



Set-Valued Evenly Convex Functions: Characterizations and C-Conjugacy

M. D. Fajardo¹ 

Received: 29 January 2021 / Accepted: 16 November 2021 / Published online: 12 March 2022
© The Author(s) 2022

Abstract

In this work we deal with set-valued functions with values in the power set of a separated locally convex space where a nontrivial pointed convex cone induces a partial order relation. A set-valued function is evenly convex if its epigraph is an evenly convex set, i.e., it is the intersection of an arbitrary family of open half-spaces. In this paper we characterize evenly convex set-valued functions as the pointwise supremum of its set-valued e -affine minorants. Moreover, a suitable conjugation pattern will be developed for these functions, as well as the counterpart of the biconjugation Fenchel-Moreau theorem.

Keywords Evenly convex sets · Set-valued functions · Partially ordered spaces · Convex conjugation · Fenchel-Moreau theorem

Mathematics Subject Classification (2010) 49N15 · 52A41 · 90C25

1 Introduction

Evenly convex sets (e -convex in brief) were introduced by Fenchel [8] in \mathbb{R}^n as those sets which can be expressed as the intersection of an arbitrary (possibly empty) family of open half-spaces. He initiated the study of this kind of sets defining a natural polarity operation and mentioning some of their properties. Later, they appeared in some works related to quasiconvex programming (see [16, 17, 22] and [23]) and mathematical economics [18]. E -convex sets are the solution sets of linear systems of either weak or strict inequalities, and from this point of view, their properties have been studied in [9] and [10], whereas some important characterizations of e -convex sets in terms of their sections and projections were given in [14]. Finally, some properties of this class of sets in locally convex spaces can be found in [1].

In a natural way, the concept of even convexity was applied to an extended real-valued function defined on \mathbb{R}^n in [24], where a function is said to be e -convex if its epigraph is e -convex. In that work, the main properties of these functions, which extend the class

✉ M. D. Fajardo
md.fajardo@ua.es

¹ Mathematics Department, Faculty of Sciences, University of Alicante, Alicante, Spain

of convex and lower semicontinuous functions, were studied. In particular, some characterizations were provided. Later, in [20], the notion of e -convex function was considered for extended real-valued functions defined on locally convex spaces, and proper e -convex functions were characterized as the pointwise supremum of their e -affine minorants. This characterization allowed, also in [20], the definition of a suitable conjugation scheme for e -convex functions. This pattern is called c -conjugation, and it was inspired by a survey written by Martínez-Legaz [19] where generalized convex duality theory is applied to quasiconvex programming. As it is shown in [20], the c -conjugation scheme allowed to set forth the e -convex counterpart of the well-known Fenchel-Moreau theorem. Recently, some works have been developed around duality theory in evenly convex optimization [2, 4–7], building different c -conjugate dual problems by means of the perturbation approach and obtaining regularity conditions expressed in terms of the even convexity of the perturbation function. A monograph presenting the state-of-art on even convexity in finite dimensional spaces has been published very recently (see [3]).

In this paper we extend the notion of even convexity to set-valued functions which map a separated locally convex space X into the power set $\mathcal{P}(Z)$ of another separated locally convex space Z , assuming that there exists a partial order in Z induced by a pointed convex cone $K \subset Z$. Every concept and result from the even convexity scalar theory can be translated in the field of set-valued functions by choosing appropriate definitions of properness, e -affine set-valued minorants, set-valued c -conjugate function, etc, and, moreover, extending operations, like the meaning of pointwise supremum or infimum. We will assume the same framework as in [12], where a characterization for proper closed convex set-valued functions by its continuous affine minorants and an appropriate Fenchel conjugate are provided. Also in that paper introduction section, different approaches which have been developed for conjugation patterns suitable for closed convex set-valued functions are highlighted.

The layout of this work is as follows. Section 2 contains preliminary results on set-valued functions, as well as fundamental definitions and results on e -convex sets and extended real-valued functions to make the paper self-contained. Section 3 is dedicated to characterizations of set-valued proper and e -convex functions in terms of different kinds of affine minorants. Section 4 is devoted to a suitable conjugation pattern for set-valued e -convex functions, which will provide a biconjugation theorem.

2 Preliminaries

Let X and Z be nontrivial separated locally convex spaces, we denote by X^* and Z^* their respective topological dual spaces, and by $\langle \cdot, \cdot \rangle$ the duality product in both cases: $\langle x, x^* \rangle = x^*(x)$ for all $x \in X, x^* \in X^*$ (analogously in Z and Z^*). In this paper we will deal with set-valued functions $f : X \rightarrow \mathcal{P}(Z)$, where $\mathcal{P}(Z)$ is the set of all the subsets of Z , including the empty set. The algebraic structure of $\mathcal{P}(Z)$ is determined by the Minkowski sum $A + B = \{a + b : a \in A, b \in B\}$, for all $A, B \in \mathcal{P}(Z)$, and the product with a nonnegative real number $tA = \{ta : a \in A\}$. We set $A + \emptyset = \emptyset + A = \emptyset$, for all $A \in \mathcal{P}(Z)$, $t\emptyset = \emptyset$, for all $t \in \mathbb{R} \setminus \{0\}$ and $0\emptyset = \{0\}$. We write $A + \{z\} = A + z$, $A + \{-z\} = A - z$. We set, for $A \neq \emptyset$ or $A \neq Z$, $(-1)A = -A = \{-a : a \in A\}$, whereas $-Z = \emptyset$ and $-\emptyset = Z$. Finally, we write $A + (-B) = A - B$.

In [13] several order relations which are used in order to formulate corresponding solution concepts for a set-valued optimization problem are introduced, allowing more practical comparisons among values of the set-valued objective function. Let $K \subset Z$ be a convex cone with $\{0\} \subsetneq K \subsetneq Z$. Then K induces in Z a quasiorder (reflexive and transitive binary

relation) in the following way: $x \leq_K y$ if and only if $y - x \in K$, for all $x, y \in Z$. Moreover, K also induces a quasiorder relation in $\mathcal{P}(Z)$ as $A \leq_K B$ if and only if $B \subseteq A + K$, see [15, Definition 2.1], called *lower set less order relation* in [13, Definition 2.6.9]. If we consider the subset of $\mathcal{P}(Z)$

$$\mathcal{P}_K^a(Z) := \{A \in \mathcal{P}(Z) : A = A + K\},$$

we will have that \leq_K induces a partial order (reflexive, antisymmetric and transitive binary relation) in $\mathcal{P}_K^a(Z)$, in fact, for $A, B \in \mathcal{P}_K^a(Z)$, $A \leq_K B$ if and only if $A \supseteq B$. With the Minkowski sum and a little modification in the scalar product, $0A = K$, for all $A \in \mathcal{P}_K^a(Z)$, we will have that $\mathcal{P}_K^a(Z)$ is a partially ordered conlinear space (see [11] for the definition and structural properties of conlinear spaces, also [12, Appendix] for a short introduction). Moreover, $(\mathcal{P}_K^a(Z), \leq_K)$ is a complete lattice, where, for $\mathcal{A} \subseteq \mathcal{P}_K^a(Z)$, $\sup\{\mathcal{A}, \leq_K\} = \bigcap_{A \in \mathcal{A}} A$ and $\inf\{\mathcal{A}, \leq_K\} = \bigcup_{A \in \mathcal{A}} A$. In the case \mathcal{A} is empty, we put $\sup\{\mathcal{A}, \leq_K\} = Z$ and $\inf\{\mathcal{A}, \leq_K\} = \emptyset$. The following definitions are well-known.

A set-valued function $f : X \rightarrow \mathcal{P}(Z)$ is said to be *proper* if its effective domain

$$\text{dom } f := \{x \in X : f(x) \neq \emptyset\}$$

is nonempty and $f(x) \neq Z$, for all $x \in \text{dom } f$. If f is not proper, we say that it is *improper*. It is said to be *convex (closed, resp.)* if its graph

$$\text{gph } f := \{(x, z) \in X \times Z : z \in f(x)\}$$

is a convex (closed with respect to the product topology, resp.) set in $X \times Z$.

We associate to any set-valued function $f : X \rightarrow \mathcal{P}(Z)$ the set-valued function

$$f_K : X \rightarrow \mathcal{P}_K^a(Z), f_K(x) := f(x) + K. \tag{1}$$

Definition 1 Let $f : X \rightarrow \mathcal{P}(Z)$ be a set-valued function. A set-valued minorant of f is $g : X \rightarrow \mathcal{P}_K^a(Z)$ verifying that, for all $x \in X$, $g(x) \leq_K f(x)$, i.e., $f_K(x) \subseteq g_K(x)$.

Definition 2 [13, page 26] A set-valued function $f : X \rightarrow \mathcal{P}(Z)$ is said to be *K-convex* if the set-valued function f_K defined in (1) is convex.

Therefore, *K-convexity* means that the *K-epigraph* of f ,

$$\text{epi}_K f := \{(x, z) \in X \times Z : z \in f(x) + K\} = \text{gph } f_K \subset X \times Z$$

is a convex set, or, equivalently, for all $t \in (0, 1)$, and $x_1, x_2 \in X$

$$f(tx_1 + (1 - t)x_2) \leq_K tf(x_1) + (1 - t)f(x_2).$$

It is easy to see that, if $f : X \rightarrow \mathcal{P}(Z)$ is *K-convex*, $f_K(x)$ is a convex set in Z , for all $x \in X$, and we can reduce the images space of f_K to

$$\mathcal{Q}_K^a(Z) := \{A \in \mathcal{P}(Z) : A = \text{conv}(A + K)\},$$

which is a partially ordered conlinear subspace of $\mathcal{P}_K^a(Z)$. Hence, for a set-valued function $f : X \rightarrow \mathcal{P}(Z)$, its *K-convex hull*

$$K\text{-conv } f : X \rightarrow \mathcal{Q}_K^a(Z),$$

defined by $z \in (K\text{-conv } f)(x)$, for $x \in X$ and $z \in Z$ if and only if $(x, z) \in \text{conv}(\text{epi}_K f)$, i.e.,

$$\text{epi}_K(K\text{-conv } f) = \text{gph}(K\text{-conv } f) = \text{conv}(\text{epi}_K f),$$

is (unique) well-defined. Moreover, $K\text{-conv } f$ is the largest K -convex minorant of f , since denoting by $\mathcal{G} = \{g : g \text{ is a } K\text{-convex minorant of } f\}$, then

$$(K\text{-conv } f)(x) = \sup \{g_K(x) : g \in \mathcal{G}\}.$$

In the following closed set-valued function definition [12, Definition 3], like in Definition 2, we have also specified that the closedness of the function is related to the convex cone K , for the sake of consistency.

