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A Constant Rank Constraint Qualification in Continuous-Time Nonlinear Programming

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Abstract
The paper addresses continuous-time nonlinear programming problems with equality and inequality constraints. First and second order necessary optimality conditions are obtained under a constant rank type constraint qualification. The first order necessary conditions are of Karush-Kuhn-Tucker type.

Keywords Nonlinear programming · Continuous-time programming · Necessary optimality conditions · Constraint qualifications · Constant rank condition

Mathematics Subject Classification (2010) 90C30 · 90C46

1 Introduction

The beginnings of continuous-time programming date from 1953 when Bellman studied in his article the so-called "bottleneck processes". He considered a certain dynamic generalization of an ordinary linear programming problem. See Bellman [7]. Since then, this

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class of problems has received important attention from the scientific community. Rigorous mathematical treatment has been given to it and the theory has been developed to include optimality conditions and duality in linear and nonlinear cases. See, for example, Farr and Hanson [11], Hanson and Mond [12], Levinson [14], Tyndall [30] and Zangwill [37].

Necessary optimality conditions of the Karush-Kuhn-Tucker (KKT) type were given, for instance, in Abrham and Buie [1], Brandão et al. [8], Hanson and Mond [12], Reiland and Hanson [27], Reiland [26] and Zalmai [35].

Sufficient optimality conditions can be found in Farr and Hanson [11], de Oliveira and Rojas-Medar [20, 21], de Oliveira et al. [22], Rojas-Medar et al. [28] and Zalmai [36], among others.

Numerical methods for solving this kind of problems were presented, for example, in Andreani et al. [3], Pullan [25], Weiss [31], Wen et al. [32] and Wu [33].

The Generalized Gordan's Transposition Theorem given in [34] was used in the proof of various results in some of the works cited above, such as [8], [20–22] and [35, 36]. It should be mentioned that the validity of such a theorem was questioned in [6], so that the proof of some results in the previously mentioned papers are not valid. Nevertheless, they are validated by changing Gordan's Theorem by another transposition theorem, the alternative theorem given in [6], for example. New assumptions may be necessary, however, since the alternative theorem provided in [6] is valid under a regularity condition along with a solvability assumption while such assumptions were absent in Gordan's Theorem in [34].

In this paper, a continuous-time nonlinear programming problem with equality and inequality constraints is considered. First and second order necessary optimality conditions are obtained under a constant rank constraint qualification. It is worth mentioning that the treating of equality constrained problems is not common in the literature for continuous-time programming. A fundamental tool in order to deal with equality constraints is a uniform implicit function theorem, given in de Pinho and Vinter [24]. Second order conditions are also not common in the literature for continuous-time programming. The first order necessary conditions are of Karush-Kuhn-Tucker type and are obtained under, as far as we know, the weakest assumptions when compared to other studies in the literature. Moreover, we show that the constant rank constraint qualification is a second order constraint qualification, generalizing a result in Andreani et al. [2] for the continuous-time context.

Generally speaking, KKT type optimality conditions are obtained in an optimization problem by means of a regularity (or constraint qualification) condition. In the specific case of continuous-time problems, Reiland [26] used a Zangwill type condition and Zalmai [36] made use of a Slater condition, for instance. Till recently, some classical constraint qualifications from mathematical programming, such as linear independence, Mangasarian-Fromovitz and constant rank conditions, had not been studied in the continuous-time context. In Monte and de Oliveira [17, 18], the authors established KKT conditions under a linear independence constraint qualification and under a Mangasarian-Fromovitz constraint qualification, respectively. Here we propose a constant rank condition.

It is important to emphasize that the constraint qualifications encountered, for example, in Reiland [26] or Zalmai [35] may be difficult to verify (even for finite dimensions) while the constant rank condition proposed here is less restrictive. In general, problems with linear constraints and problems in which the Jacobian constraint matrix has full rank satisfy such a condition. Furthermore, in [26], in addition to the Zangwill type condition, to apply a generalized Farkas' lemma [10], Reiland had assumed that the kernel of a given operator (between infinite dimensional spaces) has finite dimensions and the image of the same operator is a closed subspace. A Slater regularity condition was also required. In [35], Zalmai made use of the Generalized Gordan's Theorem (see comments above). The constraint qualification given in Monte and de Oliveira [17] is a full rank type condition, so that it is stronger than the constant rank condition proposed here. The Mangasarian-Fromovitz constraint qualification, which was extended to the continuous-time context in Monte and de Oliveira [18], is known to be neither stronger nor weaker than the constant rank one. Nevertheless, in [18], to apply the alternative theorem given in [6], an extra regularity condition is assumed along with the Mangasarian-Fromovitz constraint qualification. Therefore, the necessary optimality conditions furnished here are obtained under, to the best of our knowledge, the weakest assumptions known at the present time.

Zalmai pointed out in [35, 36] that continuous-time programming subsumes some special instances of constrained variational and optimal control problems. On the other hand, every continuous-time programming problem can be seen as an optimal control problem in which there is no state variable, the dynamics are absent and the constraints on the control are given in the functional form. Therefore, optimality conditions for continuous-time programming can be obtained from those for optimal control. For example, necessary optimality conditions of first and second order can be found in Arutyunov and Vereshchagina [5]. The optimality conditions furnished in [5] are derived by assuming that the functional constraints on the control variable either obey a regularity condition of full rank type or are affine. This assumption is, then, stronger than the constant rank constraint qualification presented in this work. As far as we know, there are no necessary optimality conditions for optimal control problems (or continuous-time problems) using the constant rank constraint qualification in the literature.

