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Distance Function Associated to a Prox-regular set

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Abstract

In this paper, we provide in a general Hilbert space new characterizations of uniform prox-regularity involving outside but sufficiently close points of considered sets. We show that the complement of a prox-regular set is nothing but the union of closed balls with common radius. We derive from this that the prox-regularity of a given closed set is equivalent to the semiconvexity property of its distance function. Various estimates involving the metric projection mapping to a prox-regular set are also established.

Keywords Variational analysis · Prox-regularity · Distance function · Semiconvexity · Subdifferential

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1 Introduction

Distance functions to prox-regular sets have been involved in the theory of sweeping processes (see, e.g., [13, 18, 20, 27–29, 32]), optimization and control problems [1, 2, 12, 21, 23] and many other domains.

Diverse properties of distance functions under prox-regularity have been established in [3–5, 9, 10, 15, 16, 18, 19, 26]. The aim of the present paper is to provide for distance functions to prox-regular sets several new properties and estimates in the general setting of Hilbert space. Estimates for the metric projection to prox-regular sets are also established.

Notation and necessary preliminaries related proximal normals and subgradients and to generalities on prox-regular sets are given in Section 2. New results on enlarged sets as well as on sets of exterior points of prox-regular sets are the subject of Section 3, while Section 4

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is devoted to some properties of the metric projection to prox-regular sets and to various new metric characterizations of such sets. Finally, Section 5 offers a general characterization of prox-regularity by means of semiconvexity of the distance function.

2 Notation and Preliminaries

Throughout the paper, $\mathcal{H}$ is a real Hilbert space endowed with its inner product $\langle \cdot, \cdot \rangle$ and its associated norm $\|\cdot\|$. The interior (resp. closure) of a subset $A \subset \mathcal{H}$ with respect to the norm $\|\cdot\|$ is denoted by $\mathrm{int}_{\mathcal{H}} A$ (resp. $\mathrm{cl}_{\mathcal{H}} A$). The letter $\mathbb{B}$ (resp. $\mathbb{S}$) stands for the closed unit ball (resp. the unit sphere) of $\mathcal{H}$ with respect to $\|\cdot\|$. The open (resp. closed) ball centered at $x \in \mathcal{H}$ with radius $r > 0$ is denoted by $B(x, r)$ (resp. $B[x, r]$).

The *metric projection multimapping* $\mathrm{Proj}_S : \mathcal{H} \rightrightarrows \mathcal{H}$ associated to a set S is defined as $\mathrm{Proj}_S(x) := \{y \in S : d_S(x) = \|x - y\|\}$ for all $x \in \mathcal{H}$, where $d_S(\cdot)$ is the *distance function* from S, that is $d_S(x) =: d(x, S) := \inf_{y \in S} \|x - y\|$. When the set $\mathrm{Proj}_S(\overline{x})$ is reduced to a singleton for some vector $\overline{x} \in \mathcal{H}$, we say that the metric projection of $\overline{x}$ on S is well defined. In such a case, the unique element of $\mathrm{Proj}_S(\overline{x})$ is denoted by $P_S(\overline{x})$ or $\mathrm{proj}_S(\overline{x})$.

2.1 Proximal Normal Cone and its Associated Subdifferential

Let us start by giving some preliminaries about normal cones and subdifferentials which will be deeply involved in the development below. For more details, we refer the reader to [14, 24, 30, 31, 33]. Throughout this subsection, we consider a nonempty closed subset S of $\mathcal{H}$ and a function $f : U \to \mathbb{R} \cup \{+\infty\}$ defined on a nonempty open subset U of $\mathcal{H}$.

A vector $\zeta \in \mathcal{H}$ is said to be a *proximal normal* to S at $x \in S$ whenever there exists a real $r > 0$ such that $x \in \mathrm{Proj}_S(x + r\zeta)$. The set $N(S; x)$ (which is a convex cone containing 0 but not necessarily closed) of all proximal normal vectors to S at $x \in S$ is called the *proximal normal cone* of S at x. By convention, if $x \in \mathcal{H} \setminus S$, we put $N(S; x) := \emptyset$. It directly follows from the definition of proximal normals that for each $u \in \mathcal{H}$ with $\mathrm{Proj}_S(u) \neq \emptyset$,

$$u - \pi \in N(S; \pi) \quad \text{for all } \pi \in \mathrm{Proj}_S(u). \tag{1}$$

A vector $\zeta \in \mathcal{H}$ is said to be a *proximal subgradient* of f at $\overline{x} \in U$ with $f(\overline{x})$ finite, provided that there are a real $\sigma \geq 0$ and a real $\eta > 0$ such that

$$\langle \zeta, y - \overline{x} \rangle \leq f(y) - f(\overline{x}) + \sigma \|y - \overline{x}\|^2 \quad \text{for all } y \in B(\overline{x}, \eta),$$

which is known to be equivalent to $(\zeta, -1) \in N\big(\mathrm{epi}\, f; (\overline{x}, f(\overline{x}))\big)$, with $\mathcal{H} \times \mathbb{R}$ endowed with its natural product structure and where $\mathrm{epi}\, f := \{(x, r) \in \mathcal{H} \times \mathbb{R} : x \in U, f(x) \leq r\}$ is the *epigraph* of f. The set $\partial f(\overline{x})$ of all such proximal subgradients is called the *proximal subdifferential* of f at $\overline{x}$. If f is not finite at $\overline{x} \in U$, it is clear that $\partial f(\overline{x}) := \emptyset$.

The proximal subgradients of $d_S(\cdot)$ is of great interest and will be at the heart of the paper. Let us first give the following description of $\partial d_S(x)$ (see [8, Theorem 4.1]) through proximal normals to S at $x \in S$:

$$\partial d_S(x) = N(S; x) \cap \mathbb{B} \quad \text{for all } x \in S. \tag{2}$$

On the other hand, for any $x \in \mathcal{H}$ with $\partial d_S(x) \neq \emptyset$, it is known (see, e.g., [16, Lemma 5, p. 114]) that $\mathrm{Proj}_S(x)$ is a singleton (i.e., $P_S(x)$ is well defined) along with

$$d_S(x)\partial d_S(x) = \{x - P_S(x)\}. \tag{3}$$

Putting together (3), (1) and (2), we then see that for every $x \in \mathcal{H} \setminus S$ with $\partial d_S(x) \neq \emptyset$,

$$\partial d_S(x) = \left\{ d_S(x)^{-1}\big(x - P_S(x)\big) \right\} \subset N\big(S; P_S(x)\big) \cap \mathbb{S} \subset \partial d_S\big(P_S(x)\big). \tag{4}$$

Besides the equality (2), we have (see, [8, 15]) a full description of $\partial d_S(x)$ for any outside point, say $x \in \mathcal{H} \setminus S$. Indeed, for $r := d_S(x) > 0$ and for the closed r-enlargement $\mathrm{Enl}_r(S) := \{u \in \mathcal{H} : d_S(u) \le r\}$ of S, it is known that

$$\partial d_S(x) = N\big(\mathrm{Enl}_r(S); x\big) \cap \mathbb{S},$$

in particular (see (2))

$$\partial d_S(x) \subset \partial d_{\mathrm{Enl}_r(S)}(x). \tag{5}$$

2.2 Semi-convexity

Let $f : U \to \mathbb{R} \cup \{+\infty\}$ be a function defined on a (not necessarily open) nonempty convex subset U of $\mathcal{H}$. One says that the function f is *σ-linearly semiconvex* on U for some real $\sigma \ge 0$ whenever for all $t \in]0, 1[$, all $x, y \in U$, one has

$$f\big(tx + (1-t)y\big) \le tf(x) + (1-t)f(y) + \frac{\sigma}{2} t(1-t) \|x - y\|^2 .$$

If $-f$ is σ-linearly semiconvex on U for some real $\sigma \ge 0$, the function f is said to be *σ-linearly semiconcave* on U. From the very definition, we easily derive that the pointwise supremum of σ-linearly semiconvex functions is σ-linearly semiconvex.

The function f is said to be *locally linearly semiconvex* (resp., *locally semiconcave*) if f is linearly semiconvex (resp., linearly semiconcave) on a neighborhood of each point of U.

It can be checked that f is σ-linearly semiconvex on U for some $\sigma \ge 0$ if and only if the function $f + \frac{\sigma}{2} \|\cdot\|^2$ is convex on U.

2.3 Prox-regular sets

This paragraph is devoted to the class of prox-regular sets. For more details, we refer the reader to [26] and to the survey [16] (see also the forthcoming monograph [33] and the references therein), where in addition to the results below and their proofs, historical comments and applications can be found.

