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Duality for Sets of Strong Slater Points

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Abstract

The strong Slater condition plays a significant role in the stability analysis of linear semi-infinite inequality systems. This piece of work studies the set of strong Slater points, whose non-emptiness guarantees the fullfilment of the strong Slater condition. Given a linear inequality system, we firstly establish some basic properties of the set of strong Slater points. Then, we derive dual characterizations for this set in terms of the data of the system, following similar characterizations provided also for the set of Slater points and the solution set of the given system, which are based on the polarity operators for evenly convex and closed convex sets. Finally, we present two geometric interpretations and apply our results to analyze the strict inequality systems defined by lower semicontinuous convex functions.

Keywords Strong Slater condition · Linear inequality system · Existence theorems · Duality

Mathematics Subject Classification (2010) 52A20 · 90C34 · 15A39

1 Introduction

This paper deals with the set of strong Slater points of a linear inequality system in $\mathbb{R}^n$ of the form

$$\sigma := \{\langle a_t, x\rangle \leq b_t,\ t \in T\}, \tag{1}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^n$, T is an arbitrary (possibly infinite) non-empty index set, and the coefficients are given by two functions $a : T \to \mathbb{R}^n$ and $b : T \to \mathbb{R}$, being $a_t := a(t)$ and $b_t := b(t)$ for all $t \in T$. It is said that $\overline{x} \in \mathbb{R}^n$ is a strong Slater point of σ if there exists $\varepsilon > 0$ such that $\langle a_t, \overline{x}\rangle + \varepsilon \leq b_t$ for all $t \in T$. We shall

Dedicated to Professor Miguel A. Goberna on the occasion of his 70th birthday.

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denote by F_{SS} the set of all the strong Slater points of σ and, when this set is non-empty, it is said that σ satisfies the strong Slater condition.

Additionally, the system σ is said to satisfy the Slater condition if there exists $\overline{x} \in \mathbb{R}^n$ such that $\langle a_t, \overline{x} \rangle < b_t$ for all $t \in T$ and, in such a case, $\overline{x}$ is called a Slater point of σ. It is obvious that any strong Slater point of σ is also a Slater point, that is, the fulfillment of the strong Slater condition is sufficient for σ to satisfy the Slater condition. However, both conditions are not equivalent in general. In [3, Lemma 3.2], Christov and Todorov proved that the equivalence holds provided that σ is a continuous linear system, that is, σ is a system as in Eq. 1 such that T is a compact Haussdorff topological space and the parameter functions a and b are continuous on T.

The Slater condition was used by Brosowski [2] and Fisher [7] in order to characterize a certain stability property (the lower semicontinuity) of the solution set mapping for the class of consistent continuous linear systems. In [18], by replacing the Slater condition by the strong Slater one, Helbig obtained the counterpart of Brosowski and Fisher characterization in a more general case, with T being an arbitrary topological space but keeping the continuity of the parameter functions a and b. In [12], the authors considered the most general setting for the system σ, when both the index set T and the parameter functions a and b are arbitrary, and they obtained six different equivalent conditions to the strong Slater condition, connecting this property with the main known stability concepts (see also [14]). One of them, introduced by Robinson [22], is related to the existence of error bounds for the distance between a solution of the system σ and the solution set of a certain system obtained by applying sufficiently small perturbations on the data of σ. More precisely, in [12, Theorem 3.1] the authors provide an error bound which depends on a strong Slater point $\overline{x}$ of σ and on the positive scalar ε associated to $\overline{x}$. We refer to [21, 25] for applications in nonlinear convex inequality systems.

The main objective of this paper is to study the set F_{SS} of strong Slater points of the linear inequality system σ. Section 2 deals with the relationships between F_{SS}, the set of Slater points, and the solution set of σ. We obtain, in Section 3, dual characterizations for the set of strong Slater points in terms of the data of σ, following similar characterizacions provided also for the set of Slater points and the solution set of the system, based on the polarity operators for evenly convex and closed convex sets, and analyze conditions under which F_{SS} is contained in a weak/strict halfspace. Section 4 is devoted to show two geometric interpretations of F_{SS}. Finally, Section 5 presents an application to systems with strict convex inequalities.

2 The Set of Strong Slater Points

We begin this section by introducing the notation and basic definitions used throughout the paper. Given a non-empty set $X \subset \mathbb{R}^n$, we denote by conv X, cone X, aff X and dim X the convex hull of X, the convex cone generated by X, the affine hull of X and the dimension of aff X, respectively. We consider cone $\emptyset := \{0_n\}$ where 0_n is the zero vector in $\mathbb{R}^n$. By $\mathbb{R}_+$ and $\mathbb{R}_{++}$ we denote the sets of non-negative and positive real numbers, respectively, being $\mathbb{R}_+ X := \{\lambda x : \lambda \geq 0,\ x \in X\}$ and $\mathbb{R}_{++} X := \{\lambda x : \lambda > 0,\ x \in X\}$ cones in $\mathbb{R}^n$ with $0_n \in \mathbb{R}_+ X$. The smallest convex cone containing $X \cup \{0_n\}$ is cone $X = \mathbb{R}_+ \operatorname{conv} X$. For T, the index set of σ, the space of generalized finite sequences, $\mathbb{R}^{(T)}$, is the linear space of those functions $\lambda : T \to \mathbb{R}$ whose support, $\operatorname{supp} \lambda := \{t \in T : \lambda_t \neq 0\}$, is finite. The convex cone of the non-negative generalized finite sequences is denoted by $\mathbb{R}_+^{(T)}$. From the topological side, given $X \subset \mathbb{R}^n$, we denote by $\operatorname{cl} X$, $\operatorname{int} X$ and $\operatorname{rint} X$, the closure, the interior and the relative interior of X, respectively.

A set $X \subset \mathbb{R}^n$ is said to be *evenly convex* (see [6]) if it is the intersection of some family, possibly empty, of open halfspaces. The *evenly convex hull* of a set $X \subset \mathbb{R}^n$, denoted by $\operatorname{eco} X$, is the smallest evenly convex set which contains X, or equivalently, it is the intersection of all the open halfspaces containing X. From the definition, given $\overline{x} \in \mathbb{R}^n$, $\overline{x} \notin \operatorname{eco} X$ if and only if there exists $z \in \mathbb{R}^n$ such that $\langle z, x - \overline{x} \rangle < 0$ for all $x \in X$. Further properties of the evenly convex hull operator can be found in [4, Chapter 1].

The set $A \subset \mathbb{R}^n$ is called *radiant* (see, e.g., [26]) if $x \in A$, $\lambda \in [0, 1]$ imply that $\lambda x \in A$. The set $B \subset \mathbb{R}^n$ is called *coradiant* if its complement $\mathbb{R}^n \backslash B$ is radiant, that is, if either $B = \mathbb{R}^n$ or $0_n \notin B$ and $x \in B$, $\lambda \geq 1$ imply that $\lambda x \in B$. Given a proper (different from both $\emptyset$ and from $\mathbb{R}^n$), closed, convex and coradiant set $C \subset \mathbb{R}^n$, the function $\phi_C : \mathbb{R}^n \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ given by $\phi_C(x) := \inf\{\langle v, x\rangle : \langle v, c\rangle \geq 1, \forall c \in C\}$ is called the *concave gauge* of C (see [1]). This function ϕ_C characterizes the set C since it holds

$$C = \{x \in \mathbb{R}^n : \phi_C(x) \geq 1\}. \tag{2}$$

Now we recall two polar operators in the literature. For a proper set $X \subset \mathbb{R}^n$, we consider

$$\begin{aligned} X^{\circ} &:= \{y \in \mathbb{R}^n : \langle x, y\rangle \leq 0,\ \forall x \in X\}, \\ X^{e} &:= \{y \in \mathbb{R}^n : \langle x, y\rangle < 0,\ \forall x \in X\}, \end{aligned}$$

assuming that $\mathbb{R}^n$ is the polar in the first sense of $\{0_n\}$ (and in the second sense of $\emptyset$), and conversely. X° is a closed convex cone containing the origin, while X^e is an evenly convex cone omitting the origin. Furthermore, one has $X^{\circ\circ} = \operatorname{cl} \operatorname{cone} X$, and so $X = X^{\circ\circ}$ if and only if X is a closed convex cone. In the same way, one has $X = X^{ee}$ if and only if X is an evenly convex cone omitting the origin.

For a function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, its effective domain is $\operatorname{dom} f := \{x \in \mathbb{R}^n : f(x) < +\infty\}$ and its epigraph is $\operatorname{epi} f := \{(x, r) \in \mathbb{R}^{n+1} : f(x) \leq r\}$. The Legendre-Fenchel conjugate of f is the function f^* defined, for every $x^* \in \mathbb{R}^n$, by $f^*(x^*) = \sup_{x \in \mathbb{R}^n}\{\langle x^*, x\rangle - f(x)\}$. If f is a proper lower semicontinuous convex function, then $f = f^{**}$.

We shall denote by F, F_S and F_{SS}, the solution set, the set of *Slater points*, and the set of *strong Slater points* of σ, respectively, that is,

$$\begin{aligned} F &:= \{x \in \mathbb{R}^n : \langle a_t, x\rangle \leq b_t,\ t \in T\}, \\ F_S &:= \{x \in \mathbb{R}^n : \langle a_t, x\rangle < b_t,\ t \in T\}, \\ F_{SS} &:= \{x \in \mathbb{R}^n : \exists\, \varepsilon > 0,\ \langle a_t, x\rangle + \varepsilon \leq b_t,\ t \in T\}. \end{aligned}$$

From these definitions, one clearly has that $F_{SS} \subset F_S \subset F$. It is well-known that F is a closed convex set and F_S is an evenly convex set. However, we do not know in general about the geometry of F_{SS} beyond its convexity and that, in some particular cases that are described ahead (cf. Proposition 2.5), it coincides with F_S.

A weak or strict inequality is said to be a *consequence of* the system σ in Eq. 1 (equivalently, it is a *consequent relation* of F) provided that every solution of σ satisfies such an inequality (that is, F is contained in the closed/open halfspace defined by such an inequality).

