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Algebraic Approach to Duality in Optimization and Applications

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Abstract
This paper studies duality of optimization problems in a vector space without topological structure. A strong duality relation is established by means of algebraic subdifferential and algebraic conjugate functions. Topological duality relations are obtained by the algebraic approach without lower semicontinuity or quasicontinuity hypothesis on perturbation functions. Applications are given for the sum of two convex functions, monotropic problems, infinite convex or linear problems. Attention is also made on the algebraic constraint qualification for problems with countably infinitely many inequality constraints.

Keywords Duality · Constraint qualification · Infinite convex problem · Monotropic problem

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1 Preliminaries

Let X be a linear space over the reals and $h : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ an extended real valued function. Recall that h is convex if its epigraph epi$h := \{(x, r) \in X \times \mathbb{R} : h(x) \leq r\}$ is a convex subset of the product vector space $X \times \mathbb{R}$, or equivalently, for any $x, y \in X$ and $t \in (0, 1)$, one has

$$h(tx + (1 - t)y) \leq th(x) + (1 - t)h(y) \tag{1}$$

in which we adopt the convention $(+\infty) + (-\infty) = (-\infty) + (+\infty) = +\infty$. The effective domain of h is denoted dom$h := \{x \in X : h(x) < +\infty\}$. The function h is proper if h does not take the value $-\infty$ and dom$h \neq \emptyset$.

To Professor R. T. Rockafellar on the occasion of his 85th birthday

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The algebraic relative interior, or intrinsic core, of a convex subset A of X is given by

$$\mathrm{icr}(A) := \{a \in A : \forall x \in A, \exists \lambda > 0 \text{ such that } a - \lambda(x - a) \in A\}.$$

For each $b \in X$ one has $\mathrm{icr}\{b\} = \{b\}$ and

$$b \in \mathrm{icr}(A) \iff \mathrm{cone}(A - b) := \mathbb{R}_+(A - b) \text{ is a linear subspace of } X.$$

The following lemma shows that if the domain of an extended real valued function has a nonempty intrinsic core, then so does its epigraph (see also [8]).

Lemma 1 *Let $h : X \to \overline{\mathbb{R}}$ be convex and let $a \in X$ be such that $h(a) \neq -\infty$ and $a \in icr(domh)$. Then h is proper and $(a, h(a) + \theta) \in icr(epih)$ for every $\theta > 0$.*

Proof Let us first prove that h is proper. Suppose to the contrary that there is some $x \in X$ such that $h(x) = -\infty$. Then $x \in \mathrm{dom}h$. Since $a \in \mathrm{icr}(\mathrm{dom}h)$ there exists $y \in \mathrm{dom}h$ such that $a \in (x, y)$, that is, $a = tx + (1 - t)y$ for some $t \in (0, 1)$. Because $h(y) < +\infty$ and $h(x) = -\infty$, in view of (1), one would have $h(a) = -\infty$, a contradiction. Thus, h is proper. Now let $(x, r) \in \mathrm{epi}h$ and $\theta > 0$. We wish to find $\bar{\lambda} > 0$ such that

$$(a, h(a) + \theta) - \bar{\lambda}((x, r) - (a, h(a) + \theta)) \in \mathrm{epi}h. \tag{2}$$

Consider the restriction $h|_L$ of h on the straight line $L := \{a + t(a - x) : t \in \mathbb{R}\}$. Then $h|_L$ is a proper convex function on L and a is an interior point of its domain with respect to the usual topology on a straight line. In view of Lemma 7.3 [17], $(a, h(a) + \theta)$ is an interior point of $\mathrm{epi}h|_L$ in $L \times \mathbb{R}$. Hence, for $(x, r) \in \mathrm{epi}h|_L$ we can find $\bar{\lambda} > 0$ such that $(a, h|_L(a) + \theta) - \bar{\lambda}((x, r) - (a, h|_L(a) + \theta)) \in \mathrm{epi}h|_L$, which evidently gives (2). □

Let X' denote the algebraic dual space of X. We consider the algebraic conjugate and biconjugate of h defined by

$$h^{\#}(\varphi) := \sup_{x \in X}\{\varphi(x) - h(x)\} \text{ for } \varphi \in X',$$
$$h^{\#\#}(x) := \sup_{\varphi \in X'}\{\varphi(x) - h^{\#}(\varphi)\} \text{ for } x \in X.$$

The algebraic subdifferential of h at a point $a \in X$ is given by

$$\partial h(a) := \begin{cases} \{\varphi \in X' : h(x) \geq h(a) + \varphi(x - a), \forall x \in X\} & \text{if } h(a) \in \mathbb{R} \\ \emptyset & \text{if } h(a) \notin \mathbb{R} \end{cases}$$

One says that h is algebraically subdifferentiable at a if $\partial h(a) \neq \emptyset$. Moreover, if $h(a) \in \mathbb{R}$, then one has

$$\varphi \in \partial h(a) \iff h(a) + h^{\#}(\varphi) = \varphi(a). \tag{3}$$

Let us consider the case when X is a topological vector space and X^* is its topological dual. The usual Fenchel conjugate h^* of h (see [4, 16, 18, 21]) is nothing, but the restriction of $h^{\#}$ on X^*. Namely,

$$h^*(x^*) := \sup_{x \in X}(\langle x^*, x\rangle - h(x)), \quad x^* \in X^*.$$

Its biconjugate is defined on X by

$$h^{**}(x) = \sup_{x^* \in X^*}(\langle x^*, x\rangle - h^*(x^*)), \quad x \in X.$$

It is clear that

$$h^{**}(x) = (h^{\#} + \delta_{X^*})^{\#}(x) \leq h^{\#\#}(x) \leq h(x) \text{ for } x \in X.$$

Similarly, the usual subdifferential of h at a with $h(a) \in \mathbb{R}$, denoted by $\partial^* h(a)$, is given by $\partial^* h(a) = \partial h(a) \cap X^*$.

Note that when X is finite dimensional, the two dual spaces X' and X^* coincide. Therefore, the algebraic concepts of conjugate function and subdifferential are the same as the usual ones. This, however, is not true in infinite dimension.

Example 1 Let X be an infinite dimensional separated locally convex space and $\varphi \in X'$ be a discontinuous linear function. Then $\partial\varphi(0_X) = \{\varphi\}$, while $\partial^*\varphi(0_X) = \emptyset$. Consider h defined by $h(x) = |\varphi(x)|$, $x \in X$ with φ as above. We have for every $\psi \in X'$ that

$$\begin{aligned} h^{\#}(\psi) &= \sup_{x\in X}(\psi(x) - |\varphi(x)|) \\ &= \sup_{x\in X}(\psi(x) - \sup_{-1\le t\le 1} t\varphi(x)) \\ &= \sup_{x\in X}\inf_{-1\le t\le 1}(\psi(x) - t\varphi(x)) \\ &= \inf_{-1\le t\le 1}\sup_{x\in X}(\psi(x) - t\varphi(x)) \\ &= \begin{cases} 0 & \text{if } \psi = t\varphi \text{ for some } t \in [-1, 1] \\ +\infty & \text{else} \end{cases} \end{aligned}$$

in which the fourth equality is obtained by the minimax theorem ([21], Theorem 2.10.2). It follows that h^* is the indicator function $\delta_{\{0_{X^*}\}}$ of the set $\{0_{X^*}\}$ and $h^{**}(x) = 0$ for every $x \in X$, while $h^{\#\#} = h$.

Remark 1 Note that equality $h^{\#\#} = h$ is not always true for h proper convex. For instance, the function $h : \mathbb{R} \to \overline{\mathbb{R}}$ given by

$$h(x) = \begin{cases} 0 & \text{for } x > 0 \\ +\infty & \text{for } \le 0, \end{cases}$$

has $h^{\#\#}(0) = 0 < h(0)$. A criterion for equality $h^{\#\#} = h$ can be established by equipping X with a weak topology associated with X'. Namely, let $\mathcal{T}$ be a locally convex topology such that the topological dual of X coincides with X'. Such topologies exist, for instance the topology generated by the family of seminorms of type $|\varphi|$, $\varphi \in X'$, or the topology generated by the family of all seminorms on X. These topologies are all locally convex. In view of Proposition 5.3 [16] equality $h^{\#\#} = h$ holds for a proper convex function h on X if and only if h is lower semicontinuous with respect to $\mathcal{T}$.

Two subsets A, B of the linear space X are said to be properly separated if there exists a nonzero linear functional $\xi \in X'$ and $(\overline{a}, \overline{b}) \in A \times B$ such that

$$\varphi(a) \le \varphi(b) \text{ for } (a, b) \in A \times B, \text{ and } \varphi(\overline{a}) < \varphi(\overline{b}).$$

The next separation criterion for convex sets in a linear space is crucial for our analysis (see [1, 6, 8, 9, 13]).

