



Calmness of the Solution Mapping of Navier-Stokes Problems Modeled by Hemivariational Inequalities

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Abstract

The main purpose of this paper is to find conditions for Hölder calmness of the solution mapping, viewed as a function of the boundary data, of a hemivariational inequality governed by the Navier-Stokes operator. To this end, a more abstract model is studied first: a class of parametric equilibrium problems defined by trifunctions. The presence of trifunctions allows the extension of the monotonicity notions in the theory of equilibrium problems.

Keywords Navier-Stokes equation · Calmness · Parametric equilibrium problems with trifunctions · Hemi-variational inequalities

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1 Introduction

In the papers [30] and [31] a class of hemivariational inequalities with the Navier-Stokes operator has been studied, where the nonslip boundary condition together with a Clarke sub-differential relation between the total pressure and the normal components of the velocity was assumed. The main feature of such a hemivariational inequality is that it is governed by a nonmonotone and nonlinear operator, and possibly by a multivalued boundary condition defined by the Clarke derivative of a locally Lipschitz superpotential. The problem under consideration comes from fluid flow control problems and flow problems for semipermeable walls and membranes. It describes a model in which the boundary orifices in a channel are regulated to reduce the pressure of the fluid on the boundary when the normal velocity reaches a prescribed value. For the particular case when the superpotential is a lower semi-continuous convex functional the problem reduces to a variational inequality governed by a maximal monotone operator (see [10, 11, 23]).

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Existence results for nonconvex locally Lipschitz superpotentials were given in the above mentioned papers [30] (stationary case), [31] (evolution case) and [3, 8, 9] (periodic or antiperiodic case), for instance.

The main purpose of this paper is to study the calmness (in the sense of [13] and [16]) of the set of solutions of a hemivariational inequality governed by the Navier-Stokes operator, when the Clarke derivative is substituted by a more general control function, depending also on the state and time variables.

Calmness is an important stability property since it gives a bound on the distance between perturbed solutions and unperturbed solutions. For real functions this property is weaker than the usual local Lipschitz continuity since one of the two points considered for comparison is required to be fixed, but it is stronger than the continuity at that point.

In the case of set-valued mappings, calmness is defined using the excess function defined by the Romanian mathematician D. Pompeiu (see [16]). In this case calmness is a generalization of the Aubin property, which in its turn is a generalization of the local Lipschitz property for set-valued mappings (see [16]). Calmness properties of solutions to parameterized equilibrium problems formulated with bifunctions have been studied widely. Most of this study has focused on particular models such as optimization problems (see [7, 24, 25] and [22]) and variational or hemivariational inequalities (see [27, 34]). Hölder calmness and Hölder continuity of solution mappings of general parametric equilibrium problems for bifunctions have been studied, for instance, in the papers [1, 2, 5, 26, 28]. The calmness property is strongly connected to the Hölder metric subregularity of the inverse mapping (see [16, 36] and the references therein).

In the last years many papers about hemivariational inequalities similar to that of [30] and [31], with important applications, were published. As an example we mention only the recently appeared article [29]. Since the study of calmness of the solution sets for similar problems is undoubtedly important, we will embed our problem in an abstract model of parametric equilibrium problem defined by trifunction and we will apply the abstract results obtained for the hemivariational inequalities governed by the Navier-Stokes operator, where the parameters are functions with some properties similar to those of Clarke generalized derivatives.

Our main motivation to study equilibrium problems defined by trifunctions is the important role that the monotonicity has in existence and stability results for equilibrium problems defined by bifunctions on one hand, and the existence of bifunctions that are not monotone, on the other hand. In [19] we have shown that for trifunctions it is possible to define a monotonicity notion such that the monotone bifunctions that have value zero on the diagonal, generate monotone trifunctions and every bifunction is monotone as a trifunction. Therefore, for instance, the so-called mixed equilibrium problems can be formulated by monotone trifunctions. This makes possible to extend the duality principle to a large class of equilibrium problems, and to use it, for instance, by proving existence and stability of the solutions.

This paper is structured in five sections, as follows:

Section 2 contains several notions and results needed in the sequel.

In Section 3 we prove our main calmness result (Theorem 1), in a general setting, for parametric equilibrium problems with trifunctions. It gives sufficient conditions for Hölder calmness based on an a priori estimation for the dual problem, which is, in fact, a gradual uniform partial calmness property (see [16]). To our best knowledge, calmness for such general problems is studied here for the first time. Existence theorems for equilibrium problems with trifunctions were given in [19].

In Section 4, we apply the previous abstract theorem to mixed equilibrium problems. Finally, in Section 5, returning to the main purpose of the paper, we focus on the Navier-Stokes problems modeled by hemivariational type inequalities with boundary control, obtaining sufficient conditions for Hölder calmness of the solution mapping. Since calmness is stronger than continuity, our results can be seen as sharpening of other results obtained before on the continuity of the solution set as a function of parameters (see, for example, Theorem 21 in [30] and the convergence results in [33]).

2 Preliminaries

For $a, b \in \mathbb{R}$ we denote by $]a, b[$ the open interval with the endpoints a and b . Let $\mathbb{R}_+ = [0, +\infty[$ and $\mathbb{R}_- =]-\infty, 0]$. We denote $a_- := d(a, \mathbb{R}_+) = \max\{-a, 0\}$ and $a_+ := d(a, \mathbb{R}_-) = \max\{a, 0\}$. The following properties hold:

- (a) $(-a)_+ = a_-$, $(-a)_- = a_+$;
- (b) If $b \geq 0$ then $a_+ \leq (a + b)_+$;
- (c) $(a + b)_+ \leq a_+ + b_+$. The equality holds if and only if $ab \geq 0$;
- (d) $(a + b)_+ \leq a_+ + |b|$;
- (e) If $\alpha > 0$, then $a_+ \leq \alpha$ if and only if $a \leq \alpha$.

In this paper, unless otherwise mentioned, M and X will be metric spaces, and - for convenience - both distances will be denoted by d . For $u \in X$ and $r > 0$, by $B(u, r)$ will be denoted the open ball centered at u , of radius r . The Euclidean norm on \mathbb{R}^d ($d = 1, 2, \dots$) will be denoted by $|\cdot|$, and the scalar product of $u, v \in \mathbb{R}^d$ by $u \cdot v$.

A function $f : M \rightarrow X$ is said to satisfy the *Hölder condition* of rank k and exponent ε if $k \geq 0, \varepsilon > 0$, and

$$d(f(\mu), f(\mu')) \leq kd^\varepsilon(\mu, \mu'), \tag{1}$$

for all $\mu, \mu' \in M$. We say that f is a (k, ε) -Hölder function near $\bar{\mu}$ if there exists a neighbourhood $U(\bar{\mu})$ of $\bar{\mu}$ such that (1) is verified for all $\mu, \mu' \in U(\bar{\mu})$. A property between this and the continuity at $\bar{\mu}$ is the calmness at $\bar{\mu}$.

A function $f : M \rightarrow X$ is said to be *Hölder calm* at $\bar{\mu}$ if there exist $k \geq 0, \varepsilon > 0$ and a neighborhood $U(\bar{\mu})$ of $\bar{\mu}$ such that

$$d(f(\mu), f(\bar{\mu})) \leq kd^\varepsilon(\mu, \bar{\mu}),$$

for all $\mu \in U(\bar{\mu})$.