Definition 1 Let S be a nonempty closed subset of $\mathcal{H}$ and $r \in]0, +\infty]$. One says that S is r-prox-regular whenever, for every $x \in S$, for every $v \in N(S; x) \cap \mathbb{B}$ and for every real $t \in]0, r]$, one has

$$x \in \mathrm{Proj}_S(x + tv).$$

The following proposition collects some fundamental characterizations and properties of prox-regular sets (see, e.g., [16]). Before stating it, recall that for any extended real $r > 0$, the *r-open enlargement* and *r-open tube* around a subset $S \subset \mathcal{H}$ are respectively defined as

$$U_r(S) := \{x \in \mathcal{H} : d_S(x) < r\} \quad \text{and} \quad \mathrm{Tube}_r(S) = U_r(S) \setminus S.$$

Proposition 1 *Let S be a nonempty closed subset of $\mathcal{H}$. The following assertions are equivalent.*

(a) *The set S is r-prox-regular;*

(b) *for all* $x_1, x_2 \in S$, *for all* $\xi \in N(S; x_1) \cap \mathbb{B}$, *one has*

$$\langle \xi, x_2 - x_1 \rangle \leq \frac{1}{2r} \|x_1 - x_2\|^2 ;$$

(c) *the multimapping* $\mathrm{Proj}_S(\cdot)$ *is single-valued on* $U_r(S)$ *and for all* $x, x' \in U_r(S)$, *one has*

$$\left\| P_S(x) - P_S(x') \right\| \leq \left(1 - \frac{d_S(x)}{2r} - \frac{d_S(x')}{2r}\right)^{-1} \left\| x - x' \right\| ;$$

(d) *for any* $s \in]0, r[$, *for all* $x, x' \in U_s(S)$, *one has*

$$\left\| P_S(x) - P_S(x') \right\| \leq \frac{1}{1 - s/r} \left\| x - x' \right\| ;$$

(e) *for all* $x \in \mathrm{Tube}_r(S)$ *such that* $u := P_S(x)$ *is well defined, one has*

$$u = P_S\big(u + t \frac{x - u}{d_S(x)}\big) \quad \text{for all } t \in [0, r[;$$

(f) *the function* d_S^2 *is* $C^{1,1}$ *on* $U_r(S)$ *and its gradient is given by*

$$\nabla d_S^2(x) = 2\big(x - P_S(x)\big) \quad \text{for all } x \in U_r(S);$$

(g) *the function* d_S *is* C^1 *on* $\mathrm{Tube}_r(S)$;
(h) *for all* $x \in U_r(S)$, *one has* $\partial d_S(x) \neq \emptyset$.

Remark 1 We point out that assertions (c) and (e) guarantee that for any $x \in \mathrm{Tube}_r(S) \setminus S$ where S is an r-prox-regular set of $\mathcal{H}$ for some real $r > 0$, the vector $u := P_S(x)$ is well defined along with $u \in \mathrm{Proj}_S\big(u + r \frac{x-u}{d_S(x)}\big)$.

Let us end this section with a result in [2] concerning the prox-regularity of sublevel sets. For the proof and other developements on preservation of prox-regularity, we refer the reader to [2, 33–35] and the references therein.

Proposition 2 *Let* $g_1, \ldots, g_m : \mathcal{H} \to \mathbb{R}$ *such that the set*

$$C = \{x \in \mathcal{H} : g_1(x) \leq 0, \ldots, g_m(x) \leq 0\}$$

is nonempty. Assume that there is an extended real $\rho \in]0, +\infty]$ *such that:*

(i) *for all* $k \in K := \{1, \ldots, m\}$, g_k *is* C^1 *on* $U_\rho(C)$;
(ii) *there is a real* $\gamma \geq 0$ *such that for any* $k \in K$, *for all* $x_1, x_2 \in U_\rho(C)$,

$$\langle \nabla g_k(x_1) - \nabla g_k(x_2), x_1 - x_2 \rangle \geq -\gamma \|x_1 - x_2\|^2 .$$

Assume also that there is a real $\delta > 0$ *such that for each* $x \in \mathrm{bdry}\, C$, *there exists* $\overline{v} \in \mathbb{B}$ *for which for every* $k \in K(x) := \{j \in K : g_j(x) = \max_{i \in K} g_i(x)\}$,

$$\langle \nabla g_k(x), \overline{v} \rangle \leq -\delta.$$

Then, the set C *is* r*-prox-regular with* $r = \min\left\{\rho, \frac{\delta}{\gamma}\right\}$.

3 Levels and Sublevel sets Associated to Distance Functions

In this present section, we establish the uniform prox-regularity of the following level and sublevel sets

$$\mathrm{Enl}_r(S) := \{d_S \le r\},\ D_r(S) := \{d_S = r\} \quad \text{and} \quad \mathrm{Exte}_r(S) := \{d_S \ge r\}.$$

Besides its own interest, such a development will be greatly involved in Section 5 which is devoted to semiconvexity property of the distance function. The first result provides various important links between the aforementioned sets. Before stating it, let us mention here that the assertion (a) below has already been established in [8] in the context of general normed spaces.

Proposition 3 *Let S be a nonempty closed subset of $\mathcal{H}$. The following hold.*

(a) *For every real $s > 0$, one has*

$$d\big(x, \mathrm{Enl}_s(S)\big) = d(x, S) - s = d\big(x, D_s(S)\big) \quad \textit{for all } x \in \mathcal{H} \setminus \mathrm{Enl}_s(S).$$

(b) *For all reals $0 < s < r$, one has*

$$U_r(S) = U_{r-s}\big(\mathrm{Enl}_s(S)\big).$$

(c) *For every real $r > 0$, one has*

$$\mathrm{cl}_{\mathcal{H}}\big(U_r(S)\big) = \mathrm{Enl}_r(S), \tag{6}$$

or equivalently

$$\mathrm{int}_{\mathcal{H}}\big(\mathrm{Exte}_r(S)\big) = \mathcal{H} \setminus \mathrm{Enl}_r(S) = \{u \in \mathcal{H} : d_S(u) > r\};$$

from (6), one also has

$$D_r(S) = \mathrm{bdry}_{\mathcal{H}}\big(U_r(S)\big).$$

If in addition the set S is r-prox-regular for some $r \in]0, +\infty]$, then the following assertions hold true:

(d) *For every $s \in]0, r]$, one has*

$$d\big(x, \mathrm{Exte}_s(S)\big) = s - d(x, S) = d\big(x, D_s(S)\big) \quad \textit{for all } x \in \mathrm{Tube}_s(S).$$

(e) *For every real $s \in]0, r]$, one has*

$$\mathrm{Tube}_s\big(\mathrm{Exte}_s(S)\big) = \mathrm{Tube}_s(S).$$

(f) *For every $s \in]0, r[$, one has*

$$\mathrm{cl}_{\mathcal{H}}\big(\mathcal{H} \setminus \mathrm{Enl}_s(S)\big) = \{u \in \mathcal{H} : d_S(u) \ge s\} = \mathrm{Exte}_s(S), \tag{7}$$

or equivalently

$$\mathrm{int}_{\mathcal{H}}\big(\mathrm{Enl}_s(S)\big) = \{u \in \mathcal{H} : d_S(u) < s\} = U_s(S);$$

further, one also has

$$D_s(S) = \mathrm{bdry}_{\mathcal{H}}\big(\mathrm{Enl}_s(S)\big).$$

Proof Let $s \in]0, +\infty[$ and let $r \in]0, +\infty]$.

(a) Let $x \in \mathcal{H} \setminus \mathrm{Enl}_s(S)$. Fix any $y \in \mathrm{Enl}_s(S)$. Pick any real $\varepsilon > 0$. There is $y_\varepsilon \in S$ such that

$$\|y - y_\varepsilon\| \le d_S(y) + \varepsilon \le s + \varepsilon$$

and then $\|x-y\| \geq \|x-y_\varepsilon\| - \|y_\varepsilon - y\| \geq d_S(x) - s - \varepsilon$. Thus, we obtain

$$d\big(x, D_s(S)\big) \geq d\big(x, \mathrm{Enl}_s(S)\big) \geq d(x,S) - s. \tag{8}$$

To confirm the equalities in (a) we must show the inequality $d(x,S) - s \geq d\big(x, D_s(S)\big)$. Fix any $z \in S$. We consider the continuous function $h : [0,+\infty[\to \mathbb{R}$ defined by

$$h(t) := d_S\big(tx + (1-t)z\big) \quad \text{for all } t \geq 0.$$

We have $h(0) = 0$ and $h(1) > s$, so we can find $t_0 \in]0,1[$ such that $h(t_0) = s$. Set $\omega := t_0 x + (1-t_0)z$ and observe that $d_S(\omega) = h(t_0) = s$ along with $\|x-z\| = \|x-\omega\| + \|\omega - z\|$. According to the inclusion $z \in S$, we have $\|x-z\| \geq \|x-\omega\| + d_S(\omega) = \|x-\omega\| + s$. Thanks to the inclusion $\omega \in D_s(S)$, we get $\|x-z\| \geq d\big(x, D_s(S)\big) + s$. Since $z \in S$ has been arbitrarily choosen, the latter inequality entails the following one

$$d(x,S) \geq d\big(x, D_s(S)\big) + s. \tag{9}$$

The equalities in (a) then follow from (8) and (9).