Recently, new constraint qualifications have appeared in optimal control theory. Considering a non-smooth optimal control problem of an implicit system and its particular cases, Clarke and Pinho [9] obtained necessary optimality conditions using the calibrated constraint qualification. Li and Ye [15] proposed the so-called weak basic constraint qualification and obtained necessary optimality conditions for non-smooth optimal control problems with mixed state and control constraints (in the autonomous case). An additional assumption is imposed, namely, the calmness of a certain perturbed constraint mapping. Although the calibrated constraint qualification can be applied in the non-autonomous case, it is stronger than the weak basic constraint qualification plus the calmness. In Li and Ye [16], the authors used the weak basic constraint qualification along with the calmness and derived necessary optimality conditions for optimal control problems with implicit and semi-explicit control systems (in which the dynamic system is autonomous). In that paper, various scenarios are presented where the weak basic constraint qualification plus the calmness are satisfied; for example when the constant rank and the weak basic constraint qualifications are both valid. Nevertheless, it is not possible to consider the weak basic constraint qualification when there is no state variable and the dynamics are absent. Then, the results in [15, 16] cannot be applied to the continuous-time problems considered here. Regarding the calibrated constraint qualification, even though it is one of the most general ones in the non-smooth setting, it is equivalent to the full rank condition in the smooth case.

The paper is organized as follows. Some preliminaries are given in Section 2. In Section 3, three important technical lemmas are proved. These lemmas will be used in later sections. The main results are contained in the last two sections: Section 4 is devoted to problems with equality constraints only and Section 5 to the general case. Moreover, some illustrative examples are given in Sections 4 and 5.

2 Preliminaries

The paper deals with the continuous-time nonlinear programming problem posed as

$$\begin{aligned} &\text{maximize} \;\; P(z) = \int_0^T \phi(z(t), t)\, dt \\ &\text{subject to} \;\; h(z(t), t) = 0 \text{ a.e. } t \in [0, T], \\ &\qquad\qquad\;\; g(z(t), t) \geq 0 \text{ a.e. } t \in [0, T], \\ &\qquad\qquad\;\; z \in L^\infty([0, T]; \mathbb{R}^n), \end{aligned} \tag{CTP}$$

where $\phi : \mathbb{R}^n \times [0, T] \to \mathbb{R}$, $h : \mathbb{R}^n \times [0, T] \to \mathbb{R}^p$, $g : \mathbb{R}^n \times [0, T] \to \mathbb{R}^m$ are given functions.

Set the index sets as $I = \{1, \ldots, p\}$ and $J = \{1, \ldots, m\}$. All vectors are column vectors. A prime denotes transposition. All integrals are in the Lebesgue sense. Inequality signs between vectors should be read component-wise.

The feasible set of problem (CTP) is denoted by

$$\Omega = \{z \in L^\infty([0, T]; \mathbb{R}^n) : h(z(t), t) = 0,\;\; g(z(t), t) \geq 0 \text{ a.e. } t \in [0, T]\}.$$

A feasible solution $\bar{z} \in \Omega$ is said to be a *local optimal solution* for (CTP) if there exists $\varepsilon > 0$ such that $P(\bar{z}) \geq P(z)$ for all $z \in \Omega$ satisfying $z(t) \in \bar{z}(t) + \varepsilon \bar{B}$ a.e. $t \in [0, T]$, where $\bar{B}$ denotes the closed unit ball with center at the origin in $\mathbb{R}^n$.

Given $\varepsilon > 0$ and a reference solution $\bar{z} \in \Omega$, consider the following hypotheses:

(H1) $\phi(z, \cdot)$ is measurable for each z; $\phi(\cdot, t)$ is twice continuously differentiable on $\bar{z}(t) + \varepsilon \bar{B}$ a.e. $t \in [0, T]$; there exists $K_\phi > 0$ such that

$$\|\nabla \phi(\bar{z}(t), t)\| \leq K_\phi \text{ a.e. } t \in [0, T];$$

(H2) $h(z, \cdot)$ and $g(z, \cdot)$ are measurable for each z; $h(\cdot, t)$ and $g(\cdot, t)$ are twice continuously differentiable on $\bar{z}(t) + \varepsilon \bar{B}$ a.e. $t \in [0, T]$; $g(\bar{z}(\cdot), \cdot)$ is essentially bounded in $[0, T]$;

(H3) There exists an increasing function $\theta : (0, \infty) \to (0, \infty)$, $\theta(s) \downarrow 0$ as $s \downarrow 0$, such that for all $\tilde{z}, z \in \bar{z}(t) + \varepsilon \bar{B}$,

$$\|\nabla(h, g)(\tilde{z}, t) - \nabla(h, g)(z, t)\| \leq \theta(\|\tilde{z} - z\|) \text{ a.e. } t \in [0, T];$$

There exists $K_0 > 0$ such that

$$\|\nabla(h, g)(\bar{z}(t), t)\| \leq K_0 \text{ a.e. } t \in [0, T].$$

Later in Section 4, the following auxiliary result about necessary optimality conditions for the unconstrained continuous-time problem will be needed, namely,

$$\begin{aligned} &\text{maximize} \;\; P(z) = \int_0^T \phi(z(t), t)\, dt \\ &\text{subject to} \;\; z \in L^\infty([0, T]; \mathbb{R}^n). \end{aligned} \tag{UP}$$

