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Limits of Eventual Families of Sets with Application to Algorithms for the Common Fixed Point Problem

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Abstract

We present an abstract framework for asymptotic analysis of convergence based on the notions of eventual families of sets that we define. A family of subsets of a given set is called here an "eventual family" if it is upper hereditary with respect to inclusion. We define accumulation points of eventual families in a Hausdorff topological space and define the "image family" of an eventual family. Focusing on eventual families in the set of the integers enables us to talk about sequences of points. We expand our work to the notion of a "multiset" which is a modification of the concept of a set that allows for multiple instances of its elements and enable the development of "multifamilies" which are either "increasing" or "decreasing". The abstract structure created here is motivated by, and feeds back to, our look at the convergence analysis of an iterative process for asymptotically finding a common fixed point of a family of operators.

Keywords Common fixed-points · Hausdorff topological space · Eventual families · Multiset · Multifamily · Set convergence · Cutters · Firmly nonexpansive operators

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1 Introduction

In this paper we present an abstract framework for asymptotic analysis of convergence based on the notions of eventual families of sets that we define. A family $\mathcal{F}$ of subsets of a set X is called here an "eventual family" if $S \in \mathcal{F}$ and $S' \supseteq S$ implies $S' \in \mathcal{F}$, i.e., if it is upper hereditary with respect to inclusion. If $S \in \mathcal{F}$ and $S' \subseteq S$ implies $S' \in \mathcal{F}$, i.e., if it is

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lower hereditary with respect to inclusion, then we call it a "co-eventual family"**.** We define accumulation points of eventual families in a Hausdorff topological space and define the "image family" $\mathcal{G}$ of an eventual family $\mathcal{F}$ under a given mapping f, called "the push of $\mathcal{F}$ by f" via $\mathcal{G}$ =Push$(f, \mathcal{F}) := \{S \subseteq Y \mid f^{-1}(S) \in \mathcal{F}\}$.

Focusing on eventual families in the set $\mathbb{N}$ of the integers enables us to talk about sequences of points, particularly, points that are generated by repeated application of an operator $T : X \to X$. We then define the notion of an "$\mathcal{E}$-limit of a sequence $(A_n)_{n\in\mathbb{N}}$ of subsets of a set X" as the set of all $x \in X$ such that the set of n with $x \in A_n$ belongs to $\mathcal{E}$, i.e., $\mathcal{E}\text{-}\lim_{n\to\infty} A_n := \{x \in X \mid \{n \mid x \in A_n\} \in \mathcal{E}\}$ where $\mathcal{E}$ is an eventual family in $\mathbb{N}$. The relationship of this notion with the classical notion of limit of a sequence of sets is studied.

In the sequel we expand our work to the notion of a "multiset" which is a modification of the concept of a set that allows for multiple instances of its elements. The number of instances given for each element is called the multiplicity of that element in the multiset. With multisets in hand we define and develop "multifamilies" which are either "increasing" or "decreasing", connecting with the earlier notions via the statement that a family of subsets of X is an eventual (resp. co-eventual) family if the multifamily that defines it is increasing (resp. decreasing).

The abstract structure created here is motivated by, and feeds back to, our look at the convergence analysis of an iterative process for asymptotically finding a common fixed point of a family of operators. This particular case serves as an example of the possible use of our theory. The work presented here adds a new angle to the theory of set convergence, see, e.g., the books by Rockafellar and R.J.-B. Wets [15, Chapter 4] and by Burachik and Iusem [6].

2 Eventual Families and Their Use in Limiting Processes

2.1 Eventual Families

We introduce the following notion of eventual families of subsets.

Definition 1 Let X be a set and let $\mathcal{F}$ be a family of subsets of X. The family $\mathcal{F}$ is called an "**eventual family"** if it is *upper hereditary with respect to inclusion*, i.e., if

$$S \in \mathcal{F},\ S' \supseteq S \Rightarrow S' \in \mathcal{F}. \tag{1}$$

The family $\mathcal{F}$ is called a "**co-eventual family"** if it is *lower hereditary with respect to inclusion*, i.e., if

$$S \in \mathcal{F},\ S' \subseteq S \Rightarrow S' \in \mathcal{F}. \tag{2}$$

We mention in passing that Borg [4] uses the term "hereditary family", in his work in the area of combinatorics, for exactly what we call here "co-eventual family". Several simple observations regarding such families can be made.

