (1.3) we mean a triplet $(x(u;v)(\cdot), u(\cdot), v(\cdot))$ such that

1) $x(u;v)(\cdot)$ is a right continuous function of bounded variation from T to H, $x(u;v)(0) = x_0$, $x(u;v)(t) \in C(t)$, $t \in T$ and $u(\cdot), v(\cdot) \in L^1(T,H)$;

2) there exists a positive Radon measure ν absolutely continuously equivalent to the measure $\mu + \lambda$ such that the differential measure $dx(u;v)$ generated by the function $x(u;v)(\cdot)$ is absolutely continuous with respect to the measure ν and the density $\frac{dx(u;v)}{d\nu}(\cdot)$ of the measure $dx(u;v)$ with respect to the measure ν and functions $u(\cdot)$, $v(\cdot)$ satisfy the inclusions

$$-\frac{dx(u;v)}{d\nu}(t) - (u(t)+v(t))\frac{d\lambda}{d\nu}(t) \in -\mathcal{N}(C(t);x(u;v)(t))\ \nu \text{ a.e.}, \tag{1.5}$$

$$u(t) \in U(t,x(u;v)(t))\ \lambda \text{ a.e.}, \tag{1.6}$$

$$v(t) \in V(t,x(u;v)(t))\ \lambda \text{ a.e.} \tag{1.7}$$

Positive Radon measures are absolutely continuously equivalent, if each of them is absolutely continuous with respect to the other.

In the definition of a solution to a measurable sweeping process we follow [3, 4] and others.

We note that the definition of solution does not depend on the measure ν in the sense that a mapping $x(u; v) : T \to H$ with the property 1) is a solution of the inclusion (1.3) if and only if the inclusion (1.5) holds for any positive Radon measure ν that is absolutely continuously equivalent to the measure $\lambda + \mu$. The solution set of the inclusion (1.3) we denote by $\mathcal{R}(x_0)$. In what follows, a solution in the sense of Definition (1.1) is called a BV solution.

Let $\overline{B}$ be the unit closed ball in H centered at the point Θ. For a bounded set $D \subset H$ we denote

$$\|D\| = \{\sup \|x\|;\ x \in D\}.$$

By haus$(\cdot,\cdot)$ we denote the Hausdorff metric on the space of all nonempty, closed, bounded sets from H.

We make the following assumptions.

Hypothesis H(U) *The multivalued mapping $U : T \times H \rightrightarrows H$ with closed, not necessary convex values has the properties:*

1) *the mapping $t \to U(t, x)$ is measurable;*

2) *the following inequalities hold*

$$\text{haus}(U(t,x), U(t,y)) \le k(t)\|x - y\|\ \lambda\ a.e., \tag{1.8}$$

$x, y \in H$, $k(\cdot) \in L^1(T, R^+)$,

$$\|U(t,x)\| = \sup\{\|u\|;\ u \in U(t,x)\} \le m_1(t) + n_1(t)\|x\|, \tag{1.9}$$

$x \in H,\ m_1(\cdot), n_1(\cdot) \in L^1(T, R^+)$.

Hypothesis H(V) *The multivalued mapping $V : T \times H \to H$ with closed convex values has the properties:*

1) *the following inequality holds*

$$d(\Theta, V(t,x)) < m_2(t) + n_2(t)\|x\|\ \lambda\ a.e., \tag{1.10}$$

$x \in H,\ m_2(\cdot), n_2(\cdot) \in L^1(T, R^+)$;

1)* *the following inequality holds*

$$\|V(t,x)\| \le m_2(t) + n_2(t)\|x\|\ \lambda\ a.e., \tag{1.11}$$

$x \in H,\ m_2(\cdot), n_2(\cdot) \in L^1(T, R^+)$;

2) *the mapping*

$$t \to V(t,x) \cap (m_2(t) + n_2(t)\|x\|)\overline{B},\ x \in H$$

has a λ-measurable selector and the mapping $x \to V(t,x)$ has closed graph for λ almost every $t \in T$;

3) *for every bounded set $D \subset H$ the set*

$$V(t, D) \cap (m_2(t) + n_2(t)\|D\|)\overline{B}$$

is relatively compact for λ almost every $t \in T$, where $V(t, D) = \{\cup V(t,x);\ x \in D\}$.

Since the inequality (1.10) is strict, for λ almost every t the set $V(t,x) \cap (m_2(t) + n_2(t)\|x\|)\overline{B}$, $x \in H$ is not empty. Consequently, Hypothesis H(V) 2) makes sense.

1.1 Main Results

Theorem 1.1 *Let Hypotheses $H(U)$ 1), 2), 3) and $H(V)$ 1), 2), 3) hold. Then, the set $\mathcal{R}(x_0)$ is not empty and*

$$\|x(u;v)(t) - x(u;v)(t-0)\| \le \mu(\{t\}),\ t \in]0,a], \tag{1.12}$$

$$\left\| \frac{dx(u;v)}{d\nu}(t) + (u(t)+v(t))\frac{d\lambda}{d\nu}(t) \right\| \le$$

$$\le \frac{d\mu}{d\nu}(t) + (\|u(t)\| + \|v(t)\|)\frac{d\lambda}{d\nu}(t)\ \nu\ a.e. \tag{1.13}$$

for any solution $(x(u;v)(\cdot), u(\cdot), v(\cdot)) \in \mathcal{R}(x_0)$ and any Radon measure ν absolutely continuously equivalent to the measure $\mu + \lambda$.

Theorem 1.2 *Let for multivalued mappings $U, V : T \times H \rightrightarrows H$ with closed convex values Hypotheses $H(U)$ 1), 2), 3) and $H(V)$ 1)*, 2), 3) hold. Then, the set $\mathcal{R}(x_0)$ is a closed subset of the space*

$$BV_+(T,H) \times \omega\text{-}L^1(T,H) \times \omega\text{-}L^1(T,H).$$

If, in addition, the set $C(t) \cap r\overline{B}$, $t \in T$ is relatively compact for $r > d(\Theta, C(t))$, then the set $\mathcal{R}(x_0)$ is compact in the space $BV_+(T,H) \times \omega\text{-}L^1(T,H) \times \omega\text{-}L^1(T,H)$.

The existence of solutions to the inclusion (1.3) with a perturbation was studied in the works [2–6].

In the works [2, 3, 5], the values of the multivalued mapping $C : T \rightrightarrows H$ are closed r-prox regular set and the inequalities (1.1), (1.2) hold.

In the work [2], one proved the existence of a unique BV solution in the case when the perturbation $f : T \times H \to H$ is single-valued. This perturbation is measurable in the first variable, Lipschitz continuous in the second variable and satisfies conventional linear growth conditions. A priori estimates for a solution were given.

A multivalued perturbation $F : T \times H \rightrightarrows H$ with convex compact values, that is scalarly upper semicontinuous and satisfies the inclusion

$$F(t,x) \subset \beta(t)(1 + \|x\|K),\ t \in T,\ x \in U_{s\in T}C(s),$$

where $\beta(\cdot) \in L^1(T,R)$ and $K \subset \overline{B}$ is a compact set, was considered in the work [3]. In this work, one proved the existence of a BV solution.

In the work [4], the inclusion (1.3) with a multivalued mapping $C : T \rightrightarrows H$ with the values being closed r-prox regular sets satisfying the inequality

$$|d(y, C(t)) - d(y, C(s))| \le |v(t) - v(s)|,$$

$y \in H,\ s, t \in T$, where $v : T \to R$ is an absolutely continuous function, was studied.

The multivalued perturbation $G : T \times H \rightrightarrows H$ in this work has the form

$$G(t,x) = F(t,x) + f(t,x), \tag{1.14}$$

where $F(t,x)$ is a multivalued mapping with convex compact values and with the properties as in the work [3], and the single-valued perturbation $f(t,x)$ has the same properties as in the work [2]. In this work, one proved the existence of an absolutely continuous solution. The existence of a BV solution with a perturbation of the form (1.14) and with the same properties for the mappings $F(t,x)$ and $f(t,x)$ as in the work [4], was proved in the work [5].

In the work [6], one proved the existence of a unique BV solution with a multivalued mapping $C: T \rightrightarrows H$ having closed r-prox regular values and a single-valued perturbation $f: T \times H \to H$. It was supposed that the inequality (1.2) holds and:

(i) the mapping $t \to f(t,x)$ is measurable for every $x \in \cup_{t\in T} C(t)$, for every bounded set $D \subset H$ the mapping $x \to f(t,x)$ is uniformly continuously on D for every $t \in T$ and there exists a function $l_D(\cdot) \in L^1(T, R^+)$ such that

$$\langle f(t,x_1) - f(t,x_2), x_1 - x_2 \rangle \ge -l_D(t)\|x_1 - x_2\|^2,$$

$t \in T,\ x_1, x_2 \in D$;

(ii) there exists a function $\alpha(\cdot) \in L^1(T, R^+)$ with $1 - 2\int_T \alpha(s)d\lambda(s) > 0$ such that

$$\|f(t,x)\| \le \alpha(t)(1 + \|x\|),\ t \in T,\ x \in H;$$

(iii) there exists $\rho_0 > \|x_0\| + \mu(]0,a])$ and $\rho \in]0,+\infty]$, $\rho \ge (\rho_0 + 2\int_T \alpha(s)d\lambda(s))/(1 - 2\int_T \alpha(s)d\lambda(s))$ and $\eta > 0$ such that

$$\widehat{\text{haus}}_\rho(C(s), C(t)) \le \mu(]s,t]),\ t,s \in T,\ s \le t,$$

for $\mu(]s,t]) < \eta$.