Lemma 2 *Two convex subsets A, B of a vector space such that $icr(A) \neq \emptyset$ and $icr(B) \neq \emptyset$ are properly separated if and only if*

$$icr(A) \cap icr(B) = \emptyset.$$

The following criterion for algebraic subdifferentiability is a slight improvement of the subdifferentiability theorem of [13] (page 27) (see also [9], page 23), stating that a real valued convex function on a convex subset of a linear space is algebraically subdifferentiable at each point of its intrinsic core. This improvement is quite easy to prove by a standard argument, but is useful in applications because the value functions associated with perturbation functions we are going to study often take the values $+\infty$ and $-\infty$.

Lemma 3 *Let $h : X \to \overline{\mathbb{R}}$ be convex, $a \in X$ such that $h(a) \neq -\infty$ and $\mathbb{R}_+(domh - a)$ is a linear subspace of X. Then h is algebraically subdifferentiable at a.*

Proof By Lemma 1 we have $\mathrm{icr}(\mathrm{epi}h) \neq \emptyset$. Moreover, $h(a) \in \mathbb{R}$ and $(a, h(a)) \notin \mathrm{icr}(\mathrm{epi}h)$ because otherwise $(a, h(a) - \varepsilon) \in \mathrm{epi}h$ for some $\varepsilon > 0$, which is impossible. We apply Lemma 2 to properly separate the convex sets $\{(a, h(a))\}$ and $\mathrm{epi}h$ in the linear product space $X \times \mathbb{R}$. Thus, there exists $(\varphi, s) \in X' \times \mathbb{R}$, $(\varphi, s) \neq (0_{X'}, 0)$ such that

$$\varphi(a) + sh(a) \leq \varphi(x) + sr \quad \text{for all} \quad (x, r) \in \mathrm{epi}h \tag{4}$$

and strict inequality holds for some $(\bar{x}, \bar{r}) \in \mathrm{epi}h$. It follows that $s > 0$. By dividing both sides of (4) by s and setting $\varphi_0 = -\varphi/s$, $r = h(x)$ for $x \in \mathrm{dom}h$, we obtain

$$-\varphi_0(x) + h(x) \geq -\varphi_0(a) + h(a) \quad \forall x \in \mathrm{dom}h.$$

This relation is true for all $x \in X$ because $h(x) = +\infty$ for $x \notin \mathrm{dom}h$. Hence $\varphi_0 \in \partial h(a)$ and the proof is complete. □

In the present paper we aim at establishing strong duality for abstract optimization problems in a vector space without topological structure (Section 2). Applications are made to obtain a formula of the algebraic conjugate of the sum of two convex functions (Section 3), strong duality for problems with inequality constraints and strong duality for infinite convex problems in Sections 4 and 5 respectively, in which particular attention is given to clarify the algebraic constraint qualification. The results of Sections 4 and 5 concerning problems in topological spaces show that in certain circumstances algebraic duality approach is useful to establish topological duality without lower semicontinuity or quasicontinuity hypothesis on the perturbation functions.

2 Algebraic Duality in Optimization

We consider an abstract optimization problem

$$\text{(P)} \quad \begin{array}{ll} \inf & f(x) \\ \text{subject to} & x \in S, \end{array}$$

where S is a nonempty set and f is a function on S with values in $\mathbb{R} \cup \{+\infty\}$. We wish to construct a dual of (P) and establish duality relations between (P) and its dual without any topological structure on S. To this purpose, we consider a perturbation function $F : S \times Y \to \overline{\mathbb{R}}$, where Y is a linear space, such that $F(x, 0_Y) = f(x)$ for all $x \in S$. The associated algebraic Lagrangian function $L : S \times Y' \to \overline{\mathbb{R}}$, is defined by

$$L(x, \psi) := \inf_{y \in Y}\{F(x, y) - \psi(y)\} \quad \text{for} \quad \psi \in Y'.$$

The algebraic Lagrangian dual problem of (P) is given as

$$\text{(AD)}\quad \begin{array}{ll} \sup & \inf_{x\in S} L(x,\psi) \\ \text{subject to} & \psi \in Y'. \end{array}$$

The optimal values of (P) and (AD) are respectively denoted by $\inf(P)$ and $\sup(AD)$. We write min and max instead of inf and sup if those values are attained. It is clear that

$$\sup(AD) \le \inf_{x\in S} \sup_{\psi\in Y'} L(x,\psi) \le \inf(P),$$

which is known as weak duality relation. We have strong duality if $\max(AD) = \inf(P)$, and zero duality gap if $\sup(AD) = \inf(P)$.
We define the value function $v : Y \to \overline{\mathbb{R}}$ associated with F by

$$v(y) := \inf_{x\in S} F(x,y) \quad \text{for} \quad y \in Y.$$

It follows from the definitions that $v^{\#}(\psi) = -\inf_{x\in S} L(x,\psi)$ for $\psi \in Y'$ and

$$\begin{aligned} v(0_Y) &= \inf(P) \\ v^{\#\#}(0_Y) &= \sup(AD). \end{aligned}$$

Weak duality relation between (P) and (AD) is nothing, but inequalities $-\infty \le v^{\#\#}(0_Y) \le v(0_Y) \le +\infty$. Note that the strict epigraphs and the domains of v and F are linked by

$$\begin{aligned} \text{epi}_s v &= \text{proj}_{Y\times\mathbb{R}} \text{epi}_s F \\ \text{dom} v &= \text{proj}_Y \text{dom} F. \end{aligned}$$

Moreover, since a function is convex if and only if its strict epigraph is convex, in the case S is a vector space, the value function v is convex if and only if $\text{proj}_{Y\times\mathbb{R}}\text{epi}_s F$ is convex. In particular, v is convex if $\text{proj}_{Y\times\mathbb{R}}\text{epi} F$ is convex, which is verified when F is convex. The following condition, a kind of algebraic constraint qualification, will be used throughout:

(CQ) *The set* $\mathbb{R}_+(\textit{proj}_Y \textit{dom} F)$ *is a linear subspace of* Y.

Theorem 1 *Assume that v is convex and (CQ) holds. Then we have*

$$-\infty \le \inf(P) = \max(AD) < +\infty.$$

In other words, there exists $\psi \in Y'$ *such that*

$$\textit{inf}(P) = \textit{inf}_{x\in S} L(x,\psi) \in [-\infty, +\infty). \tag{5}$$

Proof If $\inf(P) = -\infty$, then the conclusion of the theorem is evident and (5) holds for any $\psi \in Y'$. Let $\inf(P) > -\infty$. We have $v(0_Y) > -\infty$ and by the hypothesis, $0_Y \in \text{icr}(\text{dom} v)$. In view of Lemma 3, v is algebraically subdifferentiable at 0_Y, that is, there exists $\psi \in Y'$ such that $v^{\#}(\psi) \in \mathbb{R}$ and (3) holds. Hence

$$\inf(P) = v(0_Y) = -v^{\#}(\psi) = \inf_{x\in X} L(x,\psi) \le \sup(AD) \le \inf(P),$$

and (5) follows. □

Observe that the dual construction by the Lagrangian function allows us to obtain strong duality (Theorem 1) in rather general setting in which the constraint set S is an arbitrary set. We shall, however, focus on the case where S is a subset of a vector space X. By defining $f(x) = +\infty$ for $x \notin S$ we may assume $S = X$. In this case, the algebraic conjugate of

f is defined on X' and the algebraic conjugate of F is defined on $X' \times Y'$. It is clear that $\inf(P) = -f^{\#}(0_{X'})$. Moreover, since

$$\inf_{x\in X} L(x,\psi) = \inf_{x\in X}\inf_{y\in Y}(F(x,y) - \psi(y)) = -F^{\#}(0_{X'},\psi),$$

the dual (AD) takes the form

$$\begin{array}{ll} \sup & -F^{\#}(0_{X'},\psi) \\ \text{subject to} & \psi \in Y', \end{array}$$

which is known as the algebraic conjugate dual of (P).