Definition 3 A set-valued function $f : X \rightarrow \mathcal{P}(Z)$ is said to be K -closed if the set-valued function f_K is closed, i.e., $\text{epi}_K f$ is a closed set with respect to the product topology on $X \times Z$.

It is not difficult to prove that, if $f : X \rightarrow \mathcal{P}(Z)$ is K -closed, $f_K(x)$ is a closed set in Z , for all $x \in X$, and we can reduce the images space of f_K to

$$\mathcal{P}'_K(Z) := \{A \in \mathcal{P}(Z) : A = \text{cl}(A + K)\},$$

which is a partially ordered conlinear space if we consider the modified Minkowski sum

$$A \oplus B := \text{cl}(A + B),$$

and the modified scalar product $0A := \text{cl } K$.

Hence, for a set-valued function $f : X \rightarrow \mathcal{P}(Z)$, its K -closed hull

$$K\text{-cl } f : X \rightarrow \mathcal{P}'_K(Z),$$

is defined by $z \in (K\text{-cl } f)(x)$, for $x \in X$ and $z \in Z$ if and only if $(x, z) \in \text{cl}(\text{epi}_K f)$, i.e.,

$$\text{epi}_K(K\text{-cl } f) = \text{gph}(K\text{-cl } f) = \text{cl}(\text{epi}_K f).$$

Clearly, the K -closed K -convex hull of a set-valued function $f : X \rightarrow \mathcal{P}(Z)$,

$$K\text{-cl conv } f : X \rightarrow \mathcal{Q}'_K(Z),$$

is defined by $z \in (K\text{-cl conv } f)(x)$, for $x \in X$ and $z \in Z$ if and only if $(x, z) \in \text{cl conv}(\text{epi}_K f)$, i.e.,

$$\text{epi}_K(K\text{-cl conv } f) = \text{gph}(K\text{-cl conv } f) = \text{cl conv}(\text{epi}_K f),$$

where

$$\mathcal{Q}'_K(Z) := \{A \in \mathcal{P}(Z) : A = \text{cl conv}(A + K)\},$$

which is a partially ordered conlinear subspace of $\mathcal{P}'_K(Z)$.

We will use the following standard notation (see, for instance, [25]) for open half-spaces in locally convex spaces, either in X or Z , for $y^* \in X^*$ ($z^* \in Z^*$) and $\alpha \in \mathbb{R}$ ($\beta \in \mathbb{R}$):

$$H_{y^*, \alpha}^- := \{x \in X : \langle x, y^* \rangle < \alpha\}, H_{z^*, \beta}^- := \{z \in Z : \langle z, z^* \rangle < \beta\}.$$

In what follows, we denote by K^* the negative polar cone of K , i.e.,

$$K^* = \{z^* \in Z^* : \langle z, z^* \rangle \leq 0, \text{ for all } z \in K\},$$

and, for $(x^*, z^*) \in X^* \times (K^* \setminus \{0\})$, we will use the following set-valued functions, $S_{(x^*, z^*)}, \bar{S}_{(x^*, z^*)} : X \rightarrow \mathcal{Q}^a_K(Z)$, with open and closed half-spaces in Z as images, respectively,

$$\begin{aligned} S_{(x^*, z^*)}(x) &:= \{z \in Z : \langle x, x^* \rangle + \langle z, z^* \rangle < 0\}, \\ \bar{S}_{(x^*, z^*)}(x) &:= \{z \in Z : \langle x, x^* \rangle + \langle z, z^* \rangle \leq 0\}. \end{aligned} \tag{2}$$

As it was said in the introduction, Fenchel defined an e -convex set $C \subseteq \mathbb{R}^n$ as a set which can be expressed as the intersection of an arbitrary family of open half-spaces. In [1]

this kind of sets are defined in locally convex spaces in the following equivalent way, which will be very useful in the sequel.

Proposition 1 *A set $C \subseteq X$ is e-convex if for every point $x_0 \notin C$, there exists $x^* \in X^*$ such that $\langle x - x_0, x^* \rangle < 0$, for all $x \in C$.*

For a set $C \subseteq X$, the e-convex hull of C , $e\text{-conv } C$, is the smallest e-convex set in X containing C . For a convex subset $C \subseteq X$, it always holds $C \subseteq e\text{-conv } C \subseteq \text{cl } C$. This operator is well defined because the class of e-convex sets is closed under arbitrary intersections. Since X is a separated locally convex space, $X^* \neq \{0\}$. As a consequence of the Hahn-Banach theorem, it also holds that X is e-convex and every closed or open convex set is e-convex as well.

Definition 4 [24] A function $f : X \rightarrow \overline{\mathbb{R}} := \mathbb{R} \cup \{\mp\infty\}$ is e-convex if its epigraph $\text{epi } f = \{(x, \alpha) : f(x) \leq \alpha\}$ is e-convex in $X \times \mathbb{R}$.

Clearly, any lower semicontinuous convex function $f : X \rightarrow \overline{\mathbb{R}}$ is e-convex, but the converse does not hold, as the following example shows.

Example 1 Let $f : \mathbb{R} \rightarrow \overline{\mathbb{R}}$,

$$f(x) := \begin{cases} x^2 & \text{if } x < 0, \\ +\infty & \text{otherwise.} \end{cases}$$

Its epigraph is an e-convex set, since we can find, for any point not belonging to $\text{epi } f$, a hyperplane passing through the point but with empty intersection with $\text{epi } f$. Nevertheless, the convex set $\text{epi } f$ is not closed.

The e-convex hull of a function $f : X \rightarrow \overline{\mathbb{R}}$, $e\text{-conv } f$, is defined as the largest e-convex minorant of f . It holds that, if \tilde{f} is the lower semicontinuous convex hull of any function $f : X \rightarrow \overline{\mathbb{R}}$, then $\tilde{f}(x) \leq e\text{-conv } f(x) \leq f(x)$, for all $x \in X$.

Now we extend the concept of even convexity to set-valued functions. Like in convexity and closedness definitions, we will give a general definition and another one depending on the chosen convex cone K .

Definition 5 A set-valued function $f : X \rightarrow \mathcal{P}(Z)$ is e-convex if its graph is an e-convex set in $X \times Z$.

Definition 6 A set-valued function $f : X \rightarrow \mathcal{P}(Z)$ is K -e-convex if f_K is an e-convex function. It is equivalent to saying that its K -epigraph is an e-convex set in $X \times Z$.

Remark 1 The definition of K -e-convex set-valued function collapses into the e-convex function definition $f : X \rightarrow \overline{\mathbb{R}}$ if we consider a set-valued function $f^s : X \rightarrow \mathcal{P}(\mathbb{R})$, with $K = \mathbb{R}_+$, which can be associated to f , having the same epigraph, i.e., $\text{epi } f = \text{epi}_K f^s$. It is enough to take $f^s(x) = \{f(x)\}$, when $f(x) \in \mathbb{R}$, $f^s(x) = \emptyset$ if $f(x) = +\infty$ and $f^s(x) = \mathbb{R}$ if $f(x) = -\infty$. It holds that f is e-convex if and only if f^s is K -e-convex. Indeed, we have

$$\text{epi}_K f^s = \{(x, z) : z \in f^s(x) + K\} = \{(x, z) : f(x) \leq z\} = \text{epi } f.$$

The class of K -e-convex functions contains the class of K -closed K -convex functions, but this inclusion is strict.

Example 2 Let $f : \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$, with $K = \mathbb{R}_+$,

$$f(x) := \begin{cases} [x^2, 2x^2] & \text{if } x > 0, \\ \emptyset & \text{otherwise.} \end{cases}$$

We have that

$$\text{epi}_K f = \{(x, z) : x > 0, z \geq x^2\},$$

which is an e-convex nonclosed set.

Proposition 2 *If $f : X \rightarrow \mathcal{P}(Z)$ is K -e-convex, then $f_K(x)$ is an e-convex set in Z for all $x \in X$.*

Proof Let us show that, for $\bar{x} \in \text{dom } f$, $f(\bar{x}) + K$ is e-convex (it is evident when $\bar{x} \notin \text{dom } f$ and $f(\bar{x}) + K = \emptyset$).

Take a point $\bar{z} \notin f(\bar{x}) + K$. Then $(\bar{x}, \bar{z}) \notin \text{epi}_K f$ which is e-convex, and according to Proposition 1, there exists $0 \neq (x^*, z^*) \in X^* \times Z^*$ such that

$$\langle x - \bar{x}, x^* \rangle + \langle z - \bar{z}, z^* \rangle < 0,$$

for all $(x, z) \in \text{epi}_K f$. Now, for any point $z \in f(\bar{x}) + K$, since $(\bar{x}, z) \in \text{epi}_K f$, we have

$$\langle z - \bar{z}, z^* \rangle < 0,$$

and it shows, again in virtue of Proposition 1, that $f(\bar{x}) + K$ is e-convex. □

According to the previous result, if $f : X \rightarrow \mathcal{P}(Z)$ is K -e-convex, we can reduce the images space of f_K to

$$\mathcal{R}_K(Z) := \{A \in \mathcal{P}(Z) : A = \text{e-conv}(A + K)\}.$$

Remark 2 Actually the set-valued functions $S_{(x^*, z^*)}$ and $\bar{S}_{(x^*, z^*)}$ introduced in (2) map X to $\mathcal{R}_K(Z)$.