Calmness can be generalized for set-valued functions too. For some sets A, B in the metric space (X, d) and $a \in X$, denote by

$$d(a, B) := \inf_{b \in B} d(a, b)$$

the distance between the point a and the set B , and by

$$e(A, B) := \sup_{a \in A} d(a, B)$$

the Pompeiu excess of A with respect to B , where the convention

$$e(\emptyset, B) = \begin{cases} 0, & \text{when } B \neq \emptyset \\ +\infty, & \text{otherwise} \end{cases}$$

is used and $e(A, \emptyset) = +\infty$, for any set A , including \emptyset .

The Pompeiu-Hausdorff distance is defined by

$$h(A, B) = \max(e(A, B), e(B, A)).$$

As it is known, it does not furnish a metric on the space of all subsets, but it does on the space of nonempty closed and bounded subsets of X .

A mapping $S : M \rightarrow 2^X$ is said to be (k, ε) -Hölder calm at $(\bar{\mu}, \bar{u})$ if $(\bar{\mu}, \bar{u}) \in \text{gph}S$, and there exist neighbourhoods $U(\bar{\mu})$ of $\bar{\mu}$ and $V(\bar{u})$ of \bar{u} such that

$$e(S(\mu) \cap V(\bar{u}), S(\bar{\mu})) \leq kd^\varepsilon(\mu, \bar{\mu}) \text{ for all } \mu \in U(\bar{\mu}).$$

A mapping $S : M \rightarrow 2^X$ is said to have the *isolated Hölder calmness* property at $(\bar{\mu}, \bar{u}) \in \text{gph}S$ if there exist $k \geq 0, \varepsilon > 0$, and the neighbourhoods $U(\bar{\mu})$ of $\bar{\mu}$ and $V(\bar{u})$ of \bar{u} such that

$$d(u, \bar{u}) \leq kd^\varepsilon(\mu, \bar{\mu}) \text{ for all } \mu \in U(\bar{\mu}) \text{ and } u \in S(\mu) \cap V(\bar{u}).$$

If $\bar{\mu}$ is an isolated element of M then all the continuity properties defined before are verified.

If the above properties take place for $\varepsilon = 1$, then instead of $(k, 1)$ -Hölder we say k -Lipschitz. For more information on these notions we propose the book [16].

In the whole paper, we say that a property holds *near* u if there exists a neighborhood of u where that property is verified (see for instance [14]).

Suppose Y is a normed space and D is an open subset of Y . If $f : D \rightarrow \mathbb{R}$ is k -Lipschitz near $u \in D$, then the Clarke generalized directional derivative $f^0(u; v)$ at u in the direction $v \in Y$ is defined as follows:

$$f^0(u; v) = \limsup_{w \rightarrow u, t \searrow 0} \frac{f(w + tv) - f(w)}{t}.$$

Some of the basic properties of the function f^0 are summarised in the following

Proposition 1 (see [14], Proposition 2.1.1 and the proof of Theorem 2.3.10) *Let f be k -Lipschitz near $u \in D$. Then*

- The function $v \rightarrow f^0(u; v)$ is finite, sublinear, and k -Lipschitz on Y ;*
- The function $(u, v) \rightarrow f^0(u; v)$ is upper semi-continuous;*
- $f^0(u; -v) = (-f)^0(u; v)$;*
- If Z is a normed space, $T : Y \rightarrow Z$ is a surjective linear continuous operator and $g : T(D) \rightarrow \mathbb{R}$ is k -Lipschitz near Tu , then the function $g \circ T$ is $k\|T\|$ -Lipschitz near u and $(g \circ T)^0(u; v) = g^0(Tu, Tv)$ for $u, v \in Y$.*

Usually, a bifunction $f : X \times X \rightarrow \mathbb{R}$ is said to be *monotone* iff $0 \leq -f(u, v) - f(v, u)$, for all $u, v \in X$. It is called *strongly monotone* iff there exists $m > 0$ such that $m d^2(u, v) \leq -f(u, v) - f(v, u)$, for all $u, v \in X$. It is called *Hölder strongly monotone* iff there exist $m, \beta > 0$ such that $m d^\beta(u, v) \leq -f(u, v) - f(v, u)$, for all $u, v \in X$. In the following definition, we extend these notions for trifunctions as follows:

Definition 1 The trifunction $F : X \times X \times X \rightarrow \mathbb{R}$ is said to be *monotone* iff $F(u, v, u) \leq F(u, v, v)$ for every $u, v \in X$.

We say that F is *Hölder strongly monotone* iff there exist $m, \beta > 0$ such that

$$m d^\beta(u, v) \leq F(u, v, v) - F(u, v, u), \text{ for every } u, v \in X.$$

- Remark 1* (a) If the trifunction F has the particular form $F(u, v, w) = f(w, v) - f(w, u)$, with $f(u, u) = 0$ for any $u \in X$, then F is monotone (Hölder strongly monotone) if and only if the bifunction f is monotone (Hölder strongly monotone).
- (b) If $F(u, v, w) = g(u, v)$, with $g : X \times X \rightarrow \mathbb{R}$, obviously F is monotone as a trifunction. In consequence, if $G : X \times X \times X \rightarrow \mathbb{R}$ is a monotone trifunction and $g : X \times X \rightarrow \mathbb{R}$ is arbitrary, then the trifunction $F(u, v, w) = G(u, v, w) + g(u, v)$ is also monotone. This fact simplifies, for instance, the theory of mixed equilibrium problems, and makes it more transparent (see [19]).
- (c) If X is a normed space with the dual X^* , then an operator $T : X \rightarrow X^*$ is called *monotone* if

$$\langle T(u_1) - T(u_2), u_1 - u_2 \rangle \geq 0, \text{ for every } u_1, u_2 \in X.$$

An operator $T : X \times X \rightarrow X^*$ is called *semimonotone* if it is monotone with respect to the second variable, that is

$$\langle T(u, w_1) - T(u, w_2), w_1 - w_2 \rangle \geq 0, \text{ for every } u, w_1, w_2 \in X.$$

The function $F : X \times X \times X \rightarrow \mathbb{R}$ defined by $F(u, v, w) = \langle T(u, w), v - u \rangle$ is monotone, while the function $f(u, v) = \langle T(u, u), v - u \rangle$ is not necessarily monotone, so, for the variational inequality governed by the operator $u \mapsto T(u, u)$, the duality principle is not applicable.

Variational inequalities governed by such operators were studied, for instance, in [12] for single-valued functions and in [21] for set-valued mappings.

3 An Abstract Model

In this section we consider a general equilibrium problem where the objective function is a trifunction that depends on a parameter μ . In the papers [19] and [20] we gave existence results for such problems, motivated by the fact that the classical theory for equilibrium problems with bifunctions can not be used for some problems that appear in applications.

For a parameter $\mu \in M$, the problem that we study is

$$(PE)(\mu) \quad \text{Find } \bar{u} \in K \text{ such that } F(\bar{u}, z, \bar{u}; \mu) \geq 0, \text{ for every } z \in K,$$

where K and M are metric spaces, $F : K \times K \times K \times M \rightarrow \mathbb{R}$ is a given function.