(b) It is a straightforward consequence of (a).

(c) Assume that $r < +\infty$, otherwise there is nothing to establish. The inclusion $\mathrm{cl}_{\mathcal{H}}\big(U_r(S)\big) \subset \mathrm{Enl}_r(S)$ comes from the continuity of $d_S(\cdot)$. Let us establish the converse inclusion. Let $u \in \mathrm{Enl}_r(S)$. We may suppose that $d_S(u) = r$. Let $\varepsilon > 0$ be a real. Pick any sequence $(z_n)_{n\in\mathbb{N}}$ of S such that $r_n := \|u - z_n\| \to r$. Choose any $N \in \mathbb{N}$ such that $r_N \neq 0$ and $r_N - r < \varepsilon$. Fix any $t \in [0,1]$ such that $1 - \frac{r}{r_N} < t < \frac{\varepsilon}{r_N}$ and observe that

$$\|(1-t)u + tz_N - u\| = tr_N < \varepsilon$$

and

$$d\big((1-t)u + tz_N, S\big) \leq \|(1-t)u + tz_N - z_N\| \leq (1-t)r_N < r.$$

Consequently, we have $B(u,\varepsilon) \cap U_r(S) \neq \emptyset$ and this translates the inclusion $u \in \mathrm{cl}_{\mathcal{H}}\big(U_r(S)\big)$. The desired equality is then established.

Now, we assume for the rest of the proof that S is r-prox-regular.

(d) We may suppose that $r < +\infty$. Assume that $s \in]0,r]$ and $u \in \mathrm{Tube}_s(S)$. Set $p := \mathrm{proj}_S(u)$ and $v := p + s\frac{u-p}{\|u-p\|}$. According to Proposition 1(e) (see also Remark 1) we have the inclusion $p \in \mathrm{Proj}_S(v)$. Therefore, $d_S(v) = s$ and this allows us to write

$$\begin{aligned} d(u,S) + d\big(u, \mathrm{Exte}_s(S)\big) &\leq d(u,S) + d\big(u, D_s(S)\big) \\ &\leq \|u-p\| + \|u-v\| = s. \end{aligned} \tag{10}$$

On the other hand, we observe that for every $x \in \mathrm{Exte}_s(S)$

$$\|x-u\| \geq \|x-p\| - \|u-p\| \geq d(x,S) - d(u,S) \geq s - d(u,S),$$

which gives $d\big(u, \mathrm{Exte}_s(S)\big) \geq s - d(u,S)$, hence

$$d\big(u, \mathrm{Exte}_s(S)\big) + d(u,S) \geq s. \tag{11}$$

It remains to put together (10) and (11) to finish the proof of (d).

(e) Assume that $s \in]0,r]$. Set $E := \mathrm{Exte}_s(S)$. The inclusion $\mathrm{Tube}_s(S) \subset \mathrm{Tube}_s(E)$ directly follows from (d). Let $u \in \mathrm{Tube}_s(E)$. Obviously, we observe that $u \notin E$, i.e., $d_S(u) < s$. On the other hand, the inequality $d_E(u) < s$ furnishes $v \in E$ such that $\|u-v\| < s$. If $u \in S$, we would have $s \leq d_S(v) \leq \|u-v\| < s$, which cannot hold true. Then, we have $0 < d_S(u) < s$, i.e., $u \in \mathrm{Tube}_s(S)$.

(f) Assume that $s \in]0, r[$. First, note that we always have the inclusion $U_s(S) \subset \mathrm{int}_{\mathcal{H}}\big(\mathrm{Enl}_s(S)\big)$, or equivalently

$$\mathrm{cl}_{\mathcal{H}}\big(\mathcal{H} \setminus \mathrm{Enl}_s S\big) \subset \mathcal{H} \setminus U_s(S) = \mathrm{Ext}_s(S).$$

Let us establish the converse inclusion. Fix any $u \in \mathcal{H} \setminus U_s(S)$. We may assume that $d_S(u) = s$. Let $(s_n)_{n\in\mathbb{N}}$ be a sequence of $]s, r[$ with $s_n \to s$. Since S is r-prox-regular, the set $\mathrm{Proj}_S(u)$ is reduced to a singleton, i.e., $p := \mathrm{proj}_S(u)$ is well defined. Set for each $n \in \mathbb{N}$, $u_n := p + s_n \frac{u-p}{\|u-p\|}$. By virtue of Proposition 1(e), we have the inclusion $p \in \mathrm{Proj}_S(u_n)$ for every integer $n \geq 1$. We also see that $d_S(u_n) = s_n > s$, so $u_n \in \mathcal{H} \setminus \mathrm{Enl}_s S$ for each $n \in \mathbb{N}$. Further, $(u_n)_{n\in\mathbb{N}}$ converges to $p + s\frac{u-p}{\|u-p\|}$. Since $s = d_S(u) = \|u - p\|$, we have

$$p + s\frac{u-p}{\|u-p\|} = u.$$

Consequently, we get $u \in \mathrm{cl}_{\mathcal{H}}\big(\mathcal{H} \setminus \mathrm{Enl}_s S\big)$. The proof is then complete.

□

Remark 2 If $s \in \{0, r\}$, then (7) does not hold in general. Indeed, in the case $s = 0$, (7) means $\mathrm{int}_{\mathcal{H}}(S) = \emptyset$. Now, let us focus on the case $s = r$. Consider the set $S = \{t \in \mathbb{R} : |t| \geq 1\}$ which is r-prox-regular with $r := 1$. It is readily seen that $\mathrm{Enl}_r(S) = \mathbb{R}$ and $U_r(S) = \mathbb{R} \setminus \{0\}$, hence

$$\emptyset = \mathrm{cl}_{\mathbb{R}}(\mathbb{R} \setminus \mathrm{Enl}_r(S)) \neq \mathrm{Ext}_r(S) = \{0\}.$$

Now, we can prove the prox-regularity of enlarged and exterior sets.

Theorem 1 *Let S be an r-prox-regular subset of $\mathcal{H}$ for some real $r > 0$. Let also $s \in]0, r[$. The following hold.*
(a) The closed s-enlargement $\mathrm{Enl}_s(S)$ of S is $(r - s)$-prox-regular.
(b) If $S \neq \mathcal{H}$, then $D_s(S)$ is a C^1-submanifold which is $\min\{r - s, s\}$-prox-regular.
(c) If $S \neq \mathcal{H}$, then the r-exterior $\mathrm{Exte}_r(S)$ is r-prox-regular.

Proof Let us consider the function $\varphi : \mathcal{H} \to \mathbb{R}$ defined by

$$\varphi(x) := \frac{1}{2}(d_S^2(x) - s^2) \quad \text{for all } x \in \mathcal{H}.$$

(a) Fix any $s' \in]s, r[$. It is readily seen that $E := \mathrm{Enl}_s(S) = \{\varphi \leq 0\}$. According to the r-prox-regularity of S, we know from Proposition 1(f) that $\varphi(\cdot)$ is continuously differentiable on $U_r(S)$, or equivalently continuously differentiable on $U_{r-s}(E) \supset U_{s'-s}(E)$ (since $U_{r-s}(E) = U_r(S)$ by Proposition 3(b)) along with

$$\nabla\varphi(x) = x - \mathrm{P}_S(x) \quad \text{for all } x \in U_{r-s}(E).$$

On the other hand, using the Lipschitz property of P_S on $U_s(S)$ with Lipschitz constant $\frac{1}{1-s/r}$ therein (see Proposition 1(d)) and putting $\gamma := \frac{1}{1-s/r} - 1 \geq 0$, we get for all $x, y \in U_{s'-s}(E) \subset U_s(S)$,

$$\begin{aligned} \langle \nabla\varphi(x) - \nabla\varphi(y), x - y\rangle &= \big\langle x - P_S(x) - \big(y - P_S(y)\big), x - y\big\rangle \\ &= \|x - y\|^2 + \langle P_S(y) - P_S(x), x - y\rangle \\ &\geq -\gamma\, \|x - y\|^2. \end{aligned}$$

Now, let u be a boundary point of E, i.e., $d_S(u) = s$ by Proposition 3(f). Set $v_u := -\frac{s}{d_S^2(u)}\big(u - P_S(u)\big) \in \mathbb{B}$ and observe that $\langle u - P_S(u), v_u\rangle = -s$. By virtue of Proposition 2, the set E is $\min\left\{s' - s, \frac{s}{\gamma}\right\}$-prox-regular. It remains to observe that

$$\frac{s}{\gamma} = s\left(\frac{1}{1 - s/r} - 1\right)^{-1} = r - s$$

and to let $s' \uparrow r$ to get the desired $(r - s)$-prox-regularity.