Proposition 1 *Let $\bar{z}$ be a local optimal solution of (UP) and assume that (H1) is valid. Then,*

$$\nabla \phi(\bar{z}(t), t) = 0 \text{ a.e. } t \in [0, T]$$

and

$$\int_0^T \gamma(t)' \nabla^2 \phi(\bar{z}(t), t) \gamma(t)\, dt \leq 0 \;\forall \gamma \in L^\infty([0, T]; \mathbb{R}^n).$$

Proof The proof is standard from calculus in Banach spaces, based on Taylor expansions of the functional P by means of the Fréchet derivative. It can be found in Monte and de Oliveira [17]. □

Let $\{F_a : \mathbb{R}^n \to \mathbb{R}^n\}_{a\in A}$ be a family of maps parameterized by points a in a subset $A \subset \mathbb{R}^k$. If ∇F_a is nonsingular at some point x_0 for all $a \in A$, one knows from the classic inverse mapping theorem that, for each a, there exists some neighborhood of x_0 in which F_a is smoothly invertible. The following uniform inverse mapping theorem (de Pinho and Vinter [24], Proposition 4.1) gives conditions under which the same neighbourhood of x_0 can be chosen for all $a \in A$. Its corollary, the uniform implicit function theorem (de Pinho and Vinter [24], Corollary 4.2), will play a fundamental role here in the proof of the results in Sections 4 and 5. The theorem itself will be used in the proof of an important technical lemma in the next section.

Theorem 1 (**Uniform Inverse Mapping**) *Consider a set $A \subset \mathbb{R}^k$, a number $\alpha > 0$, n-vectors x_0 and y_0, and a family of functions $\{F_a : \mathbb{R}^n \to \mathbb{R}^n\}_{a\in A}$ satisfying $y_0 = F_a(x_0)$ for all $a \in A$. It is assumed that:*

(i) F_a *is continuously differentiable on $x_0 + \alpha B$ for all $a \in A$;*
(ii) *there exists a monotone increasing function $\theta : (0,\infty) \to (0,\infty)$, with $\theta(s) \downarrow 0$ as $s \downarrow 0$, such that*

$$\|\nabla F_a(x) - \nabla F_a(\tilde{x})\| \le \theta(\|x - \tilde{x}\|)\ \forall x, \tilde{x} \in x_0 + \alpha B,\ a \in A;$$

(iii) $\nabla F_a(x_0)$ *is nonsingular for each $a \in A$ and there exists $c > 0$ such that*

$$\|[\nabla F_a(x_0)]^{-1}\| \le c\ \forall a \in A.$$

Then there exist numbers $\varepsilon \in (0,\alpha)$ and $\delta > 0$, and a family of continuously differentiable functions $\{G_a : y_0 + \delta B \to x_0 + \alpha B\}_{a\in A}$ which are Lipschitz continuous with a common Lipschitz constant K such that

$$F_a(G_a(y)) = y\ \forall y \in y_0 + \delta B,\ a \in A,$$
$$G_a(F_a(x)) = x\ \forall x \in x_0 + \varepsilon B,\ a \in A.$$

The numbers ε and δ depend only on α, $\theta(\cdot)$ and c. Furthermore, if A is a Borel set and $a \mapsto F_a(x)$ is Borel measurable for each $x \in x_0+\alpha B$, then $a \mapsto G_a(y)$ is Borel measurable for each $y \in y_0 + \delta B$.

Corollary 1 (**Uniform Implicit Function Theorem**) *Consider a set $A \subset \mathbb{R}^k$, a number $\alpha > 0$, a family of functions $\{\psi_a : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n\}_{a\in A}$, and a point $(u_0, v_0) \in \mathbb{R}^m \times \mathbb{R}^n$ such that $\psi_a(u_0, v_0) = 0$ for all $a \in A$. Assume that:*

(i) ψ_a *is continuously differentiable on $(u_0, v_0) + \alpha B$ for all $a \in A$;*
(ii) *there exists a monotone increasing function $\theta : (0,\infty) \to (0,\infty)$, with $\theta(s) \downarrow 0$ as $s \downarrow 0$, such that*

$$\|\nabla\psi_a(\tilde{u}, \tilde{v}) - \nabla\psi_a(u, v)\| \le \theta(\|(\tilde{u}, \tilde{v}) - (u, v)\|)$$

for all $a \in A$, $(\tilde{u}, \tilde{v}),\ (u, v) \in (u_0, v_0) + \alpha B$;
(iii) $\nabla_v\psi_a(u_0, v_0)$ *is nonsingular for each $a \in A$ and there exists $c > 0$ such that*

$$\|[\nabla_v\psi_a(u_0, v_0)]^{-1}\| \le c\ \forall a \in A.$$

Then there exist $\delta \geq 0$ *and a family of continuously differentiable functions* $\{\phi_a : u_0+\delta B \to v_0+\alpha B\}_{a\in A}$ *which are Lipschitz continuous with a common Lipschitz constant* K *such that*

$$v_0 = \phi_a(u_0)\ \forall a \in A,$$
$$\psi_a(u, \phi_a(u)) = 0\ \forall u \in u_0 + \delta B,\ a \in A,$$
$$\nabla_u \phi_a(u_0) = -[\nabla_v \psi_a(u_0, v_0)]^{-1} \nabla_u \psi_a(u_0, v_0)\ \forall a \in A.$$

The numbers δ *and* k *depend only on* $\theta(\cdot)$, c *and* α. *Furthermore, if* A *is a Borel set and* $a \mapsto \psi_a(u, v)$ *is a measurable Borel function for each* $(u, v) \in (u_0, v_0) + \alpha B$, *then* $a \mapsto \phi_a(u)$ *is a measurable Borel function for each* $u \in u_0 + \delta B$.