Proposition 2 (i) *A family $\mathcal{F}$ of subsets of X is co-eventual iff its complement, i.e., the family of subsets of X which are not in $\mathcal{F}$, is eventual.*

(ii) *The empty family and the family of all subsets of X are each both eventual and co-eventual, and they are the only families with this property.*

Proof (i) This follows from the definitions. (ii) That the empty family and the family of all subsets of X are each both eventual and co-eventual is trivially true. We show that if $\mathcal{F}$ is eventual and co-eventual and is nonempty then it must contain all subsets of X. Let $S \in \mathcal{F}$ and distinguish between two cases. If $S = \emptyset$ then $\mathcal{F}$ must contain all subsets of X because $\mathcal{F}$ is eventual. If $S \neq \emptyset$ let $x \in S$, then, since $\mathcal{F}$ is co-eventual it must contain the singleton $\{x\}$. Consequently, the set $\{x, y\}$, for any y, is also in $\mathcal{F}$ and so $\{y\} \in \mathcal{F}$, thus, all subsets of X are contained in $\mathcal{F}$. Alternatively, if we look at $S \in \mathcal{F}$, then for any subset S' of X, $\mathcal{F}$ contains $S \cup S'$ since $\mathcal{F}$ is eventual. Then since $\mathcal{F}$ is co-eventual, it must contain S', leading to the conclusion that it contains all subsets. □

Remark 3 An eventual family $\mathcal{F}$ need not contain the intersection of two of its members. If it does so for every two of its members then it is a *filter*.

Similar to the notion used in [13] and [14] in the finite-dimensional space setting, we make here the next definition.

Definition 4 Given a family $\mathcal{F}$ of subsets of a set X, the "**star set associated with** $\mathcal{F}$", denoted by Star($\mathcal{F}$), is the subset of X that consists of *all* $x \in X$ *such that the singletons* $\{x\} \in \mathcal{F}$, namely,

$$\text{Star}(\mathcal{F}) := \{x \in X \mid \{x\} \in \mathcal{F}\}. \tag{3}$$

2.2 Accumulation Points as Limits of Eventual Families

Suppose now that X is a *Hausdorff* topological space.

Definition 5 Let $\mathcal{F}$ be an eventual family of subsets of X. A point $x \in X$ is called an "**accumulation (or limit) point** of $\mathcal{F}$" if every (open) neighborhood[1] of x belongs to $\mathcal{F}$. The set of all accumulation points of $\mathcal{F}$ is called the "**limit set** of $\mathcal{F}$".

Proposition 6 *The limit set of an eventual family* $\mathcal{F}$ *is always closed.*

Proof We show that the complement of the limit set, i.e., the set of all non-accumulation points, is open. The point y is a non-accumulation point iff it has an open neighborhood which does not belong to $\mathcal{F}$, i.e., when it is a member of some open set not in $\mathcal{F}$. Hence the complement of the limit set is the union of all open sets not in $\mathcal{F}$, and by definition, in a topological space, the union of any family of open sets is open. □

We turn our attention now to sequences in X, i.e., maps $\mathbb{N} \to X$, where $\mathbb{N}$ denotes the positive integers.

Definition 7 Given are a family $\mathcal{F}$ of subsets of X and a mapping between sets $f : X \to Y$. The family $\mathcal{G}$ of subsets of Y whose inverse image sets $f^{-1}(S)$ belong to $\mathcal{F}$ will be denoted by $\mathcal{G}$ =Push($f, \mathcal{F}$) and called the "**push** of $\mathcal{F}$ by f", namely,

$$\mathcal{G} = \text{Push}(f, \mathcal{F}) := \{S \subseteq Y \mid f^{-1}(S) \in \mathcal{F}\}. \tag{4}$$

[1] Since, by definition, a neighborhood always contains an *open* neighborhood, considering all neighborhoods or just the open ones does not make a difference here.

Combining Definitions 5 and 7 the following remark is obtained.

Remark 8 Let $\mathcal{E}$ be an eventual family of subsets *of* $\mathbb{N}$ and let $f : \mathbb{N} \to X$ be defined by some given sequence $(x_n)_{n\in\mathbb{N}}$ in X. The accumulation points and the limit set of $(x_n)_{n\in\mathbb{N}}$ with respect to $\mathcal{E}$ are those defined with respect to the push of $\mathcal{E}$ by f.

The next examples emerge by using two different eventual families in $\mathbb{N}$. The same 'machinery' yields both 'cases' via changing the eventual family $\mathcal{E}$ in $\mathbb{N}$.

Examples 9 (1) *Let $\mathcal{E}$ be the family of complements of finite sets in $\mathbb{N}$. Then accumulation points (i.e., limits with respect to $\mathcal{E}$) are the usual limits, and if there is a limit point then it is unique. This is the case, as one clearly sees, in a Hausdorff space X whenever $\mathcal{E}$ is a filter, as here $\mathcal{E}$ clearly is.*

(2) *Let $\mathcal{E}$ be the family of infinite subsets in $\mathbb{N}$. Then being an accumulation point means being some accumulation point of the sequence in the usual sense, which in general, need not be unique. Indeed, here $\mathcal{E}$ is not a filter.*

2.3 Operators and Seeking Fixed Points

Continuing to consider a Hausdorff topological space X, call any continuous self-mapping $T : X \to X$ "**an operator** ".