Lemma 2.1 *Assume that F_{SS} is non-empty. Then,*

(i) $\operatorname{cl} F_{SS} = \operatorname{cl} F_S = F$.
(ii) $\operatorname{rint} F_{SS} = \operatorname{rint} F_S = \operatorname{rint} F$.

Proof (i) Since $F_{SS} \subset F_S$, we have $\mathrm{cl}\, F_{SS} \subset \mathrm{cl}\, F_S = F$ where the last equality follows from [9, Proposition 1.1]. Now, let $\overline{x} \in F$ and $\widetilde{x} \in F_{SS}$. Thus, by definition, $\langle a_t, \overline{x}\rangle \leq b_t$ for all $t \in T$ and there exists $\varepsilon > 0$ such that $\langle a_t, \widetilde{x}\rangle + \varepsilon \leq b_t$ for all $t \in T$. For every $\lambda \in]0,1[$ it follows that $\langle a_t, \lambda\widetilde{x} + (1-\lambda)\overline{x}\rangle + \lambda\varepsilon \leq b_t$ for all $t \in T$, which means that $\lambda\widetilde{x} + (1-\lambda)\overline{x} \in F_{SS}$ for all $\lambda \in]0,1[$. Hence, by taking limits one gets

$$\overline{x} = \lim_{\lambda \downarrow 0} \lambda\widetilde{x} + (1-\lambda)\overline{x} \in \mathrm{cl}\, F_{SS}$$

which concludes the proof.

(ii) If F_{SS} is non-empty, then from (i) one has

$$\mathrm{rint}(\mathrm{cl}\, F_{SS}) = \mathrm{rint}(\mathrm{cl}\, F_S) = \mathrm{rint}\, F.$$

The conclusion follows from [23, Theorem 6.3] which ensures $\mathrm{rint}(\mathrm{cl}\, F_{SS}) = \mathrm{rint}\, F_{SS}$ and $\mathrm{rint}(\mathrm{cl}\, F_S) = \mathrm{rint}\, F_S$. □

As a consequence of Lemma 2.1 we get $\mathrm{rint}\, F \subset F_{SS}$ when F_{SS} is non-empty, and so

$$\mathrm{rint}\, F \subset F_{SS} \subset F_S \subset F. \tag{3}$$

A direct proof for the first inclusion can be found in [11, Proposition 1]. All the set containments in Eq. 3 can be strict, as the following example shows.

Example 2.2 Let $\sigma = \{-tx_1 - x_2 \leq |t|,\ t \in \mathbb{R}\setminus\{0\}\}$ (this linear system is a slight modification of the one given in [13, Example 5.2] where the index set is $\mathbb{R}$). It is easy to see that $F = [-1,1] \times \mathbb{R}_+$ and $F_S = F\setminus\{(-1,0),(1,0)\}$ (see Fig. 1). Since

$$\mathrm{rint}\, F =]-1,1[\times \mathbb{R}_{++} \subset F_{SS} \subset F_S,$$

we need to check whether the points in $(\{\pm 1\} \times \mathbb{R}_{++}) \cup (]-1,1[\times \{0\})$ are strong Slater points or not. On the one hand, for each $a > 0$, it can be easily checked that the points $(-1,a)$ and $(1,a)$ belong to F_{SS} by considering $\varepsilon \in]0,a]$. On the other hand, the points $(a,0)$ with $-1 < a < 1$ are not strong Slater points since, for every $\varepsilon > 0$, one has $-ta + \varepsilon > |t|$ for $t \in \left]0, \frac{\varepsilon}{1+a}\right]$. Consequently, $F_{SS} = [-1,1] \times \mathbb{R}_{++}$.

The solution set of the linear system $\{-1 \leq x_1 \leq 1, x_2 \geq 0\}$ coincides with the set F in Example 2.2. However, both the set of Slater points and the set of strong Slater points associated to this system coincide with $\mathrm{int}\, F$. This shows that both the set of Slater points and the set of strong Slater points of a linear system, depend on the inequalities defining the system, not on the solution set of such a system.

Remark 2.3 Lemma 2.1 (i) states that the closure of F_{SS} (recall that it is always convex) coincides with the closed convex set F. However, the evenly convex hull of F_{SS} does not

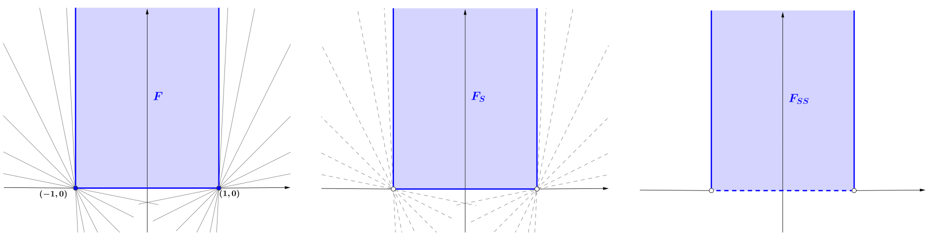

Fig. 1 The sets F, F_S and F_{SS} asociated to σ in Example 2.2

coincide, in general, with the evenly convex set F_S. Particularly, the above example shows that eco $F_{SS} = F_{SS} \subsetneq F_S$.

Although F_{SS} is evenly convex in Example 2.2, this is not always the case, as the following example shows.

Example 2.4 Let us consider the linear system $\sigma = \{x_1 - tx_2 \leq \frac{1}{4t}, t > 0\}$ in $\mathbb{R}^2$. One can check that the solution set of σ is $F = \{x \in \mathbb{R}^2 : x_1 - \sqrt{x_2} \leq 0,\ x_2 \geq 0\}$, the set of Slater points is $F_S = \{(0,0)\} \cup \{x \in \mathbb{R}^2 : x_1 - \sqrt{x_2} < 0,\ x_2 \geq 0\}$ and the set of strong Slater points is $F_{SS} = \{x \in \mathbb{R}^2 : x_1 - \sqrt{x_2} < 0,\ x_2 \geq 0\}$ (see Fig. 2). Clearly, F_{SS} is not evenly convex as the open separation property from outside points fails at the origin (observe that $(0,0) \in (\text{eco}\, F_{SS}) \backslash F_{SS}$).

There exist some cases in which the set of strong Slater points coincides with either the set of Slater points or the relative interior of the solution set of σ. Next results provide conditions guaranteeing these extreme cases.

Proposition 2.5 *Assume that F_S is non-empty. The following statements hold:*

(i) $F_{SS} = F_S$ *if and only if* $\sup\{\langle a_t, x\rangle - b_t,\ t \in T\} < 0$ *for all* $x \in F_S$.
(ii) *If* $F_{SS} \neq \emptyset$ *and* $\inf\{\langle a_t, x\rangle - b_t,\ t \in T\} > -\infty$ *for all* $x \in F_S$*, then* $F_{SS} = \text{rint}\, F$.

Proof (i) This statement follows immediately from the definition of strong Slater point.

(ii) According to Eq. 3, we just need to show that $F_{SS} \subset \text{rint}\, F$. Given $x \in F_{SS}$, we shall prove that $x \in \text{rint}\, F_S = \text{rint}\, F$ by using the characterization of the relative interior given in [23, Theorem 6.4]. Thus, we shall prove that for every $z \in F_S$, there exists $\mu > 1$ such that $z + \mu(x - z) = (1 - \mu)z + \mu x \in F_S$.

Since $x \in F_{SS}$, there exists $\varepsilon > 0$ such that $\langle a_t, x\rangle + \varepsilon \leq b_t$ for all $t \in T$. Now, pick any $z \in F_S$, which means $\langle a_t, z\rangle < b_t$ for all $t \in T$. Let $\alpha := \inf\{\langle a_t, z\rangle - b_t,\ t \in T\} \in -\mathbb{R}_+$ (this value is not $-\infty$ by assumption). Next we distinguish three cases:

1. If $\varepsilon + \alpha > 0$, then by taking any $\mu > 1$ we get $\mu(\varepsilon + \alpha) > 0 \geq \alpha$ and so $\mu\varepsilon > (1 - \mu)\alpha$. Then, for every $t \in T$,

$$\begin{aligned}\langle a_t, (1-\mu)z + \mu x\rangle + \mu b_t - b_t &= \mu\langle a_t, x\rangle + (1-\mu)(\langle a_t, z\rangle - b_t) \leq \\ &\leq \mu\langle a_t, x\rangle + (1-\mu)\alpha < \mu\langle a_t, x\rangle + \mu\varepsilon \leq \mu b_t.\end{aligned}$$

   It follows that $\langle a_t, (1-\mu)z + \mu x\rangle < b_t$ for all $t \in T$, and so $(1-\mu)z + \mu x \in F_S$.
2. If $\varepsilon + \alpha = 0$, we may consider without loss of generality an altervative $\varepsilon' > 0$ such that $\varepsilon' < \varepsilon$. In this case $\varepsilon' + \alpha < 0$ and we refer to the third case.

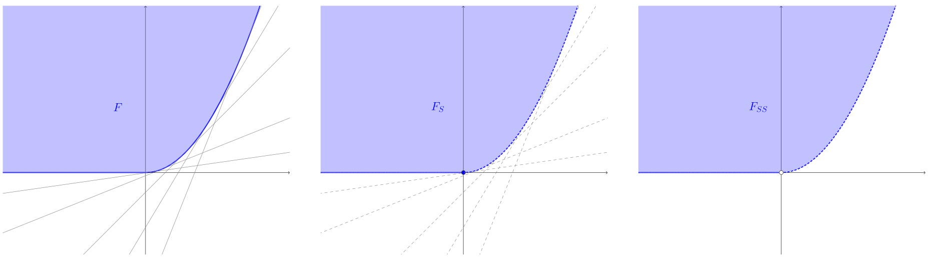

Fig. 2 The sets F, F_S and F_{SS} asociated to σ in Example 2.4

3. If $\varepsilon+\alpha < 0$, then $\alpha < 0$. By taking any μ such that $1 < \mu < \frac{\alpha}{\alpha+\varepsilon}$, we get $\mu(\varepsilon+\alpha) > \alpha$ and so $\mu\varepsilon > (1-\mu)\alpha$. Then, reasoning as in the first case, we have that $(1-\mu)z+\mu x \in F_S$.