Corollary 1 *Assume F is convex and (CQ) holds. Then for each $\varphi \in X'$ one has*

$$-\infty < f^{\#}(\varphi) = \min_{\psi\in Y'} F^{\#}(\varphi,\psi) \leq +\infty, \tag{6}$$

and, in particular,

$$-\infty \leq \inf(P) = \max_{\psi\in Y'} - F^{\#}(0_{X'},\psi) < +\infty. \tag{7}$$

Proof Let $\varphi \in X'$. Consider (P) with the objective function $f-\varphi$ instead of f. Then $F-\varphi$ is a convex perturbation function of $f-\varphi$ with $\mathrm{proj}_Y\mathrm{dom}(F-\varphi) = \mathrm{proj}_Y\mathrm{dom}F$. We apply Theorem 1 to obtain

$$-\infty \leq -(f-\varphi)^{\#}(0_{X'}) = \max_{\psi\in Y'} -(F-\varphi)^{\#}(0_{X'},\psi) < +\infty,$$

which gives (6) because $(f-\varphi)^{\#}(0_{X'}) = f^{\#}(\varphi)$ and $(F-\varphi)^{\#}(0_{X'},\psi) = F^{\#}(\varphi,\psi)$. Relation (7) is obtained from (6) by setting $\varphi = 0_{X'}$. □

If in addition X and Y are topological vector spaces, then the topological conjugate dual of (P) is nothing, but the restriction of (AD) on Y^*, that is,

$$\text{(TD)} \quad \begin{array}{ll} \sup & -F^{*}(0_{X^*},y^*) \\ \text{subject to} & y^* \in Y^*. \end{array}$$

We have

$$-\infty \leq \sup(TD) = v^{**}(0_Y) \leq v^{\#\#}(0_Y) = \sup(AD) \leq \inf(P) \leq +\infty.$$

If $\mathrm{dom}v^{\#} \subset Y^*$, then $v^{**}(y) = v^{\#\#}(y)$ for all $y \in Y$, and, consequently,

$$\sup(TD) = \sup(AD).$$

This property will be exploited in the sequel when we use algebraic constraint qualification to derive topological duality relations. The next diagram summarizes some relationships between the algebraic and the topological conjugate dualities.

F is convex, $\mathbb{R}_+\mathrm{proj}_Y(\mathrm{dom}F)$ is a linear subspace

$$\Downarrow$$

$$-\infty < f^{\#}(\varphi) = \min_{\psi \in Y'} F^{\#}(\varphi, \psi) \leq +\infty \ \forall \varphi \in X'$$

$$\Downarrow \begin{array}{l} X, Y \text{are topological vector space} \\ \text{proj}_{Y'}(\text{dom} F^{\#}) \subset Y^* \end{array}$$

$$-\infty < f^*(x^*) = \min_{y^* \in Y^*} F^*(x^*, y^*) \leq +\infty \ \forall x^* \in X^*$$

$$\Downarrow \ X, Y \text{are topological vector space}$$

$$\text{proj}_{X^* \times \mathbb{R}}(\text{epi} F^*) \text{ is } w^* - \text{closed and } \text{dom} f \neq \emptyset.$$

The first implication is due to Corollary 1. The second implication holds because the function f^* coincides with $f^{\#}$ on X^*, F^* coincides with $F^{\#}$ on $X^* \times Y^*$, and under the hypothesis $\text{proj}_{Y'}(\text{dom} F^{\#}) \subset Y^*$, one has

$$\min_{\psi \in Y'} F^{\#}(\varphi, \psi) = \min_{\psi \in Y^*} F^{\#}(\varphi, \psi).$$

The third implication comes from the fact that $\text{proj}_{X^* \times \mathbb{R}}(\text{epi} F^*) = \text{epi} f^*$. When X and Y are separated locally convex spaces and $f^{**} = F^{**}(., 0_Y)$ the converse of the third implication is also true (Corollary 2.2 [15]; see also Theorem 9.1 [4] when F is proper convex and lower semicontinuous).

Remark 2 If X and Y are vector spaces, then, as it was already discussed before, one may equip them with a locally convex topology such that $X' = X^*$ and $Y' = Y^*$. Assume that F is proper convex, lower semicontinuous with respect to the product topology on $X \times Y$. In view of Corollary 1, (6) holds, which, in its turn, by Theorem 9.1 [4], yields a closedness-type condition that $\text{proj}_{X' \times \mathbb{R}}(\text{epi} F^{\#})$ is closed with respect to the weak* topology on $X' \times \mathbb{R}$.

3 Algebraic Duality for the Sum of Convex Functions

Let W be a linear space and let $p, q : W \to \overline{\mathbb{R}}$ be functions on W. The infimal convolution of p and q is defined by

$$p \Box q(w) := \inf_{z \in W} \big(p(z) + q(w - z)\big) \ \text{ for } \ w \in W.$$

It is exact at w if the infimum is attained at some point $\bar{z} \in W$, that is $p(\bar{z}) + q(w - \bar{z}) \leq p(z) + q(w - z)$ for all $z \in W$. We wish to apply Corollary 1 to establish a formula for the algebraic conjugate of the sum $f + g$ in terms of infimal convolution of their conjugates, which is known as a generalized version of Fenchel's duality (see Remark 3).

Proposition 1 *Assume that f and g are proper convex and that $\mathbb{R}_+(dom f - dom g)$ is a linear subspace of X. Then, for each $\varphi \in X'$, one has*

$$-\infty < (f + g)^{\#}(\varphi) = f^{\#} \Box g^{\#}(\varphi) \leq +\infty,$$

in which the infimal convolution is exact.

Proof Consider (P) with the objective function $f+g$. Define a perturbation function F by

$$F(x,y)=f(x+y)+g(x) \quad \text{for} \quad (x,y)\in X\times X.$$

Then F is a proper convex function and (CQ) holds because $\text{proj}_X \text{dom} F = \text{dom} f - \text{dom} g$. Moreover, since f and g are proper, $f^{\#}$ and $g^{\#}$ do not take the value $-\infty$, and one has

$$F^{\#}(\varphi,\psi)=f^{\#}(\psi)+g^{\#}(\varphi-\psi), \quad \forall(\varphi,\psi)\in X'\times X'.$$

It remains to apply Corollary 1 (6) to complete the proof. □

Remark 3 When X is finite dimensional, the hypothesis of Proposition 1 reads $0 \in \text{ri}(\text{dom} f - \text{dom} g)$ or, equivalently, $\text{ri}(\text{dom} f)\cap \text{ri}(\text{dom} g)\neq \emptyset$, where "ri" denotes the relative interior of a set. Proposition 1 then gives Fenchel's duality Theorem 31.1 [17], which states that for proper convex functions f and g on $\mathbb{R}^n$ satisfying the above intersection condition, one has $\inf_{x\in\mathbb{R}^n}(f(x)+g(x)) = \max_{x^*\in\mathbb{R}^n} -(f^*(x^*)+g^*(-x^*))$. Note further that the latter equality remains true without f and g being lower semicontinuous even in a locally convex separated space provided that f is finite and continuous at a point of $\text{dom} g$ (see Theorem 2.8.7 (iii) [21]).

In the remaining part of this section we apply Proposition 1 to obtain a formula for the topological conjugate $(f+g)^*$.

Lemma 4 *Let $f,g: X\to\overline{\mathbb{R}}$ be two proper convex functions on a topological vector space X. Assume that $\mathbb{R}_+(\text{dom} f - \text{dom} g)$ is a linear subspace of X and that either $\text{dom} f^{\#}\subseteq X^*$ or $\text{dom} g^{\#}\subseteq X^*$. Then, for each $x^*\in X^*$ one has*

$$-\infty < (f+g)^*(x^*) = \min_{z^*\in X^*}(f^*(z^*)+g^*(x^*-z^*)) \leq +\infty. \tag{8}$$

Proof Let $x^*\in X^*$. By Proposition 1 there exists $\varphi\in X'$ such that

$$-\infty < (f+g)^*(x^*) = (f+g)^{\#}(x^*) = f^{\#}(\varphi)+g^{\#}(x^*-\varphi)\leq +\infty.$$

If $(f+g)^*(x^*)=+\infty$, then $f^*(z^*)+g^*(x^*-z^*)=+\infty$ for any $z^*\in X^*$ and (8) holds. If $(f+g)^*(x^*)\neq+\infty$, then $\varphi\in\text{dom} f^{\#}\cap(x^*-\text{dom} g^{\#})$. It is immediate from the hypothesis that $\varphi\in X^*$. Hence

$$(f+g)^*(x^*) = f^*(\overline{z}^*)+g^*(x^*-\overline{z}^*) \geq \inf_{z^*\in X^*}(f^*(z^*)+g^*(x^*-z^*)) \geq (f+g)^*(x^*),$$

which implies (8). □

The hypothesis that $\text{dom} f^{\#}\subseteq X^*$, can be ensured when X is partially ordered by a convex cone and all positive linear functionals with respect to that cone are continuous. Let us assume that X is a topological vector space equipped with a partial order generated by a convex cone $C\subset X$. The algebraic dual cone and the topological dual cone of C are respectively denoted by C°, C^+ and given by

$$C^\circ := \{\psi\in X' : \varphi(x)\geq 0, \forall x\in C\},$$
$$C^+ := \{x^*\in X^* : \langle x^*,x\rangle\geq 0\ \forall x\in C\}) = C^\circ\cap X^*.$$

A function $f: X\to\overline{\mathbb{R}}$ is said to be monotone if it is either increasing in the sense that $x_1\preceq x_2 \implies f(x_1)\leq f(x_2)$ for all $x_1,x_2\in Y$ or it is decreasing in the sense that $x_1\preceq x_2 \implies f(x_1)\geq f(x_2)$ for all $x_1,x_2\in X$.