Definition 10 Let X be a Hausdorff topological space, $T : X \to X$ an operator, $(x_n)_{n\in\mathbb{N}}$ a sequence in X, and $\mathcal{E}$ an eventual family of subsets of $\mathbb{N}$. We say that "**the sequence** $(x_n)_{n\in\mathbb{N}}$ **follows** T **with respect to** $\mathcal{E}$**"** if, for every $S \in \mathcal{E}$, there are integers p, q in S so that $x_p = T(x_q)$.

Theorem 11 *In a Hausdorff topological space X, if a sequence $(x_n)_{n\in\mathbb{N}}$ follows a continuous operator T with respect to some eventual family $\mathcal{E}$ in $\mathbb{N}$, and if y is an accumulation point of the sequence with respect to $\mathcal{E}$ then y is a fixed point of T.*

Proof Assume to the contrary that $T(y) \neq y$. Then, since the space is Hausdorff, $T(y)$ and y have disjoint open neighborhoods U_y and $U_{T(y)}$. Continuity of T guarantees that there is an open neighborhood V_y of y so that $T(V_y) \subset U_{T(y)}$. Hence,

$$U_y \cap T(V_y) = \emptyset, \tag{5}$$

meaning that $T(z) \neq z$ for $z \in U_y \cap V_y$. But $U_y \cap V_y$ is also an open neighborhood of y, and y is an accumulation point of the sequence with respect to $\mathcal{E}$, hence, the set

$$S := \{n \in \mathbb{N} \mid x_n \in U_y \cap V_y\} \tag{6}$$

is in $\mathcal{E}$. Since the sequence follows T with respect to $\mathcal{E}$, there must be p and q in S so that $x_p = T(x_q)$. This point must belong to both U_y and $T(V_y)$, which contradicts (5). □

2.4 Finitely-Insensitive Eventual Families in $\mathbb{N}$

When considering eventual families in $\mathbb{N}$ it is often desirable to assume that they are *finitely-insensitive*, as we define next. All our examples have this property.

Definition 12 A family $\mathcal{E}$ of subsets of $\mathbb{N}$ is called a "**finitely-insensitive family**" if for any $S \in \mathcal{E}$, finitely changing S, which means here adding and/or deleting a finite number of its members, will result in a set $S' \in \mathcal{E}$.

2.5 Limits of Sequences of Sets

In [13, 14] and [16] the notions of *upper limit* and *lower limit* of a sequence of subsets $(A_n)_{n\in\mathbb{N}}$ of some X are considered, in the framework of a locally compact metric space, the Euclidean space, or a normed linear space of finite dimension, respectively. When these upper limit $\limsup_{n\to\infty} A_n$ and lower limit $\liminf_{n\to\infty} A_n$ coincide one says that the sequence of sets has their common value as a *limit*, denoted by $\lim_{n\to\infty} A_n$. Thus, a function defined on sets, or taking values in sets, may be said to be *continuous* when it respects limits of sequences.

Here we define the notion of an "$\mathcal{E}$-limit of a sequence $(A_n)_{n\in\mathbb{N}}$ of subsets of a set X" and state its relationship with the classical notion of limit mentioned above.

Definition 13 Let X be a set, let $(A_n)_{n\in\mathbb{N}}$ be a sequence of subsets of X, let $\mathcal{E}$ be an eventual family in $\mathbb{N}$ and assume that $\mathcal{E}$ is finitely-insensitive. The "$\mathcal{E}$-limit of the sequence $(A_n)_{n\in\mathbb{N}}$", denoted by $\mathcal{E}\text{-}\lim_{n\to\infty} A_n$, is the set of all $x \in X$ such that the set of n with $x \in A_n$ belongs to $\mathcal{E}$, namely,

$$\mathcal{E}\text{-}\lim_{n\to\infty} A_n := \{x \in X \mid \{n \mid x \in A_n\} \in \mathcal{E}\}. \tag{7}$$

Strict logic tells us that the $\mathcal{E}$-limit is well-defined also for an empty $\mathcal{E}$ or if $\mathcal{E}$ contains all subsets. Indeed, if $\mathcal{E} = \emptyset$ then $\mathcal{E}\text{-}\lim_{n\to\infty} A_n = \emptyset$, and if $\mathcal{E}$ is the family of all subsets then $\mathcal{E}\text{-}\lim_{n\to\infty} A_n = X$.

Theorem 14 *Let X be a set, let $(A_n)_{n\in\mathbb{N}}$ be a sequence of subsets of X, and let $\mathcal{E}$ be an eventual family in $\mathbb{N}$. If $\mathcal{E}$ is a finitely-insensitive family which is not trivial, i.e., is not either empty or containing all subsets, and if the (classical) $\lim_{n\to\infty} A_n$ exists then*

$$\mathcal{E}\text{-}\lim_{n\to\infty} A_n = \lim_{n\to\infty} A_n. \tag{8}$$

Proof Note that, for a given sequence of sets $(A_n)_{n\in\mathbb{N}}$, the 'larger' the eventual family $\mathcal{E}$ is, the 'larger' is its $\mathcal{E}$-limit.