In general, the Minkowski sum of two e-convex sets is not e-convex, see, for instance, [9, Example 3.1]. Therefore, we consider the modified Minkowski sum

$$A \boxplus B := \text{e-conv}(A + B),$$

Proposition 3 $(\mathcal{R}_K(Z), \boxplus)$ is a commutative monoid with neutral element K .

Proof Being evident the commutativity of the operation \boxplus and the neutrality of K in $\mathcal{R}_K(Z)$, let us prove the associative property. It will follow from the fact that, for $A, B \subseteq Z$ nonempty sets,

$$\text{e-conv}(A + \text{e-conv } B) = \text{e-conv}(A + B). \tag{3}$$

To show this equality, is enough to see that

$$\text{e-conv}(A + \text{e-conv } B) \subseteq \text{e-conv}(A + B).$$

According to [10, Cor.2.1], ¹

$$e\text{-conv } A + e\text{-conv } B \subseteq e\text{-conv}(A + B),$$

hence $A + e\text{-conv } B \subset e\text{-conv } A + e\text{-conv } B \subseteq e\text{-conv}(A + B)$ and $e\text{-conv}(A + e\text{-conv } B) \subseteq e\text{-conv}(A + B)$.

Now let us take $A_1, A_2, A_3 \in \mathcal{R}_K(Z)$ nonempty sets (If at least one of them is the emptyset, the equality is trivial). We have, applying (3) when it is necessary,

$$\begin{aligned} A_1 \boxplus (A_2 \boxplus A_3) &= e\text{-conv}(A_1 + e\text{-conv}(A_2 + A_3)) = e\text{-conv}(A_1 + (A_2 + A_3)) = \\ &e\text{-conv}((A_1 + A_2) + A_3) = e\text{-conv}(e\text{-conv}(A_1 + A_2) + A_3) = (A_1 \boxplus A_2) \boxplus A_3 \quad \square \end{aligned}$$

Therefore, considering the modified Minkowski sum \boxplus in $\mathcal{R}_K(Z)$ and the modified scalar product

$$0A := e\text{-conv } K,$$

we conclude that $\mathcal{R}_K(Z)$ is a partially ordered conlinear space.

Definition 7 For a set-valued function $f : X \rightarrow \mathcal{P}(Z)$, its *K-e-convex hull*

$$K\text{-e-conv } f : X \rightarrow \mathcal{R}_K(Z),$$

is defined as the set-valued function whose graph is the e-convex hull of the *K*-epigraph of f ,

$$\text{epi}_K(K\text{-e-conv } f) = \text{gph}(K\text{-e-conv } f) = e\text{-conv}(\text{epi}_K f).$$

Let us observe that the *K*-e-convex hull of a set-valued function is (unique) well-defined. Moreover, $K\text{-e-conv } f$ is the largest *K*-e-convex minorant of f , since denoting by $\mathcal{G} = \{g : g \text{ is a } K\text{-e-convex minorant of } f\}$, then

$$(K\text{-e-conv } f)(x) = \sup \{g_K(x), g \in \mathcal{G}\},$$

for all $x \in X$.

Remark 3 In Remark 1, we saw how a set-valued function $f^s : X \rightarrow \mathcal{P}(\mathbb{R})$, with $K = \mathbb{R}_+$, can be associated to any extended real-valued function $f : X \rightarrow \mathbb{R}$, having the same epigraph, and f is e-convex if and only if f^s is *K*-e-convex. Nevertheless the *K*-e-convex hull of f^s is not necessarily equal to the associated set-valued function of the e-convex hull of f , even their epigraphs cannot be equal, as the following example shows.

Example 3 Let $f : \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$ defined as

$$f(x) := \begin{cases} +\infty & \text{if } x < 0, \\ 1 & \text{if } x = 0, \\ x^2 & \text{otherwise.} \end{cases}$$

Then

$$f^s(x) = \begin{cases} \emptyset & \text{if } x < 0, \\ \{1\} & \text{if } x = 0, \\ \{x^2\} & \text{otherwise} \end{cases}$$

¹This result is established in a finite dimensional framework, and it can be easily extended to the locally convex spaces context.

and

$$(\text{e-conv } f)^s(x) = \begin{cases} \emptyset & \text{if } x < 0, \\ \{0\} & \text{if } x = 0, \\ \{x^2\} & \text{otherwise,} \end{cases}$$

but

$$(K\text{-e-conv } f^s)(x) = \begin{cases} \emptyset & \text{if } x < 0, \\ (0, +\infty) & \text{if } x = 0, \\ [x^2, +\infty) & \text{otherwise.} \end{cases}$$

Moreover,

$$\text{epi}_K(\text{e-conv } f)^s = \{(x, z) : x \geq 0, z \geq x^2\}$$

and

$$\text{epi}_K(K\text{-e-conv } f^s) = \text{gph}(K\text{-e-conv } f^s) = \{(x, z) : x \geq 0, z \geq x^2\} \setminus \{(0, 0)\}.$$

We see that $\text{epi}_K(K\text{-e-conv } f^s) \subsetneq \text{epi}(\text{e-conv } f)$. The reason is that the definition of the K -e-convex hull of a set-valued function as that function whose epigraph is the e-convex hull of the epigraph of the function does not work for an extended real valued function. As it happens in this example, the e-convex hull of an epigraph is not necessarily an extended real-valued function epigraph. For the function f , one has $\text{e-conv}(\text{epi } f) = \{(x, z) : x \geq 0, z \geq x^2\} \setminus \{(0, 0)\}$.

It is worthy to mention, for a better understanding of the next sections, that a set-valued function $f : X \rightarrow \mathcal{P}(Z)$ is K -convex (K -closed, K -e-convex, resp.) iff $\text{epi}_K(K\text{-conv } f)$ ($\text{epi}_K(K\text{-cl } f)$, $\text{epi}_K(K\text{-e-conv } f)$, resp.) is equal to $\text{epi}_K f$, or, equivalently, $K\text{-conv } f = f_K$, ($K\text{-cl } f = f_K$, $K\text{-e-conv } f = f_K$, resp.). Hence, the behaviour of this kind of hulls differs from what is usually understood for a hull of any kind for an extended real-valued function, in the sense that, for instance, a K -e-convex set-valued function is not necessarily equal to its K -e-convex hull.

Finally, let us observe that, for all $x \in X$,

$$(K\text{-cl conv } f)(x) \leq_K (K\text{-e-conv } f)(x) \leq_K f(x),$$

and the first inequality can be strict. For instance, in the previous example, we have

$$(K\text{-cl conv } f^s)(x) = \begin{cases} \emptyset & \text{if } x < 0, \\ [x^2, +\infty) & \text{otherwise.} \end{cases}$$

3 Characterizing K -e-Convex Set-Valued Functions

It is a very well-known result that a proper convex lower semicontinuous function $f : X \rightarrow \overline{\mathbb{R}}$, where X is a locally convex space, is the pointwise supremum of its affine minorants, and [12, Section 3] shows that this approach can be developed for set-valued functions with a suitable definition of a set-valued affine minorant.

Again in the scalar case, in [20] different kinds of affine functions were defined, and proper e-convex functions were characterized as the pointwise supremum of the different sets of its affine minorants. Our aim in the following section is to characterize proper

K -e-convex set-valued functions by means of its set-valued affine minorants. We will define, like in [20], the sets of M_f -affine, \mathcal{C}_f -affine and e-affine minorants of a set-valued function.

Previous definitions have been linked to the chosen convex cone K , but in what follows, we assume that this cone is fixed, so it is not included in the next definitions, in order to avoid too long names.

Definition 8 Let $C \subseteq X$. A set-valued function $a : X \rightarrow \mathcal{P}(Z)$ is C -affine if there exist $x^* \in X^*$, $z^* \in K^* \setminus \{0\}$ and $\tilde{z} \in Z$ such that

$$a(x) = \begin{cases} S_{(x^*, z^*)}(x) + \tilde{z} & \text{if } x \in C, \\ \emptyset & \text{otherwise.} \end{cases} \tag{4}$$

Remark 4 It is easy to see that $a(x) + K = a(x)$, for all $x \in C$ (trivially if $x \notin C$), since $z \in a(x)$ if and only if $\langle x, x^* \rangle + \langle z - \hat{z}, z^* \rangle < 0$, and $z^* \in K^*$. Then, a C -affine function maps any $x \in X$ to $a(x) \in \mathcal{P}_K^a(Z)$, i.e., $a : X \rightarrow \mathcal{P}_K^a(Z)$.

In the next proposition we will use the fact that $C, D \subseteq X$ are e-convex iff $C \times D$ is e-convex. See [24, Proposition 1.2] for the proof in the finite dimensional case, which can easily be generalized to locally convex spaces.

Proposition 4 Let $a : X \rightarrow \mathcal{P}(Z)$ be a C -affine set-valued function. Then if C is e-convex, a is K -e-convex. Moreover, in this case, $a : X \rightarrow \mathcal{R}_K(Z)$.

Proof Let C be e-convex in (4). We have

$$\text{epi}_K a = \{(x, z) : \langle x, x^* \rangle + \langle z, z^* \rangle < \langle \tilde{z}, z^* \rangle\} \cap \{C \times Z\},$$

which is an e-convex set, since it is the intersection of e-convex sets. It follows immediately that $a(x) \in \mathcal{R}_K(Z)$, for all $x \in X$. □

Remark 5 It is well-known that if $C \subseteq X$ is a convex set and $T : X \rightarrow Y$ is a linear operator, (in this case, X and Y are real linear vector spaces), then $T(C)$ is also a convex set, and from this result one can derive that the effective domain of a K -convex set-valued function is convex, also a well-known property. But this is not true, in general, either for e-convex sets or for K -e-convex functions. See, for instance, [24, Example 2.4].