Here,

$$\widehat{\text{haus}}_\rho(C(s), C(t)) = \max\{\sup_{x\in C(s)\cap\rho\overline{B}} d(x, C(t)), \sup_{x\in C(t)\cap\rho\overline{B}} d(x, C(s))\}.$$

All the main results of the works [2–5] follow from our Theorem 1.1. A distinctive feature of Theorem 1.1, as compared to similar theorems in [2–6], is that the values of the multivalued perturbation can be closed nonconvex sets. We have been able to consider such perturbations employing methods different from those used in the works [2–6]. In these works, the proofs are based on various versions of the catching-up algorithm originated in the work [7]. Such an approach is applicable only in the case when the values of perturbation are closed convex sets, because in the proof one uses the Mazur theorem for weakly converging sequences.

Our approach is based on the classical Ky Fan fixed point theorem and the author's theorem on parameter-continuous selectors whose values are fixed points of parameter-dependent multivalued maps with closed, convex, decomposable values in the space of integrable functions.

Regarding the results of Theorem 1.2, the author is not aware of any work studying properties of BV solutions of sweeping processes with r-prox regular sets and with multivalued perturbations.

2 Main Notation, Definitions and Preliminaries

Let Y be a metric space, cY the family of all nonempty closed sets from Y, cb Y the family of all bounded sets from cY with the Hausdorff metric $\text{haus}_Y(\cdot,\cdot)$. For a topological vector space Z by ω-Z we denote the space Z endowed with the weak topology. If $D \subset Z$, then ω-D means that the set D is endowed with the topology induced by the topology of the space ω-Z. By $\overline{\text{co}}\, D$ we denote the closed convex hull of a set $D \subset Z$.

Let W be a topological space. A multivalued mapping $F : W \rightrightarrows Y$ is called lower semicontinuous, if for any open set $E \subset Y$ the set $F^{-1}(E) = \{w \in W;\ F(w) \cap E \neq \varnothing\}$ is open.

If W is a metric space, the definition of lower semicontinuity is equivalent to the following one: for any $w \in W$, $y \in F(w)$ and any sequence $w_n \in W$, $n \geq 1$, $w_n \to w$, there exists a sequence $y_n \in F(w_n)$, $n \geq 1$, converging to y.

A multivalued mapping $F : W \to Y$ is called upper semicontinuous, if for any open set $E \subset Y$ the set $F^{+}(E) = \{w \in W;\ F(w) \subset E\}$ is open.

If Y is a compact metric space and $F : W \to Y$ is a multivalued mapping with closed values, then the upper semicontinuity equivalent to the closedness of the graph of the mapping.

A multivalued mapping $F : T \to cH$ is called measurable [8], if for any closed set $E \subset H$ the set $F^{-1}(E) = \{t \in T;\ F(t) \cap E \neq \varnothing\}$ is an element of the σ-algebra Σ of Lebesgue measurable sets from T.

A set $\mathcal{K}$ of measurable mappings $u : T \to H$ is called decomposable, if for any $u, v \in \mathcal{K}$, $\Lambda \in \Sigma$ the element $\chi(\Lambda)u + \chi(T \setminus \Lambda)v$ belongs to the set $\mathcal{K}$, where $\chi(\Lambda)$ is the characteristic function of the set Λ.

A measurable multivalued mapping $\Gamma : T \to \text{cb}\, H$ is called integrally bounded, if there exists a function $m(\cdot) \in L^1(T, R^+)$ such that

$$\|\Gamma(t)\| = \sup\{\|u\|;\ u \in \Gamma(t)\} \leq m(t) \text{ a.e.}$$

The space of all measurable, integrally bounded mappings $\Gamma : T \to \text{cb}\, H$ with closed values we denote by $L^1(T, \text{cb}\, H)$, and by $\text{dcb}\, L^1(T, H)$ we denote the family of all closed, bounded, decomposable sets from $L^1(T, H)$.

If $\Gamma(\cdot) \in L^1(T, \text{cb}\, H)$, then by S_Γ we denote the family of all integrable with respect to the measure λ selectors of the mapping $t \to \Gamma(t)$. It is known that it is an element of the space $\text{dcb}\, L^1(T, H)$.

Denote by $BV_+(T, H)$ the space of all right continuous functions $x : T \to H$ of bounded variation with the topology of uniform convergence on T. The topology of the space $BV_+(T, H)$ is generated by the norm

$$\|x(\cdot)\|_{BV_+} = \sup\{\|x(t)\|;\ t \in T\}.$$

It is known that a function $x(\cdot) \in BV_+(T, H)$ has the left limit $x(t - 0)$ at every point $t \in]0, a]$.

Let $U \subset BV_+(T, H)$. The set U is called right equicontinuous at a point $s \in [0, a[$, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\|x(s) - x(t)\| \leq \varepsilon$ for all $x(\cdot) \in U$ and $t \in [s, s + \delta[$.

A set $U \subset BV_+(T, H)$ is called left equicontinuous at a point $s \in]0, a]$, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\|x(s - 0) - x(t)\| \leq \varepsilon$ for all $x(\cdot) \in U$ and $t \in]s - \delta, s[$.

A set $U \subset BV_+(T, H)$ is called unilaterally equicontinuous, if it is both right and left equicontinuous at every point $s \in]0, a[$, right equicontinuous at the point 0 and left equicontinuous at the point a.

The variation of a function $x(\cdot) \in BV_+(T, H)$ we denote by $var\, x(\cdot)$. A set $U \subset BV_+(T, H)$ is called uniformly bounded in norm and in variation, if there exists a constant $M > 0$ such that

$$\|x(t)\| \leq M, \quad t \in T,\ x(\cdot) \in U,$$

$$var\, x(\cdot) \leq M, \quad x(\cdot) \in U.$$

Let $\mathcal{B}$ be the σ-algebra of Borel sets from T. A positive Radon measure is a scalar positive measure defined on the σ-algebra $\mathcal{B}$. In what follows, by a Radon measure we mean a scalar positive Radon measure. The variation of the measure $m : \mathcal{B} \to H$ we denote by $|m|(\cdot)$. If $|m|(T) < \infty$, then the measure m is called a measure with bounded variation. In this case, the variation $|m|(\cdot)$ of the measure $m(\cdot)$ is a Radon measure.

A Radon measure ν is absolutely continuous with respect to a Radon measure μ, if $\mu(A) = 0,\ A \in \mathcal{B}$ implies that $\nu(A) = 0$. If the converse is also true, then the measures μ and ν are called absolutely continuously equivalent.

A measure $m : \mathcal{B} \to H$ is absolutely continuous with respect to a Radon measure $\mu(\cdot)$, if the measure $|m|(\cdot)$ is absolutely continuous with respect to the measure $\mu(\cdot)$.

For a Radon measure ν on T by $L^1_\nu(T, H)$ we denote the set of equivalency classes of all ν measurable mappings $g : T \to H$ such that the function $t \to \|g(t)\|$ is an element of the space $L^1_\nu(T, R^+)$. If a measure of bounded variation $m : \mathcal{B} \to H$ is absolutely continuous with respect to a Radon measure μ, then according to the Radon-Nikodym theorem there exists a function $\hat{m} \in L^1_\mu(T, H)$ such that

$$m(A) = \int_A \hat{m}(\tau)\, d\mu(\tau), \quad A \in \mathcal{B}.$$

The function $t \to \hat{m}(t)$ is called the density of the measure m with respect to the measure μ and is denoted by $\frac{dm}{d\mu}(\cdot)$.

Let $S \subset H$ be a nonempty subset,

$$d_S(x) = \inf_{s \in S} \|x - s\|,\ x \in H$$

and

$$\operatorname{Proj}_S(x) = \{y \in S;\ d_S(x) = \|x - y\|\}.$$

The proximal normal cone to the set S at a point $x \in S$ is the set

$$N^P(S; x) = \{v \in H;\ \exists r > 0,\ x \in \operatorname{Proj}_S(x + rv)\},$$

which is, evidently, a cone containing Θ. One sets $N^P(S; x) = \varnothing$, if $x \in H \backslash S$.

Definition 2.1 A nonempty closed set $S \subset H$ is called r-prox regular, if for any $x \in S$ and for all $v \in N^P(S; x) \cap \overline{B}$ and all $t \in]0, r[$ the inclusion $x \in \operatorname{Proj}_S(x + tv)$ holds.

Lemma 2.1 *Let* $U \subset BV_+(T, H)$ *and*:

1) *the set* U *is unilaterally equicontinuous*;

2) *the set*

$$U(t) = \{x(t);\ x(\cdot) \in U\}, \quad t \in T$$

is relatively compact;

3) *the set* U *is uniformly bounded in variation.*

Then, the set U is relatively compact in the space $BV_+(T, H)$, i.e. from any sequence $x_n(\cdot) \in U$, $n \geq 1$ one can extract a subsequence $x_{n_k}(\cdot) \in U$, $k \geq 1$, uniformly converging to some function $x(\cdot) \in BV_+(T, H)$.

Lemma 2.1 follows from Theorem 2.3 [9].

Lemma 2.2 ([9]) *Let Hypothesis H(V) hold. Then, for any function $x(\cdot) \in BV_+(T, H)$ the mapping*

$$t \to V(t, x(t)) \cap (m_2(t) + n_2(t)\|x(t)\|)\overline{B}$$

has a selector which is an element of the space $L^1 T, H)$.

Lemma 2.3 *If a function of bounded variation $x : T \to H$ is right continuous, then there exists a unique measure $m : \mathcal{B} \to H$ such that for any $0 \le c \le d \le a$ we have*

$$m(]c, d]) = x(d) - x(c), \tag{2.1}$$

$$m(]c, d[) = x(d - 0) - x(c). \tag{2.2}$$

Lemma 2.3 follows from Theorem 1 [10, p. 358].

The measure $m(\cdot)$ is usually called the differential measure (Stieltjes measure) generated by the function $x(\cdot)$ and is denoted by dx.