(b) Note that $D := D_s(S) \neq \emptyset$ is a C^1-submanifold in $\mathcal{H}$ since d_S is C^1 on the (open) tube $\mathrm{Tube}_r(S)$ with its gradient nonzero therein. Set $\rho := \min\{s, r - s\}$, $U_1 := \{0 < d_S < s\}$ and $U_2 := \{s < d_S < r\}$. According to (a) and (d) in Proposition 3, we have

$$d_D(x) = s - d_S(x) \quad \text{for all } x \in U_1 \tag{12}$$

and

$$d_D(x) = d_S(x) - s \quad \text{for all } x \in U_2. \tag{13}$$

We claim that $T := \mathrm{Tube}_\rho(D) \subset U_1 \cup U_2$. Fix any $x \in T$. By the very definition of T we have $x \notin D$, i.e., $d_S(x) \neq s$. Therefore, it suffices to show that $0 < d_S(x) < r$. Assume for a moment that $d_S(x) = 0$. Since $d_D(x) < \rho \leq s$, we can find some $y \in D$ such that $\|x - y\| < s$ and this cannot hold true since

$$s = d_S(y) - d_S(x) \leq \|x - y\| < s.$$

Now, assume that $d_S(x) \geq r$. Since $d_D(x) < \rho \leq r - s$ there is some $z \in D$ such that $\|x - z\| < r - s$. Fix any real $\varepsilon > 0$ small enough such that $\|x - z\| < r - s - \varepsilon$. Using the equality $d_S(z) = s$, we get $\zeta \in S$ such that $\|z - \zeta\| < s + \varepsilon$. We are then able to write

$$r \leq d_S(x) \leq \|x - \zeta\| \leq \|x - z\| + \|z - \zeta\| < r - s - \varepsilon + s + \varepsilon = r,$$

which is the desired contradiction. So, it is established that $T \subset U_1 \cup U_2$. Coming back to (12) and (13) and noting that $d_D(\cdot)$ is C^1 on $U_1 \cup U_2$ (see Proposition 1(g)) we see that $d_D(\cdot)$ is C^1 on T. It remains to invoke Proposition 1(g) again to obtain the desired ρ-prox-regularity of the set D. The proof of (b) is then complete.

(c) Assume that $S \neq \mathcal{H}$. Set $C := \mathrm{Exte}_r(S)$. According to Proposition 3(d), we have

$$d_C(u) = r - d_S(u) \quad \text{for all } u \in \mathrm{Tube}_r(S).$$

Then, by virtue of Proposition 1(g), we see that $d_C(\cdot)$ is continuously differentiable on the open set $\mathrm{Tube}_r(S)$. On the other hand, we know (see again Proposition 3(e)) that the r-open tube around S coincides with the r-open tube around $C = \mathrm{Exte}_r(S)$. Therefore, the distance function $d_C(\cdot)$ is continuously differentiable on $\mathrm{Tube}_r(C)$ and this translates the r-prox-regularity of the set C. □

Remark 3 (i) Note that the assertions (a) and (b) of the latter theorem fail for $s = r$. This can be seen with the 1-prox-regular set $S := \{(-1, 0), (1, 0)\} \subset \mathbb{R}^2$.

(ii) The constant $(r - s)$ in the assertion (a) above is sharp. Indeed, if $S = \mathcal{H} \setminus B(0, r)$ (which is r-prox-regular), then the set $\mathrm{Enl}_s(S) = \mathcal{H} \setminus B(0, r - s)$ is $(r - s)$-prox-regular.

The following proposition complements property (a) in Theorem 1.

Proposition 4 *Let three reals $0 < s < r \leq r'$ and let S be an r-prox-regular subset of $\mathcal{H}$. If $\mathrm{Enl}_s(S)$ is r'-prox-regular, then S is r'-prox-regular.*

Proof Assume that $C := \mathrm{Enl}_s(S)$ is r'-prox-regular. Fix any $x \in U_{r'}(S)$. We claim that $\partial d_S(x) \neq \emptyset$. In view of Proposition 1(h), we may suppose that $d_S(x) \geq r$. First, observe that Proposition 3(a) says that

$$d(\cdot, C) = d(\cdot, S) - s \quad \text{on } \mathcal{H} \setminus C, \tag{14}$$

in particular $d_C(x) = d_S(x) - s < r' - s$, so $x \in U_{r'}(C)$. Combining the inclusion $x \in U_{r'}(C)$ with the r'-prox-regularity of C then yields $\partial d_C(x) \neq \emptyset$. Using (14) again, we can write $\partial d_C(x) = \partial(d_S - s)(x) = \partial d_S(x) \neq \emptyset$. Consequently, the set S is r'-prox-regular by (h) in Proposition 1. The proof is finished. □

As a direct consequence, we derive the fact that a nonconvex prox-regular set does not possess a convex enlargement. More precisely:

Corollary 1 *Let S be an r-prox-regular subset of $\mathcal{H}$ with $r \in]0, +\infty[$. The following assertions are equivalent.*

(a) *The set S is convex;*
(b) *there exists $s \in]0, r[$ such that $\mathrm{Enl}_s(S)$ is convex;*
(c) *there exists $s \in]0, r[$ such that $U_s(S)$ is convex.*

Proof Obviously, the assertion (a) implies (b). The converse implication $(b) \Rightarrow (a)$ follows from Proposition 4. It remains to observe that the equivalence $(b) \Leftrightarrow (c)$ is a direct consequence of the equalities $\mathrm{int}_{\mathcal{H}}\big(\mathrm{Enl}_s(S)\big) = U_s(S)$ and $\mathrm{cl}_{\mathcal{H}}\big(U_s(S)\big) = \mathrm{Enl}_s(S)$ in Proposition 3. The proof is complete. □

4 Characterizations of *r*-prox-regular sets via Distance from Outside Points

From Proposition 1, we know that the r-prox-regularity of a (nonempty closed) subset S of $\mathcal{H}$ is equivalent to the inequality

$$\langle \xi, x' - x \rangle \leq \frac{1}{2r} \left\| x' - x \right\|^2 \quad \text{for all } x, x' \in S, \xi \in N(S; x) \cap \mathbb{B}, \tag{15}$$

which translates some hypomonotonicity property of the truncated normal cone multimapping $x \mapsto N(S; x) \cap \mathbb{B}$. Such a characterization involves only inside points of the considered set, namely $x, x' \in S$. The crucial role of the open r-enlargement $U_r(S) := \{d_S < r\}$ in various characterizations of r-prox-regular sets (see, e.g., [16, 33]) naturally leads to develop several extensions relaxing the inclusion $x, x' \in S$. This can be done by replacing $N(S; x) \cap \mathbb{B}$ (which is empty if $x \notin S$) in (15) by $\partial d_S(x)$ (see the equality (2)). There are very few results availaible in that direction: we refer to [9, Theorem 3.4] for the inequality satisfied for any r-prox-regular set $S \subset \mathcal{H}$

$$\langle \xi, x' - x \rangle \leq \frac{8}{r - d_S(x)} \|x' - x\|^2 + d_S(x') - d_S(x) \tag{16}$$

and

$$\langle \xi, x' - x \rangle \leq \frac{2}{r} \|x' - x\|^2 + d_S(x') \tag{17}$$

for any $x, x' \in U_r(S)$ and any $\xi \in \partial d_S(x)$. In the same vein, we also mention [18, Lemma 2.1] where the following estimate is provided for $x \in S$,

$$\left\langle \xi, x' - x \right\rangle \leq \frac{1}{2r} \|x' - x\|^2 + \frac{1}{2r} d_S^2(x') + \left(\frac{1}{r} \|x' - x\| + 1 \right) d_S(x'). \tag{18}$$

While (18) characterizes the prox-regularity of S with r as constant of prox-regularity, (16) and (17) entail the prox-regularity of S with constant $r/16$ and $r/4$ respectively. Estimates of constant of prox-regularity are often involved in the context of existence of solutions for prox-regular sweeping processes through Moreau's catching-up algorithm (see, e.g., [28] and the references therein).