For a set-valued function $f : X \rightarrow \mathcal{P}(Z)$ we denote by $M_f = \text{e-conv}(\text{dom } f)$, and by \mathcal{H}_f the set of all the M_f -affine minorants of f :

$$\mathcal{H}_f = \{a : X \rightarrow \mathcal{R}_K(Z) : a \text{ is } M_f\text{-affine and } a(x) \leq_K f(x), \text{ for all } x \in X\}.$$

Remark 6 Clearly, according to Proposition 4, every M_f -affine minorant of f is K -e-convex. Moreover, $\mathcal{H}_f = \mathcal{H}_{f_K}$.

Remark 7 Let us see what happens with \mathcal{H}_f if f is an improper set-valued function. If $f \equiv \emptyset$, then $\text{dom } f = \emptyset$ and $\mathcal{H}_f = \{f\}$. On the other hand, if $f(\bar{x}) = Z$, for some $\bar{x} \in X$, in the case there exists $a \in \mathcal{H}_f$, it will follow that $S_{(x^*, z^*)}(\bar{x}) + \hat{z} = Z$ with $(x^*, z^*) \in X^* \times (K^* \setminus \{0\})$ and $\hat{z} \in Z$, and this is impossible, it must be $\mathcal{H}_f = \emptyset$.

The proof of the following lemma is similar to the scalar case [20, Lemma 7].

Lemma 1 *Let $f : X \rightarrow \mathcal{P}(Z)$ be a set-valued function. Then $\mathcal{H}_f = \mathcal{H}_{K\text{-e-conv } f}$.*

Lemma 2 *Let $f : X \rightarrow \mathcal{P}(Z)$ be a set-valued function. If $K\text{-e-conv } f \equiv Z$ then necessarily $\mathcal{H}_f = \emptyset$.*

Proof If there exists $a \in \mathcal{H}_f$ as it is described in Eq. 4, then $a(x) \leq_K (K\text{-e-conv } f)(x)$, for all $x \in X$, and $a \equiv Z$, in particular, $a(0) = Z$, and it would imply that, taking into account that $z^* \neq 0$,

$$\{z : \langle z, z^* \rangle < \langle \tilde{z}, z^* \rangle\} = Z. \quad \square$$

Theorem 1 *Let $f : X \rightarrow \mathcal{P}(Z)$ be a set-valued function. The following statements are equivalent:*

- (i) $\mathcal{H}_f \neq \emptyset$.
- (ii) Either $K\text{-e-conv } f$ is proper or $f \equiv \emptyset$.
- (iii) f has a proper $K\text{-e-convex}$ minorant.

Proof (i) \Rightarrow (ii) According to Remark 7 and Lemma 1, $(K\text{-e-conv } f)(x) \neq Z$ for all $x \in X$, and therefore, in the case $K\text{-e-conv } f$ is improper, it would be $\text{dom}(K\text{-e-conv } f) = \emptyset$, and then

$$\emptyset = (K\text{-e-conv } f)(x) \leq_K f(x),$$

for all $x \in X$, which means that $f \equiv \emptyset$.

(ii) \Rightarrow (iii) $K\text{-e-conv } f$ will be a proper $K\text{-e-convex}$ minorant of f in one case, and if $f \equiv \emptyset$, any proper $K\text{-e-convex}$ set-valued function will be a minorant of f .

(iii) \Rightarrow (i) If $f \equiv \emptyset$ then $\mathcal{H}_f = \{f\}$. Now, consider the case where $\text{dom } f \neq \emptyset$. Let g be a proper $K\text{-e-convex}$ minorant of f . Take a point $x_0 \in \text{dom } g$ and $z_0 \in Z$ such that $(x_0, z_0) \notin \text{epi}_K g$. Since it is an $e\text{-convex}$ set in $X \times Z$, by Proposition 1, there exists $(x^*, z^*) \in (X^* \times Z^*) \setminus \{0\}$ such that

$$\langle x_0, x^* \rangle + \langle z_0, z^* \rangle > \langle x, x^* \rangle + \langle z, z^* \rangle, \quad (5)$$

for all $(x, z) \in \text{epi}_K g$, hence $z^* \neq 0$. In particular, for any fix point $z_1 \in g(x_0)$ and for all $k \in K$,

$$\langle x_0, x^* \rangle + \langle z_0, z^* \rangle > \langle x_0, x^* \rangle + \langle z_1 + k, z^* \rangle$$

and $z^* \in K^* \setminus \{0\}$. We consider a point $\tilde{z} \in Z$ such that $\langle \tilde{z}, z^* \rangle \geq \langle x_0, x^* \rangle + \langle z_0, z^* \rangle$ and take

$$a(x) = \begin{cases} S_{(x^*, z^*)}(x) + \tilde{z} & \text{if } x \in M_f, \\ \emptyset & \text{otherwise.} \end{cases}$$

Since $\text{epi}_K f \subseteq \text{epi}_K g$, we will have, for all $(x, z) \in \text{epi}_K f$, $\langle x, x^* \rangle + \langle z, z^* \rangle < \langle \tilde{z}, z^* \rangle$, according to (5), and $z - \tilde{z} \in S_{(x^*, z^*)}(x)$, then for all $x \in \text{dom } f$, $f(x) - \tilde{z} \subseteq S_{(x^*, z^*)}(x)$ and $a \in \mathcal{H}_f$. □

Taking into account that $K\text{-e-conv } f = f_K$ whenever f is $K\text{-e-convex}$ and $f \equiv \emptyset$ is equivalent to $f_K \equiv \emptyset$, the following corollary comes directly.

Corollary 1 *Let $f : X \rightarrow \mathcal{P}(Z)$ be a $K\text{-e-convex}$ set-valued function. Then $\mathcal{H}_f \neq \emptyset$ if and only if f_K is proper or $f_K \equiv \emptyset$.*

We recall that, if $f : X \rightarrow \mathcal{P}(Z)$ is a set-valued function and $\{a_i : i \in I\}$ is an arbitrary collection of set-valued functions where $a_i : X \rightarrow \mathcal{P}(Z)$, for all $i \in I$, saying that f is the pointwise supremum of $\{a_i : i \in I\}$,

$$f(x) + K = \sup_I \{a_i(x) + K, \leq_K\},$$

means that $f(x) + K = \bigcap_I \{a_i(x) + K\}$, for all $x \in X$. It is clear that we can also take a collection $\{a_i : i \in I\}$ where $a_i : X \rightarrow \mathcal{P}_K^a(Z)$.

Remark 8 Let us observe that f is the pointwise supremum of $\{a_i : i \in I\}$ if $\text{epi}_K f = \bigcap_I \text{epi}_K a_i$.

Theorem 2 *Let $f : X \rightarrow \mathcal{P}(Z)$ be a set-valued function. The following statements are equivalent:*

- (i) f is the pointwise supremum of its M_f -affine minorants.
- (ii) f is K -e-convex and either f_K proper, $f_K \equiv \emptyset$ or $f_K \equiv Z$.

Proof (i) \Rightarrow (ii) In the case $\mathcal{H}_f = \emptyset$, then $f_K \equiv Z$ (recall that the supremum of an empty collection of sets in $\mathcal{P}_K^a(Z)$ is Z). Clearly, $\text{epi}_K f = X \times Z$, which is e-convex.

Otherwise, if $\mathcal{H}_f \neq \emptyset$, since $\text{epi}_K f = \bigcap_{\mathcal{H}_f} \text{epi}_K a$ and it is an e-convex set (recall Remark 6), then f is K -e-convex, and by Corollary 1, either f_K is proper or $f_K \equiv \emptyset$.

(ii) \Rightarrow (i) If $f_K \equiv Z$, by Remarks 6 and 7, $\mathcal{H}_f = \emptyset$ and f is the pointwise supremum of an empty family of minorants.

If $f_K \equiv \emptyset$, then $\mathcal{H}_f = \{f\}$. Finally, assume that f is K -e-convex, with f_K proper. According again to Corollary 1, $\mathcal{H}_f \neq \emptyset$. Since $a(x) \leq_K f(x)$, for all $x \in X, a \in \mathcal{H}_f$, it follows $\text{epi}_K f \subseteq \bigcap_{\mathcal{H}_f} \text{epi}_K a$. For the converse inclusion, we will show that if $(x_0, z_0) \notin \text{epi}_K f$, then there exists $a \in \mathcal{H}_f$ such that $(x_0, z_0) \notin \text{epi}_K a$.

We can assume that $x_0 \in M_f$, otherwise $a(x_0) = \emptyset$, for all $a \in \mathcal{H}_f$.

Since $\text{epi}_K f$ is e-convex, according to Proposition 1, we can find $(x^*, z^*) \in (X^* \times Z^*) \setminus \{0\}$ such that

$$\langle x_0, x^* \rangle + \langle z_0, z^* \rangle > \langle x, x^* \rangle + \langle z, z^* \rangle, \tag{6}$$

for all $(x, z) \in \text{epi}_K f$. We ensure that $z^* \in K^* \setminus \{0\}$.

Firstly, we will see that $z^* \neq 0$. If not, $\langle x_0, x^* \rangle > \langle x, x^* \rangle$, for all $x \in \text{dom } f$, and it follows that $x_0 \notin M_f$.