If $\hat{x}(\cdot) \in L^1_\nu(T, H)$ and $x(t) = x(0) + \int_{]0,t]} \hat{x}(\tau)\, d\nu(\tau)$, $t \in T$, then $x(t)$ is a right continuous function of bounded variation, the differential measure dx generated by the function $x(\cdot)$ is absolutely continuous with respect to the measure ν and $\hat{x}(\cdot)$ is the density of the measure dx with respect to the measure ν, i.e.

$$\frac{dx}{d\nu}(t) = \hat{x}(t) \ \nu \text{ a.e.} \tag{2.3}$$

and

$$\hat{x}(t) = \lim_{s \uparrow t} \frac{dx(]s, t] \cap T)}{\nu(]s, t] \cap T)} \quad \nu \text{ a.e.} \tag{2.4}$$

It is known [11, chapter V, p. 43, theorem 1] that if the Lebesgue measure λ is absolutely continuous with respect to a Radon measure ν, then the function $\hat{x} : T \to H$ is integrable with respect to the measure λ if and only if the function $t \to \hat{x}(t)\frac{d\lambda}{d\nu}(t)$ is ν integrable. In this case, we have

$$\int_{]0,t]} \hat{x}(t)\, d\lambda(t) = \int_{]0,t]} \hat{x}(t) \frac{d\lambda}{d\nu}(\tau)\, d\nu(\tau), \quad t \in T, \tag{2.5}$$

and if $t \in]0, a]$ and $\nu(\{t\}) > 0$, then

$$\frac{d\lambda}{d\nu}(t)\, \nu(\{t\}) = 0. \tag{2.6}$$

Proposition 2.1 *Let $x : T \to H$ be a right continuous function of bounded variation with the differential measure dx absolutely continuous with respect to a Radon measure ν. Then, $t \to \|x(t)\|^2$ is a right continuous function of bounded variation and*

$$\frac{\|x(t)\|^2}{2} \le \frac{\|x(0)\|^2}{2} + \int_{]0,t]} \langle x(\tau), \frac{dx}{d\nu}(\tau) \rangle\, d\nu(\tau). \tag{2.7}$$

The proposition follows from Proposition 3.3 in [3].

Theorem 2.1 ([5]) *Let $S \subset H$ be a nonempty closed set. Then, the following conditions are equivalent*:

(a) *the set S is r-prox regular*;

(b) *for any $x_1, x_2 \in S$ and for all $v \in N^P(S; x_1)$ we have*

$$\langle v, x_2 - x_1\rangle \leq \frac{1}{2r}\|v\|\,\|x_1 - x_2\|^2;$$

(c) *for all $x_1, x_2 \in S$ and $v_1 \in N^P(S; x_1)$, $v_2 \in N^P(S, x_2)$ the inequality*

$$\langle v_1 - v_2, x_1 - x_2\rangle \geq -\frac{1}{2r}(\|v_1\| + \|v_2\|)\|x_1 - x_2\|^2$$

holds.

In the sequel, following [2], the cone $N^P(S, x)$ is denoted by $N(S, x)$.

Lemma 2.4 ([12]) *Let ν be a Radon measure on T and $g, \varphi : T \to R^+$ are two functions such that*

(i) *for some $\theta \in R^+$ and for all $t \in]0, a]$*

$$0 \leq g(\nu)\nu(\{t\}) \leq \theta < 1,\ g(\cdot) \in L^1_\nu(T, R^+);$$

(ii) *for some fixed $\alpha \in R^+$ and all $t \in T$ we have*

$$\varphi(t) \leq \alpha + \int\limits_{]0,t]} g(s)\varphi(s)d\nu(s),\ \varphi \in L^\infty_\nu(T, R^+).$$

Then, for all $t \in T$

$$\varphi(t) \leq \alpha \exp\left(\frac{1}{1-\theta}\int\limits_{]0,t]} g(s)d\nu(s)\right).$$

Lemma 2.5 *Let μ be a Radon measure on T, $m(\cdot), n(\cdot) \in L^1_\mu(T, R^+)$ and $u : T \to R^+$ is a right continuous function of bounded variation. If*

$$\frac{1}{2}u^2(t) \leq \frac{1}{2}c^2 + \int\limits_{]0,t]} (m(\tau) + n(\tau)u(\tau))u(\tau)d\mu(\tau), \tag{2.8}$$

$t \in T$, $c \geq 0$, then

$$u(t) \leq c + 2\int\limits_{]0,t]} (m(\tau) + n(\tau)u(\tau))d\mu(\tau),\ t \in T. \tag{2.9}$$

Proof We assume that the functions $m(t), n(t)$ are defined for all $t \in T$. Let $\varepsilon > 0$ be arbitrary and

$$v_\varepsilon(t) = \frac{1}{2}(c + \varepsilon)^2 + \int\limits_{]0,t]} (m(\tau) + n(\tau)u(\tau))u(\tau)d\mu(\tau). \tag{2.10}$$

Then, $v_\varepsilon(t)$ is a right continuous function of bounded variation, the differential measure dv_ε of which is absolutely continuous with respect to the measure μ and

$$\frac{dv_\varepsilon}{d\mu}(t) = (m(t) + n(t)u(t))u(t) \ \ \mu \text{ a.e.} \tag{2.11}$$

Consider the function $t \to \sqrt{v_\varepsilon(t)}$, which is positive, right continuous and increasing. Consequently, it is a function of bounded variation. The differential measure $d\sqrt{v_\varepsilon}$ has positive values and, thus, it is a Radon measure.

From the inequality

$$d\sqrt{v_\varepsilon(]s,t])} = \sqrt{v_\varepsilon(t)} - \sqrt{v_\varepsilon(s)} \leq$$

$$\frac{1}{2}\frac{1}{\sqrt{v_\varepsilon(0)}}(v_\varepsilon(t) - v_\varepsilon(s)) = \frac{1}{2\sqrt{v_\varepsilon(0)}} dv_\varepsilon(]s,t]),$$

$$s \leq t, \quad s, t \in T,$$

the absolute continuity of the measure dv_ε with respect to the measure μ and Theorem 1 [13] it follows that the measure $d\sqrt{v_\varepsilon}$ is absolutely continuous with respect to the measure μ. Using (2.4), we obtain

$$\frac{d\sqrt{v_\varepsilon}}{d\mu}(t) = \lim_{s\uparrow t} \frac{1}{\sqrt{v_\varepsilon(s)} + \sqrt{v_\varepsilon(t)}} \cdot \lim_{s\uparrow t} \frac{dv_\varepsilon(]s,t])}{\mu(]s,t])} \leq$$

$$\leq \frac{1}{\sqrt{v_\varepsilon(t-0)} + \sqrt{v_\varepsilon(t)}} \cdot \frac{dv_\varepsilon}{d\mu}(t) \tag{2.12}$$

for μ almost all $t \in]0,a]$. Since $dv_\varepsilon(0) = 0$, the inequality (2.12) holds for almost all $t \in [0,a]$. From (2.8), (2.10), (2.11) we see that

$$\frac{dv_\varepsilon}{d\mu}(t) \leq \sqrt{2}(m(t) + n(t)u(t))\sqrt{v_\varepsilon(t)},$$

for μ almost all $t \in [0,a]$.

Using this inequality and (2.12), we obtain

$$\frac{d\sqrt{v_\varepsilon}}{d\mu}(t) \leq \sqrt{2}(m(t) + n(t)u(t)),$$

μ a.e. on $[0,a]$.

Consequently,

$$\sqrt{v_\varepsilon(t)} \leq \sqrt{v_\varepsilon(0)} + \sqrt{2}\int_{]0,t]} (m(\tau) + n(\tau)u(\tau))d\mu(\tau), \quad t \in T. \tag{2.13}$$

Now the inequality (2.9) follows from the arbitrariness of $\varepsilon > 0$ and the inequalities (2.8), (2.10), (2.13). □

Remark 2.1 If the measure μ is not atomic, then from the inequality (2.8) we infer that

$$u(t) \leq c + \int_{]0,t]} (m(\tau) + n(\tau)u(\tau))d\mu(\tau). \tag{2.14}$$

3 Auxiliary Results

Lemma 3.1 *Let $C : T \rightrightarrows H$ be a multivalued mapping with r-prox regular values and the inequalities (1.1), (1.2) hold. Then, for any $u(\cdot), v(\cdot) \in L^1(T, H)$ the sweeping process*

$$-dx \in \mathcal{N}(C(t); x(t)) + u(t) + v(t), \tag{3.1}$$

$x(0) = x_0 \in C(0)$

has a unique BV solution $x(u; v)(\cdot)$, satisfying the inclusion (1.5) and the inequalities

$$\|x(u; v)(t) - x(u; v)(t - 0)\| \le \mu(\{t\}),\ t \in]0, a], \tag{3.2}$$

$$\|\frac{dx(u; v)}{d\nu}(t) + (u(t) + v(t))\frac{d\lambda}{d\nu}(t)\| \le$$

$$\le \frac{d\mu}{d\nu}(t) + \|u(t) + v(t)\|\frac{d\lambda}{d\nu}(t)\ \ \nu\ a.e. \tag{3.3}$$

for any Radon measure ν absolutely continuously equivalent to the measure μ.

The lemma follows from Theorem 5.1 [2].

In the rest of the paper, unless otherwise specified, we assume that for the multivalued mapping $C : T \rightrightarrows H$ with r-prox regular values the inequalities (1.1), (1.2) hold.

Consider the differential equation

$$\dot{r}(t) = 2(m(t) + n(t)r(t)),\ \ r(0) = \|x_0\| + \mu([0, a]), \tag{3.4}$$

that has a unique solution $r(t)$, where

$$m(t) = m_1(t) + n_1(t),\ \ n(t) = n_1(t) + n_2(t). \tag{3.5}$$

In what follows, we assume that Hypothesis H(U) 1)–3) and H(V) 1)–3) hold.