Denote by $\mathcal{G}$ the family of all *infinite* subsets of $\mathbb{N}$ and by $\mathcal{H}$ the family of all subsets of $\mathbb{N}$ with *finite complement*.[2] Then clearly (cf. Examples 9) The upper limit (resp. lower limit) of A_n is obtained as $\mathcal{E}\text{-}\lim_{n\to\infty} A_n$ for $\mathcal{E} := \mathcal{G}$ (resp. $\mathcal{E} := \mathcal{H}$.)

Now, The family $\mathcal{G}$ is the largest finitely-insensitive family which is not the set of all subsets. This is so because if $\mathcal{G}$ would contain a finite set then it would have to contain the empty set, hence, all subsets.

And the family $\mathcal{H}$ is the smallest finitely-insensitive family which is not empty. This is so because if $\mathcal{H}$ is not empty, it has a member S, thus, must contain the whole $\mathbb{N}$, hence, all subsets with finite complement.

Consequently, for a sequence $(A_n)_{n\in\mathbb{N}}$ for which $\lim_{n\to\infty} A_n$ exists, that limit will be also the $\mathcal{E}$-limit for any finitely-insensitive eventual family $\mathcal{E}$ which is not trivial, i.e., is not either empty or containing all subsets. □

[2] The families $\mathcal{G}$ and $\mathcal{H}$ were denoted by $\mathcal{N}_\infty^{\#}$ and $\mathcal{N}_\infty$, respectively, in [15, page 108].

2.6 Topological vs. Purely Set-Theoretical

Note that in contrast to Sections 2.2 and 2.3, the notions in Section 2.5 are purely set-theoretic and do not involve any topology in X. Yet, one can distill the topological aspect via the next definition.

Definition 15 Let X be a Hausdorff topological space and let $\mathcal{F}$ be an eventual family in X. The "**closure of an eventual family $\mathcal{F}$ in X**", denoted by $\text{cl}\mathcal{F}$, consists of *all subsets $S \subseteq X$ such that all the open subsets $U \subseteq X$ which contain S belong to $\mathcal{F}$.*

Clearly, $\mathcal{F}$ is always a subfamily of $\text{cl}\mathcal{F}$, and the set of limit points of an eventual family $\mathcal{F}$, in a Hausdorff topological space X, is just $\text{Star}(\text{cl}\mathcal{F})$, given in Definition 4.

3 Multisets and Multifamilies

A **multiset** (sometimes termed **bag**, or **mset**) is a modification of the concept of a set that allows for multiple instances for each of its elements. The number of instances given for each element is called the multiplicity of that element in the multiset. The multiplicities of elements are any number in $\{0, 1, \ldots, \infty\}$, see the corner-stone review of Blizard [3].

Definition 16 (i) A **multiset** M in a set X is represented by a function $\varphi_M : X \to \{0, 1, \ldots, \infty\}$ such that for any $x \in X$, $\varphi_M(x)$ is the multiplicity of x in M. We refer to this function as the "**representing function of the multiset**". If $\varphi_M(x) = 0$ then the multiplicity 0 means 'not belonging to the set'. A subset $S \subseteq X$ is a multiset represented by ι_S, the "*indicator function*" of S, i.e.,

$$\iota_S(x) := \begin{cases} 1, & \text{if } x \in S, \\ 0, & \text{if } x \notin S. \end{cases} \tag{9}$$

(ii) A **multifamily** $\mathcal{M}$ on a set X is a multiset in the powerset 2^X of X (i.e., all the subsets of X). Its representing function, denoted by $\varphi_{\mathcal{M}} : 2^X \to \{0, 1, \ldots, \infty\}$, is such that for any $S \subseteq X$, $\varphi_M(S)$ is the multiplicity of S in $\mathcal{M}$. A family $\mathcal{F}$ of subsets of X is a multifamily on X represented by $\iota_{\mathcal{F}}$, the "*indicator function" of* $\mathcal{F}$, i.e.,

$$\iota_{\mathcal{F}}(f) := \begin{cases} 1, & \text{if } f \in \mathcal{F}, \\ 0, & \text{if } f \notin \mathcal{F}. \end{cases} \tag{10}$$

(iii) A multifamily $\mathcal{M}$ on a set X with a representing function $\varphi_{\mathcal{M}}$ is called **increasing** if

$$S, S' \subseteq X,\ S \subseteq S' \Rightarrow \varphi_{\mathcal{M}}(S) \le \varphi_{\mathcal{M}}(S'), \tag{11}$$

and called **decreasing** if

$$S, S' \subseteq X,\ S \subseteq S' \Rightarrow \varphi_{\mathcal{M}}(S) \ge \varphi_{\mathcal{M}}(S'). \tag{12}$$

Clearly, a *family* of subsets of X is an *eventual (resp. co-eventual) family if the multifamily that defines it is increasing (resp. decreasing)*. The next example shows why these notions may be useful.