Lemma 5 *Let $f: X\to\overline{\mathbb{R}}$ be monotone with $\text{dom} f\neq\emptyset$. Then $\text{dom} f^{\#}\subset C^\circ\cup(-C^\circ)$.*

Proof Let $\varphi \in \operatorname{dom} f^{\#}$. Choose $b \in \operatorname{dom} f$. If h is increasing, then for every $x \in C$, we have

$$+\infty > f^{\#}(\varphi) \geq \varphi(b-x) - f(b-x) \geq \varphi(b) - \varphi(x) - f(b).$$

If $\varphi(x_0) < 0$ for some $x_0 \in C$, then for $t > 0$, we would have $tx_0 \in C$ and

$$+\infty > f^{\#}(\varphi) \geq \varphi(b) - t\varphi(x) - f(b),$$

which leads to a contradiction when t tends to $+\infty$. Therefore, $\varphi(x) \geq 0$ for all $x \in C$, which means that $\varphi \in C^{\circ}$. If h is decreasing, then for every $x \in C$, we have

$$+\infty > f^{\#}(\varphi) \geq \varphi(b+x) - f(b+x) \geq \varphi(b) + \varphi(x) - f(b).$$

By a similar argument as above, we deduce $\varphi(x) \leq 0$ for all $x \in C$, by which $\varphi \in -C^{\circ}$. □

We shall need the following hypothesis:

(Q) *All positive linear functionals on X are continuous, that is,* $C^{\circ} = C^{+}$.

When this property is true, we also say that X has the property (Q). Here are some examples of spaces having the property (Q).

a) Every finite dimensional space has property (Q). This, however, is not true in infinite dimension. To see this, it suffices to consider a Banach space and a cone corresponding to the positive part of a discontinuous linear functional. The topological dual of this cone contains only the zero element while its algebraic dual contains the functional defining it.
b) Let X be a Banach space and $C \subset X$ a convex cone such that $C - C$ is a closed linear subspace of finite codimension, then Y has the property (Q) (see [20]). In particular, the space $C_0(T)$ of continuous real valued functions vanishing at infinity on a locally compact topological space T equipped with the norm $||x||_{\infty} = \sup_{t \in T} |x(t)|$ and the partially ordering cone $C = \{x \in C_0(T) : x(t) \geq 0 \ \forall t \in T\}$ has property (Q) (see [14]).
c) Let $X = \mathbb{R}^{\mathbb{N}}$ equipped with the product topology, and let $C = \mathbb{R}^{\mathbb{N}}_{+}$. Then $\mathbb{R}^{\mathbb{N}}$ has the property (Q) (Lemma 2.1 [2]).

Remark 4 If X is a topological vector space such that $C^{\circ} = C^{+}$ for every convex cone C, then its convergence is almost of finite dimension in the sense that from a certain term, all elements of a convergent net belong to a finite dimensional subspace. Indeed, let $(x_{\alpha})_{\alpha \in I}$ be a convergent net, where I is a directed index set. Without loss of generality we may assume that it converges to 0_X. By contradiction we suppose that the net has a subnet $(x_{\beta})_{\beta \in J}$ of linearly independent vectors, where J is a directed index subset of I. Let X_1 be the linear subspace generated by x_{β}, $\beta \in J$, and X_2 its algebraic complement. Define a linear functional ξ on X by $\xi(x_{\beta}) = 1$ for $\beta \in J$, and $\xi(x) = 0$ for $x \in X_2$. Consider the cone $C = \{x \in X : \xi(x) \geq 0\}$. It is clear that C is a convex cone, $\xi \in C^{\circ}$ but it is not continuous.

Here is a formula for the conjugate of the sum of two functions one of which is monotone.

Corollary 2 *Let $f, g : X \to \overline{\mathbb{R}}$ be two proper convex functions on a topological vector space X partially ordered by a convex cone $C \subset X$ with property (Q). Assume that $\mathbb{R}_{+}(\operatorname{dom} f - \operatorname{dom} g)$ is a linear subspace of X and that either f or g is monotone. Then, for* *each* $x^* \in X^*$ *one has*

$$-\infty < (f+g)^*(x^*) = \min_{z^* \in X^*} (f^*(z^*) + g^*(x^* - z^*)) \leq +\infty. \tag{9}$$

Proof By Lemma 5 and by condition (Q), we have either $\operatorname{dom} f^\# \subseteq X^*$ or $\operatorname{dom} g^\# \subseteq X^*$. It remains to apply Lemma 4 to complete the proof. □

Remark 5 In finite dimensional spaces the monotonicity condition is unnecessary (see Remark 3). This, however, is not true in the infinite dimensional case as shown by the next example.

Example 2 Let $X = \mathbb{R}^{\mathbb{N}}$, $C = \mathbb{R}^{\mathbb{N}}_+$ and let f be a discontinuous linear functional on X. Set $g = -f$. Then X satisfies (Q) and $\mathbb{R}_+(\operatorname{dom} f - \operatorname{dom} g) = X$. It is clear that (9) does not hold because $(f+g)^* = \delta_{\{0_{X^*}\}}$ while $f^*(x^*) = g^*(x^*) = +\infty$ for all $x^* \in X^*$. Notice that f and g are not monotone with respect to C because otherwise, being positive or negative on C, in view of Lemma 2.1 [2], they should be continuous.

We now apply Corollary 2 to an extended monotropic problem. Let $(t_n)_{n\geq 1}$ be a sequence from the set $\mathbb{R} \cup \{+\infty\}$. If some of its terms is equal to $+\infty$, then we agree that the infinite sum $\sum_{n\in\mathbb{N}} t_n$ is equal to $+\infty$, and it is equal to $\alpha \in \overline{\mathbb{R}}$ if the unconditional limit of the finite sums $\sum_{i\in J} t_i$, where J is a non-empty finite subset of $\mathbb{N}$, exists and is equal to α.
Let $(f_n)_{n\geq 1}$ be a sequence of proper convex functions from $\mathbb{R}$ to $\overline{\mathbb{R}}$ such that

$$f((x_n)_n) := \sum_{n\in\mathbb{N}} f_n(x_n) \quad \text{exists} \quad \text{for} \quad \text{each} \quad (x_n)_n \in \mathbb{R}^{\mathbb{N}}.$$

It is clear that f is a convex function on $\mathbb{R}^{\mathbb{N}}$. We assume that it is proper and consider the constrained problem, known as *an extended monotropic problem* (see [3, 5, 7, 15, 19]):

$$\text{(P)} \qquad \begin{array}{ll} \inf & f((x_n)_n) \\ \text{subject to} & (x_n)_n \in K, \end{array}$$

where $K \subset \mathbb{R}^{\mathbb{N}}$ is a convex cone. This problem can be written in the form

$$\begin{array}{ll} \inf & f((x_n)_n) + \delta_K((x_n)_n) \\ \text{subject to} & (x_n)_n \in \mathbb{R}^{\mathbb{N}}. \end{array}$$

The space $\mathbb{R}^{\mathbb{N}}$ equipped with the product topology is a locally convex space. Its topological dual is the space $\mathbb{R}^{[\mathbb{N}]}$ of real sequences with finite support. The standard bilinear coupling between them is given by

$$\langle (x_n)_n, (\lambda_n)_n \rangle = \sum_{n\in\mathbb{N}} \lambda_n x_n \quad \text{for } (x_n)_n \in \mathbb{R}^{\mathbb{N}}, (\lambda_n)_n \in \mathbb{R}^{[\mathbb{N}]}.$$

Fenchel's algebraic dual of (P) is given by

$$\begin{array}{ll} \sup & -\big(f^\#(\varphi) + \delta_K^\#(-\varphi)\big) \\ \text{subject to} & \varphi \in (\mathbb{R}^{\mathbb{N}})'. \end{array}$$

According to Proposition 1 [15],

$$-\infty < f^*((\lambda_n)_n) = \sum_{n\in\mathbb{N}} f_n^*(\lambda_n) \leq +\infty \quad \forall (\lambda_n)_n \in \mathbb{R}^{[\mathbb{N}]}.$$

This and the fact that $\delta_K^{\#}(-\varphi) = 0$ for $\varphi \in K^+$ and $\delta_K^{\#}(-\varphi) = +\infty$ for $\varphi \notin K^+$ yield the topological dual of (P):

$$\text{(D)} \quad \begin{array}{ll} \sup & -\sum_{n\in\mathbb{N}} f^*((\lambda_n)_n) \\ \text{subject to} & (\lambda_n)_n \in K^+. \end{array}$$

Corollary 3 *Assume that* $\mathbb{R}_+(\mathrm{dom} f - K)$ *is a linear subspace of* $\mathbb{R}^{\mathbb{N}}$ *and one of the following conditions holds:*

i) $K^\circ = K^+$.
ii) f *is monotone with respect to* $\mathbb{R}_+^{\mathbb{N}}$.
iii) *Either* $\mathbb{R}_+^{\mathbb{N}} \subseteq K$ *or* $\mathbb{R}_+^{\mathbb{N}} \subseteq -K$.