□

As a consequence of the above proposition one has that, if σ is continuous, then $F_{SS} = F_S$ (cf. [15, Theorem 6.9]). Furthermore, if $|T| < \infty$, then $\operatorname{rint} F = F_{SS} = F_S$.

Proposition 2.6 *The following statements hold:*

(i) *If* $F_{SS} \neq \emptyset$ *and* $\sup\{\|a_t\|,\ t \in T\} < +\infty$, *then* $F_{SS} = \operatorname{int} F$.
(ii) *If* $\dim F = n$ *and* $\inf\{\|a_t\|,\ t \in T\} > 0$, *then* $F_{SS} \neq \emptyset$ *and* $\operatorname{int} F \subset F_{SS}$.

Proof (i) The proof of the inclusion $F_{SS} \subset \operatorname{int} F$ can be found in [19, Lemma 2.5]. The equality can be obtained taking into account Eq. 3 and that $\operatorname{rint} F = \operatorname{int} F$ when $\operatorname{int} F \neq \emptyset$.
(ii) Since $\dim F = n$, we can take $x \in \operatorname{int} F$. Then, there exists $\varepsilon > 0$ such that

$$x + \varepsilon\mathbb{B} \subset F, \tag{4}$$

being $\mathbb{B}$ the closed unit ball. Now, let $\alpha := \inf\{\|a_t\|,\ t \in T\} > 0$. Then, for all $t \in T$, $\|a_t\| \geq \alpha > 0$, $\frac{a_t}{\|a_t\|} \in \mathbb{B}$ and, from Eq. 4, one obtains $x + \varepsilon\frac{a_t}{\|a_t\|} \in F$. Thus,

$$\langle a_t, x\rangle + \varepsilon\alpha \leq \langle a_t, x\rangle + \varepsilon\|a_t\| = \langle a_t, x + \varepsilon\frac{a_t}{\|a_t\|}\rangle \leq b_t,$$

for all $t \in T$. Therefore, $x \in F_{SS}$ and $F_{SS} \neq \emptyset$. Finally, by Eq. 3, $\operatorname{int} F \subset F_{SS}$. □

The converse statements in Proposition 2.5 (ii) and 2.6 are not fulfilled in general, as we can see in the following examples.

Example 2.7 Let $\sigma = \{-x_1 - x_2 \leq 0,\ x_1 - x_2 \leq 0,\ -kx_2 \leq 1,\ k \in \mathbb{N}\}$. It is easy to see that $F = \{(x_1, x_2) \in \mathbb{R}^2 : -x_1 - x_2 \leq 0,\ x_1 - x_2 \leq 0\}$ and $F_S = \operatorname{int} F$. Since $\dim F = 2$ and $\inf\{\|a_t\|,\ t \in T\} = 1 > 0$, by Proposition 2.6$(ii)$, we have that $F_{SS} \neq \emptyset$ and $F_S = \operatorname{int} F \subset F_{SS}$. Therefore, $\operatorname{int} F = F_S = F_{SS}$. Nevertheless, for $\overline{x} = (0, 1) \in F_S$ one has $\inf\{\langle a_t, \overline{x}\rangle - b_t,\ t \in T\} = \inf\{-1;\ -k - 1,\ k \in \mathbb{N}\} = -\infty$. Thus, the converse of Proposition 2.5(ii) is not true in this case.

On the other hand, we have that $\sup\{\|a_t\|,\ t \in T\} = \sup\{\sqrt{2};\ k,\ k \in \mathbb{N}\} = +\infty$, so the converse of statement (i) in Proposition 2.6 also fails.

Example 2.8 Let $\sigma = \{-tx \leq 1,\ t \geq 0\}$ be the system in $\mathbb{R}$ whose solution set is $F = [0, +\infty[$. It is easy to see that $F_{SS} = F$, so that $F_{SS} \neq \emptyset$ and $\operatorname{int} F =]0, +\infty[\subset F_{SS}$. However, $\inf\{\|a_t\|,\ t \in T\} = \inf\{t,\ t \geq 0\} = 0$, so the converse of Proposition 2.6(ii) is not true.

The supremum and the infimum conditions in Proposition 2.6 are used in [17] in order to provide conditions for the stability of the strong uniqueness of the optimal solution of a given linear semi-infinite optimization problem. There, σ is said to be LHS-upper bounded (resp. LHS-positively lower bounded) if $\sup\{\|a_t\|,\ t \in T\} < +\infty$ (resp. $\inf\{\|a_t\|,\ t \in T\} > 0$).

3 Dual Characterizations

In this section, we are interested in obtaining dual characterizations for the strong Slater condition, or equivalently, for the non-emptiness of the set of strong Slater points. Our objective is to obtain the counterpart, for F_{SS}, of some results about the solution set F and the set of Slater points F_S, which can be easily obtained as extensions of those proved in [10, Lemma 3.2 and Theorem 3.7] for the finite case.

We associate to the linear inequality system $\sigma = \{\langle a_t, x\rangle \leq b_t,\ t \in T\}$ the sets

$$C(\sigma) := \{(a_t, b_t), t \in T\}, \text{ and}$$
$$D(\sigma) := C(\sigma) \cup \{(0_n, 1)\}.$$

These sets allow to define two prominent cones in linear semi-infinite programming (see, e.g., [15]) associated to σ, which are the *second order moment cone*, defined by $N(\sigma) := \operatorname{cone} C(\sigma)$, and the *characteristic cone*, defined by $K(\sigma) := \operatorname{cone} D(\sigma)$. It easily follows from the definition of $D(\sigma)$ that $K(\sigma) = N(\sigma) + \mathbb{R}_+\{(0_n, 1)\}$. Applying the two polar operators defined in the previous section to $D(\sigma)$, we get

$$D(\sigma)^\circ = \{(x, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R} : \langle a_t, x\rangle + b_t x_{n+1} \leq 0, t \in T;\ x_{n+1} \leq 0\},$$
$$D(\sigma)^e = \{(x, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R} : \langle a_t, x\rangle + b_t x_{n+1} < 0, t \in T;\ x_{n+1} < 0\}.$$

Theorem 3.1 (Characterizations of the solution set F) *The following statements hold:*

- (i) $\overline{x} \in F$ *if and only if* $(\overline{x}, -1) \in D(\sigma)^\circ$.
- (ii) $F = \emptyset$ *if and only if* $x_{n+1} = 0$ *for all* $(x, x_{n+1}) \in D(\sigma)^\circ$.
- (iii) $F \neq \emptyset$ *if and only if* $(0_n, -1) \notin \operatorname{cl} K(\sigma)$ *(equivalently,* $(0_n, -1) \notin \operatorname{cl} N(\sigma)$*).*

Proof (i) Since F is the solution set of the system σ in Eq. 1, we have that $\overline{x} \in F$ if and only if $\langle a_t, \overline{x}\rangle \leq b_t$ for all $t \in T$ or, equivalently, $\langle a_t, \overline{x}\rangle + b_t(-1) \leq 0$ for all $t \in T$, meaning that $(\overline{x}, -1) \in D(\sigma)^\circ$.

(ii) If $(\overline{x}, \overline{x}_{n+1}) \in D(\sigma)^\circ$ and $\overline{x}_{n+1} \neq 0$, then $\overline{x}_{n+1} < 0$ and

$$\langle a_t, \overline{x}\rangle + b_t \overline{x}_{n+1} \leq 0 \tag{5}$$

for all $t \in T$. Now, dividing Eq. 5 by $-\overline{x}_{n+1} > 0$, we obtain $\langle a_t, -\frac{\overline{x}}{\overline{x}_{n+1}}\rangle - b_t \leq 0$ for all $t \in T$, so that $-\frac{\overline{x}}{\overline{x}_{n+1}} \in F$ and $F \neq \emptyset$. Conversely, if $F \neq \emptyset$ and $\overline{x} \in F$, then $(\overline{x}, -1) \in D(\sigma)^\circ$ by (i).

(iii) Suppose that $F \neq \emptyset$ and $\overline{x} \in F$. By (i), $(\overline{x}, -1) \in D(\sigma)^\circ$ and so

$$(0_n, -1) \notin D(\sigma)^{\circ\circ} = \operatorname{cl} \operatorname{cone} D(\sigma) = \operatorname{cl} K(\sigma).$$

Conversely, if $F = \emptyset$, then, by (ii), $D(\sigma)^\circ \subset \{(x, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} = 0\}$ and so

$$(0_n, -1) \in \{0_n\} \times \mathbb{R} = \{(x, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} = 0\}^\circ \subset D(\sigma)^{\circ\circ} = \operatorname{cl} K(\sigma).$$

Finally, it has been proved in [15, Lemma 4.1] that $(0_n, -1) \in \operatorname{cl} K(\sigma)$ is equivalent to $(0_n, -1) \in \operatorname{cl} N(\sigma)$. □

The $K(\sigma)$ and $N(\sigma)$ versions of Theorem 3.1(iii) are due to Zhu [27] and Fan [5], respectively. A different version of the proof of (iii) can be found in [15, Corollary 3.1.1]. We recall that, according to the non-homogeneous Farkas Lemma for linear semi-infinite systems (see, e.g., [15, Corollary 3.1.2]), $\operatorname{cl} K(\sigma)$ coincides with the so-called (see [8]) *weak dual cone* of F, provided that F is non-empty, which is the set of coefficients of all the weak inequalities which are consequence of σ. Thus, statement (iii) above asserts that F is non-empty if and only if $\langle 0_n, x\rangle \leq -1$ is not a consequence of σ.

Theorem 3.2 (Characterizations of the set F_S of Slater points) *The following statements hold:*

- (i) $\overline{x} \in F_S$ *if and only if* $(\overline{x}, -1) \in D(\sigma)^e$.
- (ii) $F_S = \emptyset$ *if and only if* $D(\sigma)^e = \emptyset$.
- (iii) $F_S \neq \emptyset$ *if and only if* $0_{n+1} \notin \operatorname{eco} D(\sigma)$ *(equivalenly,* $0_{n+1} \notin \operatorname{eco} C(\sigma)$ *and* $(0_n, -1) \notin \operatorname{cl} N(\sigma)$*).*

Proof The proofs of statements (i) and (ii) are similar to those of Theorem 3.1.