Our first aim here is to provide in Theorem 2 a full characterization of r-prox-regularity encompassing (15) for possibly outside points, say $x, x' \in U_r(S)$. Before stating it, let us establish the following lemma via the elementary equality

$$\|u\|^2 - \|v\|^2 = 2\langle u, u - v\rangle - \|u - v\|^2 \leq 2\langle u, u - v\rangle \quad \text{for all } u, v \in \mathcal{H}. \tag{19}$$

Lemma 1 *Let S be an r-prox-regular subset of $\mathcal{H}$ for some extended real $r \in]0, +\infty]$. The following hold.*
(a) For all $x \in U_r(S)$ and $x' \in S$, one has

$$\left(1 - \frac{d_S(x)}{r}\right) \left\| P_S(x) - x' \right\|^2 \leq \left\| x - x' \right\|^2 - d_S^2(x), \tag{20}$$

in particular

$$\sqrt{1 - \frac{d_S(x)}{r}} \left\| P_S(x) - x' \right\| \leq \left\| x - x' \right\|.$$

(b) For all $x, x' \in U_r(S)$, one has

$$\left(1 - \frac{d_S(x)}{r}\right) \left\| P_S(x) - P_S(x') \right\|^2 \leq \left\| x - P_S(x') \right\|^2 - d_S^2(x),$$

in particular

$$\sqrt{1 - \frac{d_S(x)}{r}} \left\| P_S(x) - P_S(x') \right\| \leq \left\| x - P_S(x') \right\|.$$

(c) For all $x, x' \in U_r(S)$, one has with $p := P_S(x)$ and $p' := P_S(x')$

$$\left(1 - \frac{d_S(x)}{2r} - \frac{d_S(x')}{2r}\right) \left\| p - p' \right\|^2 \leq \frac{1}{2}\left(\left\| x - p' \right\|^2 - d_S^2(x) + \left\| x' - p \right\|^2 - d_S^2(x')\right).$$

Proof First, note that (b) and (c) can be directly derived from the inequality (20) in (a). So, let us prove the assertion (a). Fix any $x \in U_r(S)$ and $x' \in S$. By virtue of Proposition 1(c), $y := \mathrm{proj}_S(x)$ is well defined. A direct application of (19) then gives

$$\left\| x' - y \right\|^2 - \left\| x' - x \right\|^2 = 2\left\langle x' - y, x - y \right\rangle - \|x - y\|^2. \tag{21}$$

Putting the inclusion $x - y \in N(S; y)$ and the r-prox-regularity of S together, we observe (see Proposition 1(b))

$$2\left\langle x - y, x' - y \right\rangle \leq \frac{\|x - y\|}{r} \left\| x' - y \right\|^2 = \frac{d_S(x)}{r} \|x' - y\|^2. \tag{22}$$

It remains to combine (21) with (22) to complete the proof. □

Theorem 2 *Let S be a nonempty closed subset of $\mathcal{H}$ and $r \in]0, +\infty]$. The following assertions are equivalent.*

(*a*) *The set S is r-prox-regular;*
(*b*) *for any $x' \in U_r(S)$, any $x \in U_r(S)$ with $P_S(x)$ well-defined and any $\xi \in \partial d_S(x)$, one has*

$$\langle \xi, x' - x \rangle \le \frac{1}{2\big(r - d_S(x')\big)} \big(\|x' - P_S(x)\|^2 - d_S^2(x')\big) + d_S(x') - d_S(x);$$

(*c*) *for any $x \in S$, for any $x' \in U_r(S)$ and any $\xi \in \partial d_S(x)$, one has*

$$\langle \xi, x' - x \rangle \le \frac{1}{2\big(r - d_S(x')\big)} \big(\|x' - x\|^2 - d_S^2(x')\big) + d_S(x');$$

(*d*) *for any $x' \in S$, any $x \in U_r(S)$ with $P_S(x)$ well-defined and any $\xi \in \partial d_S(x)$, one has*

$$\langle \xi, x' - x \rangle \le \frac{1}{2r} \|x' - P_S(x)\|^2 - d_S(x).$$

Proof Through Proposition 1(b), we see that anyone of the assertions (b), (c), (d) implies (a), i.e., the r-prox-regularity of S. On the other hand, it is clear that (b) entails the assertions (c) and (d). It remains to establish $(a) \Rightarrow (b)$. Fix any $x' \in U_r(S)$, any $x \in U_r(S)$ with $P_S(x)$ well-defined and $\xi \in \partial d_S(x)$. Let us distinguish two cases.

Case 1. $x \in S$. Put $y := P_S(x')$. According to (2), we know that $\xi \in N(S; x) \cap \mathbb{B}$. Then, the r-prox-regularity of S gives

$$\langle \xi, x' - x \rangle = \langle \xi, y - x \rangle + \langle \xi, x' - y \rangle \le \frac{1}{2r} \|y - x\|^2 + d_S(x'). \tag{23}$$

On the other hand, thanks to Lemma 1(a), we get

$$\left(1 - \frac{d_S(x')}{r}\right) \|y - x\|^2 \le \|x' - x\|^2 - d_S^2(x'),$$

or equivalently,

$$\frac{1}{2r} \|y - x\|^2 \le \frac{1}{2\big(r - d_S(x')\big)} \big(\|x' - x\|^2 - d_S^2(x')\big). \tag{24}$$

Putting together (23), (24) and the equality $d_S(x) = 0$ yields

$$\langle \xi, x' - x \rangle \le \frac{1}{2\big(r - d_S(x')\big)} \big(\|x' - P_S(x)\|^2 - d_S^2(x')\big) + d_S(x') - d_S(x).$$

Case 2. $x \in U_r(S) \setminus S$. First, observe that (see (4)) $\partial d_S(x) = \{\xi\}$ where

$$\xi := \frac{x - P_S(x)}{d_S(x)} \in N\big(S; P_S(x)\big) \cap \mathbb{B} = \partial d_S\big(P_S(x)\big).$$

From the above expression of ξ, we see that

$$\langle \xi, x' - x \rangle = \langle \xi, x' - P_S(x) \rangle + \langle \xi, P_S(x) - x \rangle = \langle \xi, x' - P_S(x) \rangle - d_S(x). \tag{25}$$

Using the r-prox-regularity of S we also have

$$\begin{aligned} \langle \xi, x' - P_S(x) \rangle &= \langle \xi, x' - P_S(x') \rangle + \langle \xi, P_S(x') - P_S(x) \rangle \\ &\le d_S(x') + \frac{1}{2r} \|P_S(x') - P_S(x)\|^2. \end{aligned} \tag{26}$$

Further, Lemma 1 gives that

$$\left(1-\frac{d_S(x')}{r}\right)\left\|P_S(x')-P_S(x)\right\|^2 \le \left\|x'-P_S(x)\right\|^2 - d_S^2(x'). \tag{27}$$

Putting together (25), (26) and (27) we arrive to

$$\langle \xi, x'-x\rangle \le \frac{1}{2\big(r-d_S(x')\big)}\left(\left\|x'-P_S(x)\right\|^2 - d_S^2(x')\right) + d_S(x') - d_S(x).$$

The proof is complete. □

Theorem 2 brough to light the interest to estimate the quantity $\left\|P_S(x)-x'\right\|$ with $x, x' \in U_r(S)$. This is the aim of the next result which can be seen as an extension to the prox-regular framework of a result due to J.J. Moreau [25, Lemma 1(2a)] (see also [22] for similar results under convexity). It should be noted that both quantities $\left\|P_S(x)-x'\right\|$ and $\langle \xi, x'-x\rangle$ with $x \notin S$, $\xi \in \partial d_S(x)$ and $x' \in U_r(S)$ are strongly connected according to the elementary computation

$$\begin{aligned}\left\|P_S(x)-x'\right\|^2 &= \left\|(x'-x)+(x-P_S(x))\right\|^2\\ &= \left\|x'-x\right\|^2 + d_S^2(x) + 2d_S(x)\langle \xi, x'-x\rangle,\end{aligned} \tag{28}$$

where the latter equality is due to $\xi = \frac{x-P_S(x)}{d_S(x)}$ (see (3)).