Remark 1 If it is assumed that ψ_a is twice continuously differentiable on $(u_0, v_0)+\alpha B$, by analyzing the proof provided by de Pinho and Vinter [24] one can deduce that ϕ_a is twice continuously differentiable as well.

3 Technical Lemmas

In this section, some auxiliary results are given and proved. They will be used to establish the main results.

Lemma 1 *Consider a subset* $A \subset \mathbb{R}^k$ *and a family of* $p \times p$ *matrices given by* $\{M_a\}_{a\in A}$. *If there exists* $K, L > 0$ *such that*

$$\det(M_a) \geq K \quad \textit{and} \quad \|M_a\| \leq L$$

for all $a \in A$, *then*

$$\|[M_a]^{-1}\| \leq C\ \forall a \in A,$$

for some $C > 0$.

Proof The proof which is based on the singular value decomposition is straightforward. It can be found in Monte and de Oliveira [17]. □

Next we have some lemmas which are continuous-time versions of Lemmas 1 and 2 in Andreani et al. [4]. They are a consequence of the uniform inverse mapping theorem.

Lemma 2 *Consider a set* $A \subset \mathbb{R}^k$, *a number* $\alpha > 0$, n*-vectors* x_0 *and* y_0, *and a family of functions* $\{F_a = (f_1^a, \ldots, f_n^a) : \mathbb{R}^n \to \mathbb{R}^n\}_{a\in A}$ *satisfying* $y_0 = F_a(x_0)$ *for all* $a \in A$ *and all the assumptions of Theorem 1. Let* $\{f^a : \mathbb{R}^n \to \mathbb{R}\}_{a\in A}$ *be a second family of functions which are continuously differentiable on* $x_0 + \alpha B$ *for all* $a \in A$. *Assume that, for each* $a \in A$, $\nabla f^a(x)$ *is a linear combination of* $\nabla f_1^a(x), \ldots, \nabla f_r^a(x)$ *for all* $x \in x_0 + \alpha B$, *for some integer* $0 < r \leq n$. *Then there exists* $\delta > 0$ *such that* $\varphi^a : y_0 + \delta B \to \mathbb{R}$ *given by*

$$\varphi^a(u) = f^a(F_a^{-1}(u)),\ a \in A, \tag{1}$$

satisfies

$$\frac{\partial \varphi^a}{\partial u_j}(u) = 0\ \forall u \in y_0 + \delta B,\ j = r+1, \ldots, n,\ a \in A. \tag{2}$$

Proof By Theorem 1, there exist $0 < \varepsilon < \alpha$ and $\delta > 0$ and a family of continuously differentiable functions $\{G_a = F_a^{-1} : y_0 + \delta B \to x_0 + \varepsilon B\}_{a \in A}$ such that, if $F_a^{-1}(u) = (g_1^a(u), \ldots, g_n^a(u))$, then

$$u = (F_a \circ F_a^{-1})(u) = F_a(g_1^a(u), \ldots, g_n^a(u))$$
$$= (f_1^a(g_1^a(u), \ldots, g_n^a(u)), \ldots, f_n^a(g_1^a(u), \ldots, g_n^a(u))),$$

so that

$$f_q^a(g_1^a(u), \ldots, g_n^a(u)) = u_q, \; q = 1, \ldots, n, \; a \in A. \tag{3}$$

Using the chain rule in (1), one has, for $u \in y_0 + \delta B$,

$$\frac{\partial \varphi^a}{\partial u_j}(u) = \sum_{i=1}^{n} \frac{\partial f^a}{\partial x_i}(F_a^{-1}(u)) \frac{\partial g_i^a}{\partial u_j}(u), \; j = 1, \ldots, n, \; a \in A. \tag{4}$$

By hypothesis, for each $x \in x_0 + \alpha B$ and $a \in A$, there exist scalars $\lambda_1^a(x), \ldots, \lambda_r^a(x)$ such that

$$\nabla f^a(x) = \sum_{q=1}^{r} \lambda_q^a(x) \nabla f_q^a(x).$$

So,

$$\frac{\partial f^a}{\partial x_i}(x) = \sum_{q=1}^{r} \lambda_q^a(x) \frac{\partial f_q^a}{\partial x_i}(x), \; x \in x_0 + \alpha B, \; i = 1, \ldots, n, \; a \in A. \tag{5}$$

Replacing (5) in (4), one gets

$$\frac{\partial \varphi^a}{\partial u_j}(u) = \sum_{i=1}^{n} \left[\sum_{q=1}^{r} \lambda_q^a(F^{-1}(u)) \frac{\partial f_q^a}{\partial x_i}(F^{-1}(u)) \right] \frac{\partial g_i^a}{\partial u_j}(u)$$
$$= \sum_{q=1}^{r} \lambda_q^a(F^{-1}(u)) \left[\sum_{i=1}^{n} \frac{\partial f_q^a}{\partial x_i}(F^{-1}(u)) \frac{\partial g_i^a}{\partial u_j}(u) \right].$$

For $q \in \{1, \ldots, r\}$ and $j \in \{r+1, \ldots, n\}$, by (3), one has

$$0 = \frac{\partial}{\partial u_j} u_q = \frac{\partial}{\partial u_j} \Big(f_q^a(g_1^a(u), \ldots, g_n^a(u)) \Big) = \sum_{i=1}^{n} \frac{\partial f_q^a}{\partial x_i}(F^{-1}(u)) \frac{\partial g_i^a}{\partial u_j}(u).$$