Example 17 Considering the set $\mathbb{N}$, for a, finite or infinite, subset $S \subseteq \mathbb{N}$ write S as

$$S = \{n_1^S, n_2^S, \ldots\}, \tag{13}$$

where $n_\ell^S \in \mathbb{N}$ for all ℓ, and the sequence $(n_\ell^S)_{\ell=1}^L$ (where L is either finite or ∞) is strictly increasing, i.e., $n_1^S < n_2^S < \ldots$. We consider the **gaps** between consecutive elements in S as the sequence of differences

$$n_2^S - n_1^S - 1, n_3^S - n_2^S - 1, \ldots, \tag{14}$$

where, if S is finite add ∞ at the end. Defining

$$\mathrm{Gap}(S) := \limsup_k (n_{k+1}^S - n_k^S - 1), \tag{15}$$

makes Gap a multifamily on $\mathbb{N}$, thus taking values in $\{0, 1, \ldots, \infty\}$, in particular, taking the value ∞ for (among others) any finite S.

Note that if $\mathrm{Gap}(S)$ is finite then there must be an infinite number of differences $(n_{k+1}^S - n_k^S - 1)$ equal to $\mathrm{Gap}(S)$, but this is not true for any larger integer - because by the definition of lim sup and because we are dealing with integer-valued items, a finite lim sup must actually be attained an infinite number of times.

Observe further that the larger the set S is – the smaller (or equal) is $\mathrm{Gap}(S)$. Thus, Gap is a decreasing multifamily.

Define the complement-multifamily for some multifamily $\mathcal{G}$ on the subsets of a set X by

$$\mathcal{G}^c(S) := \mathcal{G}(S^c), \quad \forall S \subseteq X, \tag{16}$$

where S^c is the complement of S in X.

We will focus on coGap := Gap^c. For any $S \subseteq \mathbb{N}$, let us denote by c_S the maximal number of integers between consecutive elements of S, namely, between $n_\ell^S \in S$ and $n_{\ell+1}^S \in S$. If S has arbitrarily big such 'intervals' between consecutive elements then we write $c_S = \infty$. With this in mind, coGap $=$ Gap^c is an increasing multifamily equal to $(c_S)_{\forall S \subseteq \mathbb{N}}$.

3.1 Extensions to Multifamilies

We now extend some of the notions of Section 2.1 to multifamilies.

Definition 18 Given a multifamily $\mathcal{M}$ on the subsets of a set X whose representing function is $\varphi_\mathcal{M}$. The "**star set associated with** $\mathcal{M}$ ", denoted by Star($\mathcal{M}$), is the multiset M on X whose representing function φ_M is related to $\varphi_\mathcal{M}$ in the following manner

$$\mathrm{Star}(\mathcal{M}) := M, \text{ such that } \varphi_M(x) = \varphi_\mathcal{M}(\{x\}). \tag{17}$$

Definition 19 Given a multifamily $\mathcal{M}$ on the subsets of X whose representing function is $\varphi_\mathcal{M}$ and a mapping between sets $f : X \to Y$. The multifamily $\mathcal{G}$ on the subsets of Y, denoted by $\mathcal{G}$ =Push(f, $\mathcal{M}$), with representing function $\varphi_\mathcal{G}$, will be called the "**push of** $\mathcal{M}$ **by** f" if its representing function is related to the representing function of $\mathcal{M}$ in the following manner

$$\mathcal{G} = \mathrm{Push}(f, \mathcal{M}) \text{ such that } \varphi_\mathcal{G}(S) = \varphi_\mathcal{M}(f^{-1}(S)). \tag{18}$$

Definition 20 A multifamily $\mathcal{M}$ of subsets of $\mathbb{N}$ whose representing function is $\varphi_\mathcal{M}$ is called a "**finitely-insensitive multifamily"** if for any $S \in \mathcal{M}$, finitely changing S, i.e., adding and/or deleting a finite number of its members, will not change its multiplicity, i.e., will result in a set $S' \in \mathcal{M}$ such that $\varphi_\mathcal{M}(S) = \varphi_\mathcal{M}(S')$.

Definition 21 Let X be a Hausdorff topological space and let $\mathcal{M}$ be an *increasing* multifamily whose representing function is $\varphi_{\mathcal{M}}$. The "**closure of an increasing multifamily** M **in** X", denoted by cl$\mathcal{M}$, is defined to be the (increasing) multifamily such that for any $S \subseteq X$ it holds that

$$\varphi_{\text{cl}\mathcal{M}}(S) = \min\{\varphi_{\mathcal{M}}(U) \mid \text{ all } \textit{open subsets } U \subseteq X \text{ such that } S \subseteq U\}. \tag{19}$$

Definition 22 Let X be a Hausdorff topological space and let $\mathcal{M}$ be an *increasing* multifamily whose representing function is $\varphi_{\mathcal{M}}$. The multiset $M := \text{Star}(\text{cl}\mathcal{M})$ will be called the "**multiset-limit of** $\boldsymbol{\mathcal{M}}$" and denoted by $\lim \mathcal{M}$. Its representing function is for any $x \in X$,

$$\varphi_M(x) = \min\{\varphi_{\mathcal{M}}(U) \mid \text{all open subsets } U \subseteq X \text{ such that } x \in U\}. \tag{20}$$