Denote by $S(\mu)$ the set of solutions of the problem that depends on the parameter $\mu \in M$. Throughout the paper we suppose that it is nonempty. For a fixed value of the parameter μ , an existence result for the solutions of $(PE)(\mu)$ was proved in [19].

Theorem 1 *Let $\bar{\mu} \in M$ be a nonisolated point and $\bar{u} \in S(\bar{\mu})$ be fixed. Suppose that there exist the neighborhoods $U(\bar{\mu})$ of $\bar{\mu}$, $\tilde{V}(\bar{u})$ of \bar{u} and the numbers $a, c, \theta \geq 0$, and $b, m, \beta, \xi, \omega > 0$ such that*

- (i) $md^\beta(\bar{u}, v) \leq F(\bar{u}, v, v; \bar{\mu})_+ + F(\bar{u}, v, \bar{u}; \bar{\mu})_-$, for every $v \in S(\mu) \cap \tilde{V}(\bar{u})$ and $\mu \in U(\bar{\mu})$;
- (ii) The estimation $F(\bar{u}, v, v; \bar{\mu}) \leq cd^\beta(\bar{u}, v) + d^\theta(\bar{u}, v)[ad^\omega(\bar{u}, v) + bd^\xi(\mu, \bar{\mu})]$ holds for every $\mu \in U(\bar{\mu})$ and $v \in S(\mu) \cap \tilde{V}(\bar{u})$, with $v \neq \bar{u}$;
- (iii) $\theta < \beta$ and $c < m$.

Suppose that one of the following conditions is verified:

- 1) $a = 0$;
- 2) $\beta = \omega + \theta$, $a > 0$ and $a + c < m$;
- 3) $\beta < \omega + \theta$ and $a > 0$.

Then the mapping $S : M \rightarrow 2^K$ is Hölder calm at $(\bar{\mu}, \bar{u})$. Moreover we have the isolated Hölder calmness property at $(\bar{\mu}, \bar{u})$. In the cases 1) or 2) the solution \bar{u} is unique in the neighborhood $\tilde{V}(\bar{u})$. In the case 3) \bar{u} is unique in a neighborhood of \bar{u} .

The parameters from the definition of the calmness are, in each of the cases above:

- 1) $V(\bar{u}) = \tilde{V}(\bar{u})$, $\delta = \frac{\xi}{\beta - \theta}$, $k = \left(\frac{b}{m - c}\right)^{\frac{1}{\beta - \theta}}$, for $a = 0$,
- 2) $V(\bar{u}) = \tilde{V}(\bar{u})$, $\delta = \frac{\xi}{\beta - \theta}$, $k = \left(\frac{b}{m - c - a}\right)^{\frac{1}{\beta - \theta}}$, for $a > 0$ and $\beta = \omega + \theta$.
- 3) $V(\bar{u}) = B(\bar{u}, r) \cap \tilde{V}(\bar{u})$, $\delta = \frac{\xi}{\beta - \theta}$, $k = r \left(r^{\beta - \theta} - \frac{a}{m - c} r^\omega\right)^{\frac{1}{\theta - \beta}} \left(\frac{b}{m - c}\right)^{\frac{1}{\beta - \theta}}$,
 where $0 < r < \left(\frac{m - c}{a}\right)^{\frac{1}{\omega + \theta - \beta}}$, for $a > 0$ and $\beta < \omega + \theta$.

For the proof, we will need the following:

Lemma 1 Let $p > 0$, $q > 0$, and $l > 0$ be given real numbers, with $p < q$. Then, for every $\varepsilon \in]0, l^{\frac{1}{p-q}}[$, for all $x \in]0, \varepsilon[$ and $y > 0$ with $x^p - lx^q \leq y$, the inequality $x \leq ky^\delta$ holds, with $\delta = \frac{1}{p} > 0$ and $k = \varepsilon(\varepsilon^p - l\varepsilon^q)^{-\frac{1}{p}} > 0$.

Proof Let $p < q$ and $l > 0$. Let $\varphi : [0, \infty) \rightarrow \mathbb{R}$ be the function defined by $\varphi(\xi) = \xi - l\xi^{\frac{q}{p}}$. Let $\varepsilon \in]0, l^{\frac{1}{p-q}}[$. It is easy to see that on the interval $]0, \varepsilon^p[$ the function φ has strictly positive values and is strictly concave. From this, the quotient function $\xi \mapsto \frac{\varphi(\xi)}{\xi}$ is strictly decreasing, so for $\xi \in]0, \varepsilon^p[$, we have $\frac{\varphi(\varepsilon^p)}{\varepsilon^p} < \frac{\varphi(\xi)}{\xi}$ that is $\xi < \frac{\varepsilon^p}{\varphi(\varepsilon^p)}\varphi(\xi)$. Now consider $x \in]0, \varepsilon[$ with $x^p - lx^q \leq y$ and let $\xi = x^p$. Then we get

$$x^p < \frac{\varepsilon^p}{\varphi(\varepsilon^p)}\varphi(x^p) \leq \frac{\varepsilon^p}{\varepsilon^p - l\varepsilon^q}y.$$

and the conclusion is proved with

$$k = \varepsilon(\varepsilon^p - l\varepsilon^q)^{-\frac{1}{p}} \text{ and } \delta = \frac{1}{p}.$$

□

Proof of Theorem 1 1) In the case $a = 0$, we can choose $V(\bar{u}) = \tilde{V}(\bar{u})$. Let $\mu \in U(\bar{\mu})$ and $v \in S(\mu) \cap V(\bar{u})$, with $v \neq \bar{u}$. Since $\bar{u} \in S(\bar{\mu})$ and $v \in K$, we have

$$F(\bar{u}, v, \bar{u}; \bar{\mu}) \geq 0. \tag{2}$$

From (i), (ii), and (2) follows that

$$\begin{aligned} md^\beta(v, \bar{u}) &\leq F(\bar{u}, v, \bar{u}; \bar{\mu})_- + F(\bar{u}, v, v; \bar{\mu})_+ = F(\bar{u}, v, v; \bar{\mu})_+ \\ &= F(\bar{u}, v, v; \bar{\mu}) \leq cd^\beta(v, \bar{u}) + bd^\theta(v, \bar{u})d^\xi(\mu, \bar{\mu}). \end{aligned}$$

Therefore,

$$(m - c)d^{\beta-\theta}(v, \bar{u}) \leq bd^{\xi}(\mu, \bar{\mu}),$$

so directly

$$d(v, \bar{u}) \leq \left(\frac{b}{m - c}\right)^{\frac{1}{\beta-\theta}} d^{\frac{\xi}{\beta-\theta}}(\mu, \bar{\mu}).$$