Then $-\infty \leq \inf(P) = \max(D) < +\infty$.

Proof Under i) we equip $\mathbb{R}^{\mathbb{N}}$ with a partial order generated by K. It clearly satisfies (Q). Moreover, the indicator function δ_K is monotone with respect to this order, and $\mathrm{dom}\delta_K = K$. By applying Corollary 2 we obtain

$$-\infty < (f + \delta_K)^*((\lambda_n)_n) = \min\left(\sum_{n\in\mathbb{N}} f_n^*(\mu_n) : (\mu_n)_n \in (\lambda_n)_n + K^+\right) < +\infty,$$

Setting $\lambda_n = 0$ for all n, we deduce $-\infty \leq \inf(P) = \max(D) < +\infty$.

Under ii) or iii) we equip $\mathbb{R}^{\mathbb{N}}$ with a partial order generated by $\mathbb{R}_+^{\mathbb{N}}$. It is plain that under iii) the function δ_K is monotone. Hence, in both cases, we may apply Corollary 2 to achieve the proof. □

Remark 6 As an application of Proposition 1 we can also obtain a version of Theorem 23.8 [17] on the algebraic subdifferential of the sum of two convex functions. Namely, assume that f and g are proper convex and that $\mathbb{R}_+(\mathrm{dom} f - \mathrm{dom} g)$ is a linear subspace of X. Then for each $x \in X$, one has

$$\partial(f + g)(x) = \partial f(x) + \partial g(x). \tag{10}$$

Using the perturbation function F defined in the proof of Proposition 1, we also have $\mathrm{proj}_{X'\times R}\mathrm{epi}F^{\#} = \mathrm{epi} f^{\#} + \mathrm{epi} g^{\#}$. The closedness-type condition mentioned in Remark 2 implies the formula of Proposition 1, and hence equality (10) as well. Furthermore, under the hypothesis of Corollary 2, (10) gives a formula for the topological subdifferential of the sum of two functions : $\partial^*(f + g)(x) = \partial^* f(x) + \partial^* g(x)$, because we have then either $\partial f(x)$ or $\partial g(x)$ a subset of the topological dual space X^*.

4 Problems with Inequality Constraints

Assume that the vector space Y is equipped with a partial order generated by a convex cone $C \subset Y$: $y_1 \preceq y_2 \iff y_2 - y_1 \in C$. Let $f : X \to \overline{\mathbb{R}}$ be a proper convex function, where X is a linear space. Let $G : X \to Y \cup \{+\infty_Y\}$, where $+\infty_Y$ is a symbol with $y \preceq \infty_Y$ for all $y \in Y$. Consider the following problem with inequality constraint

$$\text{(CP)} \quad \begin{array}{ll} \inf & f(x) \\ \text{subject to} & G(x) \preceq 0_Y. \end{array}$$

The domain of G is given by $\mathrm{dom}G := \{x \in X : G(x) \neq +\infty_Y\}$ and its epigraph is given by $\mathrm{epi}G := \{(x, y) \in X \times Y : G(x) \preceq y\}$.

Let us denote the feasible set of (CP) by E, that is, $E := \{x \in X : G(x) \preceq 0_Y\}$ and $\Delta := \mathrm{dom} f \cap \mathrm{dom}\, G$. For each $\psi \in Y'$ we consider

$$(\psi \circ G)(x) = \begin{cases} \psi(G(x)) & \text{if } x \in \mathrm{dom}G \\ +\infty & \text{else.} \end{cases}$$

A perturbation function F can be given by

$$F(x, y) = \begin{cases} f(x) & \text{if } G(x) \preceq -y \\ +\infty & \text{else.} \end{cases} \tag{11}$$

The algebraic dual cone of the cone C is denoted by C°, that is,

$$C^\circ := \{\psi \in Y' : \psi(y) \geq 0, \forall y \in C\}.$$

Lemma 6 *Assume $\Delta \neq \emptyset$. For each $(\varphi, \psi) \in X' \times Y'$ we have*

$$F^\#(\varphi, \psi) = \begin{cases} (f + \psi \circ G)^\#(\varphi) & \textit{if } \psi \in C^\circ \\ +\infty & \textit{else.} \end{cases}$$

Proof We compute $F^\#(\varphi, \psi)$ by definition

$$\begin{aligned} F^\#(\varphi, \psi) &= \sup_{x \in \Delta,\, G(x) \preceq -y} (\varphi(x) + \psi(-y) - f(x)) \\ &= \sup_{x \in \Delta, u \in C} (\varphi(x) - f(x) - \psi(G(x) + u)) \\ &= \sup_{x \in \Delta, u \in C} (\varphi(x) - f(x) - (\psi \circ G)(x)) - \psi(u)) \\ &= \begin{cases} (f + \psi \circ G)^\#(\varphi) & \text{if } \psi \in C^\circ \\ +\infty & \text{else.} \end{cases} \end{aligned}$$

The proof is complete. □

Remark 7 When $\Delta = \emptyset$ and $C^\circ \neq Y'$, the conclusion of Lemma 6 is not true. In this case, we have $F^\#(\varphi, \psi) = -\infty$ for all $(\varphi, \psi) \in X' \times Y'$.

In view of Lemma 6 we obtain the associated algebraic dual of (CP) in the form:

$$\text{(ACD)} \qquad \begin{array}{ll} \sup & -(f + \psi \circ G)^\#(0_{X'}) \\ \text{subject} & \text{to } \psi \in C^\circ. \end{array}$$

Proposition 2 *Assume f is proper convex, epi$G$ is convex and the following condition holds:*

(CQ1) $\mathbb{R}_+(G(\Delta) + C)$ *is a linear subspace of Y.*

Then for each $\varphi \in X'$ one has

$$-\infty < (f + \delta_E)^\#(\varphi) = \min_{\psi \in C^\circ} (f + \psi \circ G)^\#(\varphi) \leq +\infty,$$

where δ_E *denotes the indicator function of* E*. In particular,*

$$-\infty \leq \mathit{inf}(CP) = \mathit{max}(ACD) < +\infty.$$

Proof We have epi $F = \{(x, y, r) \in X \times Y \times \mathbb{R} : (x, r) \in \text{epi} f\} \cap \{(x, y, r) \in X \times Y \times \mathbb{R} : (x, -y) \in \text{epi}G\}$. It follows that epi$F$ is convex because epi$f$ and epi$G$ are both convex, by which F is convex. Moreover, (CQ) holds due to (CQ1) and the fact that

$$\text{proj}_Y \text{dom}F = \{y \in Y : -y \in G(x) + C \text{ for some } x \in \Delta\} = -(G(\Delta) + C).$$

It remains to apply Corollary 1 and Lemma 6 to complete the proof. □

Remark 8 When $Y = \mathbb{R}^m$ and $C = \mathbb{R}^m_+$, problem (CP) has m inequality constraints. The constraint qualification (CQ1) is clearly equivalent to the well known Slater condition. In a general case we have the following characterization of (CQ1).

Proposition 3 *If (CQ1) holds, then* $C - C \subseteq \mathbb{R}_+(G(\Delta) + C)$*. Moreover, if* $G(\Delta) \subseteq C - C$*, then*

$$(CQ1) \Leftrightarrow C - C = \mathbb{R}_+(G(\Delta) + C).$$

Proof Observe that under (CQ1), the subspace $\mathbb{R}_+(G(\Delta) + C)$ contains the sets $G(\Delta) + C$, $-G(\Delta) - C$ and their sum as well. By picking an element $p \in G(\Delta)$, we have

$$C - C = p + C - p - C \subseteq G(\Delta) + C - (G(\Delta) + C) \subseteq \mathbb{R}_+\big(G(\Delta) + C\big). \tag{12}$$

Now, if $G(\Delta) \subseteq C - C$, then

$$\mathbb{R}_+\big(G(\Delta) + C\big) \subseteq \mathbb{R}_+\big((C - C) + C\big) = C - C.$$

This and (12) prove implication " $\Rightarrow$ ". The implication " $\Leftarrow$ " is clear because $C - C$ is a linear subspace. □

Remark 9 In applications, as we will see later, it is often the case that $C - C = Y$. In such cases, it follows from Proposition 3 that (CQ1) is equivalent to equality $\mathbb{R}_+(G(\Delta)+C) = Y$, which is, in particular, satisfied if $-G(\Delta)$ meets the algebraic interior core$C$ of C. Note also that if there exists $b \in Y$ such that $G(\Delta) \subseteq b + C$ with $-b \notin$ core$C$, then (CQ1) does not hold.