(iii) Assume that $F_S \neq \emptyset$. By (ii), $D(\sigma)^e \neq \emptyset$ and so $0_{n+1} \notin D(\sigma)^{ee}$. Since $\operatorname{eco} D(\sigma) \subset D(\sigma)^{ee}$, one has $0_{n+1} \notin \operatorname{eco} D(\sigma)$. Conversely, if $0_{n+1} \notin \operatorname{eco} D(\sigma)$, then there exists $(u, u_{n+1}) \in \mathbb{R}^n \times \mathbb{R}$ such that $\langle u, x\rangle + u_{n+1}x_{n+1} < 0$ for all $(x, x_{n+1}) \in D(\sigma)$. Then, $(u, u_{n+1}) \in D(\sigma)^e$ and $D(\sigma)^e \neq \emptyset$, which implies $F_S \neq \emptyset$ by (ii).

Finally, $0_{n+1} \notin \operatorname{eco} D(\sigma)$ is equivalent to $0_{n+1} \notin \operatorname{eco} C(\sigma)$ and $(0_n, -1) \notin \operatorname{cl} N(\sigma)$, thanks to [16, Lemma 3.1]. □

A different version of the proof of (iii) can be found in [4, Corollary 1.2]. We also observe that in (iii), we may equivalently write $\operatorname{eco} \mathbb{R}_{++} D(\sigma)$ instead of $\operatorname{eco} D(\sigma)$. Furthermore, the set $\operatorname{eco} \mathbb{R}_{++} D(\sigma)$ coincides with the so-called *strict dual cone* of F_S, provided that F_S is non-empty (see [8, Proposition 5.4]), which is the set of coefficients of all the strict inequalities which are consequence of $\sigma_S := \{\langle a_t, x\rangle < b_t, t \in T\}$. Thus, statement (iii) above asserts that F_S is non-empty if and only if $\langle 0_n, x\rangle < 0$ is not a consequence of σ_S.

Next, we seek for a set depending on the coefficients of the system σ that, by employing some polarity operator, could lead to dual characterizations of the non-emptiness of the set F_{SS} of strong Slater points of σ. For that purpose, we observe that, in the definition of F_{SS}, we may assume without loss of generality that $\varepsilon \leq 1$. That is, we may equivalently write

$$F_{SS} = \{x \in \mathbb{R}^n : \exists\, \varepsilon \in]0, 1], \langle a_t, x\rangle + \varepsilon \leq b_t,\ t \in T\}. \tag{6}$$

Taking this into account, we introduce further sets associated to σ,

$$\begin{aligned} \mathcal{C}(\sigma) &:= \{(a_t, b_t, -1), t \in T; (0_n, 1, -1)\}, \\ \mathcal{D}(\sigma) &:= \mathcal{C}(\sigma) \cup \{(0_n, 0, 1)\}. \end{aligned}$$

In this case, one has

$$\begin{aligned} \mathcal{D}(\sigma)^\circ = \{(x, x_{n+1}, x_{n+2}) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} :\ & \langle a_t, x\rangle + b_t x_{n+1} - x_{n+2} \leq 0,\ t \in T; \\ & x_{n+1} - x_{n+2} \leq 0;\ x_{n+2} \leq 0\}. \end{aligned}$$

Inspired by [24] and proceeding similarly as in former lines, we will consider the following cones associated to σ, $\mathcal{N}(\sigma) := \operatorname{cone} \mathcal{C}(\sigma)$ and $\mathcal{K}(\sigma) := \operatorname{cone} \mathcal{D}(\sigma)$, having that $\mathcal{K}(\sigma) = \mathcal{N}(\sigma) + \mathbb{R}_+\{(0_n, 0, 1)\}$.

Theorem 3.3 (Characterizations of the set F_{SS} of strong Slater points) *The following statements hold:*

- (i) $\overline{x} \in F_{SS}$ *if and only if* $(\overline{x}, -1, -\overline{\varepsilon}) \in \mathcal{D}(\sigma)^\circ$ *for some* $\overline{\varepsilon} \in]0, 1]$.
- (ii) $F_{SS} = \emptyset$ *if and only if* $x_{n+2} = 0$ *for all* $(x, x_{n+1}, x_{n+2}) \in \mathcal{D}(\sigma)^\circ$.
- (iii) $F_{SS} \neq \emptyset$ *if and only if* $(0_n, 0, -1) \notin \operatorname{cl} \mathcal{K}(\sigma)$ *(equivalently,* $(0_n, 0, -1) \notin \operatorname{cl} \mathcal{N}(\sigma)$*).*

Proof (i) This statement follows easily from Eq. 6 and the expression of $\mathcal{D}(\sigma)^\circ$.

(ii) Let us assume that $\overline{x}_{n+2} < 0$ for some $(\overline{x}, \overline{x}_{n+1}, \overline{x}_{n+2}) \in \mathcal{D}(\sigma)^\circ$. Since $\mathcal{D}(\sigma)^\circ$ is a cone and $\overline{x}_{n+1} \leq \overline{x}_{n+2} < 0$, then $-\overline{x}_{n+1} > 0$ and so $\left(\frac{\overline{x}}{-\overline{x}_{n+1}}, -1, \frac{\overline{x}_{n+2}}{-\overline{x}_{n+1}}\right) \in \mathcal{D}(\sigma)^\circ$ with $\frac{\overline{x}_{n+2}}{\overline{x}_{n+1}} \in]0, 1]$. Then, in virtue of (i), $\frac{-1}{\overline{x}_{n+1}}\overline{x} \in F_{SS}$ and so $F_{SS} \neq \emptyset$. The converse statement is a straightforward consequence of (i).

(iii) Assume that $F_{SS} \neq \emptyset$ and let $\overline{x} \in F_{SS}$. By (i), there exists $\overline{\varepsilon} \in]0, 1]$ such that $(\overline{x}, -1, -\overline{\varepsilon}) \in \mathcal{D}(\sigma)^\circ$. If $(0_n, 0, -1) \in \operatorname{cl} \mathcal{K}(\sigma) = \mathcal{D}(\sigma)^{\circ\circ}$, then

$$\langle (0_n, 0, -1), (\overline{x}, -1, -\overline{\varepsilon}) \rangle \leq 0,$$

which implies $\overline{\varepsilon} \leq 0$ and so, a contradiction. Thus, $(0_n, 0, -1) \notin \operatorname{cl} \mathcal{K}(\sigma)$.

Conversely, if $F_{SS} = \emptyset$, then $x_{n+2} = 0$ for all $(x, x_{n+1}, x_{n+2}) \in \mathcal{D}(\sigma)^\circ$ according to (ii). Since $\langle (0_n, 0, -1), (x, x_{n+1}, x_{n+2}) \rangle = 0$ for all $(x, x_{n+1}, x_{n+2}) \in \mathcal{D}(\sigma)^\circ$, one has $(0_n, 0, -1) \in \mathcal{D}(\sigma)^{\circ\circ} = \operatorname{cl} \mathcal{K}(\sigma)$.

Finally, the fact that $(0_n, 0, -1) \notin \operatorname{cl} \mathcal{K}(\sigma)$ is equivalent to $(0_n, 0, -1) \notin \operatorname{cl} \mathcal{N}(\sigma)$ follows from [15, Lemma 4.1]. □

Remark 3.4 Given $\overline{x} \in F_{SS}$, we can take $\varepsilon_{\overline{x}} := \min\{1, \inf\{b_t - \langle a_t, x \rangle : t \in T\}\}$ so that $\varepsilon_{\overline{x}} \in]0, 1]$ and $\langle a_t, \overline{x} \rangle - b_t + \varepsilon \leq 0$ for all $t \in T$ and for all $\varepsilon \in]0, \varepsilon_{\overline{x}}]$. Therefore, in Theorem 3.3(i) we have that $(\overline{x}, -1, -\varepsilon) \in \mathcal{D}(\sigma)^\circ$ for all $\varepsilon \in]0, \varepsilon_{\overline{x}}]$, if $\overline{x} \in F_{SS}$.

Observe that in Theorem 3.3 (iii) we are not explicitly assuming the consistency of σ to characterize the existence of strong Slater points, although it is a condition which is implicit as we can see in the following result, that can be found in [15, Theorem 6.1] with a different proof.

Proposition 3.5 *The following statements are equivalent:*

(i) $F_{SS} \neq \emptyset$.
(ii) $F \neq \emptyset$ *and* $0_{n+1} \notin \operatorname{cl} \operatorname{conv} C(\sigma)$.

Proof $(i) \Rightarrow (ii)$ If $F_{SS} \neq \emptyset$, then $F \neq \emptyset$ by Eq. 3. Now assume, on the contrary, that $0_{n+1} \in \operatorname{cl} \operatorname{conv} C(\sigma)$. Then, there exists $\{\lambda^k\}_{k\in\mathbb{N}} \subset \mathbb{R}_+^{(T)}$ with $\sum_{t\in T} \lambda_t^k = 1$ for every $k \in \mathbb{N}$, such that $0_{n+1} = \lim\limits_{k\to\infty} \sum_{t\in T} \lambda_t^k (a_t, b_t)$. Hence, one can write $(0_n, 0, -1) = \lim\limits_{k\to\infty} \sum_{t\in T} \lambda_t^k (a_t, b_t, -1)$, and so $(0_n, 0, -1) \in \operatorname{cl} \mathcal{N}(\sigma)$. This implies by Theorem 3.3 (iii) that $F_{SS} = \emptyset$ and so, a contradiction.