- 2) In the case $\beta = \omega + \theta$ and $a + c < m$, we choose $V(\bar{u}) = \tilde{V}(\bar{u})$. For $\mu \in U(\bar{\mu})$ and $v \in S(\mu) \cap \tilde{V}(\bar{u})$ we have, similar to case 1),

$$md^{\beta}(v, \bar{u}) \leq cd^{\beta}(v, \bar{u}) + d^{\theta}(v, \bar{u})[ad^{\omega}(v, \bar{u}) + bd^{\xi}(\mu, \bar{\mu})],$$

and further,

$$(m - c - a)d^{\beta-\theta}(v, \bar{u}) \leq bd^{\xi}(\mu, \bar{\mu})$$

which implies

$$d(v, \bar{u}) \leq \left(\frac{b}{m - c - a}\right)^{\frac{1}{\beta-\theta}} d^{\frac{\xi}{\beta-\theta}}(\mu, \bar{\mu}).$$

- 3) In the case $\beta < \omega + \theta$ and $a > 0$, let r be such that $0 < r < (\frac{m-c}{a})^{\frac{1}{\omega+\theta-\beta}}$, let the neighborhood of \bar{u} be $V(\bar{u}) = B(\bar{u}, r) \cap \tilde{V}(\bar{u})$, let $\mu \in U(\bar{\mu})$ and $v \in S(\mu) \cap V(\bar{u})$ with $v \neq \bar{u}$. In the same way as before, we get

$$(m - c)d^{\beta-\theta}(v, \bar{u}) \leq ad^{\omega}(v, \bar{u}) + bd^{\xi}(\mu, \bar{\mu})$$

and further on, since $m - c > 0$,

$$d^{\beta-\theta}(v, \bar{u}) \leq \frac{a}{m - c}d^{\omega}(v, \bar{u}) + \frac{b}{m - c}d^{\xi}(\mu, \bar{\mu}).$$

We apply Lemma 1 with $x := d(v, \bar{u})$, $p := \beta - \theta$, $q := \omega$, $y := \frac{b}{m-c}d^{\xi}(\mu, \bar{\mu})$, $l := \frac{a}{m-c}$, $\varepsilon := r$. Since $v \in B(\bar{u}, r) \cap \tilde{V}(\bar{u})$ and $v \neq \bar{u}$, the conditions of the lemma are verified and we get

$$d(v, \bar{u}) \leq kd^{\delta}(\mu, \bar{\mu}), \tag{3}$$

where $\delta = \frac{\xi}{\beta-\theta}$ and $k = r \left(r^{\beta-\theta} - \frac{a}{m-c}r^{\omega}\right)^{\frac{1}{\theta-\beta}} \left(\frac{b}{m-c}\right)^{\frac{1}{\beta-\theta}}$. If $v = \bar{u}$, (3) is verified.

So, in all cases there exist k and δ such that, for every $v \in S(\mu) \cap V(\bar{u})$,

$$d(v, \bar{u}) \leq kd^{\delta}(\mu, \bar{\mu}).$$

This implies

$$e(S(\mu) \cap V(\bar{u}), S(\bar{\mu})) = \sup_{v \in S(\mu) \cap V(\bar{u})} d(v, S(\bar{\mu})) \leq \sup_{v \in S(\mu) \cap V(\bar{u})} d(v, \bar{u}) \leq kd^{\delta}(\mu, \bar{\mu}).$$

If $u \in S(\bar{\mu}) \cap V(\bar{u})$, then we get $u = \bar{u}$, so \bar{u} is the unique solution in the neighborhood $V(\bar{u})$. □

Remark 2 (a) If the function $F(\cdot, \cdot, \cdot; \bar{\mu})$ is Hölder strongly monotone, then condition (i) from Theorem 1 is verified. This follows directly from the fact that $F(u, v, v) - F(u, v, u) \leq F(u, v, v)_+ + F(u, v, u)_-$. The converse is not true (see [2] for the case of bifunctions).

- (b) In Section 5 we will see how properties (i) and (ii) appear for hemivariational inequalities governed by the Navier-Stokes operator.
- (c) For bifunctions in the cases 1) and 2), Theorem 1 is well-known (see [1, 2]). In this particular case, the set K was considered to depend also on a parameter λ . Theorem 1 can be extended in this sense, but it is not our aim in this paper.

4 Parametric Mixed Equilibrium Problems

Mixed equilibrium problems have an important role in applied mathematics. They were first studied in the paper [6].

Consider the function F having the particular form

$$F(u, v, w; \mu) = f(w, v; \mu) - f(w, u; \mu) + g(u, v; \mu)$$

where $f : K \times K \times M \rightarrow \mathbb{R}$ is such that the bifunction $f(\cdot, \cdot; \mu)$ is monotone, $f(u, u; \mu) = 0$, for all $u \in K, \mu \in M$ and $g : K \times K \times M \rightarrow \mathbb{R}$ is an arbitrary function. In this case we have

$$F(u, v, u; \mu) = f(u, v; \mu) + g(u, v; \mu) \text{ and } F(u, v, v; \mu) = -f(v, u; \mu) + g(u, v; \mu).$$

The problem $(PE)(\mu)$ becomes the mixed parametric equilibrium problem defined by f and g :

$$(PME)(\mu) \text{ Find } \bar{u} \in K \text{ such that } f(\bar{u}, z; \mu) + g(\bar{u}, z; \mu) \geq 0, \text{ for every } z \in K.$$

We denote by $S(\mu)$ the set of solutions of the problem $(PME)(\mu)$ and suppose that it is nonempty. The next result follows directly from Theorem 1.

Theorem 2 *Let $\bar{\mu} \in M$ be nonisolated and $\bar{u} \in S(\bar{\mu})$ be fixed. Suppose that there exist some neighborhoods $U(\bar{\mu})$ of $\bar{\mu}$, $\tilde{V}(\bar{u})$ of \bar{u} , and the numbers $a, b_1, b_2, c, \theta \geq 0, m, \beta, \xi, \omega > 0$ such that*

- (i) $md^\beta(\bar{u}, v) \leq [f(\bar{u}, v; \bar{\mu}) + g(\bar{u}, v; \bar{\mu})]_- + [f(v, \bar{u}; \bar{\mu}) - g(\bar{u}, v; \bar{\mu})]_-$, for every $v \in S(\mu) \cap \tilde{V}(\bar{u})$ and $\mu \in U(\bar{\mu})$;
- (ii) $f(v, \bar{u}; \mu) - f(v, \bar{u}; \bar{\mu}) \leq b_1 d^\theta(\bar{u}, v) d^\xi(\mu, \bar{\mu})$, for every $\mu \in U(\bar{\mu}), v \in S(\mu) \cap \tilde{V}(\bar{u})$, with $v \neq \bar{u}$;
- (iii) $g(\bar{u}, v; \bar{\mu}) + g(v, \bar{u}; \mu) \leq cd^\beta(\bar{u}, v) + d^\theta(\bar{u}, v)[ad^\omega(\bar{u}, v) + b_2 d^\xi(\mu, \bar{\mu})]$, for every $\mu \in U(\bar{\mu})$ and $v \in S(\mu) \cap \tilde{V}(\bar{u})$, with $v \neq \bar{u}$;
- (iv) $0 < \beta - \theta$ and $c < m$.

Suppose that one of the conditions 1), 2), 3) from Theorem 1 is verified.