We now consider a particular case of problem (CP). Let $A : X \to Y$ be a linear map, $l \in X'$, and $b \in Y$, where X is a linear space and Y is another linear space equipped with a partial order generated by a convex cone $C \subset Y$. We consider the following linear problem

$$\text{(LP)} \quad \begin{array}{ll} \inf & l(x) \\ \text{subject to} & A(x) \succeq b. \end{array}$$

We denote the adjoint of A by A', that is, $A' : Y' \to X'$ given by $A'(\psi) = \psi \circ A$ for $\psi \in Y'$. Problem (LP) is a particular case of (CP) in which $f(x) = l(x)$ and $G(x) = b - A(x)$ for $x \in X$. Moreover, epi$G$ is convex and $\Delta = X$. By using the perturbation function (11) and Lemma 6 we obtain the following algebraic dual of (LP) :

$$\text{(ALD)} \quad \begin{array}{ll} \sup & -F^{\#}(0_{X'}, \psi) \\ \text{subject to} & \psi \in Y', \end{array}$$

in which

$$F^{\#}(\varphi,\psi)=\begin{cases}-\psi(b) & \text{if } \psi\in C^{\circ} \text{ and } A'(\psi)=l-\varphi\\ +\infty & \text{else}\end{cases}$$

because for $\psi \in C^{\circ}$ one has

$$\begin{aligned}\big(l+\psi\circ(b-A)\big)^{\#}(\phi) &= \sup_{x\in X}\big(\phi(x)-l(x)-\psi(b)-\psi\circ A(x)\big)\\ &= -\psi(b)-\sup_{x\in X}\big(\phi-l-A'(\psi)\big)(x)\\ &= \begin{cases}-\psi(b) & \text{if } \ A'(\psi)=l-\varphi\\ +\infty & \text{else.}\end{cases}\end{aligned}$$

It is clear that for $l\notin A'(C^{\circ})$, the dual objective function $-F^{\#}(0_{X'},.)$ is identically equal to $-\infty$ and $\sup(ALD)=\max(ALD)=-\infty$. For $l\in A'(C^{\circ})$, problem (ALD) is feasible and written as

(ALD)' $\quad\begin{array}{ll}\sup & \psi(b)\\ \text{subject to} & A'(\psi)=l,\ \psi\in C^{\circ}.\end{array}$

Proposition 4 *Assume that* $\mathbb{R}_+(A(X)-C-b)$ *is a linear subspace of* Y*. For each* $l\in X'$ *one has either*

$$\textit{inf}(LP)=-\infty \textit{ and } (ALD)' \textit{ unfeasible},$$

or

$$\textit{inf}(LP)\neq-\infty \textit{ and } \textit{inf}(LP)=\textit{max}(ALD)'\in\mathbb{R}.$$

Proof Observe that for $G(x)=b-A(x)$ and $\Delta=X$, one has $\mathbb{R}_+(G(\Delta)+C)=\mathbb{R}_+(C+b-A(X))$. It remains to apply Proposition 2 to complete the proof. □

5 Infinite Convex Problems

We consider the following infinite constrained problem

(ICP) $\quad\begin{array}{ll}\inf & f(x)\\ \text{subject to} & g_t(x)\le 0,\ t\in T,\end{array}$

where T is an infinite index set, $f: X\to\overline{\mathbb{R}}$ and $g_t: X\to\overline{\mathbb{R}}, t\in T$ are proper convex functions on a vector space X. Set $Y=\mathbb{R}^T$, $C=\mathbb{R}^T_+$ consisting of all families $(\lambda_t)_t\subset\mathbb{R}$ with $\lambda_t\ge 0$ for all $t\in T$, and define $G: X\to Y\cup\{\infty_Y\}$ by

$$G(x)=\begin{cases}(g_t(x))_t & \text{if } x\in\bigcap_{t\in T}\operatorname{dom}g_t\\ \infty_Y & \text{else.}\end{cases}$$

With this G, problem (ICP) takes the form of (CP) studied in Section 4. Its algebraic dual is given by

(ICD) $\quad\begin{array}{ll}\sup & -(f+\psi\circ G)^{\#}(0_{X'})\\ \text{subject to} & \psi\in(\mathbb{R}^T_+)^{\circ}.\end{array}$

We have the following duality result.

Corollary 4 *Assume that f and g_t, $t \in T$ are proper convex and that $\mathbb{R}_+(\bigcup_{x \in dom f \cap (\cap_{t\in T} dom g_t)} g_t(x) + \mathbb{R}^T_+)$ is a linear subspace of $\mathbb{R}^T$. Then*

$$-\infty \leq inf(ICP) = max(ICD) < +\infty.$$

Proof Observe that the set Δ is given by $\Delta = \text{dom} f \cap (\cap_{t\in T} \text{dom} g_t)$. It remains to apply Proposition 2 to achieve the proof. □

We are now interested in two particular problems corresponding to $T = \mathbb{N}$ and T being a locally compact Hausdorff topological space. Since the topological dual space of $\mathbb{R}^T$ is well known for these T, we focus our study on the topological dual of (ICD) and specify the condition (CQ1) for these problems. By so doing we show that in some cases usual topological conditions for strong duality are not applied, but the algebraic ones are.

5.1 Countably Infinite Convex Problem

We consider (ICP) with $T = \mathbb{N}$, that is,

$$\text{(CICP)} \quad \begin{array}{ll} \inf & f(x) \\ \text{subject to} & g_n(x) \leq 0, n \in \mathbb{N}, \end{array}$$

where $f : X \to \overline{\mathbb{R}}$ and $g_n : X \to \overline{\mathbb{R}}$, $n \in \mathbb{N}$ are proper convex functions on a vector space X (see [10–12]). By using (ICD) we obtain the associated topological dual of (CICP):

$$\text{(CICD)} \quad \begin{array}{ll} \sup & \inf_{x\in\Delta} \left(f(x) + \sum_{n\in\mathbb{N}} \lambda_n g_n(x)\right) \\ \text{subject to} & (\lambda_n)_n \in \mathbb{R}^{[\mathbb{N}]}_+, \end{array}$$

where $\Delta = \text{dom} f \bigcap \left(\cap_{n\in\mathbb{N}} \text{dom} g_n\right)$.

Let us first analyze condition (CQ1). The next result is a kind of Slater's constraint qualification in infinite dimension the proof of which requires some effort.

Lemma 7 *The following conditions are equivalent:*

(i) $\mathbb{R}_+\left(\bigcup_{x\in\Delta} \prod_{n\in\mathbb{N}}[g_n(x), +\infty)\right)$ *is a linear subspace;*
(ii) $\mathbb{R}_+\left(\bigcup_{x\in\Delta} \prod_{n\in\mathbb{N}}[g_n(x), +\infty)\right) = \mathbb{R}^{\mathbb{N}}$;
(iii) $\Delta^- := \{x \in \Delta : g_n(x) < 0$ *for all* $n \in \mathbb{N}\} \neq \emptyset$ *and there is some* $k \geq 1$ *such that*

$$\mathbb{R}^{\mathbb{N}\setminus\{1,...,k-1\}} = \bigcup_{x\in\Delta^-} \prod_{n\geq k} [g_n(x), \infty). \tag{13}$$

Proof The equivalence between (i) and (ii) follows from Proposition 3 and from the fact that $\mathbb{R}^{\mathbb{N}}_+ - \mathbb{R}^{\mathbb{N}}_+ = \mathbb{R}^{\mathbb{N}}$. We prove implication (iii) ⇒ (ii). Let $a = (a_n)_n \in \mathbb{R}^{\mathbb{N}}$. By (13), for the element $(-|a_n|)_n$, there is some $\bar{x} \in \Delta^-$ such that

$$-|a_n| \geq g_n(\bar{x}) \text{ for } n \geq k. \tag{14}$$

Choose $\epsilon > 0$ such that $g_n(\bar{x}) \leq -\epsilon$ for $n = 1, ..., k-1$. Such ϵ exists because $\bar{x} \in \Delta^-$. Set $t = \max_{n=1,...,k-1} |a_n|/\epsilon$. Then $-(t+1)\epsilon \leq -|a_n|$, $n = 1, ..., k-1$. It follows that

$$(t+1)g_n(\bar{x}) \leq -(t+1)\epsilon \leq -|a_n| \leq a_n \text{ for } n = 1, ..., k-1.$$