Given a multifamily $\mathcal{M}$ on the subsets of $\mathbb{N}$ whose representing function is $\varphi_{\mathcal{M}}$, the 'limiting notions' with respect to $\mathcal{M}$ for a sequence $(x_n)_{n\in\mathbb{N}}$, are defined as those with respect to Push$(f, \mathcal{M})$ of $\mathcal{M}$ to X by the function $f : \mathbb{N} \to X$ which represents the sequence (x_n) . In particular, for an increasing multifamily $\mathcal{M}$ on the subsets of $\mathbb{N}$ whose representing function is $\varphi_{\mathcal{M}}$, the multiset limit of Push$(f, \mathcal{M})$ will be called the "**multiset-limit of** $\boldsymbol{(x_n)}$", denoted by $\lim_{\mathcal{M}} x_n$.

Denoting the representing function of this multiset $\mathcal{G}$ on X by $\varphi_{\mathcal{G}}$, we can describe it as follows. Given a point $x \in X$, consider the following subsets of $\mathbb{N}$

$$S(U) := \{n \in \mathbb{N} \mid x_n \in U\}, \text{ for open neighborhoods } U \text{ of } x. \tag{21}$$

Then,

$$\varphi_{\mathcal{G}}(x) = \min\{\varphi_{\mathcal{M}}(S(U)) \mid \text{all open subsets } U \subseteq X \text{ such that } x \in U\}. \tag{22}$$

Remark 23 Note, that for a set S not to belong to coGap, i.e., to have coGap$(S) = 0$, just means that S is finite - as a 'family, ignoring multiplicities' and coGap is just the family of *infinite* sets of natural numbers.

Thus, when we turn to the *limit* of a sequence $(x_n)_{n\in\mathbb{N}}$ in a Hausdorff space X (a notion which is obviously dependent on the topology. In a Banach or Hilbert space we will have strong and weak limits etc.); and we take the coGap-limit (it will be a multiset on X, to which for some x in X to belong (at least) n times, one must have, for every neighborhood U of x, that the x_n stay in U for some n consecutive places as far as we go); then the coGap-limit of $(x_n)_{n\in\mathbb{N}}$, 'forgetting the multiplicities' is just the set of accumulation points of $(x_n)_{n\in\mathbb{N}}$ (which is, recalling the examples in Section 2, just its $\mathcal{G}$-limit for $\mathcal{G}$ the eventual family of the infinite subsets of $\mathbb{N}$).

Note that, in general, if the sequence has a limit x^* (in the good old sense) then its coGap-limit 'includes x^* infinitely many times and does not include any other point'. This sort of indicates to what extent the coGap-limit may be viewed as 'more relaxed' than the usual limit.

The inverse implication does not always hold (it holds however in a compact space) as the following counterexample shows. In $\mathbb{R}$ (the reals), define a sequence by

$$x_{2n} := n \quad \text{and} \quad x_{2n-1} := -1 \tag{23}$$

then its coGap-limit contains -1 infinitely often and does not contain others, but -1 is not a limit.

4 Convergence of Algorithms for Solving the Common Fixed-Point Problem

Given a finite family of self-mapping operators $\{T_i\}_{i=1}^m$ acting on the Hilbert space H with $\mathrm{Fix}\, T_i \neq \emptyset$, $i = 1, 2, \ldots, m$, where $\mathrm{Fix}\, T_i := \{x \in H \mid T_i(x) = x\}$ is the fixed points set of T_i, the "**common fixed point problem**" (CFPP) is to find a point

$$x^* \in \cap_{i=1}^m \mathrm{Fix}\, T_i. \tag{24}$$

This problem serves as a framework for handling many important aspects of solving systems of nonlinear equations, feasibility-seeking of systems of constraint sets and optimization problems, see, e.g., the excellent books by Berinde [2] and by Cegielski [7] and references therein. In particular, iterative algorithms for the CFPP form an ever growing part of the field. There are many algorithms around for solving CFPPs, see, e.g., Zaslavski's book [17]. To be specific, we use the "Almost Cyclic Sequential Algorithm (ACSA) for the common fixed-point problem", which is Algorithm 5 in Censor and Segal [9], which is, in turn, a special case of an algorithm in the paper by Combettes [11, Algorithm 6.1]. The abstract study of limits of eventual families developed here can serve as a unifying convergence analysis of many iterative processes. It grew out of our look at the almost cyclic sequential algorithm and, therefore, we describe this algorithm and its relation with the present work next.