Then the mapping $S : M \rightarrow 2^X$ is Hölder calm at $(\bar{\mu}, \bar{u})$. Moreover, we have the isolated Hölder calmness property at $(\bar{\mu}, \bar{u})$. In the cases 1), 2) the solution \bar{u} is unique in the neighborhood $\tilde{V}(\bar{u})$. In the case 3) \bar{u} is unique too in a neighborhood of \bar{u} .

Proof We only have to check condition (ii) from Theorem 1. Let $\mu \in U(\bar{\mu})$ and $v \in S(\mu) \cap \tilde{V}(\bar{u})$. Then,

$$f(v, \bar{u}; \mu) + g(v, \bar{u}; \mu) \geq 0.$$

We have, for $b = b_1 + b_2$,

$$\begin{aligned} F(\bar{u}, v, v; \bar{\mu}) &= -f(v, \bar{u}; \bar{\mu}) + g(\bar{u}, v; \bar{\mu}) \\ &\leq -f(v, \bar{u}; \bar{\mu}) + g(\bar{u}, v; \bar{\mu}) + f(v, \bar{u}; \mu) + g(v, \bar{u}; \mu) \\ &\leq cd^\beta(\bar{u}, v) + d^\theta(\bar{u}, v)[ad^\omega(\bar{u}, v) + bd^\xi(\mu, \bar{\mu})]. \end{aligned}$$

Therefore Theorem 1 can be applied. □

For stability results in the case of parametric mixed problems we mention also [27].

5 Navier-Stokes Problems Modeled by Hemivariational Inequalities

In the papers [30] and [31], Migórski and Ochal studied a class of hemivariational problems for the Navier-Stokes operators, in the stationary and evolution case, respectively. When Ω is a bounded simply connected domain of \mathbb{R}^d , $d \in \{2, 3, 4\}$, with boundary Γ of class C^2 , the Navier-Stokes equations that describe the flow of a viscous incompressible constant density fluid in the domain Ω are the following:

$$u' - \alpha \Delta u + (u \cdot \nabla)u + \nabla p = \phi, \tag{4}$$

$$\nabla \cdot u = 0 \text{ on } Q = \Omega \times]t_0, t_1[. \tag{5}$$

Here $u : \Omega \times [t_0, t_1] \rightarrow \mathbb{R}^d$ is the velocity, $\alpha > 0$ is the kinematic viscosity of the fluid, $p : \Omega \times [t_0, t_1] \rightarrow \mathbb{R}$ is the pressure, $\phi : Q \rightarrow \mathbb{R}^d$ is a vector field given by the external forces.

To obtain a variational formulation of the previous equations, it is convenient to rewrite the problem in the equivalent Leray form (see [32]).

For this let us consider the set

$$W = \{w \in C^\infty(\Omega, \mathbb{R}^d) : \text{div } w = 0 \text{ on } \Omega\}.$$

Denote by V and H the closure of W in the norms of $W_2^1(\Omega, \mathbb{R}^d)$ (the usual Sobolev space) and $L^2(\Omega, \mathbb{R}^d)$, respectively. We have $V \subset H \simeq H^* \subset V^*$ with the embeddings being dense, continuous, and compact.

Consider the spaces

$$\mathcal{V} = L^2(t_0, t_1; V), \mathcal{H} = L^2(t_0, t_1; H) \text{ and } \mathcal{W} = \{w \in \mathcal{V} : w' \in \mathcal{V}^*\},$$

where the time derivative w' is understood in the sense of vector valued distributions. In this case $\mathcal{W} \subset \mathcal{V} \subset \mathcal{H} \subset \mathcal{V}^*$ and the embeddings are continuous.

The pairing between V and V^* will be denoted by $\langle \cdot, \cdot \rangle$, and the pairing between \mathcal{V} and \mathcal{V}^* will be denoted by $\langle\langle \cdot, \cdot \rangle\rangle$. The space \mathcal{W} is a separable reflexive Banach space with the norm $\|w\|_{\mathcal{W}} = \|w\|_{\mathcal{V}} + \|w'\|_{\mathcal{V}^*}$ and is continuously embedded in $C([t_0, t_1]; H)$ (see [35]). The norm on \mathcal{V} will be denoted by $\|\cdot\|$, for other norms the space will be mentioned if necessary.

To write the weak formulation of the problem (4)-(5), consider the operators $A : V \rightarrow V^*$ and $B : V \times V \rightarrow V^*$ defined by

$$\langle Au, v \rangle = \alpha \int_{\Omega} \sum_{i=1}^d D_i u \cdot D_i v \, dx, \tag{6}$$

$$\langle B(u, v), w \rangle = \int_{\Omega} \sum_{i,j=1}^d u_i (D_i v_j) w_j \, dx, \quad B[u] = B(u, u), \tag{7}$$

where D_i is the operator $\frac{\partial}{\partial x_i}$, and denote

$$\langle \Phi(t), v \rangle := \int_{\Omega} \phi(x, t) v(x) dx,$$

for $u, v, w \in V$. It is well known (see [32], p. 162) that the operator B is well defined only if $d \in \{2, 3, 4\}$. For problem (4)-(5) to be well posed it is necessary to assign some boundary conditions. Let us consider, for instance, in the case $d = 3$, the Neumann condition $u|_{\Gamma} = h$, where $h = pn - \alpha \frac{\partial \gamma u}{\partial n}$, n is the outward unit normal vector to $\partial\Omega$, $\frac{\partial}{\partial n}$ is the normal derivative operator, and $\gamma : W_2^1(\Omega, \mathbb{R}^d) \rightarrow L^2(\Gamma, \mathbb{R}^d)$ is the trace operator. If we multiply

the equation (4) by a test function $v \in V$, then using the Gauss formulae, we obtain the weak formulation of the Navier-Stokes equation with the Neumann boundary condition:

$$(u'(t) + Au(t) + B[u(t)], v) + \int_{\Gamma} h \cdot \gamma v d\sigma = \langle \Phi(t), v \rangle, \text{ a.e. } t \in]t_0, t_1[, v \in V,$$

where σ is the Hausdorff measure on Γ .

Similar to [30] and [31], the Neumann boundary condition can be generalized by the subdifferential condition

$$h(x, t) \in \partial j(x, t, \gamma u(x, t)) \text{ on } \Gamma \times [t_0, t_1],$$

where ∂j denotes the Clarke subdifferential of the locally Lipschitz function $j : \Gamma \times [t_0, t_1] \times \mathbb{R}^d \rightarrow \mathbb{R}$ with respect to the third variable. In this case the problem becomes the following evolution hemivariational inequality: For a given $\mathcal{K} \subseteq \mathcal{W}$, find $\bar{u} \in \mathcal{K}$ such that

$$\langle \bar{u}'(t) + A\bar{u}(t) + B[\bar{u}(t)] - \Phi(t), v - \bar{u}(t) \rangle + \int_{\Gamma} j^0(x, t, \gamma \bar{u}(x, t); \gamma v(x) - \gamma \bar{u}(x, t)) d\sigma \geq 0, \tag{8}$$

for all $v \in V$, a.e. $t \in [t_0, t_1]$, where j^0 is the Clarke directional derivative of $j(x, t, \cdot)$.