Moreover, as $g_n(\bar{x}) < 0$, we deduce from (14) that

$$(t+1)g_n(\bar{x}) \le g_n(\bar{x}) \le -|a_n| \le a_n \text{ for } n \ge k.$$

Consequently, $(a_n)_n \in (t+1)\big((g_n(\bar{x}))_n + \mathbb{R}^{\mathbb{N}}_+\big)$. This proves (ii).
Conversely, assume (ii). For $(a_n)_n$ with $a_n = -1, n \in \mathbb{N}$, there exist some $t > 0$ and $x \in \Delta$ such that $a_n \ge tg_n(x)$ for all $n \in \mathbb{N}$. In particular, $g_n(x) < 0, n \in \mathbb{N}$, which shows that $x \in \Delta^-$ and Δ^- is nonempty. Further, suppose to the contrary that (13) does not hold. For each $k \ge 1$ there exists an element $a^k = (a^k_n)_{n \ge k}$ that does not belong to the set $\bigcup_{x \in \Delta^-} \prod_{n \ge k}[g_n(x), \infty)$. Let $b^k = (b^k_n)_{n \ge k}$ with $b^k_n = -|a^k_n| - 1$. Since $b^k_n < a^k_n$, we have

$$b^k \notin \bigcup_{x \in \Delta^-} \prod_{n \ge k}[g_n(x), \infty) \text{ for all } k \ge 1,$$

that is, for every $x \in \Delta^-$ there is some index $n(k,x) \ge k$ such that

$$b^k_{n(k,x)} < g_{n(k,x)}(x). \tag{15}$$

Let $(c_n)_{n \ge 1} \in \mathbb{R}^{\mathbb{N}}$ be defined by $c_n := \min\{b^1_n, ..., b^n_n\}, \quad n \ge 1$. It is clear that

$$c_n \le b^i_n \le -1 \quad \text{for all} \quad i \le n. \tag{16}$$

Now consider the element $(nc_n)_n \in \mathbb{R}^{\mathbb{N}}$. By hypothesis, there are some $t > 0$ and $\bar{x} \in \Delta$ such that

$$nc_n \ge tg_n(\bar{x}) \quad \text{for all } n \in \mathbb{N}. \tag{17}$$

In particular, $\bar{x} \in \Delta^-$ because $c_n, n \in \mathbb{N}$ are all negative. In view of (15) and (16) we have

$$c_{n(k,\bar{x})} \le b^k_{n(k,\bar{x})} < g_{n(k,\bar{x})}(\bar{x}) \text{ for all } k \ge 1.$$

This implies that $c_{n(k,\bar{x})}/g_{n(k,\bar{x})}(\bar{x}) > 1$, which together with (17) yields

$$t \ge n(k,\bar{x}) \frac{c_{n(k,\bar{x})}}{g_{n(k,\bar{x})}(\bar{x})} > n(k,\bar{x}) \ge k \text{ for all } k \ge 1.$$

We arrive at a contradiction when $k > t$. The proof is complete. □

We are now in position to give conditions for strong duality

Proposition 5 *Assume that there is some $x \in \Delta$ such that $g_n(x) < 0$ for all $n \in \mathbb{N}$ and there is some $k \ge 1$ such that*

$$\mathbb{R}^{\mathbb{N} \setminus \{1,...,k-1\}} = \bigcup_{x \in \Delta} \prod_{n \ge k}[g_n(x), \infty). \tag{18}$$

Then

$$-\infty \le \inf(ICP) = \max(ICD) < +\infty.$$

Proof Observe that $G(\Delta) + C = \bigcup_{x \in \Delta} \prod_{n \in \mathbb{N}}[g_n(x), +\infty)\big)$. It is clear that (18) is equivalent to (13). In view of Lemma 7, (CQ1) holds. Applying Proposition 2 and Lemma 2.1 [2] we find some $(\lambda_n)_n \in \mathbb{R}^{\mathbb{N}}$ such that

$$-\infty < (f + \delta_E)^\#(0_{X'}) = \inf_{x \in \Delta} \Big(f(x) + \sum_{n \in \mathbb{N}} \lambda_n g_n(x)\Big) \le +\infty.$$

This yields $-\infty \le \inf(ICP) = \max(ICD) < +\infty$ as requested. □

A simple class of problems that satisfy the hypothesis of Proposition 5 consists of convex problems in which $X = \mathbb{R}^{\mathbb{N}}$ and the constraints are separated in the sense that $g_n((x_n)_n) = h_n(x_n)$ for $(x_n)_n \in \mathbb{R}^{\mathbb{N}}$ with $h_n : \mathbb{R} \to \overline{\mathbb{R}}, n \geq 1$ being proper convex. We have $\Delta = \text{dom f} \cap \prod_{n\in\mathbb{N}} \text{dom} h_n$ and denote $P_k(\text{dom } f) := \{x_k \in \mathbb{R} : (x_n)_n \in \text{dom } f\}$.

Corollary 5 *Assume that* f *is proper convex and there are some* $(x_n)_n \in \Delta$ *such that* $h_n(x_n) < 0$ *for all* $n \in \mathbb{N}$ *and* $k \geq 1$ *such that* $\inf_{x\in P_n(dom\ f)} h_n(x) = -\infty$ *for all* $n \geq k$. *Then*

$$\inf_{h_n(x_n)\leq 0, n\in\mathbb{N}} f((x_n)_n) = \max_{(\lambda_n)_n \in \mathbb{R}_+^{[\mathbb{N}]}} \inf_{(x_n)_n\in\Delta} \{f((x_n)_n) + \sum_{n\in\mathbb{N}} \lambda_n h_n(x_n)\} < +\infty.$$

Proof Observe that (18) is satisfied because by the hypothesis, we have

$$\bigcup_{(x_n)_n\in\Delta} \prod_{n\geq k} \{h_n(x_n)\} = \mathbb{R}^{\mathbb{N}\setminus\{1,\ldots,k-1\}}.$$

It remains to apply Proposition 5 to complete the proof. □

Remark 10 We notice that the hypothesis requested in Proposition 5 is sufficient, but not necessary for strong duality $\inf(ICP) = \max(ICD)$. For instance with $g_n(x) = 0$ for all $x \in \text{dom} f$ and $n \in \mathbb{N}$, the above mentioned equality holds trivially. However, the set $\mathbb{R}_+\big(\bigcup_{x\in\Delta} \prod_{n\in\mathbb{N}}[g_n(x), +\infty)\big) = \mathbb{R}_+^{\mathbb{N}}$ is not a linear subspace. Moreover, as we have already noticed in Remark 8 that for problems with finite number of inequality constraints $g_i(x) \leq 0, i = 1, ..., m$, the constraint qualification is equivalent to the Slater condition: there exists some $\bar{x} \in \text{dom} f$ such that $g_i(\bar{x}) < 0, i = 1, ..., m$. This, however, is not true for (ICP).

Example 3 Let the constraints of (ICP) be defined by $g_n(x) = -n$ for all $n \in \mathbb{N}$ and let $\Delta \neq \emptyset$. It is clear that the Slater condition holds. Consider an element $(a_n)_n \in \mathbb{R}^{\mathbb{N}}$ given by $a_n = -n^2, n \in \mathbb{N}$. We claim that $(a_n)_n \notin \mathbb{R}_+\big(\bigcup_{x\in\Delta} \prod_{n\in\mathbb{N}}[g_n(x), +\infty)\big)$. Indeed, by supposing the contrary, we may find some $t > 0$ and $x \in \Delta$ such that $a_n \geq t g_n(x)$ for all $n \in \mathbb{N}$. It follows that $-n^2 \geq t(-n)$, or equivalently, $t \geq n$ for all $n \in \mathbb{N}$, a contradiction. Hence, the constraint qualification does not hold.

Remark 11 When X is a locally convex space and the functions $f, g_n, n \in \mathbb{N}$ are proper convex, lower semicontinuous, and inf(ICP) $\neq +\infty$, a necessary and sufficient condition for strong duality has been established in Theorem 1 [12], which demands that the set $\cup_{(\lambda_n)_n\in\mathbb{R}_+^{[\mathbb{N}]}} \text{epi}(f + \delta_{\text{dom}G} + \sum_{n\in\mathbb{N}} \lambda_n g_n)^*$ is w*-closed regarding $\{0_{X^*}\} \times \mathbb{R}$ in the sense that its intersection with $\{0_{X^*}\} \times \mathbb{R}$ coincides with the intersection of its closure with $\{0_{X^*}\} \times \mathbb{R}$.

Finally we apply Proposition 5 to infinite linear problems. Consider the following algebraic infinite linear problem

$$\text{(AILP)} \quad \begin{array}{ll} \inf & l(x) \\ \text{subject to} & l_n(x) \geq b_n, n \in \mathbb{N}, \end{array}$$

where $l, l_n \in X'$ and $b_n \in \mathbb{R}, n \in \mathbb{N}$. By setting $A(x) = (l_n(x))_n$ and $C = \mathbb{R}_+^{\mathbb{N}}$ we can see that (AILP) is exactly (LP) studied in Section 3.3. The algebraic dual (ALD)' yields the following associated topological dual

$$\text{(TILD)}\quad \begin{array}{ll} \sup & \sum_{n\in\mathbb{N}} \lambda_n b_n \\ \text{subject to} & (\lambda_n)_n \in \mathbb{R}_+^{[\mathbb{N}]}, \sum_{n\in\mathbb{N}} \lambda_n l_n = l. \end{array}$$

Corollary 6 *Assume that there is some $x \in X$ such that $b_n < l_n(x)$ for all $n \in \mathbb{N}$ and there is some $k \geq 1$ such that*

$$\mathbb{R}^{\mathbb{N}\setminus\{1,\ldots,k-1\}} = \bigcup_{x\in X} \prod_{n\geq k} [l_n(x), \infty). \tag{19}$$

Then, either $\inf(AILP) = -\infty$ *and (TILD) is unfeasible, or* $\inf(ALIP) \neq -\infty$ *and* $\inf(ALIP) = \max(TILD) \in \mathbb{R}$.