Therefore, one concludes that

$$\frac{\partial \varphi^a}{\partial u_j}(u) = 0, \; j = r+1, \ldots, n, \; a \in A.$$

□

Lemma 3 *Consider a set $A \subset \mathbb{R}^k$, a number $\alpha > 0$, n-vectors x_0 and y_0, and a family of functions $\{\tilde{F}_a = (f_1^a, \ldots, f_r^a) : \mathbb{R}^n \to \mathbb{R}^r\}_{a \in A}$, $0 < r \le n$, satisfying $\tilde{F}_a(x_0) = z_0$ for all $a \in A$, where $y_0 = (z_0, w_0) \in \mathbb{R}^r \times \mathbb{R}^{n-r}$. Suppose that the following conditions hold:*

(a) *$\tilde{F}_a$ is continuously differentiable on $x_0 + \alpha B$ for all $a \in A$;*
(b) *There exists $\theta : (0, \infty) \to (0, \infty)$, $\theta(s) \downarrow 0$ when $s \downarrow 0$, such that*

$$\|\nabla \tilde{F}_a(x) - \nabla \tilde{F}_a(\bar{x})\| \le \theta(\|x - \bar{x}\|),$$

for all $x, \bar{x} \in x_0 + \alpha B$, $a \in A$; there exists $\tilde{K} > 0$ such that

$$\|\nabla \tilde{F}_a(x_0)\| \le \tilde{K}, \; a \in A;$$

(c) There exists $K > 0$ *such that*

$$\det\{[\nabla \tilde{F}_a(x_0)][\nabla \tilde{F}_a(x_0)]'\} \geq K, \; a \in A.$$

Then there exists a family of continuously differentiable functions $\{\hat{F}_a : \mathbb{R}^n \to \mathbb{R}^{n-r}\}_{a\in A}$ *such that* $F_a = (\tilde{F}_a, \hat{F}_a)$ *satisfies (i)* $F_a(x_0) = y_0$ *for all* $a \in A$*; and (ii) all assumptions of Theorem 1.*

Proof For each $a \in A$, let M_a be a matrix whose columns form an orthonormal basis for the orthogonal complement to the subspace generated by the rows of $\nabla \tilde{F}_a(x_0)$.

For each $a \in A$, define $\hat{F}_a : \mathbb{R}^n \to \mathbb{R}^{n-r}$ and $F_a : \mathbb{R}^n \to \mathbb{R}^n$ respectively as

$$\hat{F}_a(x) = {M_a}'(x - x_0) + w_0 \quad \text{and} \quad F_a(x) = (\tilde{F}_a(x), \hat{F}_a(x)).$$

Then,

$$F_a(x_0) = (\tilde{F}_a(x_0), \hat{F}_a(x_0)) = (z_0, w_0) = y_0, \; a \in A,$$

and

$$\nabla F_a(x)' = [\nabla \tilde{F}_a(x)' \; \nabla \hat{F}_a(x)'] = [\nabla \tilde{F}_a(x)' \; M_a] \; \forall x, \; a \in A.$$

Moreover,

(i) From assumption (a) and the definition of $\hat{F}_a$, it follows that F_a is continuously differentiable on $x_0 + \alpha B$ for all $a \in A$;

(ii) From (b), for $x, \bar{x} \in x_0 + \alpha B$ one has

$$\|\nabla F_a(x) - \nabla F_a(\bar{x})\| = \|\nabla \tilde{F}_a(x) - \tilde{F}_a(\bar{x})\| \leq \theta(\|x - \bar{x}\|),$$

where $\theta : (0, \infty) \to (0, \infty)$, $\theta(s) \downarrow 0$ when $s \downarrow 0$;

(iii) By construction, $\|\nabla \hat{F}_a(x_0)\| = \|{M_a}'\| = 1$. Thence,

$$\|\nabla F_a(x_0)\| = \left\| \begin{bmatrix} \nabla \tilde{F}_a(x_0) \\ {M_a}' \end{bmatrix} \right\| \leq \sqrt{\|\nabla \tilde{F}_a(x_0)\|^2 + \|{M_a}'\|^2} \leq \sqrt{\tilde{K}^2 + 1}.$$

Also by construction,

$$\det([\nabla F_a(x_0)][\nabla F_a(x_0)]') = \det([\nabla \tilde{F}_a(x_0)][\nabla \tilde{F}_a(x_0)]') \geq K, \; a \in A,$$

implying that $\det(\nabla F_a(x_0)) \geq \sqrt{K} > 0$, $a \in A$. Therefore, $\nabla F_a(x_0)$ is nonsingular. It follows, by Lemma 1, that there exists $M > 0$ such that

$$\|[\nabla F_a(x_0)]^{-1}\| \leq M, \; a \in A.$$

Thus, the family $\{F_a\}_{a\in A}$ satisfies all the assumptions of Theorem 1. □