Proposition 5 *Let S be an r-prox-regular subset of $\mathcal{H}$ for some $r \in]0,+\infty]$. The following hold.*
(a) For all $x, x' \in U_r(S)$, one has

$$\left(1-\frac{d_S(x)}{r-d_S(x')}\right)\left\|P_S(x)-x'\right\|^2 \le 2d_S(x)d_S(x')\left(1-\frac{d_S(x')}{2\big(r-d_S(x')\big)}\right) + \left\|x-x'\right\|^2 - d_S^2(x).$$

(b) For any $s \in]0,r[$ and any $x, x' \in U_s(S)$, one has

$$\left\|P_S(x)-x'\right\|^2 \le \left(1+\frac{d_S(x)}{r(1-s/r)^2}\right)\left\|x'-x\right\|^2 + 2d_S(x)d_S(x') - d_S^2(x).$$

(c) For all $x, x' \in U_r(S)$, one has

$$\left\|P_S(x)-x'\right\|^2 \le \left(1+\frac{4rd_S(x)}{\big(2r-d_S(x)-d_S(x')\big)^2}\right)\left\|x-x'\right\|^2 + 2d_S(x)d_S(x') - d_S^2(x).$$

Proof The assertion (a) follows from Theorem 2(b) and the equality (28).
Let us establish (b) (resp. (c)). Let $s \in]0,r[$ and let also $x, x' \in U_s(S)$ (resp. $x, x' \in U_r(S)$). Set $y := P_S(x)$ and $y' := P_S(x')$. Noting that $d_S(x) = \|y-x\|$ and applying the equality in (19) with $u := y-x'$ and $v := x-x'$ give

$$\left\|y-x'\right\|^2 - \left\|x-x'\right\|^2 = 2\langle x-y, x'-y\rangle - d_S^2(x).$$

From the r-prox-regularity of S and the inclusion $x-y \in N(S;y)$ we have

$$\begin{aligned}2\langle x-y, x'-y\rangle &= 2\langle x-y, x'-y'\rangle + 2\langle x-y, y'-y\rangle\\ &\le 2d_S(x)d_S(x') + \frac{d_S(x)}{r}\left\|y'-y\right\|^2.\end{aligned}$$

By virtue of Proposition 1(d) (resp. Proposition 1(c))

$$\left\| y' - y \right\|^2 \leq \frac{1}{(1 - s/r)^2} \left\| x' - x \right\|^2$$

(resp.

$$\|y' - y\|^2 \leq \left(1 - \frac{d_S(x)}{2r} - \frac{d_S(x')}{2r}\right)^{-2} \|x' - x\|^2).$$

It remains to put all together to get (b) (resp. (c)). The proof is complete. □

We can also estimate the quantity $\|P_{S_1}(x) - P_{S_2}(x)\|$ through Hausdorff distance. Recall that the *Hausdorff-Pompeiu distance* is defined for two nonempty subsets $S_1, S_2 \subset \mathcal{H}$ by

$$\text{haus}(S_1, S_2) := \max\{\text{exc}(S_1, S_2), \text{exc}(S_2, S_1)\},$$

with $\text{exc}(S_1, S_2) := \sup_{x \in S_1} d_{S_2}(x)$. The next result is essentially due to M.V. Balashov and G.E. Ivanov [6, Theorem 2]. The proof below follows for a large part their idea.

Proposition 6 *Let S_1, S_2 be r-prox-regular subsets of $\mathcal{H}$ with $r \in]0, +\infty]$. Let also $x \in \mathcal{H}$ such that* $\max\left\{d_{S_1}(x), d_{S_2}(x)\right\} \leq s < r$ *for some real s. For each $i, j \in \{1, 2\}$, assume that $P_{S_i}(x) \in \text{Enl}_r(S_j)$ and set $d_{i,j} := d\big(P_{S_j}(x), S_i\big)$. Then, one has*

$$\left\| P_{S_1}(x) - P_{S_2}(x) \right\|^2 \leq \frac{2s}{1 - s/r} \max_{i \neq j} d_{i,j} \left(1 - \frac{d_{i,j}}{2r}\right).$$

In particular, if $\text{haus}(S_1, S_2) \leq r$*, one has*

$$\left\| P_{S_1}(x) - P_{S_2}(x) \right\| \leq \left(\frac{2s}{1 - s/r} \text{haus}(S_1, S_2)\right)^{1/2}.$$

Proof For each $i \in \{1, 2\}$, $\text{Proj}_{S_i}(x)$ is reduced to a singleton $\{p_i\}$ (thanks to $x \in U_r(S_i)$ and the fact that S_i is r-prox-regular). We are going to show that

$$2\langle x - p_1, p_2 - p_1 \rangle \leq \frac{s}{r}\left(\|p_1 - p_2\|^2 + 2rd_{1,2}\left(1 - \frac{d_{1,2}}{2r}\right)\right).$$

We may suppose that $x \neq p_1$, hence $x \notin S_1$. In particular, we have $x \in U_r(S_1) \setminus S_1$, so we can apply Proposition 1(e) to get

$$p_1 \in \text{Proj}_{S_1}\left(p_1 + \frac{r(x - p_1)}{\|x - p_1\|}\right).$$

Note that for all $z \in S_1$,

$$\left\| p_1 + \frac{r(x - p_1)}{\|x - p_1\|} - p_2 \right\| \geq \left\| p_1 + \tfrac{r(x-p_1)}{\|x-p_1\|} - z \right\| - \|p_2 - z\| \geq r - \|p_2 - z\|.$$

Passing to the supremum yields

$$\left\| p_1 + \frac{r(x - p_1)}{\|x - p_1\|} - p_2 \right\| \geq \sup_{z \in S_1}(r - \|p_2 - z\|) = r - d_{S_1}(p_2) = r - d_{1,2}.$$

We deduce from this (thanks to the inequality $r \geq d_{1,2}$)

$$\|p_1 - p_2\|^2 + \frac{2r}{\|x - p_1\|}\langle x - p_1, p_1 - p_2 \rangle + r^2 \geq r^2 - 2rd_{1,2} + d_{1,2}^2,$$

or equivalently

$$2r\,\langle x - p_1, p_2 - p_1\rangle \le \|x - p_1\| \left(\|p_1 - p_2\|^2 + 2rd_{1,2}\left(1 - \frac{d_{1,2}}{2r}\right)\right).$$

Keeping in mind that $d_{S_1}(x) = \|x - p_1\| < s$, we obtain

$$2\,\langle x - p_1, p_2 - p_1\rangle \le \frac{s}{r} \left(\|p_1 - p_2\|^2 + 2rd_{1,2}\left(1 - \frac{d_{1,2}}{2r}\right)\right),$$

which is the inequality claimed above. In the same way, we show

$$2\,\langle x - p_2, p_1 - p_2\rangle \le \frac{s}{r} \left(\|p_1 - p_2\|^2 + 2rd_{2,1}\left(1 - \frac{d_{2,1}}{2r}\right)\right).$$

Adding the two latter inequalities, we have with $m := \max_{i \ne j} d_{i,j}\big(1 - \frac{d_{i,j}}{2r}\big)$

$$\|p_1 - p_2\|^2 \le \frac{s}{r}(\|p_1 - p_2\|^2 + 2rm).$$

The proof is complete. □

Remark 4 The exponent 1/2 in the above Holder property is known to be sharp even for convex sets (see, [17, p.235]).

5 Semi-convexity of Distance Function

As observed in [16, Proposition 18], a nonempty closed subset S of $\mathcal{H}$ is r-prox-regular for some extended real $r > 0$ if and only if its associated square distance function d_S^2 is $\frac{2s}{r-s}$-linearly semiconvex (or equivalently $d_S^2 + \frac{s}{r-s}\,\|\cdot\|^2$ is convex) on any open convex subset V of $U_s(S)$ for every $0 < s < r$. This can be seen through the following computation valid for any $x, y \in U_s(S)$ with $\sigma := \frac{s}{r-s}$ and $g := d_S^2 + \sigma\,\|\cdot\|^2$

$$\begin{aligned}\langle \nabla g(x) - \nabla g(y), x - y\rangle &= 2(1+\sigma)\,\|x - y\|^2 - 2\,\langle P_S(x) - P_S(y), x - y\rangle \\ &\ge 2\big(1 + \sigma - (1 - s/r)^{-1}\big)\,\|x - y\|^2 = 0.\end{aligned}$$

Our aim in the present section is to characterize the prox-regularity through the semiconvexity of its distance instead of the square distance. We start with the following result taken from [11, Proposition 2.2.2] showing that distance functions from subsets of Hilbert spaces have particular semiconcavity properties. For the convenience of the reader we provide a proof.