Now, if $z^* \notin K^*$, it would exist $k_0 \in K$ such that $\langle k_0, z^* \rangle > 0$, and considering that, for all $(x, z) \in \text{epi}_K f$ and $\lambda > 0, (x, z + \lambda k_0) \in \text{epi}_K f$, inequality (6) will not hold for λ large enough, which leads us to conclude that $z^* \in K^* \setminus \{0\}$. We consider now a point $\tilde{z} \in Z$ such that $\langle \tilde{z}, z^* \rangle = \langle x_0, x^* \rangle + \langle z_0, z^* \rangle$ and take

$$a(x) = \begin{cases} S_{(x^*, z^*)}(x) + \tilde{z} & \text{if } x \in M_f, \\ \emptyset & \text{otherwise.} \end{cases}$$

It is easy to check that $a \in \mathcal{H}_f$, nevertheless,

$$\langle x_0, x^* \rangle + \langle z_0 - \tilde{z}, z^* \rangle = 0,$$

and $z_0 \notin a(x_0)$, hence $(x_0, z_0) \notin \text{epi}_K a$. □

Corollary 2 Let $f : X \rightarrow \mathcal{P}(Z)$ having a proper K - e -convex minorant. Then, for all $x \in X$,

$$(K\text{-}e\text{-conv } f)(x) = \sup_{\mathcal{H}_f} \{a(x), \leq_K\}.$$

Proof By Theorem 1, either K - e -conv f is proper or $f \equiv \emptyset$. In the first case, taking into account that K - e -conv $f = (K$ - e -conv $f)_K$, by Theorem 2, K - e -conv f is the pointwise supremum of its M_{K - e -conv f -affine minorants, which is the same set as the set of the M_f -affine minorants of f , according to Lemma 1.

In the case $f \equiv \emptyset$, f is K - e -convex, hence K - e -conv $f = f_K = f$ and moreover $\mathcal{H}_f = \{f\}$. \square

Definition 9 Let \mathcal{C} be the set of all e -convex subsets in X . We say that a set-valued function $a : X \rightarrow \mathcal{R}_K(Z)$ is \mathcal{C} -affine if there exists $C \in \mathcal{C}$ such that a is C -affine.

For a set-valued function $f : X \rightarrow \mathcal{P}(Z)$, we will denote by \mathcal{C}_f the set of all its \mathcal{C} -affine minorants.

Theorem 3 Let $f : X \rightarrow \mathcal{P}(Z)$. The following statements are equivalent:

- (i) f is the pointwise supremum of its \mathcal{C} -affine minorants.
- (ii) f is K - e -convex and either f_K is proper, $f_K \equiv \emptyset$ or $f_K \equiv Z$.

Proof (i) \Rightarrow (ii) It is clear that f is K - e -convex. Now, if $\mathcal{C}_f = \emptyset$, then $f_K \equiv Z$. In the case $\mathcal{C}_f \neq \emptyset$, if $f_K \not\equiv \emptyset$, it must be proper, otherwise there would exist $x \in \text{dom } f_K$ satisfying $f_K(x) = Z$ and hence for all $a \in \mathcal{C}_f$ it would be $a(x) = Z$ and it is not possible (we can use the same reasoning than the one used in Remark 7 for \mathcal{H}_f).

(ii) \Rightarrow (i) In the case $f_K \equiv Z$, we will have $\mathcal{C}_f = \emptyset$ and f is the pointwise supremum of an empty family. On the other hand, in the cases f_K proper or $f_K \equiv \emptyset$, we will have $\emptyset \neq \mathcal{H}_f \subseteq \mathcal{C}_f$, and by Theorem 2, we obtain

$$\text{epi}_K f = \bigcap_{\mathcal{H}_f} \text{epi}_K a \supseteq \bigcap_{\mathcal{C}_f} \text{epi}_K a \supseteq \text{epi}_K f.$$

\square

Definition 10 We say that a function $a : X \rightarrow \mathcal{R}_K(Z)$ is e -affine if

$$a(x) = \begin{cases} S_{(x^*, z^*)}(x) + \tilde{z} & \text{if } \langle x, y^* \rangle < \alpha, \\ \emptyset & \text{otherwise,} \end{cases}$$

for some $x^*, y^* \in X^*$, $z^* \in K^* \setminus \{0\}$, $\tilde{z} \in Z$ and $\alpha \in \mathbb{R}$. For a set-valued function $f : X \rightarrow \mathcal{P}(Z)$, we will denote by \mathcal{E}_f the set of all the e -affine minorants of f .

Theorem 4 Let $f : X \rightarrow \mathcal{P}(Z)$ a set-valued function. The following statements are equivalent:

- (i) f is the pointwise supremum of its e -affine minorants.
- (ii) f is K - e -convex and either f_K is proper, $f_K \equiv \emptyset$ or $f_K \equiv Z$.

Proof (i) \Rightarrow (ii) The proof is similar to the proof of (i) \Rightarrow (ii) in Theorem 3. Just change \mathcal{C}_f by \mathcal{E}_f .

(ii) \Rightarrow (i) In the case $f_K \equiv Z$ we have $\mathcal{E}_f = \emptyset$. Let us assume then that either f_K is proper or $f_K \equiv Z$. By Corollary 1 we have $\emptyset \neq \mathcal{H}_f \subseteq \mathcal{C}_f$ and by Theorem 3, $\bigcap_{\mathcal{C}_f} \text{epi}_K a = \text{epi}_K f$. Now, take any function $a \in \mathcal{C}_f$, and write $\text{dom } a = \bigcap_{T_a} H_t^-$ expressing this e-convex set as an intersection of open half-spaces. We define, for all $t \in T_a$,

$$a_t(x) = \begin{cases} S_{(x^*, z^*)}(x) + \tilde{z} & \text{if } x \in H_t^-, \\ \emptyset & \text{otherwise,} \end{cases}$$

where x^*, z^* and \tilde{z} are the data defining the function a . It is easy to check that $\text{epi}_K a = \bigcap_{T_a} \text{epi}_K a_t$ and then, denoting by $S = \bigcup_{\mathcal{C}_f} \{a_t, t \in T_a\}$, we have $S \subset \mathcal{E}_f$ and so $\mathcal{E}_f \neq \emptyset$ and

$$\text{epi}_K f = \bigcap_{\mathcal{C}_f} \left\{ \bigcap_{T_a} \text{epi}_K a_t \right\} = \bigcap_S \text{epi}_K a \supseteq \bigcap_{\mathcal{E}_f} \text{epi}_K a \supseteq \text{epi}_K f. \quad \square$$

Remark 9 Let us observe that, as a result of the above proof, $\mathcal{E}_f \neq \emptyset$ whenever f is K -e-convex and f_K is proper.

Corollary 3 *Let $f : X \rightarrow \mathcal{P}(Z)$ be a set-valued function having a proper K -e-convex minorant. Then K -e-conv f is the pointwise supremum of its e-affine minorants.*

Proof The proof of $\mathcal{E}_f = \mathcal{E}_{K\text{-e-conv } f}$ is similar to that one of $\mathcal{H}_f = \mathcal{H}_{K\text{-e-conv } f}$ (we recall Lemma 1). Now, according to Theorem 1, either K -e-conv $f = (K\text{-e-conv } f)_K$ is proper or $f \equiv \emptyset$. In the first case, by Theorem 4 it follows that K -e-conv f is the pointwise supremum of its e-affine minorants. On the other hand, if $f \equiv \emptyset$, it would be $f = f_K = K\text{-e-conv } f$ with $\mathcal{E}_f = \{f\}$. □

4 C-Conjugating Set-Valued Functions

We will generalize the conjugation pattern for extended real-valued functions, described in [20], suitable for e-convex functions. It is based on the generalized convex conjugation theory introduced by Moreau [21]. We give a brief description of it. Let us consider the space $W := X^* \times X^* \times \mathbb{R}$ with the coupling functions $c : X \times W \rightarrow \overline{\mathbb{R}}$ and $c' : W \times X \rightarrow \overline{\mathbb{R}}$ given by

$$c(x, (x^*, y^*, \alpha)) = c'((x^*, y^*, \alpha), x) := \begin{cases} \langle x, x^* \rangle & \text{if } \langle x, y^* \rangle < \alpha, \\ +\infty & \text{otherwise.} \end{cases}$$

Given two functions $f : X \rightarrow \overline{\mathbb{R}}$ and $g : W \rightarrow \overline{\mathbb{R}}$, the c -conjugate of f , $f^c : W \rightarrow \overline{\mathbb{R}}$, and the c' -conjugate of g , $g^{c'} : X \rightarrow \overline{\mathbb{R}}$, are defined

$$f^c(x^*, y^*, \alpha) := \sup_{x \in X} \{c(x, (x^*, y^*, \alpha)) - f(x)\}, \tag{7}$$

$$g^{c'}(x) := \sup_{(x^*, y^*, \alpha) \in W} \{c'((x^*, y^*, \alpha), x) - g(x^*, y^*, \alpha)\}, \tag{8}$$

respectively, with the conventions $(+\infty) + (-\infty) = (-\infty) + (+\infty) = (+\infty) - (+\infty) = (-\infty) - (-\infty) = -\infty$.

The counterpart of the Fenchel-Moreau theorem for e-convex functions derives immediately from [19, Corollary 6.1], and a function $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is e-convex iff $f^{cc'} = f$.

We can rewrite (7) as

$$f^c(x^*, y^*, \alpha) = \begin{cases} -\inf_{x \in X} \{f(x) + \langle -x, x^* \rangle\} & \text{if } \text{dom } f \subseteq H_{y^*, \alpha}^- \\ +\infty & \text{otherwise,} \end{cases}$$

for all $(x^*, y^*, \alpha) \in W$. This formula inspires the generalization of c -conjugacy to set-valued functions. As in the previous section, we assume that the convex cone K is fix.