Using the notations (6) and (7), define the Navier-Stokes operator $N : V \rightarrow V^*$ by $Nu = Au + B[u]$, for $u \in V$. We have the following properties (see for instance [30] (p. 206), [32] (Chapter II)):

- I. $A : V \rightarrow V^*$ is linear, continuous, symmetric and $\langle Au, u \rangle \geq \alpha \|u\|_V^2$, for all $u \in V$,
- II. $B : V \times V \rightarrow V^*$ is bilinear, continuous and $\langle B(u, v), v \rangle = 0$, for all $u, v \in V$,
- III. The mapping $B[\cdot] : V \rightarrow V^*$ is weakly continuous.

From these follows that the function b , defined by $b(u, v, z) := \langle B(u, v), z \rangle$ is trilinear and continuous. From property II, we get

$$0 = \langle B(u, u + v), u + v \rangle = \langle B(u, u), v \rangle + \langle B(u, v), u \rangle,$$

so $\langle B(u, v), u \rangle = -\langle B(u, u), v \rangle$. On one hand we have

$$\langle B[u] - B[v], u - v \rangle = -\langle B(u, u), v \rangle - \langle B(v, v), u \rangle$$

and on the other hand

$$\langle B(u - v, v), u - v \rangle = \langle B(u, v), u \rangle - \langle B(v, v), u \rangle.$$

It follows that

$$\langle B[u] - B[v], u - v \rangle = \langle B(u - v, v), u - v \rangle \leq c_1 \cdot \|u - v\|_V^2 \cdot \|v\|_V,$$

where c_1 is a positive constant and $u, v \in V$. We can take

$$c_1 = \sup_{\|v\|, \|w\|=1} \langle B(w, v), w \rangle. \tag{9}$$

For $u, v, z \in \mathcal{V}$ we denote

$$\langle \langle \mathcal{A}u, v \rangle \rangle = \int_{t_0}^{t_1} \langle Au(t), v(t) \rangle dt, \quad \langle \langle \mathcal{B}(u, v), z \rangle \rangle = \int_{t_0}^{t_1} \langle B(u(t), v(t)), z(t) \rangle dt \text{ and}$$

$$\langle \langle \mathcal{N}u, v \rangle \rangle = \int_{t_0}^{t_1} \langle Nu(t), v(t) \rangle dt.$$

The generalized derivative u' defines a linear operator $\mathcal{L} : \mathcal{W} \rightarrow \mathcal{V}^*$ given by

$$\langle \langle \mathcal{L}u, v \rangle \rangle = \int_{t_0}^{t_1} \langle u'(t), v(t) \rangle dt, \text{ for all } v \in \mathcal{V}.$$

It is known that the linear operator \mathcal{L} is monotone on \mathcal{W} if $\langle \mathcal{L}u, u \rangle \geq 0$ for all $u \in \mathcal{W}$. According to

$$\langle \mathcal{L}u, u \rangle = \int_{t_0}^{t_1} \left(\frac{1}{2} \|u(t)\|_H^2 \right)' dt = \frac{1}{2} \left(\|u(t_1)\|_H^2 - \|u(t_0)\|_H^2 \right)$$

the monotonicity of \mathcal{L} on \mathcal{K} follows when, for any $u \in \mathcal{K}$, the inequality $\|u(t_0)\|_H \leq \|u(t_1)\|_H$ holds. This happens, for instance, in the periodic case $\mathcal{K} = \{u \in \mathcal{W} : u(t_0) = u(t_1)\}$, in the anti-periodic case $\mathcal{K} = \{u \in \mathcal{W} : u(t_0) = -u(t_1)\}$, and for $\mathcal{K} = \{u \in \mathcal{W} : u(t_0) = 0\}$. Existence theorems were given, for instance, in [3], for periodic solutions, and for antiperiodic solutions in [8] and [9].

Let M be a nonempty set of functions $\mu : \Gamma \times [t_0, t_1] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ with the following properties:

- (M1) The iterative integral $\int_{t_0}^{t_1} \int_{\Gamma} \mu(x, t, \gamma u(x, t); \gamma v(x, t)) d\sigma dt$ exists for all $u, v \in \mathcal{H}$;
- (M2) For every $\mu, \bar{\mu} \in M$ there exists a function $\varphi_{\mu, \bar{\mu}} \in L^2(\Gamma \times [t_0, t_1])$ for which

$$|\mu(x, t, r; s) - \bar{\mu}(x, t, r; s)| \leq \varphi_{\mu, \bar{\mu}}(x, t) |s|$$
 for all $r, s \in \mathbb{R}^d$, a.e. $(x, t) \in \Gamma \times [t_0, t_1]$;
- (M3) The function $s \mapsto \mu(x, t, r; s)$ is positively homogeneous for all $r \in \mathbb{R}^d$, a.e. $(x, t) \in \Gamma \times [t_0, t_1]$.

For $\mu, \bar{\mu} \in M$ we define the distance

$$d(\mu, \bar{\mu}) = \left(\int_{t_0}^{t_1} \int_{\Gamma} \sup_{r, s \in \mathbb{R}^d, |s|=1} |\mu(x, t, r; s) - \bar{\mu}(x, t, r; s)|^2 d\sigma dt \right)^{\frac{1}{2}}. \tag{10}$$

From (M2) it follows that $d(\mu, \bar{\mu}) < +\infty$ for all $\mu, \bar{\mu} \in M$.

For $\mu \in M$ and $u, v \in \mathcal{V}$ denote

$$\mathcal{G}_{\mu}(u; v) = \int_{t_0}^{t_1} \int_{\Gamma} \mu(x, t, \gamma u(x, t); \gamma v(x, t)) d\sigma dt.$$

Instead of (8) we consider a more general problem, for a given $\mathcal{K} \subseteq \mathcal{W}$:

(NS)(μ) Find $u \in \mathcal{K}$ such that, for all $v \in \mathcal{K}$,

$$\langle \mathcal{L}u + \mathcal{N}u - \Phi, v - u \rangle + \mathcal{G}_{\mu}(u; v - u) \geq 0.$$

We call such a problem hemivariational-like inequality with boundary control variable μ . Denote by $S(\mu)$ the set of solutions of this problem and suppose it is nonempty for all $\mu \in M$. For $\bar{\mu} \in M, \bar{u} \in S(\bar{\mu})$ and $\tau > 0$, denote

$$c(\tau) := \limsup_{v \rightarrow \bar{u}, v \in \mathcal{K}} \|v - \bar{u}\|^{-\tau} \left(\mathcal{G}_{\bar{\mu}}(\bar{u}; v - \bar{u}) + \mathcal{G}_{\bar{\mu}}(v; \bar{u} - v) \right). \tag{11}$$

Using Theorem 2 we are able to prove the following:

Theorem 3 *Let $\mathcal{K} \subseteq \mathcal{W}$ and let the operator \mathcal{L} be monotone on \mathcal{K} . Let $\bar{\mu} \in M$ be nonisolated and $\bar{u} \in S(\bar{\mu})$. Suppose that the conditions (M1) - (M3) are verified and the norms $\|\varphi_{\mu, \bar{\mu}}\|_{L^2}$ are bounded for μ near $\bar{\mu}$. Suppose further that there exists $\rho > 0$ such that $\|\bar{u}\| < \rho$ and $\rho c_1 < \alpha$, where α is the kinematic viscosity of the fluid, and c_1 is defined by (9).*

Suppose that one of the following conditions is verified:

- 1') $\mathcal{G}_{\bar{\mu}}(\bar{u}, v - \bar{u}) + \mathcal{G}_{\bar{\mu}}(v, \bar{u} - v) \leq 0$ for v near \bar{u} ;
- 2') $c(2) < \alpha - \rho c_1$;
- 3') There exists $\tau > 2$ such that $c(\tau) < \infty$.