Proposition 7 *Let S be a nonempty subset of $\mathcal{H}$. The following hold:*

(a) *The square distance function d_S^2 is 2-linearly semiconcave on $\mathcal{H}$.*
(b) *For any nonempty convex subset U of $\mathcal{H}$ and for any real $\delta > 0$ such that $U \cap \big(S + B(0,\delta)\big) = \emptyset$, d_S is δ^{-1}-semiconcave on U. So, d_S is locally linearly semiconcave on $\mathcal{H} \setminus S$.*
(c) *If S is the union of a collection of closed balls with a common radius $r > 0$, then on each convex set U included in $\mathrm{cl}_{\mathcal{H}}\big(\mathcal{H} \setminus S\big)$, the distance function d_S is r^{-1}-semiconcave.*

Proof (*a*) For all $x \in \mathcal{H}$, we have $d_S^2(x) = \|x\|^2 + \inf_{y\in S}\big(\|y\|^2 - 2\langle x, y\rangle\big)$. On the other hand, for each $y \in S$, the function $\varphi_y : \mathcal{H} \to \mathbb{R}$ defined by

$$\varphi_y(x) = \|y\|^2 - 2\langle x, y\rangle = \langle -2x + y, y\rangle \quad \text{for all } x \in \mathcal{H}$$

is concave. Thus, there is a concave function $g : \mathcal{H} \to \mathbb{R}$ such that $d_S^2(\cdot) = \|\cdot\|^2 + g(\cdot)$ and this translates the desired semiconcavity property.

(*b*) Let U be a nonempty convex subset of $\mathcal{H}$, $\delta > 0$ be a real such that $U \cap \big(S + B(0,\delta)\big) = \emptyset$. Set $f = d_S^2$ and observe that $f(U) \subset [\delta^2, +\infty[$. The function $g = \sqrt{\cdot}$ is increasing, concave and $\frac{1}{2\delta}$-Lipschitz on $[\delta^2, +\infty[$. It is then an exercise to check the δ^{-1}-semiconcavity of the chain $d_S = g \circ f$.

(*c*) Let $(a_i)_{i\in I}$ be a family of $\mathcal{H}$ such that $S = \bigcup_{i\in I} B[a_i, r]$ and let a nonempty convex set U included in $\mathrm{cl}_{\mathcal{H}}\big(\mathcal{H} \setminus S\big)$. Fix any $i \in I$. Put $S_i := B[a_i, r]$. Note also that for each $i \in I$, $d_{\{a_i\}}^2(x) \geq r^2$ for all $x \in U$, hence, by (*b*) above, the function $d_{\{a_i\}}(\cdot) = \|\cdot - a_i\|$ is r^{-1}-linearly semiconcave on U. Through the equality $d_{S_i}(\cdot) = \|\cdot - a_i\| - r$, we see that $d_{S_i}(\cdot)$ is also r^{-1}-linearly semiconcave on U. From

$$d_S(x) = \inf_{j\in I} d_{S_j}(x) \quad \text{for all } x \in U$$

we see that $-d_S(\cdot)$ is the pointwise supremum of r^{-1}-linearly semiconvex functions on U. Therefore, $d_S(\cdot)$ is r^{-1}-linearly semiconcave on U. The proof is complete. □

The next result shows that the complement of a prox-regular set is the union of a family of closed balls with a common radius.

Theorem 3 *Let S be an r-prox-regular subset of $\mathcal{H}$ with $r \in]0, +\infty[$. Then, for any $s \in]0, r[$, the set $\mathcal{H} \setminus S$ is the union of a family of closed balls of $\mathcal{H}$ of radius s.*

Proof Fix any $s \in]0, r[$. If $S = \mathcal{H}$, then $\mathcal{H} \setminus S = \emptyset$ and there is nothing to prove. Assume that $S \neq \mathcal{H}$. Fix any $y \in \mathcal{H} \setminus S$. If $d_S(y) \geq r$, then we have $B(y, r) \cap S = \emptyset$ hence $B[y, s] \subset \mathcal{H} \setminus S$. Suppose now $0 < d_S(y) < r$. According to the r-prox-regularity of S, $\mathrm{Proj}_S(y)$ is reduced to a singleton, say $\mathrm{Proj}_S(y) = \{p\}$. With $v := \frac{y-p}{\|y-p\|}$, we have (see Remark 1)

$$p \in \mathrm{Proj}_S(p + rv),$$

hence $B(p + rv, r) \cap S = \emptyset$. Observe also that

$$\|y - p - rv\| = \left\| \left(1 - \frac{r}{\|y - p\|}\right)(y - p) \right\| = |\|y - p\| - r| = r - d_S(y).$$

If $s \geq r - d_S(y)$, then $y \in B[p + rv, s]$ and $B[p + rv, s] \subset \mathcal{H} \setminus S$ since $B[p + rv, s] \subset B(p + rv, r)$. So, assume that $s < r - d_S(y)$, so in particular $y \neq p + rv$ (if $y = p + rv$, then $s < r - d_S(p + rv) = 0$). Set

$$z = y - \|y - p - rv\|^{-1} s(y - p - rv).$$

We have $y \in B[z, s]$. Fix any $u \in B[z, s]$ and observe that

$$\begin{aligned}\|u - p - rv\| &\leq \|u - z\| + \|z - p - rv\| \\ &= \|u - z\| + \left\| \left(1 - \frac{s}{\|y - p - rv\|}\right)(y - p - rv) \right\| \\ &= \|u - z\| + |\|y - p - rv\| - s| \\ &= \|u - z\| + |r - d_S(y) - s|,\end{aligned}$$

which combined with the inequality $s < r - d_S(y)$ yields

$$\|u - p - rv\| \leq \|u - z\| + r - d_S(y) - s \leq r - d_S(y).$$

Hence, the inclusion $B[z, s] \subset B(p + rv, r)$ holds true. Therefore, $y \in B[z, s] \subset \mathcal{H} \setminus S$. In conclusion, any point of $\mathcal{H} \setminus S$ belongs to some closed ball of radius s included in $\mathcal{H} \setminus S$. □

Remark 5 It is clear that the above proof of Theorem 3 utilizes only the property (e) in Proposition 1. Then Theorem 3 still holds true in any uniformly convex Banach space whose norm is uniformly smooth since it is known that the mentioned property (e) is satisfied in such spaces (see [3, 7]).

We derive from the latter result a full characterization of the prox-regularity through the semiconvexity of distance functions. Such a fact has been established in a very different way by M.V. Balashov [5, Theorem 2.7].

Theorem 4 *Let S be a nonempty closed subset of $\mathcal{H}$ and let $r \in]0, +\infty]$. The following assertions are equivalent.*

- (a) *The set S is r-prox-regular;*
- (b) *for any real $0 < s < r$, the distance function d_S is $(r-s)^{-1}$-semiconvex on any convex set included in the open s-enlargement $U_s(S)$ (resp. on the open s-tube $U_s(S) \setminus S$);*
- (c) *the distance function d_S is locally linearly semiconvex on $U_r(S)$.*

Proof (a) $\Rightarrow$ (b) Fix any $s \in]0, r[$. Let $t \in]s, r[$. Thanks to Theorem 1(a), we know that $\mathrm{Enl}_s(S)$ is $(r - s)$-prox-regular, hence Theorem 3 guarantees that $\Omega := \mathcal{H} \setminus \mathrm{Enl}_s(S)$ is the union of a family of closed balls with common radius $r - t$. Then, using Proposition 7(c) we obtain that the function $d\big(\cdot, \mathrm{cl}_{\mathcal{H}}(\Omega)\big) = d(\cdot, \Omega)$ is $(r - t)^{-1}$-linearly semiconcave on any convex set included in $\mathrm{cl}_{\mathcal{H}}(\mathcal{H} \setminus \Omega) = \mathrm{Enl}_s(S)$. By Proposition 3(f), we have $\mathrm{cl}_{\mathcal{H}}(\Omega) = \mathrm{Exte}_s(S)$. Consequently, the distance function $d\big(\cdot, \mathrm{Exte}_s(S)\big)$ is $(r - t)^{-1}$-linearly semiconcave on any convex set included in $U_s(S) \subset \mathrm{Enl}_s(S)$. Since t has been arbitrarily choosen in $]s, r[$, we see through the definition of linearly semiconcave functions that $d\big(\cdot, \mathrm{Exte}_s(S)\big)$ is $(r - s)^{-1}$-linearly semiconcave on any convex set included in $U_s(S) \subset \mathrm{Enl}_s(S)$.