Definition 11 Let $f : X \rightarrow \mathcal{P}(Z)$ be a set-valued function. We define the c -conjugate of f (respect to K), $f^c : X^* \times X^* \times (K^* \setminus \{0\}) \times \mathbb{R} \rightarrow \mathcal{P}(Z)$ as

$$f^c(x^*, y^*, z^*, \alpha) := \begin{cases} -\text{e-conv} \left[\bigcup_X \{f(x) + \bar{S}_{(x^*, z^*)}(-x)\} \right] & \text{if } \text{dom } f \subseteq H_{y^*, \alpha}^- \\ \emptyset & \text{otherwise.} \end{cases}$$

For a set-valued function $g : X^* \times X^* \times (K^* \setminus \{0\}) \times \mathbb{R} \rightarrow \mathcal{P}(Z)$, its c' -conjugate (respect to K), $g^{c'} : X \rightarrow \mathcal{P}(Z)$ is defined, if $x \in H_{y^*, \alpha}^-$, for all $(x^*, y^*, z^*, \alpha) \in \text{dom } g$:

$$g^{c'}(x) := \bigcap_{\text{dom } g} \{ \bar{S}_{(x^*, z^*)}(x) - g(x^*, y^*, z^*, \alpha) \},$$

and $g^{c'}(x) := \emptyset$ otherwise.

Remark 10 Actually,

$$\inf\{ \{f(x) + \bar{S}_{(x^*, z^*)}(-x)\}_X, \leq_K \} = \bigcup_X \{f(x) + \bar{S}_{(x^*, z^*)}(-x)\} \text{ and}$$

$$\sup\{ \{ \bar{S}_{(x^*, z^*)}(x) - g(x^*, y^*, z^*, \alpha) \}_{\text{dom } g}, \leq_K \} = \bigcap_{\text{dom } g} \{ \bar{S}_{(x^*, z^*)}(x) - g(x^*, y^*, z^*, \alpha) \}$$

in the previous definitions, since $\bar{S}_{(x^*, z^*)}(-x) + K = \bar{S}_{(x^*, z^*)}(-x)$, for all $x \in X$.

Definition 12 The function $\sigma_f : X^* \times X^* \times Z^* \times \mathbb{R} \rightarrow \bar{\mathbb{R}}$ defined as

$$\sigma_f(x^*, y^*, z^*, \alpha) := \begin{cases} \sup_{\text{epi}_K f} \{ \langle x, x^* \rangle + \langle z, z^* \rangle \} & \text{if } \text{dom } f \subseteq H_{y^*, \alpha}^- \\ +\infty & \text{otherwise.} \end{cases}$$

is called the *support function* of the K -epigraph of the set-valued function $f : X \rightarrow \mathcal{P}(Z)$ relative to open half-spaces.

Remark 11 Let $f : X \rightarrow \mathcal{P}(Z)$ be a set-valued function and let us consider the indicator function of $\text{epi}_K f$, $\delta_{\text{epi}_K f} : X \times Z \rightarrow \bar{\mathbb{R}}$,

$$\delta_{\text{epi}_K f}(x, z) = \begin{cases} 0 & \text{if } z \in f(x) + K, \\ +\infty & \text{otherwise.} \end{cases}$$

It is easy to check that $\delta_{\text{epi}_K f}^c(x^*, z^*, y^*, 0, \alpha) = \sigma_f(x^*, y^*, z^*, \alpha)$.

Remark 12 Let us observe that, if f is K -e-convex and $f_K \neq \emptyset$, $\text{epi}_K f$ is a proper e-convex subset of $X \times Z$ and hence it can be expressed as the intersection of a nonempty family of open half-spaces, being nontrivial at least one of them. Denote one of these half-spaces by $H = \{(x, z) \in X \times Z : \langle x, x^* \rangle + \langle z, z^* \rangle < \beta\}$. Since $\text{epi}_K f \subseteq H$, we will have that $\sigma_f(x^*, 0, z^*, \alpha) \leq \beta < +\infty$, for all $\alpha > 0$ and $\text{dom } \sigma_f \neq \emptyset$.

Definition 13 Let $C \subset X \times Z$ be a nonempty set. We define the function $\eta_C : X^* \times Z^* \rightarrow \{0, 1\}$ as

$$\eta_C(x^*, z^*) = \begin{cases} 0 & \text{if } \langle x, x^* \rangle + \langle z, z^* \rangle < \sup_C \{ \langle x, x^* \rangle + \langle z, z^* \rangle \} \text{ for all } (x, z) \in C, \\ 1 & \text{otherwise.} \end{cases}$$

Lemma 3 Let $f : X \rightarrow \mathcal{P}(Z)$ be a set-valued function. Then, for all $(x^*, y^*, z^*, \alpha) \in X^* \times X^* \times (K^* \setminus \{0\}) \times \mathbb{R}$,

$$-f^c(x^*, y^*, z^*, \alpha) = \begin{cases} \{z \in Z : \langle z, z^* \rangle < \sigma_f(x^*, y^*, z^*, \alpha)\} & \text{if } \eta_{\text{epi}_K f}(x^*, z^*) = 0, \\ \{z \in Z : \langle z, z^* \rangle \leq \sigma_f(x^*, y^*, z^*, \alpha)\} & \text{if } \eta_{\text{epi}_K f}(x^*, z^*) = 1. \end{cases}$$

Consequently, $\text{dom } f^c = \text{dom } \sigma_f$.

Proof Take any point $(x^*, y^*, z^*, \alpha) \in X^* \times X^* \times (K^* \setminus \{0\}) \times \mathbb{R}$.

In first place, the equality holds trivially in the case $\text{dom } f \not\subset H_{y^*, \alpha}^-$, because $\sigma_f(x^*, y^*, z^*, \alpha) = +\infty$ and $-f^c(x^*, y^*, z^*, \alpha) = Z$ (recall that we set $-\emptyset = Z$).

Hence, let us suppose that $\text{dom } f \subset H_{y^*, \alpha}^-$. We denote

$$D_0 = \{z \in Z : \langle z, z^* \rangle < \sigma_f(x^*, y^*, z^*, \alpha)\},$$

$$D_1 = \{z \in Z : \langle z, z^* \rangle \leq \sigma_f(x^*, y^*, z^*, \alpha)\}.$$

In the case $\eta_{\text{epi}_K f}(x^*, z^*) = 0$, take any point $x \in \text{dom } f$, $z_1 \in f(x)$ and $z_2 \in \bar{S}_{(x^*, z^*)}(-x)$. Then

$$\langle z_1 + z_2, z^* \rangle < \sigma_f(x^*, y^*, z^*, \alpha),$$

and $z_1 + z_2 \in D_0$, which means that, for all $x \in \text{dom } f$,

$$f(x) + \bar{S}_{(x^*, z^*)}(-x) \subset D_0,$$

and D_0 an e-convex set, implying that $-f^c(x^*, y^*, z^*, \alpha) \subset D_0$.

For the converse inclusion, take $z \in D_0$ and $(\hat{x}, \hat{z}) \in \text{epi}_K f$ verifying that

$$\langle z, z^* \rangle < \langle \hat{x}, x^* \rangle + \langle \hat{z}, z^* \rangle.$$

Then, since $\hat{z} = z_1 + k$, for some $z_1 \in f(\hat{x})$ and $k \in K$, we have

$$\langle -\hat{x}, x^* \rangle + \langle z - z_1, z^* \rangle < \langle k, z^* \rangle \leq 0,$$

and $z - z_1 \in \bar{S}_{(x^*, z^*)}(-\hat{x})$, obtaining $D_0 \subset -f^c(x^*, y^*, z^*, \alpha)$. The proof is similar for the case $\eta_{\text{epi}_K f}(x^*, z^*) = 1$ and D_1 ; the inclusion $-f^c(x^*, y^*, z^*, \alpha) \subset D_1$ can be shown analogously, whereas the converse inclusion holds if we take a point $(\hat{x}, \hat{z}) \in \text{epi}_K f$ verifying that

$$\langle z, z^* \rangle \leq \langle \hat{x}, x^* \rangle + \langle \hat{z}, z^* \rangle.$$

Finally, we have that $(x^*, y^*, z^*, \alpha) \in \text{dom } f^c$ iff $-f^c(x^*, y^*, z^*, \alpha) \neq Z$, which means that $\sigma_f(x^*, y^*, z^*, \alpha) < +\infty$, i.e., $(x^*, y^*, z^*, \alpha) \in \text{dom } \sigma_f$. □

Lemma 4 Let $f : X \rightarrow \mathcal{P}(Z)$ a set-valued function. Then, if $x \in \text{dom } f^{cc}$,

$$f^{cc}(x) = \bigcap_{\text{dom } \sigma_f} \{z \in Z : \langle x, x^* \rangle + \langle z, z^* \rangle < \sigma_f(x^*, y^*, z^*, \alpha), \eta_{\text{epi}_K f}(x^*, z^*) = 0, \langle x, x^* \rangle + \langle z, z^* \rangle \leq \sigma_f(x^*, y^*, z^*, \alpha), \eta_{\text{epi}_K f}(x^*, z^*) = 1\}. \tag{9}$$

Consequently, $f^{cc'}$ is an e -convex set in Z , for all $x \in X$, and $f^{cc'} : X \rightarrow \mathcal{R}_K(Z)$ is a K - e -convex set-valued function. Moreover, $f(x) \subseteq f^{cc'}(x)$ and then $f_K(x) \subseteq f^{cc'}(x)$, for all $x \in X$.