4.1 The Almost Cyclic Sequential Algorithm (ACSA)

Let $\langle x, y\rangle$ and $\|x\|$ be the Euclidean inner product and norm, respectively, in the J-dimensional Euclidean space R^J. Given $x, y \in R^J$ we denote the half-space

$$H(x, y) := \left\{u \in R^J \mid \langle u - y, x - y\rangle \leq 0\right\}. \tag{25}$$

Definition 24 An operator $T : R^J \to R^J$ is called "**a cutter**" if

$$\mathrm{Fix}\, T \subseteq H(x, T(x)), \text{ for all } x \in R^J, \tag{26}$$

or, equivalently,

$$\text{if } z \in \mathrm{Fix}\, T \text{ then } \langle T(x) - x, T(x) - z\rangle \leq 0, \text{ for all } x \in R^J. \tag{27}$$

The class of cutters was called $\Im$-class by Bauschke and Combettes [1] who first defined this notion and showed (see [1, Proposition 2.4]) (i) that the set of all fixed points of a cutter T with nonempty $\mathrm{Fix}\, T$ is closed and convex because

$$\mathrm{Fix}\, T = \cap_{x\in R^J} H(x, T(x)), \tag{28}$$

and (ii) that the following holds

$$\text{If } T \in \Im \text{ then } Id + \lambda(T - Id) \in \Im, \text{ for all } \lambda \in [0, 1], \tag{29}$$

where Id is the identity operator. This class of operators includes, among others, the resolvents of a maximal monotone operators, the firmly nonexpansive operators, namely, operators $N : R^J \to R^J$ that fulfil

$$\|N(x) - N(y)\|^2 \leq \langle N(x) - N(y), x - y\rangle, \text{ for all } x, y \in R^J, \tag{30}$$

the orthogonal projections and the subgradient projectors. Note that every cutter belongs to the class of operators $\mathcal{F}^0$, defined by Crombez [12, p. 161]. The term “cutter” was proposed in [8], see [7, pp. 53–54] for other terms that are used for these operators.

The following definition of a demiclosed operator that originated in Browder [5] (see, e.g., [11]) will be required.

Definition 25 An operator $T : R^J \to R^J$ is said to be “**demiclosed at** $y \in R^J$” if for every $\overline{x} \in R^J$ and every sequence $(x_n)_{n\in\mathbb{N}}$ in R^J, such that, $\lim_{n\to\infty} x_n = \overline{x}$ and $\lim_{n\to\infty} T(x_n) = y$, we have $T(\overline{x}) = y$.

For instance, the orthogonal projection onto a closed convex set is everywhere a demiclosed operator, due to its continuity.

Remark 26 [11] If $T : R^J \to R^J$ is nonexpansive, then $T - Id$ is demiclosed on R^J.

In sequential algorithms for solving the common fixed point problem the order by which the operators are chosen for the iterations is given by a “**control sequence”** of indices $(i(n))_{n\in\mathbb{N}}$, see, e.g., [10, Definition 5.1.1].

Definition 27 (i) **Cyclic control.** A control sequence is “**cyclic”** if $i(n) = n \bmod m + 1$, where m is the number of operators in the common fixed point problem.
(ii) **Almost cyclic control.** $(i(n))_{n\in\mathbb{N}}$ is “**almost cyclic on** $\{1, 2, \ldots, m\}$ ” if $1 \le i(n) \le m$ for all $n \ge 0$, and there exists an integer $c \ge m$ (called the “**almost cyclicality constant**”), such that, for all $n \ge 0$, $\{1, 2, \ldots, m\} \subseteq \{i(n+1), i(n+2), \ldots, i(n+c)\}$.

Consider a finite family $T_i : R^J \to R^J$, $i = 1, 2, \ldots, m$, of cutters with $\cap_{i=1}^m \operatorname{Fix} T_i \neq \emptyset$. The following algorithm for finding a common fixed point of such a family is a special case of [11, Algorithm 6.1].

Algorithm 28 ***Almost Cyclic Sequential Algorithm (ACSA) for solving common fixed point problems*** [9, Algorithm 5]

Initialization: $x_0 \in R^J$ *is an arbitrary starting point.*
Iterative Step: *Given* x_n, *compute* x_{n+1} *by*

$$x_{n+1} = x_n + \lambda_n(T_{i(n)}(x_n) - x_n). \tag{31}$$

Control: $(i(n))_{n\in\mathbb{N}}$ *is almost cyclic on* $\{1, 2, \ldots, m\}$.
Relaxation parameters: $(\lambda_n)_{n\in\mathbb{N}}$ *are confined to the interval* $[0, 2]$.

The convergence theorem of Algorithm 28 is as follows.

Theorem 29 *Let* $\{T_i\}_{i=1}^m$ *be a finite family of cutters* $T_i : R^J \to R^J$, *which satisfies*

(i) $\Omega := \cap_{i=1}^m \operatorname{Fix} T_i$ *is nonempty, and*
(ii) $T_i - Id$ *are demiclosed at 0, for every* $i \in \{1, 2, \ldots, m\}$.

Then any sequence $(x_n)_{n\in\mathbb{N}}$, *generated by Algorithm 28, converges to a point in* Ω.