$[(ii) \Rightarrow (i)]$ If $F_{SS} = \emptyset$, then $(0_n, 0, -1) \in \operatorname{cl} \mathcal{N}(\sigma)$ by Theorem 3.3 (iii). Hence, there exist sequences $\{\delta^k\}_{k\in\mathbb{N}} \subset \mathbb{R}_+$ and $\{\lambda^k\}_{k\in\mathbb{N}} \subset \mathbb{R}_+^{(T)}$ such that

$$(0_n, 0, -1) = \lim_{k\to\infty} \sum_{t\in T} \lambda_t^k (a_t, b_t, -1) + \delta^k (0_n, 1, -1). \tag{7}$$

If $\{\delta^k\}$ is unbounded, then we may assume $\lim_{k\to\infty} \delta^k = +\infty$. Hence, from Eq. 7, one has $0_{n+2} = \lim\limits_{k\to\infty} \sum_{t\in T} (\delta^k)^{-1} \lambda_t^k (a_t, b_t, -1) + (0_n, 1, -1)$, which implies

$$(0_n, -1) = \lim_{k\to\infty} \sum_{t\in T} (\delta^k)^{-1} \lambda_t^k (a_t, b_t) \in \operatorname{cl} N(\sigma)$$

and so, by Theorem 3.1 (iii), $F = \emptyset$. Assume now that $\{\delta^k\}$ is bounded. Then, it contains a convergent subsequence and, for brevity, we write $\lim_{k\to\infty} \delta^k = \delta \geq 0$. From Eq. 7, we have

$$0_{n+1} = \lim_{k\to\infty} \sum_{t\in T} \lambda_t^k (a_t, b_t) + \delta^k (0_n, 1), \tag{8}$$

$$1 = \lim_{k\to\infty} \sum_{t\in T} \lambda_t^k + \delta^k. \tag{9}$$

Since $F \neq \emptyset$, for any $\overline{x} \in F$ one has

$$0 = \langle 0_{n+1}, (\overline{x}, -1)\rangle = \lim_{k\to\infty} \sum_{t\in T} \lambda_t^k \langle (a_t, b_t), (\overline{x}, -1)\rangle + \delta^k \langle (0_n, 1), (\overline{x}, -1)\rangle$$
$$= \lim_{k\to\infty} \sum_{t\in T} \lambda_t^k (\langle a_t, \overline{x}\rangle - b_t) - \delta^k.$$

Therefore, $\lim_{k\to\infty} \sum_{t\in T} \lambda_t^k (\langle a_t, \overline{x}\rangle - b_t) = 0$ and $\delta = \lim_{k\to\infty} \delta^k = 0$. Taking this into account in Eqs. 8 and 9, we get that $0_{n+1} = \lim_{k\to\infty} \sum_{t\in T} \lambda_t^k (a_t, b_t)$ and $1 = \lim_{k\to\infty} \sum_{t\in T} \lambda_t^k$. Since $\gamma^k := \sum_{t\in T} \lambda_t^k > 0$ for k large enough, then

$$0_{n+1} = \lim_{k\to\infty} \sum_{t\in T} (\gamma^k)^{-1} \lambda_t^k (a_t, b_t)$$

with $\sum_{t\in T} (\gamma^k)^{-1} \lambda_t^k = 1$, which shows that $0_{n+1} \in \operatorname{cl}\operatorname{conv} C(\sigma)$. □

As a consequence of Proposition 3.5 and Theorem 3.1 (iii), one has that $F_{SS} \neq \emptyset$ if and only if $(0_n, -1) \notin \operatorname{cl} N(\sigma)$ and $0_{n+1} \notin \operatorname{cl}\operatorname{conv} C(\sigma)$, that is, in order to guarantee the fulfillment of the strong Slater condition, one has to check two conditions in the space $\mathbb{R}^{n+1}$. Theorem 3.3 (iii) shows that these two conditions are indeed equivalent to a unique condition, $(0_n, 0, -1) \notin \operatorname{cl} \mathcal{N}(\sigma)$, in the space $\mathbb{R}^{n+2}$.

Proposition 3.6 (Consequent weak relations of F_{SS}) *Assume that F_{SS} is non-empty. Then, $F_{SS} \subset \{x \in \mathbb{R}^n : \langle a, x\rangle \leq b\}$ if and only if $(a, b, 0) \in \operatorname{cl} \mathcal{K}(\sigma)$.*

Proof Assume that $F_{SS} \subset \{x \in \mathbb{R}^n : \langle a, x\rangle \leq b\}$. We shall prove that $(a, b, 0) \in \mathcal{D}(\sigma)^{\circ\circ} = \operatorname{cl} \mathcal{K}(\sigma)$. For that purpose, given $(x, x_{n+1}, x_{n+2}) \in \mathcal{D}(\sigma)^\circ$, we distinguish two cases:

- Case 1: $x_{n+1} \leq x_{n+2} < 0$. In this case, we see that $\left(\frac{x}{-x_{n+1}}, -1, \frac{x_{n+2}}{-x_{n+1}}\right) \in \mathcal{D}(\sigma)^\circ$ where $0 < \frac{x_{n+2}}{x_{n+1}} \leq 1$. According to Theorem 3.3 (i), $\overline{x} := \frac{-1}{x_{n+1}} x \in F_{SS}$ and so, by assumption, $\langle a, \overline{x}\rangle \leq b$, that is, $\langle a, x\rangle + b x_{n+1} \leq 0$.
- Case 2: $x_{n+1} \leq x_{n+2} = 0$. For $\overline{x} \in F_{SS}$, let $\overline{\varepsilon} \in]0, 1]$ be such that $(\overline{x}, -1, -\overline{\varepsilon}) \in \mathcal{D}(\sigma)^\circ$. Then,

  $$(x^\lambda, x_{n+1}^\lambda, x_{n+2}^\lambda) := (1-\lambda)(x, x_{n+1}, 0) + \lambda(\overline{x}, -1, -\overline{\varepsilon}) \in \mathcal{D}(\sigma)^\circ$$

  for all $\lambda \in]0, 1[$. Since $x_{n+1}^\lambda \leq x_{n+2}^\lambda < 0$, the vector $(x^\lambda, x_{n+1}^\lambda, x_{n+2}^\lambda)$ corresponds to the Case 1. Thus,

  $$(1-\lambda)(\langle a, x\rangle + b x_{n+1}) + \lambda(\langle a, \overline{x}\rangle - b) = \langle a, x^\lambda\rangle + b x_{n+1}^\lambda \leq 0$$

  for all $\lambda \in]0, 1[$. By taking limits when $\lambda \to 0$, one has $\langle a, x\rangle + b x_{n+1} \leq 0$.

Since $\langle a, x\rangle + b x_{n+1} \leq 0$ for all $(x, x_{n+1}, x_{n+2}) \in \mathcal{D}(\sigma)^\circ$, then $(a, b, 0) \in \mathcal{D}(\sigma)^{\circ\circ} = \operatorname{cl} \mathcal{K}(\sigma)$.

Now, assume that $(a, b, 0) \in \mathrm{cl}\,\mathcal{K}(\sigma)$ and let $\overline{x} \in F_{SS}$. Since $(\overline{x}, -1, -\overline{\varepsilon}) \in \mathcal{D}(\sigma)^\circ$ for some $\overline{\varepsilon} \in]0, 1]$ by Theorem 3.3 (i), and $\mathrm{cl}\,\mathcal{K}(\sigma) = \mathcal{D}(\sigma)^{\circ\circ}$, then $\langle (a, b, 0), (\overline{x}, -1, -\overline{\varepsilon})\rangle \leq 0$, and so $\langle a, \overline{x}\rangle \leq b$. Thus, $F_{SS} \subset \{x \in \mathbb{R}^n : \langle a, x\rangle \leq b\}$. □

We observe that an alternative and shorter proof of this proposition can be given as follows: in virtue of Lemma 2.1 (i), $F_{SS} \subset \{x \in \mathbb{R}^n : \langle a, x\rangle \leq b\}$ is equivalent to say that $\langle a, x\rangle \leq b$ is a consequence of σ. By the extended Farkas Theorem (see, e.g., [15, Corollary 3.1.2]), this is equivalent to $(a, b) \in \mathrm{cl}\, K(\sigma)$. Finally, it can be checked that this condition is equivalent to $(a, b, 0) \in \mathrm{cl}\,\mathcal{K}(\sigma)$.

Proposition 3.7 (Consequent strict relations of F_{SS}) *Assume that F_{SS} is non-empty. Then, $F_{SS} \subset \{x \in \mathbb{R}^n : \langle a, x\rangle < b\}$ if and only if* $(0_n, 0, -1) \in \mathrm{cl}\,\mathrm{cone}\big(\mathcal{C}(\sigma) \cup \{(-a, -b, 0)\}\big)$.

Proof Assume that $F_{SS} \subset \{x \in \mathbb{R}^n : \langle a, x\rangle < b\}$. Then,

$$\{x \in \mathbb{R}^n : \exists\, \varepsilon > 0,\ \langle a_t, x\rangle + \varepsilon \leq b_t,\ t \in T;\ \langle -a, x\rangle \leq -b\} = \emptyset.$$

Let $\mathcal{E}(\sigma) := \mathcal{D}(\sigma) \cup \{(-a, -b, 0)\}$. If $x_{n+2} < 0$ for some $(x, x_{n+1}, x_{n+2}) \in \mathcal{E}(\sigma)^\circ$, since $x_{n+1} \leq x_{n+2} < 0$, then $-x_{n+1} > 0$ and

$$\left(\frac{x}{-x_{n+1}}, -1, \frac{x_{n+2}}{-x_{n+1}}\right) \in \mathcal{E}(\sigma)^\circ \subset \mathcal{D}(\sigma)^\circ, \tag{10}$$

having that $\frac{x_{n+2}}{x_{n+1}} \in]0, 1]$. Thus, $\frac{-1}{x_{n+1}}x \in F_{SS}$ by Theorem 3.3 (i). Moreover, as $(-a, -b, 0) \in \mathcal{E}(\sigma)$, Eq. 10 implies $\langle -a, \frac{-1}{x_{n+1}}x\rangle \leq -b$, which contradicts the hypothesis. Thus, we have that $x_{n+2} = 0$ for all $(x, x_{n+1}, x_{n+2}) \in \mathcal{E}(\sigma)^\circ$.

Now, since $\langle (0_n, 0, -1), (x, x_{n+1}, x_{n+2})\rangle = 0$ for all $(x, x_{n+1}, x_{n+2}) \in \mathcal{E}(\sigma)^\circ$, one has $(0_n, 0, -1) \in \mathcal{E}(\sigma)^{\circ\circ} = \mathrm{cl}\,\mathrm{cone}\,\mathcal{E}(\sigma)$. By reasoning as in [15, Lemma 4.1], this condition is equivalent to $(0_n, 0, -1) \in \mathrm{cl}\,\mathrm{cone}\big(\mathcal{C}(\sigma) \cup \{(-a, -b, 0)\}\big)$.