Proof Take $\bar{x} \in \text{dom } f^{cc'}$ and denote by

$$D = \bigcap_{\text{dom } \sigma_f} \{z \in Z : \langle \bar{x}, x^* \rangle + \langle z, z^* \rangle < \sigma_f(x^*, y^*, z^*, \alpha), \eta_{\text{epi}_K f}(x^*, z^*) = 0, \langle \bar{x}, x^* \rangle + \langle z, z^* \rangle \leq \sigma_f(x^*, y^*, z^*, \alpha), \eta_{\text{epi}_K f}(x^*, z^*) = 1\}.$$

Take now $(\bar{x}^*, \bar{y}^*, \bar{z}^*, \bar{\alpha}) \in \text{dom } \sigma_f$. Name

$$D_0 = \{z \in Z : \langle \bar{x}, \bar{x}^* \rangle + \langle z, \bar{z}^* \rangle < \sigma_f(\bar{x}^*, \bar{y}^*, \bar{z}^*, \bar{\alpha})\},$$

$$D_1 = \{z \in Z : \langle \bar{x}, \bar{x}^* \rangle + \langle z, \bar{z}^* \rangle \leq \sigma_f(\bar{x}^*, \bar{y}^*, \bar{z}^*, \bar{\alpha})\}.$$

From the definition of the c' -conjugate function, and taking into account that, according to Lemma 3, $\text{dom } f^c = \text{dom } \sigma_f$, it holds

$$f^{cc'}(\bar{x}) = \bigcap_{\text{dom } \sigma_f} [\bar{S}_{(x^*, z^*)}(\bar{x}) - f^c(x^*, y^*, z^*, \alpha)].$$

Take $\bar{z} \in f^{cc'}(\bar{x})$ and suppose $\eta_{\text{epi}_K f}(\bar{x}^*, \bar{z}^*) = 0$. Since $(\bar{x}^*, \bar{y}^*, \bar{z}^*, \bar{\alpha}) \in \text{dom } \sigma_f$, we have $\bar{z} \in \bar{S}_{(\bar{x}^*, \bar{z}^*)}(\bar{x}) - f^c(\bar{x}^*, \bar{y}^*, \bar{z}^*, \bar{\alpha})$ and $\bar{z} = z_1 + z_2$, where $z_1 \in S_{(\bar{x}^*, \bar{z}^*)}(\bar{x})$ and $z_2 \in -f^c(\bar{x}^*, \bar{y}^*, \bar{z}^*, \bar{\alpha})$. Therefore

$$\langle \bar{x}, \bar{x}^* \rangle + \langle z_1, \bar{z}^* \rangle \leq 0,$$

and, by Lemma 3,

$$\langle z_2, \bar{z}^* \rangle < \sigma_f(\bar{x}^*, \bar{y}^*, \bar{z}^*, \bar{\alpha}).$$

Then $\langle \bar{x}, \bar{x}^* \rangle + \langle \bar{z}, \bar{z}^* \rangle < \sigma_f(\bar{x}^*, \bar{y}^*, \bar{z}^*, \bar{\alpha})$ and $\bar{z} \in D_0$. Analogously, it can be shown that, if $\eta_{\text{epi}_K f}(\bar{x}^*, \bar{z}^*) = 1$, $f^{cc'}(\bar{x}) \subseteq D_1$. Then $f^{cc'}(\bar{x}) \subseteq D$.

For the converse inclusion, take $\bar{z} \in D$ and $(\bar{x}^*, \bar{y}^*, \bar{z}^*, \bar{\alpha}) \in \text{dom } \sigma_f = \text{dom } f^c$. In the case $\eta_{\text{epi}_K f}(\bar{x}^*, \bar{z}^*) = 0$, we denote by

$$\beta = \sigma_f(\bar{x}^*, \bar{y}^*, \bar{z}^*, \bar{\alpha}) - \langle \bar{x}, \bar{x}^* \rangle - \langle \bar{z}, \bar{z}^* \rangle > 0.$$

Choose $z_2 \in Z$ such that $-\beta < \langle \bar{x}, \bar{x}^* \rangle + \langle z_2, \bar{z}^* \rangle \leq 0$, and $\bar{z} = (\bar{z} - z_2) + z_2$, with $z_2 \in \bar{S}_{(\bar{x}^*, \bar{z}^*)}(\bar{x})$ and, according to Lemma 3, $\bar{z} - z_2 \in -f^c(\bar{x}^*, \bar{y}^*, \bar{z}^*, \bar{\alpha})$.

In the case $\eta_{\text{epi}_K f}(\bar{x}^*, \bar{z}^*) = 1$, we can follow the same steps than in the previous proof, choosing $z_2 \in Z$ such that $\langle \bar{x}, \bar{x}^* \rangle + \langle z_2, \bar{z}^* \rangle = 0$.

Hence, if $\bar{x} \in \text{dom } f^{cc'}$, then $f^{cc'}(\bar{x}) = D$.

It is clear that, for all $x \in X$, $f^{cc'}(x)$ is an e -convex set in Z . Moreover, $\text{epi}_K f^{cc'}$ is the intersection of the epigraphs of all the K - e -convex set-valued functions

$$a_0(x) = \{z \in Z : \langle x, x^* \rangle + \langle z, z^* \rangle < \sigma_f(x^*, y^*, z^*, \alpha)\} \text{ if } \eta_{\text{epi}_K f}(x^*, z^*) = 0,$$

$$a_1(x) = \{z \in Z : \langle x, x^* \rangle + \langle z, z^* \rangle \leq \sigma_f(x^*, y^*, z^*, \alpha)\} \text{ if } \eta_{\text{epi}_K f}(x^*, z^*) = 1,$$

for all $(x^*, y^*, z^*, \alpha) \in \text{dom } \sigma_f$, and $f^{cc'}$ is K - e -convex.

Is is evident also from (9) that $f^{cc'}(x) + K \subseteq f^{cc'}(x)$, for all $x \in X$, therefore $e\text{-conv}(f^{cc'}(x) + K) \subseteq f^{cc'}(x)$, for all $x \in X$, and $f^{cc'} : X \rightarrow \mathcal{R}_K(Z)$. The inclusions $f(x) \subseteq f^{cc'}(x)$ and $f_K(x) \subseteq f^{cc'}(x)$, for all $x \in X$, are trivial. \square

Next we will give the biconjugation theorem.

Theorem 5 *Let $f : X \rightarrow \mathcal{P}(Z)$ be a set-valued function. Then it is K -e-convex if and only if $f^{cc'}(x) = f_K(x)$, for all $x \in X$.*

Proof Let us assume that f is K -e-convex. If $f_K \equiv Z$, from Lemma 4, we have $f_K = f^{cc'}$. In the case $f \equiv \emptyset$, it will be $\sigma_f \equiv -\infty$ and again applying Lemma 4, $f^{cc'} \equiv \emptyset$. Hence, if either $f_K \equiv Z$ or $f \equiv \emptyset$, it follows that $f^{cc'} = f_K$.

Suppose that $f_K \not\equiv Z$ and $\text{dom } f \neq \emptyset$. According to Lemma 4, it is enough to prove that $f^{cc'}(x) \subseteq f_K(x)$, for all $x \in X$. In first place, let us observe that, according again to Lemma 4, $\emptyset \neq \text{dom } f \subseteq \text{dom } f^{cc'}$ and, moreover, recalling Remark 12, $\text{dom } \sigma_f \neq \emptyset$, and for all $x \in \text{dom } f^{cc'}$, $f^{cc'}(x) \neq Z$ and $f^{cc'}$ is proper. Since $f_K(x) \subseteq f^{cc'}(x)$, for all $x \in X$, and $f_K \not\equiv \emptyset$, f_K is also proper.

Applying Theorem 4 to both functions, f and $f^{cc'}$, we have $\mathcal{E}_f \neq \emptyset$, $\mathcal{E}_{f^{cc'}} \neq \emptyset$ (see the proof of Theorem 4) and

$$\text{epi}_K f = \bigcap_{\mathcal{E}_f} \text{epi}_K a \text{ and } \text{epi}_K f^{cc'} = \bigcap_{\mathcal{E}_{f^{cc'}}} \text{epi}_K a.$$

We will show that $\mathcal{E}_f \subseteq \mathcal{E}_{f^{cc'}}$. Take any e-affine function $a \in \mathcal{E}_f$,

$$a(x) = \begin{cases} S_{(x^*, z^*)}(x) + \tilde{z} & \text{if } \langle x, y^* \rangle < \alpha, \\ \emptyset & \text{otherwise.} \end{cases}$$

Then $f(x) \subseteq a(x)$, for all $x \in X$, and it holds, for all $x \in \text{dom } f \subset H_{y^*, \alpha}^-$,

$$f(x) \subseteq S_{(x^*, z^*)}(x) + \tilde{z}.$$

Taking into account that $\bar{S}_{(x^*, z^*)}(-x) + S_{(x^*, z^*)}(x) \subseteq S_{(x^*, z^*)}(0)$, we will obtain, for all $x \in \text{dom } f$,

$$f(x) + \bar{S}_{(x^*, z^*)}(-x) \subseteq S_{(x^*, z^*)}(0) + \tilde{z}.$$

Clearly $S_{(x^*, z^*)}(0) + \tilde{z}$ is an e-convex set, hence

$$-f^c(x^*, y^*, z^*, \alpha) \subseteq S_{(x^*, z^*)}(0) + \tilde{z}, \tag{10}$$

and $(x^*, y^*, z^*, \alpha) \in \text{dom } f^c$.

Now, if for some point $\bar{x} \in X$, $f^{cc'}(\bar{x}) = \emptyset$, then $f^{cc'}(\bar{x}) \subseteq a(\bar{x})$ and $a(\bar{x}) \leq_K f^{cc'}(\bar{x})$.

Then, assuming that $f^{cc'}(\bar{x}) \neq \emptyset$, from (10) and Definition 11 we conclude that

$$f^{cc'}(\bar{x}) \subseteq \bar{S}_{(x^*, z^*)}(\bar{x}) + S_{(x^*, z^*)}(0) + \tilde{z} \subseteq S_{(x^*, z^*)}(\bar{x}) + \tilde{z} = a(\bar{x}),$$

and $a \in \mathcal{E}_{f^{cc'}}$. Hence, $\text{epi}_K f^{cc'} \subseteq \text{epi}_K f$ and $f^{cc'}(x) \subseteq f_K(x)$, for all $x \in X$. □

Theorem 6 *Let $f : X \rightarrow \mathcal{P}(Z)$ be a set-valued function. Then $f^{cc'} = K$ -e-conv f .*

Proof By Lemma 4, $f^{cc'}(x) \leq_K f(x)$, for all $x \in X$, hence $f^{cc'}(x) \leq_K K$ -e-conv $f(x)$, for all $x \in X$, since $f^{cc'}$ is a K -e-conv minorant of f .