Lemma 4 *Consider a set* $A \subset \mathbb{R}^k$*, a number* $\alpha > 0$*, n-vectors* x_0 *and* $y_0 \in \mathbb{R}^n$*, and families of functions* $\{f^a : \mathbb{R}^n \to \mathbb{R}\}_{a\in A}$ *and* $\{\tilde{F}_a = (f_1^a, \ldots, f_r^a) : \mathbb{R}^n \to \mathbb{R}\}_{a\in A}$, $0 < r \leq n$, *satisfying* $\tilde{F}_a(x_0) = z_0$ *for all* $a \in A$*, where* $y_0 = (z_0, w_0) \in \mathbb{R}^r \times \mathbb{R}^{n-r}$*. Assume that* f^a *is continuously differentiable on* $x_0 + \alpha B$ *for all* $a \in A$*, and that* $\{\tilde{F}_a\}_{a\in A}$ *satisfies assumptions (a)-(c) of Lemma 3. In addition, assume that* $\nabla f^a(x)$ *is a linear combination of* $\nabla f_1^a(x), \ldots, \nabla f_r^a(x)$ *for all* $a \in A$ *and all* $x \in x_0 + \alpha B$*. Specifically, for* $x = x_0$*,*

$$\nabla f^a(x_0) = \sum_{i=1}^{r} \lambda_i^a \nabla f_i^a(x_0), \; a \in A.$$

Then there exist $\sigma, \rho > 0$ *and a family of continuously differentiable functions* $\{\chi^a : z_0 + \rho B \to \mathbb{R}\}_{a\in A}$ *such that for all* $x \in x_0 + \sigma B$ *and each* $a \in A$*,*

$$(f_1^a(x), \ldots, f_r^a(x)) = \tilde{F}_a(x) \in z_0 + \rho B$$

and

$$f^a(x) = \chi^a(\tilde{F}_a(x)) = \chi^a(f_1^a(x), \ldots, f_r^a(x)).$$

The numbers σ and ρ depend only on α, θ and $\tilde{K}$, K. Furthermore,

$$\frac{\partial \chi^a}{\partial u_i}(\tilde{F}_a(x_0)) = \lambda_i^a, \; i = 1, \ldots, r, \; a \in A.$$

Proof By Lemma 3, for each $a \in A$, one can define $n - r$ functions $f_{r+1}^a, \ldots, f_n^a$ in such a way that $F_a = (\tilde{F}_a, \hat{F}_a) = (f_1^a, \ldots, f_r^a, f_{r+1}^a, \ldots, f_n^a)$ satisfies the hypotheses of Lemma 2. Then, by (2), the function φ^a as defined in (1) does not depend on the variables $u_{r+1}, \ldots, u_n$, for all $a \in A$. Provided $f_1^a, \ldots, f_n^a$ are continuous for all $a \in A$, there exist open balls $x_0 + \sigma B \subset x_0 + \alpha B$ and $(z_0 + \rho_1 B) \times (w_0 + \rho_2 B) \subset (z_0, w_0) + \delta B = y_0 + \delta B$ such that

$$(f_1^a(x), \ldots, f_r^a(x)) = \tilde{F}_a(x) \in z_0 + \rho_1 B \;\forall x \in x_0 + \sigma B. \tag{6}$$

For each $a \in A$, define $\chi^a : z_0 + \rho_1 B \to \mathbb{R}$ as

$$\chi^a(u) = \varphi^a(u, \hat{F}_a(x_0)) = \varphi^a(u_1, \ldots, u_r, f_{r+1}^a(x_0), \ldots, f_n^a(x_0)).$$

Clearly, χ^a is continuously differentiable for all $a \in A$. Putting $\rho = \rho_1$, using (6) and (1)–(2), one has for all $x \in x_0 + \sigma B$ and each $a \in A$ that

$$(f_1^a(x), \ldots, f_r^a(x)) = \tilde{F}_a(x) \in z_0 + \rho B$$

and

$$\begin{aligned} \chi^a(\tilde{F}_a(x)) &= \chi^a(f_1^a(x), \ldots, f_r^a(x)) \\ &= \varphi^a(f_1^a(x), \ldots, f_r^a(x), f_{r+1}^a(x_0), \ldots, f_n^a(x_0)) \\ &= \varphi^a(f_1^a(x), \ldots, f_r^a(x), f_{r+1}^a(x), \ldots, f_n^a(x)) \\ &= \varphi^a(F_a(x)) = f^a(x). \end{aligned}$$

From $f^a(x) = \chi^a(\tilde{F}_a(x)) = \chi^a(f_1^a(x), \ldots, f_r^a(x))$ one gets

$$\nabla f^a(x) = \sum_{i=1}^{r} \frac{\partial \chi^a}{\partial u_i}(\tilde{F}_a(x)) \nabla f_i^a(x), \; x \in x_0 + \sigma B.$$

On the other hand,

$$\nabla f^a(x_0) = \sum_{i=1}^{r} \lambda_i^a \nabla f_i^a(x_0), \; a \in A,$$

and, by assumption (c), $\nabla f_1^a(x_0), \ldots, \nabla f_r^a(x_0)$ are linearly independent for all $a \in A$. It follows that

$$\frac{\partial \chi_a}{\partial u_j}(\tilde{F}_a(x_0)) = \lambda_j^a, \; j = 1, \ldots, r, \; a \in A.$$

□

4 Optimality Conditions for Problems with Equality Constraints

In this section, the continuous-time programming problem with equality constraints is regarded:

$$\begin{aligned} &\text{maximize} \ \ P(z) = \int_0^T \phi(z(t), t)\, dt \\ &\text{subject to} \ \ h(z(t), t) = 0 \text{ a.e. } t \in [0, T], \\ &\qquad\qquad\quad z \in L^\infty([0, T], \end{aligned} \tag{PEC}$$

where $\phi : \mathbb{R}^n \times [0, T] \to \mathbb{R}$ and $h : \mathbb{R}^n \times [0, T] \to \mathbb{R}^p$.

We will establish first and second order necessary optimality conditions for (PEC) under the following constant rank condition.