Proof Since $\inf(AILP) \geq \sup(TILD)$, we obtain that (TILP) has no feasible solution if $\inf(AILP) = -\infty$. For the case $\inf(AILP) > -\infty$, we consider $g_n(x) = b_n - l_n(x)$, $\Delta = X$ and show that (18) and (19) are equivalent. Indeed, if (19) holds, then for every $(a_n)_{n\geq k} \in \mathbb{R}^{\mathbb{N}\setminus\{1,\ldots,k-1\}}$, one finds some $x \in X$ such that $l_n(x) + s_n = a_n - b_n$ with $s_n \geq 0, n \geq k$. Setting $y = -x$, one obtains $g_n(y)+s_n = b_n - l_n(y)+s_n = b_n + l_n(x)+s_n = a_n, n \geq k$, which establishes (18). Conversely, let (18) hold. Let $(a_n)_{n\geq k} \in \mathbb{R}^{\mathbb{N}\setminus\{1,\ldots,k-1\}}$. For $(a_n + b_n)_{n\geq k}$, there are some $x \in X$ and $s_n \geq 0$ such that $g_n(x) + s_n = a_n + b_n, n \geq k$. This implies $l_n(-x) + s_n = a_n, n \geq k$, which shows that (19) is satisfied. Now it remains to apply Proposition 5 to complete the proof. □

Remark 12 Condition (19) cannot be satisfied if X is finite dimensional. In fact, let $e^i, i = 1, \ldots, k$ be a basis of X and $y_n^i = l_n(e^i)$. Choose $(z_n)_n \in \mathbb{R}^{\mathbb{N}}$ with $z_n = -n \sum_{i=1}^k (|y_n^i| + 1)$. We claim that $(z_n)_n \notin \bigcup_{x\in X} \prod_{n\geq k} [l_n(x), \infty)$. Suppose to the contrary that there is some $x \in X$, say $x = \sum_{i=1}^k t_i e^i$ with $t_i \in \mathbb{R}$ such that $l_n(x) \leq z_n, n \in \mathbb{N}$. Then $\sum_{i=1}^k t_i y_n^i \leq -n \sum_{i=1}^k (|y_n^i| + 1)$ for $n \geq 1$. It follows that

$$\sum_{i=1}^k |t_i| \geq \frac{\sum_{i=1}^k t_i y_n^i}{\sum_{i=1}^k (|y_n^i| + 1)} \geq n,$$

which is impossible when n is sufficiently large.
In infinite dimension, (19) holds for instance when the linear operator from X to $\mathbb{R}^{\mathbb{N}\setminus\{1,\ldots,k-1\}}$ defined by $x \mapsto (l_n(x))_{n\geq k}$ for $x \in X$ is surjective.

5.2 Continuous Convex Problems

We consider the following infinite constrained problem

$$\text{(CCP)}\quad \begin{array}{ll} \inf & f(x) \\ \text{subject to} & g_t(x) \leq 0, t \in T, \end{array}$$

where the index set T is a locally compact Hausdorff topological space, $f : X \to \overline{\mathbb{R}}$ and $g_t : X \to \overline{\mathbb{R}}, t \in T$ are proper convex functions on a vector space X. To simplify the writing, we use $g(t, x)$ instead of $g_t(x)$ and assume throughout that for every $x \in M := \cap_{t\in T} \mathrm{dom}\, g(t, .)$, the function $t \mapsto g(t, x)$ belongs to the space $C_0(T)$ of continuous functions vanishing at infinity equipped with the sup norm and partially ordered by the positive cone $C := \{y \in C_0(T) : y(t) \geq 0 \text{ for all } t \in T\}$. The algebraic dual of (CCP) gives its associated topological dual in the form

$$\text{(CCD)} \quad \begin{array}{ll} \sup & \inf_{x\in\Delta} \big(f(x) + \langle y^*, g(.,x)\rangle\big) \\ \text{subject to} & y^* \in C^+, \end{array}$$

where $\Delta := M \cap \operatorname{dom} f$ and C^+ is the topological dual cone of C in the space of regular Borel measures on T.

Proposition 6 *Assume that* $\mathbb{R}_+\big(\cup_{x\in\Delta}\{g(.,x)\} + C\big) = C_0(T)$. *Then one has*

$$-\infty \le \inf(CCP) = \max(CCD) < +\infty.$$

Proof We wish to apply Proposition 2 with $Y = C_0(T)$ and

$$G(x) = \begin{cases} g(.,x) & \text{if } x \in M \\ \infty_Y & \text{else.} \end{cases}$$

It is clear that epi(G) is convex. By the hypothesis,

$$\mathbb{R}_+\big(G(\Delta) + C\big) = \mathbb{R}_+\big(\cup_{x\in\Delta}\{g(.,x)\} + C\big) = C_0(T),$$

which implies (CQ1). In view of Proposition 2,

$$-\infty \le \inf(CCP) = \max_{\psi\in C^\circ} \inf_{x\in\Delta} \big(f(x) + \psi \circ g(.,x)\big) < +\infty.$$

According to Remark 3 b), $C^\circ = C^+$. We deduce

$$-\infty \le \inf(CCP) = \max(CCD) < +\infty$$

as requested. □

When T is compact, the space $C_0(T)$ coincides with $C(T)$, the space of continuous functions on T. The cone C° of positive functionals on C is known to be the set $R^+(T)$ of positive Radon measures (regular Borel measures) on T. Thus, for each $y^* \in C^+$, there is some $\mu \in R^+(T)$ such that $\langle y^*, g(.,x)\rangle = \int g(t,x)d\mu(t)$.

Lemma 8 *Assume T is compact. Then* $\mathbb{R}_+\big(\cup_{x\in\Delta}\{g(.,x)\}+C\big) = C(T)$ *if and only if there is some $\bar{x} \in \Delta$ such that $g(t,\bar{x}) < 0$ for all $t \in T$.*

Proof The "only if" part is clear. We prove the "if" part. Let $y \in C(T)$ be given. By the definition of the sup norm, $-\|y\| \le y(t)$ for all $t \in T$. Since $t \mapsto g(t,\bar{x})$ is continuous and T is compact, there is some $\epsilon > 0$ such that $g(t,\bar{x}) \le -\epsilon$ for all $t \in T$. Choose $s \ge \|y\|/\epsilon$. We obtain $sg(t,\bar{x}) \le (\|y\|/\epsilon)g(t,\bar{x}) \le -\|y\| \le y(t)$ for all $t \in T$. This shows that $y \in \mathbb{R}_+\big(\cup_{x\in\Delta}\{g(.,x)\} + C\big)$. The proof is complete. □

Corollary 7 *Assume that T is compact and there is some $\bar{x} \in \Delta$ such that $g(t,\bar{x}) < 0$ for all $t \in T$. Then one has*

$$-\infty \le \inf(CCP) = \max_{\mu\in R^+(T)} \inf_{x\in\Delta} \Big(f(x) + \int g(t,x)d\mu(t)\Big) < +\infty.$$

Proof By Lemma 8, $\mathbb{R}_+\big(\cup_{x\in\Delta}\{g(.,x)\} + C\big) = C(T)$. The corollary is obtained by Proposition 6. □

Remark 13 The necessary and sufficient condition for the constraint qualification in Lemma 8 is not true when T is not compact as it is shown in the next example.

Example 4 Let us consider $T = [0, \infty)$ and $g(t, x) = -1/(1+t)$ for all $t \in T$ and $x \in \Delta$. We have $g(., x) \in C_0(T)$. We claim that $\mathbb{R}_+\big(\cup_{x\in\Delta}\{g(., x)\} + C\big) \neq C_0(T)$. To see this, choose $y(t) = -1/\sqrt{1+t}$. Suppose to the contrary that there is some $\alpha > 0$ such that $y(t) \geq \alpha g(t, x), t \in T$. We obtain that $\alpha \geq y(t)/g(t, x) = \sqrt{1+t}$ for all $t \in T$, which is a contradiction when t tends to infinity.

6 Conclusion

Throughout this work we have used condition (CQ) in one form or another to obtain strong duality results. When X and Y are locally convex separated spaces the topological strong duality

$$\inf_{x\in X} F(x, 0_Y) = \max_{y^*\in Y^*}(-F^*(0_{X^*}, y^*)$$

is classically established under a constraint qualification and some lower semicontinuity or quasicontinuity hypothesis on F (Theorem 7 [18], Theorem 2.7.1 [21]). The main merit of the algebraic duality approach we have developed in this work is that, in favorable circumstances, condition (CQ) is sufficient to ensure the topological strong duality without further hypothesis on F.

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