Conversely, if there exists $\overline{x} \in F_{SS}$ such that $\langle a, \overline{x}\rangle \geq b$, then, by Theorem 3.3 (i), there exists $\overline{\varepsilon} \in]0, 1]$ such that $(\overline{x}, -1, -\overline{\varepsilon}) \in \mathcal{E}(\sigma)^\circ$. If $(0_n, 0, -1) \in \mathrm{cl}\,\mathrm{cone}\big(\mathcal{C}(\sigma) \cup \{(-a, -b, 0)\}\big)$ or, equivalently, $(0_n, 0, -1) \in \mathrm{cl}\,\mathrm{cone}\,\mathcal{E}(\sigma) = \mathcal{E}(\sigma)^{\circ\circ}$, then

$$\langle (0_n, 0, -1), (\overline{x}, -1, -\overline{\varepsilon})\rangle \leq 0,$$

which implies $\overline{\varepsilon} \leq 0$ and so, a contradiction. Thus, $(0_n, 0, -1) \notin \mathrm{cl}\,\mathrm{cone}\big(\mathcal{C}(\sigma) \cup \{(-a, -b, 0)\}\big)$ and the conclusion follows. □

Proposition 3.8 *Assume that F_{SS} is non-empty. Each one of the following conditions is sufficient for the set containment $F_{SS} \subset \{x \in \mathbb{R}^n : \langle a, x\rangle < b\}$.*

(i) $F_S \subset \{x \in \mathbb{R}^n : \langle a, x\rangle < b\}$.
(ii) $(a, b, c) \in \mathrm{cl}\,\mathcal{K}(\sigma)$ *for some* $c < 0$.

Proof The first statement is obvious since $F_{SS} \subset F_S$. So, let assume that $(a, b, c) \in \mathrm{cl}\,\mathcal{K}(\sigma)$ for some $c < 0$ and let $\overline{x} \in F_{SS}$. Since $(\overline{x}, -1, -\overline{\varepsilon}) \in \mathcal{D}(\sigma)^\circ$ for some $\overline{\varepsilon} \in]0, 1]$ by Theorem 3.3 (i), and $\mathrm{cl}\,\mathcal{K}(\sigma) = \mathcal{D}(\sigma)^{\circ\circ}$, then

$$\langle a, \overline{x}\rangle - b < \langle a, \overline{x}\rangle - b - c\overline{\varepsilon} = \langle (a, b, c), (\overline{x}, -1, -\overline{\varepsilon})\rangle \leq 0.$$

Hence, $\langle a, \overline{x}\rangle < b$, and so $F_{SS} \subset \{x \in \mathbb{R}^n : \langle a, x\rangle < b\}$. □

4 Geometry

The geometry of the sets F (a closed convex set) and F_S (an evenly convex set) follows from their definitions, since a set is closed and convex (evenly convex, respectively) if and only if it is the solution set of a system containing an arbitrary number of weak (strict, respectively) inequalities (see, e.g., [15, Chapter 5]). Furthermore, this geometry can be also derived from Theorems 3.1 (i) and 3.2 (i), respectively. Next, we obtain two geometric interpretations of F_{SS} (which are indeed equivalent) by taking into account Theorem 3.3 (i).

For that purpose, we firstly recall the description in Eq. 6 for the set F_{SS}, and we shall consider the set

$$F_{SS}^{\triangle} := \{(x,\varepsilon) \in \mathbb{R}^n \times \mathbb{R} : \langle a_t, x\rangle + \varepsilon \leq b_t, t \in T;\ 0 < \varepsilon \leq 1\}, \tag{11}$$

which can be interpreted as a lifting of the set F_{SS} into $\mathbb{R}^{n+1}$. It easily follows that

$$F_{SS} = \operatorname{proj}_{\mathbb{R}^n} F_{SS}^{\triangle},$$

where $\operatorname{proj}_{\mathbb{R}^n} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is the mapping defined by $\operatorname{proj}_{\mathbb{R}^n}(x,\varepsilon) = x$. Thus, if $(x,\varepsilon) \in F_{SS}^{\triangle}$ then $x \in F_{SS}$, and conversely, if $x \in F_{SS}$ then $(x,\varepsilon) \in F_{SS}^{\triangle}$ for all $\varepsilon \in]0, \varepsilon_x]$ (recall Remark 3.4). Consequently, one has

$$F_{SS} \neq \emptyset \quad \Longleftrightarrow \quad F_{SS}^{\triangle} \neq \emptyset.$$

Clearly, $F_{SS}^{\triangle}$ is an evenly convex set. However, the projection of an evenly convex set is not, in general, an evenly convex set (see, e.g., [20]).

Theorem 4.1 (Geometry of F_{SS}) *The following statements hold:*

(i) *Assume that $F_{SS} \neq \emptyset$. Then, for each $\overline{x} \in \mathbb{R}^n \backslash F_{SS}$ and each $\overline{\varepsilon} \in]0, 1]$, there exists $(\overline{u}, \overline{v}) \in \mathbb{R}^n \times \mathbb{R}$ such that*

$$\langle \overline{u}, x\rangle + \varepsilon \leq \overline{v} < \langle \overline{u}, \overline{x}\rangle + \overline{\varepsilon} \tag{12}$$

for all $(x,\varepsilon) \in F_{SS}^{\triangle}$.

(ii) *Let $G \subset \mathbb{R}^n \times]0, 1]$ and $G^{\nabla} := \operatorname{proj}_{\mathbb{R}^n} G$. If, for every $(\overline{x}, \overline{\varepsilon}) \in (\mathbb{R}^n \backslash G^{\nabla}) \times]0, 1]$, there exists $(\overline{u}, \overline{v}) \in \mathbb{R}^n \times \mathbb{R}$ such that $\langle \overline{u}, x\rangle + \varepsilon \leq \overline{v} < \langle \overline{u}, \overline{x}\rangle + \overline{\varepsilon}$ for all $(x,\varepsilon) \in G$, then G^{∇} is the set of strong Slater points of certain linear system.*

Proof (i) As $F_{SS} \neq \emptyset$, by Lemma 2.1 (i), $\operatorname{cl} F_{SS} = F$. Let $\overline{x} \in \mathbb{R}^n \backslash F_{SS}$. Next we distinguish two cases.

Case 1. If $\overline{x} \notin \operatorname{cl} F_{SS}$, then, by the well-known strong separation theorem, there exists $(u, v) \in \mathbb{R}^n \times \mathbb{R}$ such that $\langle u, x\rangle \leq v < \langle u, \overline{x}\rangle$ for all $x \in F_{SS}$, or equivalently,

$$\langle u, x\rangle - v \leq 0 < \langle u, \overline{x}\rangle - v$$

for all $x \in F_{SS}$. Let $(\widetilde{u}, \widetilde{v}) := \frac{1}{\langle u, \overline{x}\rangle - v}(u, v)$. Then, for any $\overline{\varepsilon} \in]0, 1]$ one has

$$\langle \widetilde{u}, x\rangle - \widetilde{v} + \varepsilon \leq \langle \widetilde{u}, x\rangle - \widetilde{v} + 1 \leq 1 < 1 + \overline{\varepsilon} = \langle \widetilde{u}, \overline{x}\rangle - \widetilde{v} + \overline{\varepsilon}$$

for all $(x,\varepsilon) \in F_{SS}^{\triangle}$. Finally, by considering $(\overline{u}, \overline{v}) := (\widetilde{u}, 1 + \widetilde{v})$ one gets Eq. 12.

Case 2. If $\overline{x} \in F \backslash F_{SS}$, then one has $(\overline{x}, -1, -\varepsilon) \notin \mathcal{D}(\sigma)^{\circ}$ for every $\varepsilon \in]0, 1]$ by Theorem 3.3 (i). Fix $\overline{\varepsilon} \in]0, 1]$. Since $(\overline{x}, -1, -\overline{\varepsilon}) \notin \mathcal{D}(\sigma)^{\circ}$ and $\mathcal{D}(\sigma)^{\circ}$ is a closed convex cone, there exists $(u, v, w) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$ such that

$$\langle (u, v, w), (x, x_{n+1}, x_{n+2})\rangle \leq 0 < \langle (u, v, w), (\overline{x}, -1, -\overline{\varepsilon})\rangle \tag{13}$$

for all $(x, x_{n+1}, x_{n+2}) \in \mathcal{D}(\sigma)^{\circ}$. Theorem 3.3 guarantees that for every $x \in F_{SS}$, $(x, -1, -\varepsilon) \in \mathcal{D}(\sigma)^{\circ}$ for all $\varepsilon \in]0, \varepsilon_x]$ (see Remark 3.4). Thus, we get from Eq. 13 that

$$\langle (u, v, w), (x, -1, -\varepsilon)\rangle \leq 0 < \langle (u, v, w), (\overline{x}, -1, -\overline{\varepsilon})\rangle \tag{14}$$

for all $(x, \varepsilon) \in F_{SS}^{\triangle}$. If $w = 0$, then Eq. 14 means that $\langle u, x\rangle \leq v < \langle u, \overline{x}\rangle$ for all $x \in F_{SS}$, which is equivalent to $\overline{x} \notin \text{cl}\, F_{SS} = F$, a contradiction with the assumption. Then, we have $w \neq 0$. Now, letting $\varepsilon \downarrow 0$ in Eq. 14 we get

$$\langle (u, v, w), (x, -1, 0)\rangle \leq 0 < \langle (u, v, w), (\overline{x}, -1, -\overline{\varepsilon})\rangle \tag{15}$$

for all $x \in F_{SS}$. Since $\overline{x} \in \text{cl}\, F_{SS}$, there exists a sequence $\{x^k\}_{k\in\mathbb{N}} \subset F_{SS}$ which converges to $\overline{x}$. Thus, by Eq. 15, one has

$$\langle (u, v, w), (x^k, -1, 0)\rangle \leq 0 < \langle (u, v, w), (\overline{x}, -1, -\overline{\varepsilon})\rangle$$

for all $k \in \mathbb{N}$, and taking limits when $k \to \infty$ we obtain

$$\langle (u, v, w), (\overline{x}, -1, 0)\rangle \leq 0 < \langle (u, v, w), (\overline{x}, -1, -\overline{\varepsilon})\rangle,$$

yielding $w\overline{\varepsilon} < 0$. Since $\overline{\varepsilon} > 0$, then one has $w < 0$. Dividing Eq. 14 by $-w > 0$, and defining $(\overline{u}, \overline{v}) := \frac{-1}{w}(u, v)$, we get

$$\langle (\overline{u}, \overline{v}, -1), (x, -1, -\varepsilon)\rangle \leq 0 < \langle (\overline{u}, \overline{v}, -1), (\overline{x}, -1, -\overline{\varepsilon})\rangle \tag{16}$$

for all $(x, \varepsilon) \in F_{SS}^{\triangle}$, which is equivalent to Eq. 12.