Now, observe that a direct application of Proposition 3(d) yields

$$d(x, S) = s - d(x, \mathrm{Exte}_s(S)) \quad \text{if } x \in \mathrm{Tube}_s(S) \tag{29}$$

from which we derive the $(r-s)^{-1}$-linearly semiconvexity of d_S on any convex set included in $\mathrm{Tube}_s(S)$. On the other hand, for any $x_0 \in S$, from the inequality $\|x_0 - y\| \geq d(y, S) \geq s$ valid for all $y \in \mathrm{Exte}_s(S)$, we see that

$$s \leq d\big(x_0, \mathrm{Exte}_s(S)\big). \tag{30}$$

Putting together (29) and (30), we arrive to

$$d(x,S)=\max\left\{0, s-d\big(x, \mathrm{Exte}_s(S)\big)\right\} \quad \text{for all } x\in U_s(S).$$

This equality ensures that function d_S is $(r-s)^{-1}$-linearly semiconcave on any convex set V included in $U_s(S)$ as the pointwise maximum of two functions which are $(r-s)^{-1}$-linearly semiconvex on V. This justifies the implication $(a)\Rightarrow(b)$.
The implication $(b)\Rightarrow(c)$ being obvious, let us establish $(c)\Rightarrow(a)$. So, assume that $d_S(\cdot)$ is locally linearly semiconvex on $U_r(S)$. Fix any $x\in U_r(S)$. There are two reals $\rho,\delta>0$ such that $f:=d_S(\cdot)+\rho\|\cdot\|^2$ is convex on $B(x,\delta)\subset U_r(S)$. According to the $C^{1,1}$ property of $\|\cdot\|^2$, we have

$$\partial f(x)=\partial d_S(x)+\nabla\|\cdot\|^2(x).$$

Combining the latter equality with the nonemptiness $\partial f(x)\neq\emptyset$ (since f is convex and continuous), we get $\partial d_S(x)\neq\emptyset$. The r-prox-regularity of S follows from Proposition 1(h). The proof is complete. □

Given an r-prox-regular subset S of $\mathcal{H}$ for some $r\in]0,+\infty]$, we see through the property (b) in Theorem 4 above that for any real $0<s<r$ and any open convex set $V\subset U_s(S)$,

$$\left\langle\xi, x'-x\right\rangle\le d_S(x')-d_S(x)+\frac{1}{2(r-s)}\left\|x'-x\right\|^2,$$

for all $x,x'\in V$ and all $\xi\in\partial d_S(x)$. The next result is devoted to remove the restriction to the convex set V.

Theorem 5 *Let S be a nonempty closed subset of $\mathcal{H}$ and let $r>0$ be a real. The following assertions are equivalent.*

(a) *The set S is r-prox-regular;*
(b) *for all $s\in]0,r[$, for all $x,x'\in U_s(S)$, for all $\xi\in\partial d_S(x)$, one has*

$$\left\langle\xi, x'-x\right\rangle\le\frac{1}{2(r-s)(1-s/r)}\left\|x'-x\right\|^2+d_S(x')-d_S(x);$$

(c) *for all $x,x'\in U_r(S)$ with $d_S(x')\le d_S(x)$, for all $\xi\in\partial d_S(x)$, one has*

$$\left\langle\xi, x'-x\right\rangle\le\frac{1}{2(r-d_S(x'))}\left\|x'-x\right\|^2;$$

(d) *for all $x,x'\in U_r(S)$ with $d_S(x')\ge d_S(x)$, for all $\xi\in\partial d_S(x)$, one has*

$$\left\langle\xi, x'-x\right\rangle\le\frac{1}{2(r-d_S(x'))}\left(\left\|x'-x\right\|^2-\big(d_S(x')-d_S(x)\big)^2\right)+d_S(x')-d_S(x).$$

Proof $(a)\Rightarrow(b)$, Let $s\in]0,r[$. Fix any $x,x'\in U_s(S)$ and $\xi\in\partial d_S(x)$. In view of Theorem 2(c), we may suppose that $x\notin S$. Then, we know that $\xi=d_S(x)^{-1}(x-P_S(x))$ (see (3)) and this entails

$$\begin{aligned}\left\langle\xi, x'-x\right\rangle&=\left\langle\xi, x'-P_S(x')\right\rangle+\left\langle\xi, P_S(x')-P_S(x)\right\rangle+\langle\xi, P_S(x)-x\rangle\\&\le d_S(x')+\left\langle\xi, P_S(x')-P_S(x)\right\rangle-d_S(x).\end{aligned}\tag{31}$$

On the other hand, the inclusion $\xi\in\partial d_S(P_S(x))$ allows us to apply (b) and (d) in Proposition 1 to get

$$\left\langle\xi, P_S(x')-P_S(x)\right\rangle\le\frac{1}{2r}\left\|P_S(x')-P_S(x)\right\|^2\le\frac{1}{2r(1-s/r)^2}\left\|x'-x\right\|^2.\tag{32}$$

Putting together (31) and (32) gives the inequality claimed in (b), that is,

$$\langle \xi, x' - x \rangle \leq \frac{1}{2(r-s)(1-s/r)} \|x' - x\|^2 + d_S(x') - d_S(x).$$

$(b) \Rightarrow (a)$, Let $x, x' \in S$, $\xi \in N(S; x) \cap \mathbb{B}$. Fix any sequence $(s_n)_{n\geq 1}$ of $]0, r[$ with $s_n \to 0$. We have $\xi \in \partial d_S(x)$ (see (2)) and obviously $x, x' \in U_{s_n}(S)$ for every $n \geq 1$, hence

$$\langle \xi, x' - x \rangle \leq \frac{1}{2(r-s_n)(1-s_n/r)} \|x' - x\|^2.$$

Letting $n \to \infty$ in the latter inequality guarantees the r-prox-regularity of S according to (b) in Proposition 1.
$(a) \Leftrightarrow (c)$ The implications $(c) \Rightarrow (a)$ and $(d) \Rightarrow (a)$ are direct consequences of $(a) \Leftrightarrow (b)$ in Proposition 1 and of the equality (2).
Now, let us focus on $(a) \Rightarrow (c)$ and $(a) \Rightarrow (d)$. Fix for a moment $x, x' \in U_r(S)$. Let also $\xi \in \partial d_S(x)$. Set $C := \mathrm{Enl}_\rho(S)$ where $\rho := d_S(x) \in [0, r[$. In particular, note that $x \in C$. According to Theorem 1(a), the set C is $(r-\rho)$-prox-regular. On the other hand, using Proposition 3(a) and the inclusion (5)

$$d_C(x') < r - \rho \quad \text{and} \quad \xi \in \partial d_C(x).$$

We are then in a position to invoke (c) Theorem 2 to get

$$\langle \xi, x' - x \rangle \leq \frac{\|x' - x\|^2}{2(r - \rho - d_C(x'))} + d_C(x') \left(1 - \frac{d_C(x')}{2(r - \rho - d_C(x'))}\right). \tag{33}$$

$(a) \Rightarrow (c)$, Take any $x, x' \in U_r(S)$ with $d_S(x') \leq d_S(x)$, i.e., $x' \in C$. If $x \notin S$, it follows from the inequality (33) that

$$\langle \xi, x' - x \rangle \leq \frac{1}{2(r - d_S(x'))} \|x' - x\|^2.$$

Further, if $x \in S$, we must have $x' \in S$ and the latter inequality still holds (see Theorem 2 or Proposition 1).
$(a) \Rightarrow (d)$, Take now $x, x' \in U_r(S)$ with $d_S(x') \geq d_S(x)$. If $d_S(x') = d_S(x)$, the desired inequality follows from $(a) \Rightarrow (c)$. Assume that $d_S(x') > d_S(x)$, so $x' \notin C$. Keeping in mind Proposition 3(a), we have $d_C(x') = d_S(x') - d_S(x) = d_S(x') - \rho$ with $\rho = d_S(x) \in [0, r[$. Coming back to (33), we arrive to

$$\langle \xi, x' - x \rangle \leq \frac{1}{2(r - d_S(x'))} \|x' - x\|^2 + \left(d_S(x') - d_S(x)\right) \left(1 - \frac{d_S(x') - d_S(x)}{2(r - d_S(x'))}\right).$$

The proof is then complete. □

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