Definition 1 The constant rank constraint qualification (CRCQ) for the problems with equality constraints is said to be satisfied at $\bar{z} \in \Omega$ if there exists $\varepsilon > 0$ such that $\nabla h(z, t)$ has constant rank on $\bar{z}(t) + \varepsilon B$ a.e. $t \in [0, T]$.

Remark 2 Analogous to the classical constant rank constraint qualification given by Janin [13], if one assumes that for all $I \subset \{1, \dots, p\}$ the set $\{\nabla h_i(z, t)\}_{i \in I}$ has constant rank on $\bar{z}(t) + \varepsilon B$ a.e. $t \in [0, T]$, then (CRCQ) given in Definition 1 does hold.

Let us assume that (CRCQ) holds at $\bar{z} \in \Omega$. Then, $\operatorname{rank}(\nabla h(\bar{z}(t), t)) = r$ a.e. $t \in [0, T]$ for some positive constant r, and it is clear that there exists an index subset, say $\{i_1, \dots, i_r\}$, such that, if

$$\Upsilon(t) := [\nabla h_{i_1}(\bar{z}(t), t) \ \cdots \ \nabla h_{i_r}(\bar{z}(t), t)]' \text{ a.e. } t \in [0, T],$$

then

$$\det\{[\Upsilon(t)][\Upsilon(t)]'\} \neq 0 \text{ a.e. } t \in [0, T]. \tag{7}$$

Nevertheless, in the proof of the main result in this section, we will apply Lemma 4, and condition (7) does not allow the application of such a lemma. See condition (c) of the lemma. We will need to further assume that $\det\{[\Upsilon(t)][\Upsilon(t)]'\}$ is uniformly bounded from below almost everywhere in $[0, T]$.

(H4) Let $\bar{z} \in \Omega$. If $r = \operatorname{rank}(\nabla h(\bar{z}(t), t))$ a.e. $t \in [0, T]$, there exist an index subset, say $\{i_1, \dots, i_r\}$, and a constant $C > 0$ such that

$$\det\{[\Upsilon(t)][\Upsilon(t)]'\} \geq C \text{ a.e. } t \in [0, T],$$

where

$$\Upsilon(t) = [\nabla h_{i_1}(\bar{z}(t), t) \ \cdots \ \nabla h_{i_r}(\bar{z}(t), t)]'.$$

Before we state the KKT type optimality conditions, let us give a simple example.

Example 1 Consider the system

$$\begin{cases} h_1(z(t), t) = t z_1(t) + t^2 z_2(t) - t^3 z_3(t) = 0 \text{ a.e. } t \in [0, 1], \\ h_2(z(t), t) = -z_1(t) - t z_2(t) + t^2 z_3(t) = 0 \text{ a.e. } t \in [0, 1], \end{cases}$$

and $\bar{z}(t) = (0, 0, 0)$ a.e. $t \in [0, 1]$. Note that

$$\nabla h(z(t), t) = \begin{bmatrix} t & t^2 & -t^3 \\ -1 & -t & t^2 \end{bmatrix}$$

has rank 1 in any open neighborhood of origin in $\mathbb{R}^3$ a.e. $t \in [0, 1]$. By choosing $\Upsilon(t) = [-1 \;\; -t \;\; t^2]$ a.e. $t \in [0, 1]$, one has $\det\{[\Upsilon(t)][\Upsilon(t)]'\} \geq 1$ a.e. $t \in [0, 1]$. Thus, both (CRCQ) and (H4) are satisfied at $\bar{z} \equiv 0$.

If h_2 is changed to $h_1/2$, then (CRCQ) is valid, but $\det\{[\Upsilon(t)][\Upsilon(t)]'\}$ is not uniformly bounded below on [0, 1] for any choice of Υ.

Theorem 2 *Let $\bar{z}$ be a local optimal solution of (PEC). Assume that (H1)-(H4) and (CRCQ) hold. Then, there exists $u \in L^\infty([0, T]; \mathbb{R}^p)$ such that*

$$\nabla\phi(\bar{z}(t), t) + \sum_{i=1}^{p} u_i(t)\nabla h_i(\bar{z}(t), t) = 0 \text{ a.e. } t \in [0, T] \tag{8}$$

and

$$\int_0^T \gamma(t)' \Big[\nabla^2\phi(\bar{z}(t), t) + \sum_{i=1}^{p} u_i(t)\nabla^2 h_i(\bar{z}(t), t)\Big]\gamma(t)\, dt \leq 0 \;\forall \gamma \in N, \tag{9}$$

where

$$N = \{\gamma \in L^\infty([0, T]; \mathbb{R}^n) : \nabla h_i(\bar{z}(t), t)'\gamma(t) = 0 \text{ a.e. } t \in [0, T],\; i \in I\}.$$

Proof Let $\bar{z}$ be a local optimal solution of (PEC) on $\bar{z}(t) + \varepsilon\bar{B}$ a.e. in $[0, T]$.

Let A_0 be the largest subset of $[0, T]$ in which conditions (H1)-(H4) and (CRCQ) do not hold for every $t \in A_0$. Provided A_0 has Lebesgue measure zero, it follows that there exists a Borel set A_1 (being the intersection of a countable family of open sets) with $A_0 \subset A_1$ such that $A_1 \setminus A_0$ is of measure zero (see Rudin [29]). Set $A = [0, T] \setminus A_1$.

The proof is divided in several steps.