Then the mapping $\mu \mapsto S(\mu)$ is Hölder calm at $(\bar{\mu}, \bar{u})$. Moreover, the solution set S has the isolated calmness property at $(\bar{\mu}, \bar{u})$, and the solution \bar{u} is unique in a neighborhood of \bar{u} .

Proof Let the functions $f : \mathcal{K} \times \mathcal{K} \rightarrow \mathbb{R}$ and $g : \mathcal{K} \times \mathcal{K} \times M \rightarrow \mathbb{R}$ be defined by

$$f(u, v) := \langle \mathcal{A}u, v - u \rangle + \langle \mathcal{L}u, v - u \rangle$$

and

$$g(u, v; \mu) = \langle \mathcal{B}[u], v - u \rangle + \mathcal{G}_{\mu}(u; v - u) - \langle \Phi, v - u \rangle.$$

With these notations problem (NS)(μ) is of the form (PME)(μ) studied in Section 4.

The function f is strongly monotone, so the condition (i) of Theorem 2 is verified with $\beta = 2$ and $m = \alpha$. Indeed, for any $v \in \mathcal{K}$ we have

$$\begin{aligned} \alpha \|\bar{u} - v\|^2 &\leq -f(\bar{u}, v) - f(v, \bar{u}) = -f(\bar{u}, v) - g(\bar{u}, v; \bar{\mu}) - f(v, \bar{u}) + g(\bar{u}, v; \bar{\mu}) \\ &\leq [f(\bar{u}, v) + g(\bar{u}, v; \bar{\mu})]_- + [f(v, \bar{u}) - g(\bar{u}, v; \bar{\mu})]_- . \end{aligned}$$

Condition (ii) of Theorem 2 is trivially verified.

For $\mu \in M$ and $v \in \mathcal{K}$ we have

$$g(\bar{u}, v; \bar{\mu}) + g(v, \bar{u}; \mu) = \langle \mathcal{B}[\bar{u}] - \mathcal{B}[v], v - \bar{u} \rangle + \mathcal{G}_{\bar{\mu}}(\bar{u}; v - \bar{u}) + \mathcal{G}_{\mu}(v; \bar{u} - v).$$

As it was mentioned before,

$$\langle \mathcal{B}[\bar{u}] - \mathcal{B}[v], v - \bar{u} \rangle = \langle \mathcal{B}(\bar{u} - v), v - \bar{u} \rangle \leq c_1 \cdot \|\bar{u} - v\|^2 \cdot \|v\|.$$

Further, by $\|\bar{u}\| < \rho$ there exists $\delta > 0$ such that, for $\|v - \bar{u}\| < \delta$, we have $\|v\| < \rho$.

On the other hand, we have

$$\begin{aligned} |\mathcal{G}_{\mu}(v, \bar{u} - v) - \mathcal{G}_{\bar{\mu}}(v, \bar{u} - v)| &\leq \int_{t_0}^{t_1} \int_{\Gamma} |\mu(x, t, \gamma v(x, t); \gamma \bar{u}(x, t) - \gamma v(x, t)) \\ &\quad - \bar{\mu}(x, t, \gamma v(x, t); \gamma \bar{u}(x, t) - \gamma v(x, t))| d\sigma dt. \end{aligned}$$

On the subset of $\Gamma \times [t_0, t_1]$ where $\gamma \bar{u}(x, t) \neq \gamma v(x, t)$ holds, by (M3) it follows that

$$\begin{aligned} &|\mu(x, t, \gamma v(x, t); \gamma \bar{u}(x, t) - \gamma v(x, t)) - \bar{\mu}(x, t, \gamma v(x, t); \gamma \bar{u}(x, t) - \gamma v(x, t))| \\ &= |\gamma \bar{u}(x, t) - \gamma v(x, t)| \left| \mu \left(x, t, \gamma v(x, t); \frac{\gamma \bar{u}(x, t) - \gamma v(x, t)}{|\gamma \bar{u}(x, t) - \gamma v(x, t)|} \right) \right. \\ &\quad \left. - \bar{\mu} \left(x, t, \gamma v(x, t); \frac{\gamma \bar{u}(x, t) - \gamma v(x, t)}{|\gamma \bar{u}(x, t) - \gamma v(x, t)|} \right) \right|. \end{aligned}$$

On the other hand, on the subset of $\Gamma \times [t_0, t_1]$ where $\gamma \bar{u}(x, t) = \gamma v(x, t)$, property (M2) implies that

$$|\mu(x, t, \gamma v(x, t); \gamma \bar{u}(x, t) - \gamma v(x, t)) - \bar{\mu}(x, t, \gamma v(x, t); \gamma \bar{u}(x, t) - \gamma v(x, t))| = 0.$$

Using the above considerations, the Cauchy-Schwarz inequality and (10), it follows that there exists $c_0 > 0$ such that

$$|\mathcal{G}_{\mu}(v, \bar{u} - v) - \mathcal{G}_{\bar{\mu}}(v, \bar{u} - v)| \leq c_0 \|v - \bar{u}\| d(\mu, \bar{\mu}).$$

Then, for $v \in \mathcal{K}$ near \bar{u} , we have

$$\begin{aligned} & \mathcal{G}_{\bar{\mu}}(\bar{u}; v - \bar{u}) + \mathcal{G}_{\mu}(v; \bar{u} - v) \\ &= \mathcal{G}_{\bar{\mu}}(\bar{u}; v - \bar{u}) + \mathcal{G}_{\bar{\mu}}(v, \bar{u} - v) - \mathcal{G}_{\bar{\mu}}(v, \bar{u} - v) + \mathcal{G}_{\mu}(v; \bar{u} - v) \\ &\leq c_0 \|v - \bar{u}\| \cdot d(\mu, \bar{\mu}) + \mathcal{G}_{\bar{\mu}}(\bar{u}; v - \bar{u}) + \mathcal{G}_{\bar{\mu}}(v, \bar{u} - v). \end{aligned}$$

In case 1'), we get $\mathcal{G}_{\bar{\mu}}(\bar{u}; v - \bar{u}) + \mathcal{G}_{\mu}(v; \bar{u} - v) \leq c_0 \|v - \bar{u}\| \cdot d(\mu, \bar{\mu})$.

In case 2'), there exists $a_0 > 0$ such that $c(2) < a_0 < \alpha - \rho c_1$ and

$$\mathcal{G}_{\bar{\mu}}(\bar{u}; v - \bar{u}) + \mathcal{G}_{\bar{\mu}}(v, \bar{u} - v) \leq a_0 \|v - \bar{u}\|^2,$$

for any v near \bar{u} .