Consider the set

$$S_U = \{u(\cdot) \in L^1(T, H);\ \|u(t)\| \le m_1(t) + n_1(t)r(t)\ \text{ a.e.}\}, \tag{3.6}$$

$$S_V = \{v(\cdot) \in L^1(T, H);\ \|u(t)\| \le m_2(t) + n_2(t)r(t)\ \text{ a.e.}\} \tag{3.7}$$

and the multivalued mapping $\Gamma : T \rightrightarrows H$

$$\Gamma(t) = (m_1(t) + n_1(t)r(t))\overline{B},\ t \in T, \tag{3.8}$$

that is measurable with closed convex values. It is well known that for any measurable function $u : T \to H$ the function

$$t \to \text{Proj}_{\Gamma(t)} u(t)$$

is uniquely defined and measurable. Since $m_1(\cdot), n_1(\cdot) \in L^1(T, H)$, the operator $\mathcal{L} : L^1(T, H) \to S_U$

$$\mathcal{L}(u)(t) = \text{Proj}_{\Gamma(t)} u(t) \tag{3.9}$$

is well defined.

Lemma 3.2 *The operator* $\mathcal{L}: L^1(T, H) \to S_U$ *has the properties*

$$\|\mathcal{L}(u_1)(t) - \mathcal{L}(u_2)(t)\| \le \|u_1(t) - u_2(t)\|,\ t \in T, \tag{3.10}$$

$u_i(\cdot) \in L^1(T, H)$, $i = 1, 2$,

$$\mathcal{L}(u)(t) = u(t),\ t \in T,\ u(\cdot) \in S_U \tag{3.11}$$

and

$$\|\mathcal{L}(u)(t)\| \le m_1(t) + n_1(t)r(t)\ \ a.e.,\ u(\cdot) \in L^1(T, H). \tag{3.12}$$

The lemma follows from the properties of projection onto a closed convex set in a Hilbert space and (3.8).

Lemma 3.3 *For any* $u(\cdot) \in L^1(T, H)$, $v(\cdot) \in S_V$, *the sweeping process*

$$-dx \in \mathcal{N}(C(t), x(t)) + \mathcal{L}(u)(t) + v(t), \tag{3.13}$$

$x(0) = x_0 \in C(0)$ *has a unique* BV *solution* $x(\mathcal{L}(u); v)$ *with the properties*

$$\|x(\mathcal{L}(u); v)(t) - x(\mathcal{L}(u); v)(t-0)\| \le \mu(t),\ t \in]0, a], \tag{3.14}$$

$$\|x(\mathcal{L}(u); v)\| \le r(t),\ \ t \in T, \tag{3.15}$$

$$\left\| \frac{dx(\mathcal{L}(u); v)}{d\nu}(t) \right\| \le \frac{d\mu}{d\nu}(t) + 2(\|\mathcal{L}(u)(t)\| + \|v(t)\|)\frac{d\lambda}{d\nu}(t)\ \ \nu\ a.e. \tag{3.16}$$

for any Radon measure ν *absolutely continuously equivalent to the Radon measure* $\mu + \lambda$.

Proof The existence of a unique BV solution $x(\mathcal{L}(u); v)$ follows from Lemma 3.1 and (3.7), (3.12). The inequalities (3.14), (3.16) follow from (3.2), (3.3).

Using (3.16) and (2.5), we obtain

$$\|x(\mathcal{L}(u); v)\| \le \|x_0\| + \mu([0, a]) + 2 \int_{[0,t]} (\|\mathcal{L}(u)(\tau)\| + \|v(t)\|) d\lambda(\tau).$$

From this inequality, (3.7), (3.12) and (3.5) we deduce that

$$\|x(\mathcal{L}(u); v)(t)\| \le \|x_0\| + \mu([0, a]) + 2 \int_{[0,t]} (m(\tau) + n(\tau))r(\tau)) d\lambda(\tau).$$

The last inequality and (3.4) give the inequality (3.15). Lemma is proved. □

Denote by $\Lambda : L^1(T, H) \times S_V \to BV_+(T, H)$ the operator

$$\Lambda(u; v) = x(\mathcal{L}(u); v),\ \ u(\cdot) \in L^1(T, H),\ v \in S_U. \tag{3.17}$$

From (3.11), (3.17) we directly see that

$$\Lambda(u; v) = x(u; v),\ \ u(\cdot) \in S_U,\ v \in S_V. \tag{3.18}$$

Theorem 3.1 *There exists a constant $L > 0$ such that*

$$\|\Lambda(u_1; v_1)(t) - \Lambda(u_2; v_2)(t)\| \leq$$
$$\leq L \int_{]0,t]} (\|u_1(\tau) - u_2(\tau)\| + \|v_1(\tau) - v_2(\tau)\|) d\lambda(\tau), t \in T, \tag{3.19}$$

$u_i(\cdot) \in L^1(T, H)$, $v_i(\cdot) \in S_V$, $i = 1, 2$.

Proof Let $u_i(\cdot) \in L^1(T, H)$, $v_i(\cdot) \in S_V$, $i = 1, 2$. According to Lemma 3.3 the inclusion (3.13) has BV solutions $x(\mathcal{L}(u_i); v_i,), i = 1, 2$, which, as follows from (1.5), satisfy the inclusion

$$-\frac{dx(\mathcal{L}(u_i); v_i)}{d\nu}(t) - (\mathcal{L}(u_i)(t) + v_i(t))\frac{d\lambda}{d\nu}(t) \in$$
$$\in N(C(t); x(\mathcal{L}(u_i); v_i)(t)) \ \nu \text{ a.e.} \tag{3.20}$$

From these inclusions, Proposition 2.1, Theorem 2.1 (c) and Lemma 2.5 it follows that

$$\|x(\mathcal{L}(u_1); v_1)(t) - x(\mathcal{L}(u_2); v_2)(t)\| \leq$$
$$\leq 2 \int_{]0,t]} (\|\mathcal{L}(u_1)(\tau) - \mathcal{L}(u_2)(\tau)\| + \|v_1(\tau) + v_2(\tau)\|)\frac{d\lambda}{d\nu}(\tau)d\nu(\tau) +$$
$$+ \frac{1}{r} \int_{]0,t]} \Bigg\{ \|x(\mathcal{L}(u_1; v_1)(\tau) - x(\mathcal{L}(u_2); v_2)(\tau)\| \cdot$$
$$\cdot \left[\sum_{i=1}^{2} (\| \frac{dx(\mathcal{L}(u_i); v_i)}{d\nu}(\tau)\| + (\|\mathcal{L}(u_i)(\tau) + v_i(\tau)\|)\frac{d\lambda}{d\nu}(\tau)) \right] \Bigg\} d\nu(\tau). \tag{3.21}$$

Let $t^* \in]0, t]$ be arbitrary and fixed. Denote

$$\alpha = 2 \int_{]0,t^*]} (\|u_1(\tau) - u_2(\tau)\| + \|v_1(\tau) + v_2(\tau)\|) d\lambda(\tau), \tag{3.22}$$

$$g(t) = \frac{1}{r} \sum_{i=1}^{2} (\| \frac{dx(\mathcal{L}(u_i); v_i)}{d\nu}(t)\| + (\|\mathcal{L}(u_i)(\tau)\| + \|v_i(\tau)\|)\frac{d\lambda}{d\nu}(\tau)). \tag{3.23}$$

Using (2.5), (3.10), (3.21)–(3.23), we arrive at the inequality

$$\|x(\mathcal{L}(u_1); v_1)(t) - x(\mathcal{L}(u_2); v_2)(t)\| \leq$$
$$\alpha + \int_{]0,t]} g(\tau)\|x(\mathcal{L}(u_1); v_1)(\tau) - x(\mathcal{L}(u_2); v_2)(\tau)\| d\nu(\tau), \ t \in]0, t^*]. \tag{3.24}$$

From (3.23), (2.6) we obtain

$$g(t)\nu(\{t\}) = \frac{1}{r} \sum_{i=1}^{2} \left\| \frac{dx(\mathcal{L}(u_i); v_i)}{d\nu}(t) \right\| \nu(\{t\}). \tag{3.25}$$

Let

$$\gamma = 2 \sup_{s \in]0,a]} \mu(\{s\}).$$

As follows from (1.2), (3.14), we have

$$2 \max_{1 \le i \le 2} \sup_{\tau \in]0,a]} \|x(\mathcal{L}(u_i); v_i)(\tau) - x(\mathcal{L}(u_i); v)(\tau - 0)\| \le \gamma < r.$$

Since

$$\left\| \frac{dx(\mathcal{L}(u_i); v_i)}{d\nu}(t) \right\| \nu(\{t\}) = \|x(\mathcal{L}(u_i); v_i)(t) - x(\mathcal{L}(u_i); v_i)(\tau - 0)\|,$$

from (3.25) and the last inequality we obtain

$$0 \le g(t)\nu(\{t\}) \le \frac{\gamma}{r} < 1, \ t \in]0, a].$$

From this inequality, (3.24) and Lemma 2.4 we see that

$$\|x(\mathcal{L}(u_1); v_1)(t) - x(\mathcal{L}(u_2); v_2)(t)\| \le$$

$$\le \alpha \exp\left(\frac{1}{1-\theta} \int_{]0,t]} g(\tau) d\nu(\tau) \right), \ t \in]0, t^*], \tag{3.26}$$

where

$$\theta = \frac{\gamma}{r}. \tag{3.27}$$

Using (2.5), (3.7), (3.5), (3.12), (3.16), (3.23), we infer that

$$\int_{]0,t]} g(\tau) d\nu(\tau) \le \frac{2}{r}\mu(]0, a]) + \frac{4}{r} \int_T (m(t) + n(t)r(t)) d\lambda(t), \ t \in T. \tag{3.28}$$

Using (3.26), (3.22), (3.28) and the arbitrariness of $t^* \in]0, a]$, we derive (3.19) with the constant L,

$$L = 2 \exp\left(\frac{1}{1-\theta} \left[\frac{2}{r}\mu([0, a]) + \frac{4}{r} \int_T (m(t) + n(t)r(t)) d\lambda(t) \right] \right).$$

The theorem is proved. □

According to Hypothesis H(V) 3), the set

$$V(t, r(a))\overline{B}) \cap (m_2(t) + n_2(t)r(a))\overline{B}, \ t \in T = [0, a]$$

is relatively compact for almost every $t \in T$.