Proof This follows as a special case of [11, Theorem 6.6 (i)]. □

4.2 An Abstract Approach to The Convergence of the ACSA

Given a sequence $(x_n)_{n\in\mathbb{N}}$ in a Hausdorff topological space X, push the multiset coGap in $\mathbb{N}$ to a multiset $\mathcal{M}$ on the subsets of X, and then consider its *limit* L (see Definitions 21 and 22 above) with respect to the multiset Star(cl$\mathcal{M}$) whose representing function value at $x \in X$ is the minimum of the value of coGap on the sets $\{n \in \mathbb{N} \mid x_n \in U\}$ for (open) neighborhoods U of x.

Then, by what was said in Section 2.3, Example 17 and Theorem 11, we reach the following conclusion.

Conclusion 30 For an operator (i.e., a continuous mapping) $T : X \to X$, if $(x_n)_{n\in\mathbb{N}}$ follows T for the eventual family which is the level family, for some c,

$$\mathrm{coGap}_c := \{S \subset \mathbb{N} \mid \mathrm{coGap}(S) \geq c\}, \tag{32}$$

then the level set $\{x \in X \mid L(x) \geq c\}$, where L is the limit of the multiset $\mathcal{M}$ on the subsets of X, mentioned above, will consist of fixed points of T.

This is the case with respect to each of the operators of the CFPP, for any sequence generated by the ACSA. Thus, any sequence of iterations of the ACSA follows each of the operators of the CFPP with respect to the eventual family $\mathcal{E}_c$ in $\mathbb{N}$ consisting of *all subsets of* $\mathbb{N}$ *that, after any number N, contain some 'interval' of c consecutive numbers* for some fixed number c.

This means that the eventual family $\mathcal{E}_c$, mentioned in Section 2.3 as relevant to the sequence of iterations in the ACSA will be just *the 'level family'* $\{S \subset \mathbb{N} \mid \mathrm{coGap}(S) \geq c\}$, and clearly any such level family of an *increasing* multiset is automatically an eventual family.

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References

1. Bauschke, H.H., Combettes, P.L.: A weak-to-strong convergence principle for Fejér-monotone methods in Hilbert spaces. Math. Oper. Res. **26**, 248–264 (2001)
2. Berinde, V. Iterative Approximation of Fixed Points, 2nd edn. Springer, Berlin, Heidelberg, Germany (2007)
3. Blizard, W.: Multiset theory. Notre Dame Journal of Formal Logic **30**, 36–66 (1989)
4. Borg, P.: Cross-intersecting non-empty uniform subfamilies of hereditary families. Eur. J. Comb. **78**, 256–267 (2019)

5. Browder, F.E.: Convergence theorems for sequences of nonlinear operators in Banach spaces. Math. Z **100**, 201–225 (1967)
6. Burachik, R.S., Iusem, A.N.: Set-Valued Mappings and enlargements of monotone operators. Springer Science+Business media LLC (2008)
7. Cegielski, A.: Iterative Methods for Fixed Point Problems in Hilbert Spaces. Springer, Berlin (2012)
8. Cegielski, A., Censor, Y.: Opial-type theorems and the common fixed point problem. In: Bauschke, H.H., Burachik, R.S., Combettes, P.L., Elser, V., Luke, D.R., Wolkowicz, H. (eds.) Fixed-Point Algorithms for Inverse Problems in Science and Engineering, pp. 155–183. Springer, New York, NY, USA (2011)
9. Censor, Y., Segal, A.: On the string averaging method for sparse common fixed points problems. Int. Trans. Oper. Res. **16**, 481–494 (2009)
10. Censor, Y., Zenios, S.A.: Parallel Optimization: Theory, Algorithms, and Applications. Oxford University Press, New York, NY, USA (1997)
11. Combettes, P.L.: Quasi-Fejérian analysis of some optimization algorithms. In: Butnariu, D., Censor, Y., Reich, S. (eds.) Inherently Parallel Algorithms in Feasibility and Optimization and Their Applications, pp. 115–152. Elsevier, Amsterdam (2001)
12. Crombez, G.: A geometrical look at iterative methods for operators with fixed points. Numer. Funct. Anal. Optim. **26**, 157–175 (2005)
13. Dantzig, G.B., Folkman, J., Shapiro, N.: On the continuity of the minimum set of a continuous function. J. Math. Anal. Appl. **17**, 519–548 (1967)
14. Lent, A., Censor, Y.: The primal-dual algorithm as a constraint-set-manipulation device. Math. Program. **50**, 343–357 (1991)
15. Rockafellar, R.T., Wets, R.J.-B.: Variational Analysis. Springer, Berlin, Heidelberg, Germany (1998)
16. Salinetti, G., Wets, R.J.-B.: On the convergence of sequences of convex sets in finite dimensions. SIAM Review **21**, 18–33 (1979). Addendum: SIAM Review 22, 86, (1980)
17. Zaslavski, A.J.: Approximate Solutions of Common Fixed-Point Problems. Springer International Publishing, Switzerland (2016)