(ii) Let $G \subset \mathbb{R}^n \times]0, 1]$ and $G^{\triangledown} := \text{proj}_{\mathbb{R}^n} G$. Consider the index set $T := (\mathbb{R}^n \backslash G^{\triangledown}) \times]0, 1]$. By assumption, for every $t := (\overline{x}, \overline{\varepsilon}) \in T$, there exists $(a_t, b_t) \in \mathbb{R}^n \times \mathbb{R}$ such that $\langle a_t, x\rangle + \varepsilon \leq b_t < \langle a_t, \overline{x}\rangle + \overline{\varepsilon}$ for all $(x, \varepsilon) \in G$. On the one hand, if $x \in G^{\triangledown}$, then there exists $\varepsilon \in]0, 1]$ such that $(x, \varepsilon) \in G$, and so $\langle a_t, x\rangle + \varepsilon \leq b_t$ for all $t \in T$. On the other hand, if $\overline{x} \notin G^{\triangledown}$, then, for every $\overline{\varepsilon} \in]0, 1]$, there exists $t = (\overline{x}, \overline{\varepsilon}) \in T$ such that $\langle a_t, \overline{x}\rangle + \overline{\varepsilon} > b_t$. Consequently, we have shown that

$$G^{\triangledown} = \{x \in \mathbb{R}^n : \exists\, \varepsilon \in]0, 1], \langle a_t, x\rangle + \varepsilon \leq b_t,\ t \in T\},$$

and the conclusion follows. □

Next example provides an intuitive explanation of the geometry of a set of strong Slater points of a linear inequality system through conditions (i) and (ii) in Theorem 4.1.

Example 4.2 Let $\sigma := \{-x \leq t,\ t > 0\}$ be the system in $\mathbb{R}$ whose solution set is $F = [0, +\infty[$. By Proposition 2.6, since $\dim F = 1$ and $\inf\{\|a_t\|,\ t \in T\} = \sup\{\|a_t\|,\ t \in T\} = 1$, one has $F_{SS} = \text{int}\, F =]0, +\infty[$. Furthermore, for $x \in F_{SS}$ one has (see Remark 3.4)

$$\varepsilon_x := \min\{1, \inf\{t + x : t > 0\}\} = \begin{cases} x, & \text{if } x < 1, \\ 1, & \text{if } x \geq 1. \end{cases}$$

Thus, according to Eq. 11, $F_{SS}^{\triangle} = \left\{(x, \varepsilon) \in \mathbb{R}^2 : x > 0,\ 0 < \varepsilon \leq 1,\ \varepsilon \leq x\right\}$. For each $\overline{x} \in \mathbb{R}\backslash F_{SS} =]-\infty, 0]$ and each $\overline{\varepsilon} \in]0, 1]$, Theorem 4.1 (i) guarantees the existence of $(\overline{u}, \overline{v}) \in \mathbb{R}^2$ such that $\langle \overline{u}, x\rangle + \varepsilon \leq \overline{v} < \langle \overline{u}, \overline{x}\rangle + \overline{\varepsilon}$ for all $(x, \varepsilon) \in F_{SS}^{\triangle}$, that is, there exists a non-vertical hyperplane (see, for instance, line r in Fig. 3) that strongly separates $(\overline{x}, \overline{\varepsilon})$ from $F_{SS}^{\triangle}$.

Now, consider the sets $G_1 :=]0, +\infty[\times]0, 1]$ and $G_1^{\triangledown} := \text{proj}_{\mathbb{R}} G_1 =]0, +\infty[$. For $\overline{x} = 0 \in \mathbb{R}\backslash G_1^{\triangledown}$ and $\overline{\varepsilon} \in]0, 1[$, it can be checked that there does not exist $(\overline{u}, \overline{v}) \in \mathbb{R}^2$ such that $\langle \overline{u}, x\rangle + \varepsilon \leq \overline{v} < \overline{\varepsilon} = \langle \overline{u}, \overline{x}\rangle + \overline{\varepsilon}$ for all $(x, \varepsilon) \in G_1$. Thus, Theorem 4.1 (ii) can not be applied in this case. This fact does not mean that $]0, +\infty[$ is not the set of strong Slater

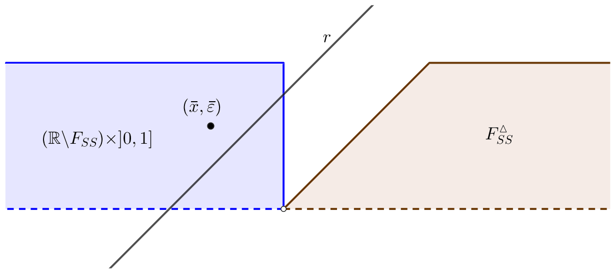

Fig. 3 Strong separation of $(\overline{x},\overline{\varepsilon})$ from $F_{SS}^{\triangle}$

points of certain linear system. Indeed, if we consider $G_2 := \{(x,\varepsilon)\in\mathbb{R}^2 : x>0, 0<\varepsilon\leq 1, \varepsilon\leq x\}$, then $G_2^{\nabla} := \operatorname{proj}_{\mathbb{R}} G_2 = G_1^{\nabla} =]0,+\infty[$ and, for $\overline{x}\in\mathbb{R}\backslash G_2^{\nabla}$ and $\overline{\varepsilon}\in]0,1]$, one has that $-x+\varepsilon\leq -\overline{x}+\frac{\overline{\varepsilon}}{2} < -\overline{x}+\overline{\varepsilon}$ for all $(x,\varepsilon)\in G_2$. Hence, Theorem 4.1 (ii) applies and the set $]0,+\infty[$ is the set of strong Slater points of the linear system obtained from G_2

$$\left\{-x\leq -\overline{x}+\frac{\overline{\varepsilon}}{2}, \overline{x}\leq 0, 0<\overline{\varepsilon}\leq 1\right\} = \{-x\leq t, t>0\}.$$

Secondly, we shall consider the set

$$F_{SS}^{\circledast} := \{(x,x_{n+1})\in\mathbb{R}^n\times\mathbb{R} : \langle a_t,x\rangle + b_t x_{n+1}\leq -1,\ t\in T;\ x_{n+1}\leq -1\}. \tag{17}$$

According to Theorems 3.1 (iii) and 3.3 (iii), the existence of strong Slater points for σ is equivalent to the consistency of the linear inequality system defining the set $F_{SS}^{\circledast}$ in Eq. 17, that is,

$$F_{SS}\neq\emptyset \quad\Longleftrightarrow\quad F_{SS}^{\circledast}\neq\emptyset.$$

More precisely, one has:

- If $\overline{x}\in F_{SS}$, that is, if there exists $\overline{\varepsilon}\in]0,1]$ such that $\langle a_t,\overline{x}\rangle+\overline{\varepsilon}\leq b_t$ for all $t\in T$ (i.e., $(\overline{x},\overline{\varepsilon})\in F_{SS}^{\triangle}$), then $\frac{1}{\overline{\varepsilon}}(\overline{x},-1)\in F_{SS}^{\circledast}$.
- Conversely, if $(x,x_{n+1})\in F_{SS}^{\circledast}$, then $\frac{-1}{x_{n+1}}x\in F_{SS}$. Indeed, one has $\frac{-1}{x_{n+1}}(x,1)\in F_{SS}^{\triangle}$.

As a consequence of these relations, we observe that

$$F_{SS} = \pi(F_{SS}^{\circledast}),$$

where π denotes the mapping $\pi:\mathbb{R}^n\times\mathbb{R}\backslash\{0\}\to\mathbb{R}^n$ defined by $\pi(x,x_{n+1}) := \frac{-1}{x_{n+1}}x$.

Lemma 4.3 *Assume that $F_{SS}\neq\emptyset$. Then, $F_{SS}^{\circledast}$ is a proper closed convex coradiant set.*

Proof Clearly, $F_{SS}^{\circledast}$ is a closed convex set in $\mathbb{R}^{n+1}$. Since $0_{n+1}\notin F_{SS}^{\circledast}$, and $\lambda(x,x_{n+1})\in F_{SS}^{\circledast}$ for all $\lambda\geq 1$ and for all $(x,x_{n+1})\in F_{SS}^{\circledast}$, then $F_{SS}^{\circledast}$ is a proper coradiant set. □

Although $F_{SS}^{\circledast}$ is a closed convex set, its image by the mapping π is not necessarily a closed convex set. Next, we exploit the properties of $F_{SS}^{\circledast}$ given in Lemma 4.3 to provide a geometric interpretation of F_{SS}.

Theorem 4.4 (Geometry of F_{SS}) *The following statements hold:*

(*i*) *Assume that* $F_{SS} \neq \emptyset$*. Then, for each* $\overline{x} \in \mathbb{R}^n \backslash F_{SS}$ *and each* $\overline{\varepsilon} \in]0, 1]$*, there exists* $(\overline{u}, \overline{v}) \in \mathbb{R}^n \times \mathbb{R}$ *such that*

$$\langle \overline{u}, x \rangle + \overline{v} x_{n+1} \leq -1 < \frac{1}{\overline{\varepsilon}} \left(\langle \overline{u}, \overline{x} \rangle - \overline{v} \right) \tag{18}$$

for all $(x, x_{n+1}) \in F_{SS}^{\circledast}$.