Let

$$W(t) = V(t, r(t)\overline{B}) \cap (m_2(t) + n_2(t)r(t))\overline{B}, \ t \in T = [0, a]. \tag{3.29}$$

Since $W(t) \subset V(t, r(a)\overline{B}) \cap (m_2(t) + n_2(t)r(a))\overline{B}$, $t \in T$, the values of multivalued mapping $t \to \overline{\text{co}}\, W(t)$ are convex compact sets and

$$\|\overline{\text{co}}\, W(t)\| \le m_2(t) + n_2(t)r(t). \tag{3.30}$$

Let

$$S_{\overline{\text{co}}\, W} = \{v(\cdot) \in L^1(T, H);\ v(t) \in \overline{\text{co}}\, W(t) \text{ a.e.}\}. \tag{3.31}$$

From Lemma 2.2 it follows that the multivalued mapping

$$t \to V(t, \Lambda(u; v)(t)) \cap (m_2(t) + n_2(t)\|\Lambda(u; v)(t)\|)\overline{B}$$

has a measurable selector that is an element of the space $L^1(T, H)$. Hence, the set $S_{\overline{\text{co}}\, W}$ is not empty.

Lemma 3.4 ([14]) *The following statements are true*:
a) $S_{\overline{\text{co}}\, W}$ *is a nonempty, convex, compact subset of the space* $L^1(T, H)$;
b) *for any* $v(\cdot) \in S_{\overline{\text{co}}\, W}$ *we have*

$$\|v(t)\| \le m_2(t) + n_2(t)r(t) \text{ a.e.}; \tag{3.32}$$

c) *the set*

$$S_{\overline{\text{co}}\, W}(t) = \{v(t);\ v(\cdot) \in S_{\overline{\text{co}}\, W}\} \subset H$$

is compact for almost all $t \in T$.

Theorem 3.2 *The operator* Λ *is continuous from* $L^1(T, H) \times \omega\text{-}S_{\overline{\text{co}}\, W}$ *to* $BV_+(T, H)$.

Proof Since $S_{\overline{\text{co}}\, W}$ is a convex metrizable compact set in the topology of the space $\omega\text{-}L^1(T, H)$, it is enough to show the sequential continuity of the operator $\Lambda(u; v)$.

Let a sequence $u_n(\cdot) \in L^1(T, H)$, $n \ge 1$ converge in the space $L^1(T, H)$ to $u_0(\cdot)$, and a sequence $v_n \in S_{\overline{\text{co}}\, W}$, $n \ge 1$ converge to $v_0(\cdot)$ in the space $\omega\text{-}L^1(T, H)$. Recalling (3.17), denote

$$x_n(\cdot) = \Lambda(u_n; v_n)(\cdot) = x(\mathcal{L}(u_n); v_n)(\cdot),\ n \ge 0, \tag{3.33}$$

where $x(\mathcal{L}(u_n); v_n)(\cdot)$, $n \ge 0$ are solutions of the inclusion (3.13) corresponding to $u_n(\cdot), v_n(\cdot), n \ge 0$.

From (3.16) it follows that

$$\left\|\frac{dx_n}{d\nu}(t)\right\| \le \frac{d\mu}{d\nu}(t) + 2(\|\mathcal{L}(u_n)(t)\| + \|v_n(t)\|\frac{d\lambda}{d\nu}(t).$$

From this inequality and (3.7), (3.12), (3.5), (2.5) we obtain

$$\|x_n(t) - x_n(s)\| \le \mu(]s, t]) + 2\int_{]s,t]} (m(\tau) + n(\tau)r(t))d\lambda(\tau),$$

$n \ge 0$, $s \le t$, $s, t \in T$.

From this inequality and (3.15) it follows that the sequence $x_n(\cdot), n \geq 1$ is uniformly bounded in norm and in variation. From Theorem 2.1 [15, Chap. 0] we know that there exists a subsequence $x_{n_m}(\cdot), m \geq 1$ of the subsequences $x_n(\cdot), n \geq 1$, pointwise converging in the space ω-H to some function $y : T \to H$ of bounded variation.

Let

$$J_1^m(t) = \int\limits_{]0,t]} \langle \mathcal{L}(u_0)(\tau) - \mathcal{L}(u_{n_m})(\tau), x_{n_m}(\tau) - x_0(\tau)\rangle d\lambda(\tau), \tag{3.34}$$

$$J_2^m(t) = \int\limits_{]0,t]} \langle v_0(\tau) - v_{n_m}(\tau), x_{n_m}(\tau) - y(\tau)\rangle d\lambda(\tau), \tag{3.35}$$

$$J_3^m(t) = \int\limits_{]0,t]} \langle v_0(\tau) - v_{n_m}(\tau), y(\tau) - x_0(\tau)\rangle d\lambda(\tau), \tag{3.36}$$

$m \geq 1$.

Since $u_{n_m}(\cdot) \to u_0(\cdot)$, $m \to \infty$ in the space $L^1(T, H)$, from (3.10), (3.15) and (3.34) we infer that

$$\lim_{m\to\infty} \sup_{t\in]0,a]} |J_1^m(t)| = 0. \tag{3.37}$$

From Lemma 3.4 it follows that the set

$$\{\cup(v_0(\tau) - v_{n_m}(\tau);\ m \geq 1\} \subset H$$

is relatively compact for almost all $t \in T$.

Since the sequence $x_{n_m}(\tau) - y(\tau), m \geq 1, \tau \in T$ is bounded, the sequence of functions

$$h \to \langle h, x_{n_m}(\tau) - y(\tau)\rangle,\ \ m \geq 1$$

is equicontinuous. It is well known that on every equicontinuous set the topology of pointwise convergence coincides with the topology of uniform convergence on compact sets. Therefore,

$$\lim_{m\to\infty} \langle v_0(\tau) - v_{n_m}(\tau), x_{n_m}(\tau) - y(\tau)\rangle = 0 \text{ a.e.}$$

From this equality, (3.15), (3.7), (3.35) and Lebesgue's dominated convergence theorem it follows that

$$\lim_{m\to\infty} \sup_{t\in]0,a]} |J_2^m(t)| = 0. \tag{3.38}$$

Consider the functions $t \to J_3^m(t)$, $m \geq 1$. Since $v_{n_m}(\cdot) \to v_0(\cdot)$, $m \to \infty$ in the space ω-$L^1(T, H)$, we have $J_3^m(t) \to 0$, $m \to \infty$, $t \in T$. From (3.32), (3.15), (3.36) we see that

$$|J_3^m(t) - J_3^m(s)| \leq 4 \int\limits_{]s,t]} (m_2(\tau) + n_2(\tau)r(\tau))r(\tau) d\lambda(\tau).$$

From this inequality it follows that the sequence of functions $t \to J_3^m(t)$, $m \geq 1$ is equicontinuous. Hence,

$$\lim_{m\to\infty} \sup_{t\in T} |J_3^m(t)| = 0. \tag{3.39}$$

Since $x_{n_m}(\cdot), m \geq 1$ is a solution of the inclusion (3.13), the inclusion (3.20) holds. From this inclusion, Proposition 2.1 and Theorem 2.1 (c) we obtain the inequality

$$\frac{1}{2}\|x_{n_m}(t) - x_0(t)\|^2 \leq J_1^m(t) + J_2^m(t) + J_3^m(t) + \frac{1}{2r}\int_{]0,t]} \|x_{n_m}(\tau) - x_0(\tau)\|^2 \cdot$$

$$\cdot(\|\frac{dx_{n_m}}{d\nu}(\tau)\| + \|\frac{dx_0}{d\nu}(\tau)\| + (\|\mathcal{L}(u_{n_m}(\tau)\| + \|\mathcal{L}(u_0)(\tau)\| +$$

$$+\|v_{n_m}(\tau)\| + \|v_0(\tau)\|)\frac{d\lambda}{d\nu}(\tau))d\nu(\tau). \tag{3.40}$$

From (3.37)–(3.40) it follows that for any $\varepsilon > 0$ there exists $m(\varepsilon) \geq 1$ such that

$$\|x_{n_m}(\tau) - x_0(\tau)\|^2 \leq \varepsilon + \int_{]0,t]} \|x_{n_m}(\tau) - x_0(\tau)\|^2 g_m(\tau) d\nu(\tau), \tag{3.41}$$

$m \geq m(\varepsilon)$, $t \in]0, a]$, where

$$g_m(t) = \frac{1}{r}(\|\frac{dx_{n_m}}{d\nu}(t)\| + \|\frac{dx_0}{d\nu}(t)\|) + \|\mathcal{L}(u_{n_m}(t)\| +$$

$$+\|\mathcal{L}(u_0)(t)\| + \|v_{n_m}(t)\| + \|v_0(t)\|\frac{d\lambda}{d\nu}(t),\ m \geq m(\varepsilon). \tag{3.42}$$

Reasoning as in the proof of Theorem 3.1, we see that

$$0 \leq g_m(t)\nu(\{t\}) \leq \theta < 1,\ t \in]0, a],$$

where θ is defined by the equality (3.27).

Using Lemma 2.4, we arrive at the inequality

$$\|x_{n_m}(t) - x_0(t)\|^2 \leq \varepsilon \exp\left(\frac{1}{1-\theta}\int_{]0,t]} g_m(\tau) d\nu(\tau)\right), \tag{3.43}$$

$m \geq m(\varepsilon)$.