Now we will show that K -e-conv $f(x) \leq_K f^{cc'}(x)$, i.e. $f^{cc'}(x) \subseteq K$ -e-conv $f(x)$, for all $x \in X$.

Since $f(x) + K \subseteq K$ -e-conv $f(x)$, for all $x \in X$, in particular, $\text{dom } f \subseteq \text{dom } K$ -e-conv f .

Take $(x^*, y^*, z^*, \alpha) \in X^* \times X^* \times (K^* \setminus \{0\}) \times \mathbb{R}$, such that $\text{dom } f \subseteq H_{y^*, \alpha}^-$. Then, by Definition 12,

$$\sigma_f(x^*, y^*, z^*, \alpha) = \sup_{\text{epi}_K f} \{(x, x^*) + \langle z, z^* \rangle\} \leq \sup_{\text{epi}_K K\text{-e-conv } f} \{(x, x^*) + \langle z, z^* \rangle\} \quad (11)$$

Therefore, if $\text{dom } f \subseteq H_{y^*, \alpha}^-$,

$$\sigma_f(x^*, y^*, z^*, \alpha) \leq \sigma_{K\text{-e-conv } f}(x^*, y^*, z^*, \alpha).$$

On the other hand, if $\text{dom } f \not\subseteq H_{y^*, \alpha}^-$, $\text{dom } K\text{-e-conv } f \not\subseteq H_{y^*, \alpha}^-$ either, and hence, $\sigma_f(x^*, y^*, z^*, \alpha) = \sigma_{K\text{-e-conv } f}(x^*, y^*, z^*, \alpha) = +\infty$. According to Lemma 4, for all $x \in X$, $f^{cc'}(x) \subseteq (K\text{-e-conv } f)^{cc'}(x) = K\text{-e-conv } f(x)$, the last equality follows from Theorem 5. \square

Example 4 Let us consider a well-known extended real-valued function, the indicator function of a set $C \subseteq X$, $\delta_C : X \rightarrow \overline{\mathbb{R}}$,

$$\delta_C(x) = \begin{cases} 0 & \text{if } x \in C, \\ +\infty & \text{otherwise.} \end{cases}$$

We will have that C is e-convex iff δ_C is e-convex. On the other hand, denoting by δ_C^* the Fenchel conjugate of δ_C , we have that the c-conjugate of δ , $\delta_C^c : X^* \times X^* \times \mathbb{R} \rightarrow \overline{\mathbb{R}}$, is

$$\delta_C^c(x^*, y^*, \alpha) = \begin{cases} \delta_C^*(x^*) & \text{if } C \subseteq H_{y^*, \alpha}^-, \\ +\infty & \text{otherwise,} \end{cases}$$

where $\delta_C^*(x^*) = \sigma_C(x^*)$ is the well-known support function of C . Moreover, $\delta_C^{cc'} = \delta_{\text{e-conv } C}$.

Coming back to the set-valued functions framework, with K a convex cone and taking into account that $\text{e-conv } K + K = \text{e-conv } K$, let us consider the set-valued indicator function of a set $C \subset X$, $\Delta_C : X \rightarrow \mathcal{R}_K(Z)$,

$$\Delta_C(x) = \begin{cases} \text{e-conv } K & \text{if } x \in C, \\ \emptyset & \text{otherwise,} \end{cases}$$

which is K -e-convex iff C is e-convex. We calculate its c-conjugate (respect to K),

$$\Delta_C^c(x^*, y^*, z^*, \alpha) = \begin{cases} -\text{e-conv} \left[\bigcup_C \{\bar{S}_{(x^*, z^*)}(-x)\} \right] & \text{if } C \subseteq H_{y^*, \alpha}^-, \\ \emptyset & \text{otherwise.} \end{cases}$$

Making a comparison with the (convex) conjugate set-valued function of Δ_C suggested in [12, Section 4], $\Delta_C^* : X^* \times (K^* \setminus \{0\}) \rightarrow \mathcal{P}(Z)$,

$$\Delta_C^*(x^*, z^*) = -\text{cl} \left[\bigcup_C \{\bar{S}_{(x^*, z^*)}(-x)\} \right],$$

we deduce that, if $C \subseteq H_{y^*, \alpha}^-$, we could have $\Delta_C^c(x^*, y^*, z^*, \alpha) \subseteq \Delta_C^*(x^*, z^*)$, this is not the case in the scalar framework, but recall that also K -convex hull concept is tighter in the set-valued context than in the scalar one. Finally, we also have that $\Delta_C^{cc'} = \Delta_{\text{e-conv } C}$.

Acknowledgements The author sincerely thanks anonymous referees for their careful reading and thoughtful comments. Their suggestions have significantly improved the quality of the paper.

Research partially supported by MINECO of Spain and ERDF of EU, Grant MTM2014-59179-C2-1-P.

Funding Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature.

Declarations

Conflict of Interests The author declares that she has no conflict of interest.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

1. Daniilidis, A., Martínez-Legaz, J.E.: Characterizations of evenly convex sets and evenly quasiconvex functions. *J. Math. Anal. Appl.* **273**, 58–66 (2002)
2. Fajardo, M.D.: Regularity conditions for strong duality in evenly convex optimization problems. An application to Fenchel duality. *J. Convex Anal.* **22**(3), 711–731 (2015)
3. Fajardo, M.D., Goberna, M.A., Rodríguez, M.M.L., Vicente-Pérez, J.: *Even Convexity and Optimization. Handling strict inequalities*. Springer, Berlin (2020)
4. Fajardo, M.D., Rodríguez, M.M.L., Vidal, J.: Lagrange duality for evenly convex optimization problems. *J. Optim. Theory Appl.* **168**(1), 109–128 (2016)
5. Fajardo, M.D., Vidal, J.: Stable strong Fenchel and Lagrange duality for evenly convex optimization problems. *Optimization* **65**(9), 1675–1691 (2016)
6. Fajardo, M.D., Vidal, J.: A comparison of alternative c-conjugate dual problems in infinite convex optimization. *Optimization* **66**(5), 705–722 (2017)
7. Fajardo, M.D., Vidal, J.: Necessary and sufficient conditions for strong fenchel-Lagrange duality via a coupling conjugation scheme. *J. Optim. Theory Appl.* **176**(1), 57–73 (2018)
8. Fenchel, W.: A remark on convex sets and polarity. *Comm. Sèm. Math. Univ. Lund (Medd. Lunds Algebra Univ. Math. Sem.)* pp. 82–89. Tome Supplémentaire (1952)
9. Goberna, M.A., Jornet, V., Rodríguez, M.M.L.: On linear systems containing strict inequalities. *Linear Algebra Appl.* **360**, 151–171 (2003)
10. Goberna, M.A., Rodríguez, M.M.L.: Analyzing linear systems containing strict inequalities via evenly convex hulls. *Eur. J. Oper. Res.* **169**, 1079–1095 (2006)
11. Hamel, A.: *Variational Principles on Metric and Uniform Spaces*. Habilitation thesis, Martin-Luther-Universität Halle-Wittenberg (2005)
12. Hamel, A.: A duality theory for set-valued functions I: Fenchel conjugation theory. *Set-Valued Anal.* **17**, 153–182 (2009)
13. Khan, A.A., Tammer, C., Zălinescu, C.: *Set-valued Optimization*. Springer, Berlin (2015)
14. Klee, V., Maluta, E., Zanco, C.: Basic properties of evenly convex sets. *J. Convex Anal.* **14**(1), 137–148 (2006)
15. Kuroiwa, D., Tanaka, T., Ha, T.: On cone convexity of set-valued maps. *Nonlinear Anal.* **30**(3), 1487–1496 (1997)
16. Martínez-Legaz, J.E.: A generalized concept of conjugation methods. In: Hiriart-Urruty, J.B., Oettli, W., Stoer, J. (eds.) *Optimization: Theory and Algorithms, Lecture Notes in Pure and Applied Mathematics*, vol. 86, pp. 45–49. Marcel Dekker, New York (1983)
17. Martínez-Legaz, J.E.: Quasiconvex duality theory by generalized conjugation methods. *Optimization* **19**, 603–652 (1988)
18. Martínez-Legaz, J.E.: Duality between direct and indirect utility functions under minimal hypotheses. *J. Math. Econ.* **20**, 199–209 (1991)
19. Martínez-Legaz, J.E.: Generalized convex duality and its economic applications. In: *Handbook of Generalized Convexity and Generalized Monotonicity, Nonconvex Optim. Appl.*, vol. 76, pp. 237–292. Springer, New York (2005)

20. Martínez-Legaz, J.E., Vicente-Pérez, J.: The e-support function of an e-convex set and conjugacy for e-convex functions. *J. Math. Anal. Appl.* **376**, 602–612 (2011)
21. Moreau, J.J.: Inf-convolution, sous-additivité, convexité des fonctions numériques. *J. Math. Pures Appl.* **49**, 109–154 (1970)
22. Passy, U., Prisman, E.: Conjugacy in quasiconvex programming. *Math. Programming* **30**, 121–146 (1984)
23. Penot, J.P., Volle, M.: On quasiconvex duality. *Math. Oper. Res.* **15**, 597–625 (1990)
24. Rodríguez, M.M.L., Vicente-Pérez, J.: On evenly convex functions. *J. Convex Anal.* **18**, 721–736 (2011)
25. Zălinescu, C.: *Convex Analysis in General Vector Spaces*. World Scientific, New Jersey (2002)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.