(*ii*) *Let* $H \subset \mathbb{R}^n \times]-\infty, -1]$ *and* $H^* := \pi(H)$*. If, for every* $(\overline{x}, \overline{\varepsilon}) \in (\mathbb{R}^n \backslash H^*) \times]0, 1]$*, there exists* $(\overline{u}, \overline{v}) \in \mathbb{R}^n \times \mathbb{R}$ *such that* $\langle \overline{u}, x \rangle + \overline{v} x_{n+1} \leq -1 < \frac{1}{\overline{\varepsilon}} (\langle \overline{u}, \overline{x} \rangle - \overline{v})$ *for all* $(x, x_{n+1}) \in H$*, then* H^* *is the set of strong Slater points of certain linear system.*

Proof (*i*) Let $\overline{x} \in \mathbb{R}^n \backslash F_{SS}$. Then, for every $\varepsilon \in]0, 1]$ one has that $\frac{1}{\varepsilon}(\overline{x}, -1) \notin F_{SS}^{\circledast}$. Fix $\overline{\varepsilon} \in]0, 1]$. Thanks to Lemma 4.3 and the characterization of proper closed convex coradiant sets by means of its concave gauge (see Eq. 2), one can write

$$F_{SS}^{\circledast} = \{(x, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R} : \phi_{F_{SS}^{\circledast}}(x, x_{n+1}) \geq 1\}.$$

Hence, if $\frac{1}{\overline{\varepsilon}}(\overline{x}, -1) \notin F_{SS}^{\circledast}$, one has that $\phi_{F_{SS}^{\circledast}}\left(\frac{1}{\overline{\varepsilon}}(\overline{x}, -1)\right) < 1$. Equivalently, there exists $(\overline{u}, \overline{v}) \in \mathbb{R}^n \times \mathbb{R}$ such that

$$\langle \overline{u}, x \rangle + \overline{v} x_{n+1} \leq -1 < \frac{1}{\overline{\varepsilon}} \left(\langle \overline{u}, \overline{x} \rangle - \overline{v} \right)$$

for all $(x, x_{n+1}) \in F_{SS}^{\circledast}$, which is precisely Eq. 18.

(*ii*) Let $H \subset \mathbb{R}^n \times]-\infty, -1]$ and $H^* := \pi(H)$. Consider the index set $T := (\mathbb{R}^n \backslash H^*) \times]0, 1]$. By assumption, for every $t := (\overline{x}, \overline{\varepsilon}) \in T$, there exists $(a_t, b_t) \in \mathbb{R}^n \times \mathbb{R}$ such that $\langle a_t, x \rangle + b_t x_{n+1} \leq -1 < \frac{1}{\overline{\varepsilon}} (\langle a_t, \overline{x} \rangle - b_t)$ for all $(x, x_{n+1}) \in H$. On the one hand, if $\overline{x} \notin H^*$, then, for every $\overline{\varepsilon} \in]0, 1]$, there exists $t = (\overline{x}, \overline{\varepsilon}) \in T$ such that $\frac{1}{\overline{\varepsilon}} (\langle a_t, \overline{x} \rangle - b_t) > -1$, that is, $\langle a_t, \overline{x} \rangle + \overline{\varepsilon} > b_t$. On the other hand, if $x \in H^*$, then there exists $x_{n+1} \in]-\infty, -1]$ such that $-x_{n+1}(x, -1) \in H$, and so $-x_{n+1}(\langle a_t, x \rangle - b_t) \leq -1$ for all $t \in T$, or equivalently, by letting $\varepsilon := -(x_{n+1})^{-1} \in]0, 1]$, $\langle a_t, x \rangle + \varepsilon \leq b_t$ for all $t \in T$. Consequently, we have shown that

$$H^* = \{x \in \mathbb{R}^n : \exists\, \varepsilon \in]0, 1], \langle a_t, x \rangle + \varepsilon \leq b_t,\ t \in T\},$$

and the conclusion follows. □

Remark 4.5 We observe that Eqs. 12 and 18 are equivalent conditions indeed. Thus, both Theorems 4.1 and 4.4 state actually the same thesis, but they are formulated by considering different sets $F_{SS}^{\triangle}$ (evenly convex) and $F_{SS}^{\circledast}$ (closed convex) in $\mathbb{R}^{n+1}$. The given proofs are different too.

5 Application to Systems with Strict Convex Inequalities

Now we apply the former results to provide necessary and sufficient conditions for the consistency of systems with strict inequalities of the form

$$\tau := \{f_t(x) < 0, t \in T\}, \tag{19}$$

determined by proper lower semicontinuous convex functions $f_t : \mathbb{R}^n \to \overline{\mathbb{R}}$. This kind of systems, including weak inequalities also, were analyzed in [8] in the context of set containments.

We firstly point out that the solution set of τ is not necessarily evenly convex (even when T is singleton).

Example 5.1 Consider again the function $f(x) = x_1 - \sqrt{x_2}$ on its effective domain $\operatorname{dom} f = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \geq 0\}$ introduced in Example 2.4. This function is proper lower semicontinuous convex (since the lower level set $\{(x, y) \in \operatorname{dom} f : x - \sqrt{y} \leq r\}$ is closed and convex for every $r \in \mathbb{R}$), and so $f = f^{**}$. Thus, since

$$f^*(y_1, y_2) = \begin{cases} \frac{-1}{4y_2}, & \text{if } y_1 = 1, y_2 < 0, \\ +\infty, & \text{otherwise,} \end{cases}$$

one has

$$\{f(x) < 0\} = \{\exists\, \varepsilon > 0,\, f(x) + \varepsilon \leq 0\} = \{\exists\, \varepsilon > 0, x_1 + y_2 x_2 + \varepsilon \leq \frac{-1}{4y_2}, y_2 < 0\}.$$

Therefore, $\{f(x) < 0\}$ coincides with the set of strong Slater points of the linear system $\{x_1 - tx_2 \leq \frac{1}{4t}, t > 0\}$, which is not evenly convex as shown in Example 2.4.

Theorem 5.2 (Necessary and sufficient conditions for the consistency of τ) *Let $f_t : \mathbb{R}^n \to \overline{\mathbb{R}}$ be proper lower semicontinuous convex functions for all $t \in T$. Consider the following statements:*

(i) $\quad (0_n, 0, -1) \notin \operatorname{cl} \operatorname{cone} \left[\left(\bigcup_{t \in T} \operatorname{epi} f_t^* \times \{-1\} \right) \cup \{(0_n, 1, -1)\} \right]$;

(ii) $\quad \tau = \{f_t(x) < 0, t \in T\}$ *is consistent;*

(iii) $\quad 0_{n+1} \notin \operatorname{eco} \left[\left(\bigcup_{t \in T} \operatorname{epi} f_t^* \right) \cup \{(0_n, 1)\} \right]$.

Then, one has $(i) \Rightarrow (ii) \Rightarrow (iii)$.

Proof $(i) \Rightarrow (ii)$ According to Theorem 3.3 (iii), (i) is equivalent to the existence of strong Slater points of the linear system $\{\langle a, x\rangle \leq b, (a, b) \in \operatorname{epi} f_t^*, t \in T\}$. Hence, there exist $\overline{x} \in \mathbb{R}^n$ and $\overline{\varepsilon} \in]0, 1]$ such that

$$\langle a, \overline{x}\rangle + \overline{\varepsilon} \leq f_t^*(a) + \delta$$

for all $\delta \in \mathbb{R}_+, a \in \operatorname{dom} f_t^*, t \in T$. Thus, $f_t(\overline{x}) < f_t(\overline{x}) + \overline{\varepsilon} = f_t^{**}(\overline{x}) + \overline{\varepsilon} \leq 0$ for all $t \in T$, which shows that τ is consistent.

[$(ii) \Rightarrow (iii)$] If τ is consistent, then there exists $\overline{x} \in \mathbb{R}^n$ such that

$$\langle a, \overline{x}\rangle - f_t^*(a) \leq f_t^{**}(\overline{x}) = f_t(\overline{x}) < 0$$

for all $a \in \operatorname{dom} f_t^*, t \in T$. Thus, the set of Slater points of the linear system $\{\langle a, x\rangle \leq b, (a, b) \in \operatorname{epi} f_t^*, t \in T\}$ is non-empty and so, by Theorem 3.2 (iii), statement (iii) holds. □

We observe that, in Theorem 5.2, the epigraphs of the functions f_t^*, $t \in T$, can be replaced by their corresponding graphs. Furthermore, one has that, if for every $t \in T$ there exists a compact set $C_t \subset \mathbb{R}^{n+1}$ such that $f_t(\cdot) = \max\{\langle a, \cdot\rangle - b : (a, b) \in C_t\}$, then the system τ is consistent if and only if

$$0_{n+1} \notin \operatorname{eco} \left[\left(\bigcup_{t \in T} C_t \right) \cup \{(0_n, 1)\} \right].$$

The proof of this fact was given in [8, Proposition 6.3], and it follows also from Theorem 3.2 (iii), since $f_t(x) < 0$ if and only if $\langle a, x\rangle < b$ for all $(a, b) \in C_t$.

We conclude by pointing out that, whenever all the functions f_t in Theorem 5.2 are linear, then one has that: statement (ii) is equivalent to the existence of Slater points of a linear inequality system σ; statement (i) is equivalent, by Theorem 3.3 (iii), to the existence of strong Slater points of σ; statements (ii) and (iii) are equivalent in this case in virtue of [16, Lemma 3.1]; statements (i) and (ii) are not equivalent in general. To see this, consider the system $\sigma = \{t_1 x \leq t_2, (t_1, t_2) \in T\}$ where $T = \{(t_1, t_2) \in \mathbb{R}^2 : t_1 \geq 0, t_2 \geq 0, t_1 + t_2 \neq 0\}$. It can checked that $F =]-\infty, 0]$, $F_S =]-\infty, 0[$ and so (ii) in Theorem 5.2 holds, and $F_{SS} = \emptyset$ and so (i) in Theorem 5.2 does not hold.

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