Using (3.42) and reasoning as in the proof of Theorem 3.1, we obtain an analogue of the inequality (3.28)

$$\int_{]0,t]} g_m(\tau) d\nu(\tau) \leq \frac{2}{r}\mu([0, a]) + \frac{4}{r}\int_T (m(t) + n(t)r(t)) d\lambda(\tau).$$

From this inequality and (3.43) we derive

$$\|x_{n_m}(t) - x_0(t)\|^2 \leq \varepsilon L_1,\ m \geq 1,\ t \in T, \tag{3.44}$$

where L_1 is defined by the equality

$$L_1 = \exp\left(\frac{1}{1-\theta}\left[\frac{2}{r}(\mu([0,a]) + 2\int\limits_T (m(t) + n(t)r(t))d\lambda(\tau)\right]\right). \tag{3.45}$$

From the arbitrariness of $\varepsilon > 0$ and (3.44) it follows that the sequence $x_{n_m}(\cdot), m \geq 1$ converges in the space $BV_+(T, H)$ to $x_0(\cdot)$.

We have thus shown that if a sequence $u_n(\cdot), n \geq 1$ converges to $u_0(\cdot)$ in $L^1(T, H)$, a sequence $v_n(\cdot) \in S_{\overline{\text{co}}\, W}, n \geq 1$ converges to v_0 in ω-$L^1(T, H)$, then there exists a subsequence $x_{n_m}(\cdot), m \geq 1$ of the sequence $x_n(\cdot), n \geq 1$, converging to x_0 in the space $BV_+(T, H)$.

Suppose that the sequence $x_n(\cdot), n \geq 1$ itself does not converge to $x_0(\cdot)$ in the space $BV_+(T, H)$. Then, there exists a subsequence $x_{n_k}(\cdot), k \geq 1$ of the sequence $x_n(\cdot), n \geq 1$ such that any subsequence of the sequence $x_{n_k}(\cdot), k \geq 1$ does not converges to x_0. Repeating the reasoning above to the sequences $x_{n_k}(\cdot)$ and $v_{n_k}(\cdot)$ and taking into account the fact that for $u_0(\cdot)$, $v_0(\cdot)$ the inclusion (3.13) has a unique solution, we arrive at a contradiction. Consequently, the sequence $x_n(\cdot), n \geq 1$ converges to $x_0(\cdot)$ in the space $BV_+(T, H)$. Recalling the notation (3.33), we obtain the statement of the theorem. The theorem is proved. □

4 Multivalued Nemytskii Operator

Let $u(\cdot) \in L^1(T, H)$, $v(\cdot) \in S_{\overline{\text{co}}\, W}$. Consider the multivalued mapping $t \to U(t, \Lambda(u; v)(t))$. Since $t \to \Lambda(u; v)(t)$ is a right continuous function of bounded variation, it is Lebesgue measurable. Hence, from Hypothesis H(U) it follows that the mapping $t \to U(t, \Lambda(u; v)(t))$ is an element of the space $L^1(T, \text{cb}\, H)$. Then, the set

$$\Phi(u; v) = \{f(\cdot) \in L^1(T, H);\ f(t) \in U(t, \Lambda(u; v)(t)) \text{ a.e.}\} \tag{4.1}$$

is an element of the space $\text{dcb}\, L^1(T, H)$. Consequently, we can define a multivalued mapping $\Phi : L^1(T, H) \times S_{\overline{\text{co}}\, W} \to \text{dcb}\, L^1(T, H)$, which is called the Nemytskii multivalued operator.

On the space $L^1(T, H)$ consider the function

$$P(x) = \int\limits_T \rho(t, x(t))\, dt,\ x(\cdot) \in L^1(T, H), \tag{4.2}$$

with

$$\rho(t, x(t)) = \exp(-2L \int\limits_0^t k(\tau)\, d\tau)\, \|x(t)\|, \tag{4.3}$$

where the function $k(\cdot)$ is from the inequality (1.8), and the constant $L > 0$ is from the inequality (3.19). It is clear that the function $P(x)$ is a norm equivalent to the standard norm of the space $L^1(T, X)$.

The Hausdorff distance between elements of the space $\text{cb}\, L^1(T, H)$, when the space $L^1(T, H)$ is endowed with the standard norm, we denote by $\text{haus}_L(\cdot, \cdot)$.

When the space $L^1(T, H)$ is endowed with the norm (4.2), the Hausdorff distance between elements from $\operatorname{cb} L^1(T, H)$ we denote by $\operatorname{haus}_P(\cdot, \cdot)$.

Theorem 4.1 *The Nemytskii operator $\Phi(u; v)$ has the properties*:

1) *the operator $\Phi(u; v)$ is continuous from $L^1(T, H) \times \omega\text{-}S_{\overline{\mathrm{co}}\, W}$ to the space* $\operatorname{dcb} L^1(T, H)$ *with the Hausdorff metric* $\operatorname{haus}_L(\cdot, \cdot)$;

2) *the following inequality holds*

$$\operatorname{haus}_L(\Phi(u_1; v), \Phi(u_2; v)) \le L_2 \|u_1 - u_2\|_{L^1}, \tag{4.4}$$

$u_i(\cdot) \in L^1(T, H)$, $i = 1, 2$, $v \in S_{\overline{\mathrm{co}}\, W}$,

$$L_2 = L\|k(\cdot)\|_{L^1}; \tag{4.5}$$

$$\operatorname{haus}_P(\Phi(u_1; v); \Phi(u_2; v)) \le \frac{1}{2} P(u_1 - u_2), \tag{4.6}$$

$u_i(\cdot) \in L^1(T, H)$, $i = 1, 2$, $v \in S_{\overline{\mathrm{co}}\, W}$.

Proof From Proposition 4.2 in [16] and (4.1) we infer that

$$\operatorname{haus}_{L^1}(\Phi(u_1; v_1), \Phi(u_2; v_2)) \le$$

$$\int_T \operatorname{haus}(U(t, \Lambda(u_1; v_1)(t)), U(t, \Lambda(u_2; v_2)(t)))dt.$$

Then, using the inequality (1.8), we obtain

$$\operatorname{haus}_{L^1}(\Phi(u_1; v_1), \Phi(u_2; v_2)) \le \|k(\cdot)\|_{L^1} \|\Lambda(u_2; v_2) - \Lambda(u_1; v_1)\|_{BV_+}. \tag{4.7}$$

And the statement 1) of Theorem 4.1 follows from this inequality and Theorem 3.2.

The inequality (4.4) follows from the inequality (4.7) and (3.19), (4.5).

Now, we prove the inequality (4.6).

From (1.8), (3.19) we deduce that

$$\operatorname{haus}(U(t, \Lambda(u_1; v)(t)), U(t, \Lambda(u_2; v)(t))) \le k(t)L \int_0^t \|u_1(\tau) - u_2(\tau)\|\, d\tau.$$

From this inequality and (4.1)–(4.3) we obtain

$$\operatorname{haus}_P(\Phi(u_1; v), \Phi(u_2; v)) \le$$

$$\le \int_T \left(\exp(-2L \int_0^t k(\tau)\, d\tau)\right) k(t))L \left(\int_0^t \|u_1(\tau) - u_2(\tau)\|\, d\tau\right) dt.$$

Integrating by parts the right-hand side of this inequality we arrive at the inequality

$$\operatorname{haus}_P(\Phi(u_1; v), \Phi(u_2; v)) \le \frac{1}{2} \int_T (\exp(-2L \int_0^t k(\tau)\, d\tau)) \|u_1(t) - u_2(t)\|\, dt. \tag{4.8}$$

Now the inequality (4.6) follows from the inequalities (4.8) and (4.2), (4.3). Theorem is proved. □

For a fixed $v(\cdot) \in \omega\text{-}S_{\overline{\text{co}}\,W}$ denote by $(\text{Fix})(v)$ the set of fixed points of the operator $\Phi(u; v)$.

Theorem 4.2 *The following statements hold*:

a) for any $v(\cdot) \in \omega\text{-}S_{\overline{\text{co}}\,W}$ *the set* $(\text{Fix}\,\Phi)(v)$ *is not empty*;

b) there exists a continuous function $u : \omega\text{-}S_{\overline{\text{co}}\,W} \to L^1(T, H)$ *such that* $u(v) \in (\text{Fix}\,\Phi)(v)$, *i.e.*

$$u(v) \in \Phi(u(v); v), \ \ v \in \omega\text{-}S_{\overline{\text{co}}\,W}; \tag{4.9}$$

c)

$$u(v)(t) \in U(t, x(u(v); v)(t)) \text{ a.e.}, \ v(\cdot) \in \omega\text{-}S_{\overline{\text{co}}\,W}. \tag{4.10}$$

Proof From the statement 1) of Theorem 4.1 it follows that for a fixed $u(\cdot) \in L^1(T, H)$ the mapping $v \to \Phi(u; v)$ is lower semicontinuous from the compact metric space $\omega\text{-}S_{\overline{\text{co}}\,W}$ to the space $L^1(T, H)$ with closed, bounded, decomposable values. Now the statements a), b) of theorem follow from the inequality (4.6) and Theorem 1.1 [17].

From (4.1) and (4.9) it follows that

$$u(v)(t) \in U(t, \Lambda(u(v); v)(t)) \text{ a.e.}, \ \ v(\cdot) \in S_{\overline{\text{co}}\,W}. \tag{4.11}$$

From this inclusions and (3.6), (3.15), (3.17), (1.9) we infer that

$$u(v) \in S_U, \ \ v \in S_{\overline{\text{co}}\,W}. \tag{4.12}$$

Then, according to (3.18) we have

$$\Lambda(u(v); v) = x(u(v); v), \ \ v \in S_{\overline{\text{co}}\,W}. \tag{4.13}$$

Using (4.11), (4.13), we obtain the inclusion (4.10). The theorem is proved. □

5 Proof of the Main Results

Consider the operator $\Lambda(u; v)$ defined by the equality (3.17). Then, as follows from the equality (4.13), Theorem 3.2 and the statement b) of Theorem 4.2, the mapping $v \to \Lambda(u(v); v)$ is continuous from $\omega\text{-}S_{\overline{\text{co}}\,W}$ to $BV_+(T, H)$.

Let

$$Q(t, x) = V(t, x) \cap (m_2(t) + n_2(t)\|x\|)\overline{B} \tag{5.1}$$

and

$$S_Q(v) = \{f(\cdot) \in L^1(T, H); \ \ f(t) \in Q(t, \Lambda(u(v); v)(t) \text{ a.e.}\}, \tag{5.2}$$

$v \in S_{\overline{\text{co}}\,W}$.