In case 3'), there exists $a_1 > 0$ such that

$$\mathcal{G}_{\bar{\mu}}(\bar{u}; v - \bar{u}) + \mathcal{G}_{\bar{\mu}}(v, \bar{u} - v) \leq a_1 \|v - \bar{u}\|^{\tau},$$

for any v near \bar{u} .

Taking account of the previous inequalities, for $\|v - \bar{u}\| < \delta$ it follows that

$$\begin{aligned} & g(\bar{u}, v; \bar{\mu}) + g(v, \bar{u}; \mu) \leq \rho c_1 \|v - \bar{u}\|^2 + a \|v - \bar{u}\|^{\zeta} + c_0 \|v - \bar{u}\| \cdot d(\mu, \bar{\mu}) \\ &= \rho c_1 \|v - \bar{u}\|^2 + \|v - \bar{u}\| (a \|v - \bar{u}\|^{\zeta-1} + c_0 d(\mu, \bar{\mu})), \end{aligned}$$

where $a = 0$ in case 1'), $a = a_0$ and $\zeta = 2$ in case 2'), $a = a_1$ and $\zeta = \tau > 2$ in case 3').

In this way, if $\|\bar{u}\| < \rho$ and $\|v - \bar{u}\| < \delta$, conditions of Theorem 2 are fulfilled with $m = \alpha, b = c_0, c = \rho c_1, \theta = 1, \omega = \zeta - 1, \beta = 2$, and $\xi = 1$. □

Remark 3 (a) Hypothesis $\rho c_1 < \alpha$ suggests that, if the viscosity coefficient α is small, then the neighbourhood $B(0, \rho)$ of 0, where the calmness property holds, is small too. If α is very small, problems may arise concerning stability and the transition towards turbulent flows (see [32]). When fluctuations of flow velocity occur at very small spatial and temporal scales, one goes towards the so called turbulent models (see, for instance [4, 15, 17]).

(b) Condition $c(\tau) < +\infty$ is verified if the bifunction $\mathcal{G}_{\bar{\mu}}$ is monotone. This is why the condition $c(\tau) < +\infty$ is sometimes named relaxed monotonicity (see for instance [30]).

(c) If the function $s \mapsto \mu(x, t, r, s)$ is linear, with $\mu(x, t, r, s) = \mu_0(x, t, r)(s)$ for all $r, s \in \mathbb{R}^d$, a.e. $(x, t) \in \Gamma \times [t_0, t_1]$, then condition (M2) can be substituted with the following:

(M2') For every $\mu, \bar{\mu} \in M$, there exists a function $\varphi_{\mu, \bar{\mu}} \in \mathcal{V}$ such that

$$|\mu_0(x, t, r) - \bar{\mu}_0(x, t, r)| \leq \varphi_{\mu, \bar{\mu}}(x, t),$$

for all $r \in \mathbb{R}^d$, a.e. $(x, t) \in \Gamma \times [t_0, t_1]$. In this case we have

$$d(\mu, \bar{\mu}) = \left(\int_{t_0}^{t_1} \int_{\Gamma} \sup_{r \in \mathbb{R}^d} |\mu_0(x, t, r) - \bar{\mu}_0(x, t, r)|^2 d\sigma dt \right)^{\frac{1}{2}},$$

where $\bar{\mu}_0$ is defined similarly to μ_0 .

To apply the previous theorem in the theory of hemivariational inequalities, let us consider a function $j : \Gamma \times [t_0, t_1] \times \mathbb{R}^d \rightarrow \mathbb{R}$ for which $j(\cdot, \cdot, r) \in L^1(\Gamma \times [t_0, t_1])$ for some $r \in \mathbb{R}^d$. As in [14], Section 2.7, consider two hypotheses on the function j for $p \in [1, 2]$, and $q = \frac{p}{p-1}$ ($q = \infty$ if $p = 1$):

Hypothesis I: There exists $k \in L^q(\Gamma \times [t_0, t_1])$ such that for all $(x, t) \in \Gamma \times [t_0, t_1]$

$$|j(x, t, r) - j(x, t, r')| \leq k(x, t)|r - r'|, \text{ for all } r, r' \in \mathbb{R}^d.$$

Hypothesis II: The function $r \mapsto j(x, t, r)$ is locally Lipschitz and for some constant $c_0 > 0$ one has $|z| \leq c_0(1 + |r|^{p-1})$, for all $(x, t) \in \Gamma \times [t_0, t_1], r \in \mathbb{R}^d, z \in \partial_r j(x, t, r)$. Here $\partial_r j$ is the Clarke generalized gradient of j with respect to the third variable $r \in \mathbb{R}^d$. These two hypotheses appear, for instance, in [14] and [18]. Hypothesis II was used to prove existence results in [3, 30], and [31].

Let J be a set of functions $j : \Gamma \times [t_0, t_1] \times \mathbb{R}^d \rightarrow \mathbb{R}$ such that at least one of the hypotheses I or II is verified. Then, for all Lebesgue integrable functions $u, v : \Omega \times [t_0, t_1] \rightarrow \mathbb{R}$, the function

$$(x, t) \mapsto j_r^0(x, t, \gamma u(x, t); \gamma v(x, t))$$

is defined and it is Lebesgue integrable. Indeed, by the previous hypotheses, the function $j(\cdot, \cdot, r)$ is Lebesgue measurable, for all $r \in \mathbb{R}^d$, and $j(x, t, \cdot)$ is continuous, for all $(x, t) \in \Omega \times [t_0, t_1]$. We may express $j_r^0(x, t, \gamma u(x, t); \gamma v(x, t))$ as the limit of

$$\frac{j(x, t, y + \lambda \gamma v(x, t)) - j(x, t, y)}{\lambda}$$

where $\lambda \searrow 0$, taking rational values and $y \rightarrow \gamma u(x, t)$, taking values in a countable dense subset $\{y_i : i \in \mathbb{N}\}$ of \mathbb{R}^d . Thus the function $(x, t) \mapsto j_r^0(x, t, \gamma u(x, t); \gamma v(x, t))$ is measurable as a countable upper limit of measurable functions (see [18], Theorem 1.2.20 and [14], pag. 78, Lemma).

Corollary 1 *Suppose that the operator \mathcal{L} is monotone on \mathcal{K} , the functions $\mu \in M$ are defined by $\mu = j_r^0, j \in J$ and $c(\tau)$ is defined by (11). Suppose that the condition (M2) is verified and the norms $\|\varphi_{\mu, \bar{\mu}}\|_{L^2}$ are bounded for μ near $\bar{\mu}$. Let $\bar{u} \in S(\bar{\mu})$. Let $\rho > 0$ be such that $\|\bar{u}\| < \rho$, and $\rho c_1 < \alpha$, where c_1 is defined by (9). Suppose further that one of the conditions 1') - 3') from Theorem 3 is verified. Then the mapping $\mu \mapsto S(\mu)$ is Hölder calm at $(\bar{\mu}, \bar{u})$. Moreover, the set-valued solution function S has the isolated calmness property and the solution \bar{u} is unique in a neighborhood of \bar{u} .*

Proof From Proposition 1 it follows that (M1)-(M3) are verified, so Theorem 3 can be applied. □

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