From Lemma 2.2 and (5.1) it follows that $S_Q(v)$ is a nonempty, convex, compact subset of the space $\omega\text{-}L^1(T, H)$. Using (5.1), (5.2), (3.29), (3.31), (4.13), (3.15), we obtain the inclusion

$$S_Q(v) \subset S_{\overline{\mathrm{co}}\,W}, \quad v(\cdot) \in S_{\overline{\mathrm{co}}\,W}. \tag{5.3}$$

Thus, the multivalued mapping $v \to S_Q(v)$ from the set ω-$S_{\overline{\mathrm{co}}\,W}$ to the set ω-$S_{\overline{\mathrm{co}}\,W}$ is well defined. Its values are nonempty, convex, compact subsets of the space ω-$L^1(T, H)$.

Let

$$\mathcal{K} = \{\Lambda(u(v); v); \; v(\cdot) \in \omega\text{-}S_{\overline{\mathrm{co}}\,W}\}.$$

Then, the set $\mathcal{K}$ is a compact subset of the space $BV_+(T, H)$. From Corollary 2.4 in [9] it follows that there exists a compact set $D \subset H$ such that

$$\Lambda(u(v); v)(t) \in D, \; t \in T, \; v(\cdot) \in S_{\overline{\mathrm{co}}\,W}. \tag{5.4}$$

According to Hypothesis H(V) 3), the values of mapping $t \to V(t, D) \cap (m_2(t) + n_2(t)\|D\|)\overline{B}$ are relatively compact sets for almost every $t \in T$.

Let

$$V^*(t) = V(t, D) \cap (m_2(t) + n_2(t)\|D\|)\overline{B}.$$

Then the values of mapping $t \to \overline{\mathrm{co}}\, V^*(t)$ are convex compact sets for almost all $t \in T$. Since $Q(t, x) \subset \overline{\mathrm{co}}\, V^*(t, x)$, $x \in D$, from $H(V)\,2)$ it follows that the mapping $x \to Q(t, x)$, $x \in D$ is upper semicontinuous with convex compact values for almost all $t \in T$.

Let a sequence $v_n \in \omega$-$S_{\overline{\mathrm{co}}\,W}$, $n \geq 1$ converge to $v(\cdot) \in \omega$-$S_{\overline{\mathrm{co}}\,W}$ in the space ω-$L^1(T, H)$. Then, the sequence $\Lambda(u(v_n); v_n)$ converges to $\Lambda(u(v); v)$ in the space $BV_+(T, H)$. According to (5.4) we have

$$\Lambda(u(v)_n; v_n)(t) \in D, \; v_n(\cdot) \in S_{\overline{\mathrm{co}}\,W}. \tag{5.5}$$

Then, from (5.5) and the upper semicontinuity of the mapping $x \to Q(t, x)$, $x \in D$ for almost every $t \in T$ we see that

$$\bigcap_{n=1}^{\infty} \overline{\mathrm{co}}\{\bigcup_{k \geq n}^{\infty} Q(t, \Lambda(u(v_n); v_n)(t))\} \subset Q(t, \Lambda(u(v); v)(t)) \text{ a.e.} \tag{5.6}$$

If a sequence $f_n(\cdot) \in S_Q(v_n)$, $n \geq 1$ converges in the space ω-$L^1(T, H)$ to $f(\cdot)$, then from (5.2), (5.6) and the Mazur lemma for weakly converging sequences it follows that

$$f(t) \in Q(t, \Lambda(u(v); v)(t)) \text{ a.e.}$$

From this inclusions and (5.2) we infer that $f(\cdot) \in S_Q(v)$. Consequently, the mapping $v \to S_Q(v)$, $v \in \omega$-$S_{\overline{\mathrm{co}}\,W}$ has closed graph in the topology of the space ω-$L^1(T, H)$. Then, from (5.3) and the metrizability of the compact set ω-$S_{\overline{\mathrm{co}}\,W}$ it follows that the mapping $v \to S_Q(v)$, $v \in \omega$-$S_{\overline{\mathrm{co}}\,W}$ is upper semicontinuous from ω-$S_{\overline{\mathrm{co}}\,W}$ to ω-$S_{\overline{\mathrm{co}}\,W}$ with convex compact values. According to the Ky Fan fixed point theorem [18] there exists a fixed point v_* of the mapping $v \to S_Q(v)$, i.e.

$$v_* \in S_Q(v_*). \tag{5.7}$$

Denote $u_* = u(v_*)$. Then, from (4.13) it follows that

$$x(u_*; v_*) = \Lambda(u(v_*); v_*). \tag{5.8}$$

Using (5.1), (5.2), (5.7), (5.8), (4.10), (3.1), (3.2), (3.3), we obtain

$$-\frac{dx(u_*;v_*)}{d\nu}(t)-(u_*(t)+v_*(t))\frac{d\lambda}{d\nu}(t)\in\mathcal{N}(C(t);x(u_*;v_*)(t))\ \nu \text{ a.e.},$$

$$u_*(t)\in U(t,x(u_*;v_*)(t)) \text{ a.e.},$$

$$v_*(t)\in V(t,x(u_*;v_*)(t)) \text{ a.e.},$$

$$\|x(u_*;v_*)(t)-x(u_*;v_*)(t-0)\|\le\mu(\{t\}),\quad t\in T.$$

Consequently, according to (1.5)–(1.7) the triplet $(x(u_*;v_*)(\cdot),u_*(\cdot),v_*(\cdot))$ is a BV solution of the inclusion (1.3). Finally, the inequalities (1.12), (1.13) follow from (3.2), (3.3). Theorem 1.1 is proved.

Proof of Theorem 1.2 Under the assumptions of Theorem 1.2, Hypotheses $H(U)$ 1), 2), 3) hold for the mapping $U:T\times H\rightrightarrows H$ with closed convex values, and Hypotheses $H(V)$ 1)*, 2), 3) hold for the mapping $V:T\times H\rightrightarrows H$ with convex compact values. Theorem 1.1 implies that the set $\mathcal{R}(x_0)$ is not empty. Now, we show that

$$\|x(u;v)\|\le M \tag{5.9}$$

for any $(x(u;v)(\cdot),u(\cdot),v(\cdot))\in\mathcal{R}(x_0)$ for some $M>0$. From (3.3), (1.9), (1.11), (3.5) and (2.5) we infer that

$$\|x(u;v)(t)\|\le\|x_0\|+\mu([0,a])+2\int_T m(t)d\lambda(t)+$$

$$+2\int_{]0,t]}\|x(u;v)(\tau)\|n(\tau)d\lambda(\tau). \tag{5.10}$$

From (5.10) and the Bellman–Gronwall inequalities we obtain

$$\|x(u;v)(t)\|\le(\|x_0\|+\mu([0,a])+2\int_T m(t)d\lambda(t))\exp(2\int_T n(\tau)d\lambda(\tau)).$$

This inequality implies directly the inequality (5.9).

Let

$$S_U^*=\{u(\cdot)\in L^1(T,H);\ \ \|u(t)\|\le m_1(t)+n_1(t)M\},$$

$$S_V^*=\{v(\cdot)\in L^1(T,H);\ \ \|v(t)\|\le m_2(t)+n_2(t)M\}.$$

Since for any $(x(u;v),u(\cdot),v(\cdot))\in\mathcal{R}(x_0)$ the inclusions $u(\cdot)\in S_U^*$, $v(\cdot)\in S_V^*$ are valid and the sets S_{U^*}, S_{V^*} are convex, metrizable, and compact in the space ω-$L^1(T,H)$, for the closedness of the set $\mathcal{R}(x_0)$ in the space $BV_+(T,H)\times\omega\text{-}L^1(T,H)\times\omega\text{-}L^1(T,H)$ it is enough to prove its sequential closedness.

Let $(x(u_n;v_n),u_n(\cdot),v_n(\cdot))\in\mathcal{R}(x_0)$, $n\ge1$ and sequences $u_n(\cdot),v_n(\cdot)$, $n\ge1$ converge in the space ω-$L^1(T,H)$ to $u_0(\cdot)$ and $v_0(\cdot)$, and the sequence $x(u_n;v_n)(\cdot)$, $n\ge1$ converges to $y(\cdot)$ in $BV_+(T,H)$.

Let $x(u_0; v_0)$ be a solution of the inclusion

$$-dx(u_0; v_0)(t) \in \mathcal{N}(C(t); x(u_0; v_0)(t)) + u_0(t) + v_0(t). \tag{5.11}$$

Denote

$$x_n(\cdot) = x(u_n; v_n)(\cdot),\, w_n(t) = u_n(t) + v_n(t),\; n \geq 0. \tag{5.12}$$

Consider the functions

$$I_1^n(t) = \int\limits_{]0,t]} \langle w_0(\tau) - w_n(\tau), x_n(\tau) - y(\tau)\rangle d\lambda(\tau), \tag{5.13}$$

$$I_2^n(t) = \int\limits_{]0,t]} \langle w_0(\tau) - w_n(\tau), y(\tau) - x_0(\tau)\rangle d\lambda(\tau), \tag{5.14}$$

$n \geq 1$.

Since $w_n(\cdot) \to w_0(\cdot)$ in ω-$L^1(T, H)$ and $x_n(\cdot) \to y(\cdot)$ in $BV_+(T, H)$ and, consequently, in $L^1(T, H)$, from (5.13) we obtain

$$\lim_{n\to\infty} \sup_{t\in T} |I_1^n(t)| = 0. \tag{5.15}$$

From the convergence $w_n(\cdot) \to w_0(\cdot)$ in ω-$L^1(T, H)$ and (5.14) we infer that

$$I_2^n(t) \to 0,\; n \to \infty,\; t \in T. \tag{5.16}$$

Using (5.14), (5.9), (3.5), (5.12) and the inclusions $u_n(\cdot) \in S_U^*$, $v_n(\cdot) \in S_V^*$, we obtain the inequality

$$|I_2^n(t) - I_1^n(s)| \leq 4 \int\limits_{[s,t]} (m(t) + n(t)M)Md\lambda(\tau),\; s \leq t.$$

From this inequality it follows that the sequence of functions $t \to I_2^n(t)$, $n \geq 1$ is equicontinuous on T. Since on every equicontinuous set the topology of pointwise convergence coincides with the topology of uniform convergence on T, from (5.16) we see that

$$\lim_{n\to\infty} \sup |I_2^n(t)| = 0. \tag{5.17}$$

By analogy with (3.40) we obtain the inequality

$$\frac{1}{2}\|x_n(t) - x_0(t)\|^2 \leq I_1^n(t) + I_2^n(t) +$$

$$+\frac{1}{2r} \int\limits_{]0,t]} \|x_n(\tau) - x_0(\tau)\|^2 \cdot (\|\frac{dx_n}{d\nu}(\tau\| + \|\frac{dx_0}{d\nu}(\tau)\| +$$

$$+(\|u_n(t)\| + \|v_n(t)\| + \|u_0(t)\| + \|v_0(t)\|\frac{d\lambda}{d\nu}(t))d\nu(\tau). \tag{5.18}$$

From (5.15), (5.17) and (5.18) it follows that for any $\varepsilon > 0$ there exists $n(\varepsilon) \geq 1$ such that

$$\|x_n(t) - x_0(t)\|^2 \leq \varepsilon + \int\limits_{]0,t]} \|x_n(\tau) - x_0(\tau)\|^2 g_n(\tau) d\nu(\tau),\ n \geq n(\varepsilon),\ t \in]0, a],$$

where

$$g_n(t) = \frac{1}{r}(\|\frac{dx_n}{d\nu}(t\| + \|\frac{dx_0}{d\nu}(t)\| +$$

$$+(\|u_n(t)\| + u_0(t)\| + \|v_n(t)\| + \|u_0(t)\|)\frac{d\lambda}{d\nu}(\tau),$$

$n \geq n(\varepsilon)$.

Using the inequality $\|x_n(t)\| \leq M$, $n \geq 0$, $u_n(\cdot) \in S^*_U$, $v_n(\cdot) \in S^*_V$, $n \geq 0$ and reasoning as in the proof of Theorem 3.2, we obtain the inequality

$$\|x_n(t) - x_0\|^2 \leq \varepsilon L^*_1,\ n \geq n(\varepsilon),\ t \in T,$$

where the constant L^*_1 is defined by the equality

$$L^*_1 = \exp\left(\frac{1}{1-\theta}\left[\frac{2}{r}(\mu([0, a]) + 2\int\limits_T (m(t) + n(t)M) d\lambda(\tau))\right]\right),$$

and θ by the equality (3.27).

From the last inequality it follows that

$$x_n(\cdot) \to x_0(\cdot) \text{ in } BV_+(T, H). \tag{5.19}$$

Since $u_n(t) \in U(t, x_n(t))$ a.e., from the convergence $u_n(\cdot) \to u_0$ in ω-$L^1(T, H)$, Hypotheses $H(V)$ and the convexity and closedness of the values of mapping $U(t, x)$, according to the Mazur theorem for weakly converging sequences, we infer that

$$u_0(t) \in \bigcap_{n=1}^{\infty} \overline{\text{co}} \bigcup_{k \geq n}^{\infty} U(t, x_k(t)) \subset U(t, x_0(t)) \text{ a.e.} \tag{5.20}$$

From the inequality (1.11) and Hypothesis H(V) 3) it follows that the values of mapping $t \to \overline{\text{co}}\, V(t, M\overline{B})$ are convex compact sets. Hence, according to Hypothesis H(V) 2) for almost every $t \in T$ the mapping $x \to V(t, x)$, $x \in M\overline{B}$ is upper semicontinuous with convex compact values.

Now, from the inclusions

$$v_n(t) \in V(t, x_n(t)),\ n \geq 1,$$

(5.19), the convergence of $v_n(\cdot)$ to v_0 in the space ω-$L^1(T, H)$ and the Mazur theorem for weakly converging sequences we obtain the inclusion

$$v_0(t) \in \bigcap_{n=1}^{\infty} \overline{\text{co}} \bigcup_{k \geq n}^{\infty} V(t, x_k(t)) \subset V(t, x_0(t)) \text{ a.e.} \tag{5.21}$$

From (5.11), (5.20), (5.21) it follows that $(x(u_0; v_0)(\cdot), u_0(\cdot), v_0(\cdot)) \in \mathcal{R}(x_0)$. Therefore, the set $\mathcal{R}(x_0)$ is closed in the space $BV_+(T, H) \times \omega$-$L^1(T, H) \times \omega$-$L^1(T, X)$.

Let $(x(u;v)(\cdot), u(\cdot), v(\cdot)) \in \mathcal{R}(x_0)$. We call the function $t \to (x(u;v)(t)$ a trajectory. The family of all trajectories we denote by $\mathcal{T}r(x_0)$. Since $\|x(u;v)(t)\| \leq M$, $t \in T$, $u(\cdot) \in S_U^*$, $v \in S_V^*$, using the inequality (3.3), we obtain the inequality

$$\|\frac{dx(u;v)}{d\nu}(t)\| \leq \frac{d\mu}{d\nu}(t) + 2(m(t) + n(t)M)\frac{d\lambda}{d\nu}dt,$$

where $m(t)$ and $n(t)$ are defined by the equality (3.5).

From this inequality we have

$$\|x(u;v)(t) - x(u;v)(s)\| \leq \mu(]s,t]) + 2\int_{]s,t]}(m(\tau) + n(\tau)M)d\lambda(\tau),\ s \leq t,\ s,t \in T,$$

$$\|x(u;v)(t-0) - x(s)\| \leq \mu(]s,t[) + 2\int_{]s,t[}(m(\tau) + n(\tau)M)d\lambda(\tau),\ s \leq t,\ s,t \in M.$$

From these inequalities and (5.9) it follows that the set $\mathcal{T}r(x_0)$ is uniformly bounded in norm and in variation and unilaterally equicontinuous.

If for any $r > d(\Theta, C(t))$ the set $C(t) \cap r\overline{B}$, $t \in T$ is relatively compact, then the set $\{\cup x(u;v)(t) : x(u;v) \in \mathcal{T}r(x_0)\}$ is relatively compact. According to Lemma 2.1, the set $\mathcal{T}r(x_0)$ is relatively compact in the space $BV_+(T,H)$. Now, from the compactness of sets S_U^*, S_V^* in the space ω-$L^1(T,H)$, the relative compactness of $\mathcal{T}r(x_0)$ in the space $BV_+(T,H)$ and the closedness of $\mathcal{R}(x_0)$ in the space $BV_+(T,H) \times \omega$-$L^1(T,H) \times \omega$-$L^1(T,H)$ we derive the compactness of the set $\mathcal{R}(x_0)$ in $BV_+(T,H) \times \omega$-$L^1(T,H) \times \omega$-$L^1(T,H)$. The theorem is proved. □

Let $C(T,H)$ be the space of continuous functions from T to H with sup-norm.

Corollary 5.1 *Let under conditions of Theorem 1.1 instead of the inequality (1.1) we have*

$$|d(y, C(t)) - d(y, C(s))| \leq |\beta(t) - \beta(s)|,\ y \in H,\ s,t \in T, \tag{5.22}$$

where $\beta : T \to R$ is an absolutely continuous function. Then, there exists an absolutely continuous function $x(u;v) : T \to H$ and functions $u(\cdot), v(\cdot) \in L^1(T,H)$ such that $x(u;v)(0) = x_0$, $x(u;v)(t) \in C(t)$, $t \in T$

$$-\frac{dx(u;v)(t)}{dt} \in \mathcal{N}(C(t); x(u;v)(t)) + u(t) + v(t)\ a.e.$$

and the inclusions (1.6), (1.7) hold.

We denote by $\mathcal{AR}(x_0)$ the set of $(x(u;v)(\cdot), u(\cdot), v(\cdot))$ having the properties indicated in Corollary 5.1.

Corollary 5.2 *Let under conditions of Theorem 1.2 instead of the inequality (1.1) the inequality (5.2) holds. Then, the set $\mathcal{AR}(x_0)$ is a closed subset of the space $C(T,H) \times \omega$-$L^1(T,H) \times \omega$-$L^1(T,H)$. If, in addition, the set $C(t) \cap r\overline{B}$, $t \in T$ is relatively compact for $r > d(\Theta, C(t))$, then the set $\mathcal{AR}(x_0)$ is a compact set in the space $C(T,H) \times \omega$-$L^1(T,H) \times \omega$-$L^1(T,H.)$*

Corollaries 5.1 and 5.2 follow directly from Theorems 1.1 and 1.2.

In fact, let

$$\mu(A) = \int_A |\beta(t)|dt, \ A \in \mathcal{B}. \tag{5.23}$$

Then $\mu(\cdot)$ is a Radon measure.

From (5.22), (5.23) it follows that the inequality (1.1) holds with the measure $\mu(\cdot)$ defined by the equality (5.23). From (5.23) it follows that the measures λ and $\nu = \mu + \lambda$ are absolutely continuously equivalent. Hence, in Definition 1.1 and in (1.5) instead of the measure ν one can take the measure λ.

It is well known [19] that the function $x(u; v)(\cdot)$ appearing in Definition 1.1, when the measure ν is replaced with the measure λ, is absolutely continuous and we have

$$\frac{dx(u;v)}{d\lambda}(t) = \frac{dx(u;v)(t)}{dt} \text{ a.e.}$$

Hence, Corollaries 5.1 and 5.2 are restatements of Theorems 1.1 and 1.2 applied to the Lebesgue measure λ instead of the measure ν.

Declarations

Competing Interests The author declares no competing interests.

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