STEP 1: (Discarding redundant constraints) Let $i_0 \in \{1, \ldots, p\} \setminus \{i_1, \ldots, i_r\}$. In Lemma 4, let us identify t with a, ε with α, $x_0 = y_0 = 0$, and

$$f^t(x) = h_{i_0}(\bar{z}(t) + x, t),$$
$$f_j^t(x) = h_{i_j}(\bar{z}(t) + x, t), \; j = 1, \ldots, r.$$

It is clear from (H2)-(H4) and (CRCQ) that f^t is continuously differentiable on εB for all $t \in A$, and that $\tilde{F}_t = (f_1^t, \ldots, f_r^t)$ satisfy the assumptions of Lemma 3, $t \in A$. It is easy to see from (H4) that $\nabla f^t(x) = \nabla h_{i_0}(\bar{z}(t) + x, t)$ is a linear combination of $\nabla f_1^t(x) = \nabla h_{i_1}(\bar{z}(t) + x, t), \ldots, \nabla f_r^t(x) = \nabla h_{i_r}(\bar{z}(t) + x, t)$ for all $x \in \varepsilon B$, $t \in A$. It follows from Lemma 4 that there exist σ, ρ and $\chi^t : \rho B \to \mathbb{R}$ such that for all $x \in \sigma B$,

$$(h_{i_1}(\bar{z}(t) + x, t), \ldots, h_{i_r}(\bar{z}(t) + x, t)) \in \rho B, \; t \in A,$$

and

$$h_{i_0}(\bar{z}(t) + x, t) = \chi^t(h_{i_1}(\bar{z}(t) + x, t), \ldots, h_{i_r}(\bar{z}(t) + x, t), t), \; t \in A. \tag{10}$$

STEP 2: (Equality constraints encompassed by an implicit function) Define $\mu : \mathbb{R}^n \times \mathbb{R}^r \times [0, T] \to \mathbb{R}^r$ as

$$\mu(\xi, \eta, t) = H(\bar{z}(t) + \xi + \Upsilon(t)'\eta, t),$$

where $H = (h_{i_1}, \ldots, h_{i_r})$. In Corollary 1, identify t with a, (ξ, η) with (u, v), $(u_0, v_0) = (0, 0)$, and $\mu(\cdot, \cdot, t)$ with $\psi^a(\cdot, \cdot)$. One has that

$$\mu(0, 0, t) = H(\bar{z}(t), t) = (h_{i_1}(\bar{z}(t), t), \ldots, h_{i_r}(\bar{z}(t), t)) = (0, \ldots, 0) \; \forall t \in A.$$

Note that, by (H3),

$$\begin{aligned}\|\Upsilon(t)'\| &= \|[\nabla h_{i_1}(\bar{z}(t),t) \ \cdots \ \nabla h_{i_r}(\bar{z}(t),t)]\| \\ &\leq \|[\nabla h_1(\bar{z}(t),t) \ \cdots \ \nabla h_p(\bar{z}(t),t))]\| \leq K_0 \text{ a.e. } t \in [0,T]. \end{aligned} \tag{11}$$

Let $\alpha = \min\{\frac{\varepsilon}{2}, \frac{\varepsilon}{2K_0}\}$ and $(\xi,\eta) \in (0,0) + \alpha B$. So,

$$\|\bar{z}(t) + \xi + \Upsilon(t)'\eta - \bar{z}(t)\| = \|\xi + \Upsilon(t)'\eta\| \leq \|\xi\| + \|\Upsilon(t)'\|\|\eta\| \leq \varepsilon$$

for almost every $t \in [0,T]$. By (H2), $\mu(\cdot,\cdot,t)$ is continuously differentiable on $(0,0) + \alpha B$, for all $t \in A$, so that assumption (i) in Corollary 1 is valid. Now checking (ii), let $(\tilde{\xi},\tilde{\eta}),\ (\xi,\eta) \in (0,0) + \alpha B$. One has that

$$\nabla\mu(\xi,\eta,t) = [\nabla H(\bar{z}(t) + \xi + \Upsilon(t)'\eta, t) \ \ \nabla H(\bar{z}(t) + \xi + \Upsilon(t)'\eta, t)\Upsilon(t)']$$

and

$$\begin{aligned}&\nabla\mu(\tilde{\xi},\tilde{\eta},t) - \nabla\mu(\xi,\eta,t) \\ &\quad = [\nabla H(\bar{z}(t) + \tilde{\xi} + \Upsilon(t)'\tilde{\eta}, t) - \nabla H(\bar{z}(t) + \xi + \Upsilon(t)'\eta, t)][I_n \ \ \Upsilon(t)'],\end{aligned}$$

where I_n denotes the identity matrix of order n. Then, by (H3) and (11)

$$\begin{aligned}&\|\nabla\mu(\tilde{\xi},\tilde{\eta},t) - \nabla\mu(\xi,\eta,t)\| \\ &\quad \leq \|[\nabla H(\bar{z}(t) + \tilde{\xi} + \Upsilon(t)'\tilde{\eta}, t) - \nabla H(\bar{z}(t) + \xi + \Upsilon(t)'\eta, t)]\|\|[I_n \ \ \Upsilon(t)']\| \\ &\quad \leq \|\nabla h(\bar{z}(t) + \tilde{\xi} + \Upsilon(t)'\tilde{\eta}, t) - \nabla h(\bar{z}(t) + \xi + \Upsilon(t)'\eta, t)\|\|[I_n \ \ \Upsilon(t)']\| \\ &\quad \leq \theta(\|(\tilde{\xi} - \xi) + \Upsilon(t)'(\tilde{\eta} - \eta)\|)(1 + K_0) \\ &\quad \leq \theta(\|(\tilde{\xi} - \xi)\| + K_0\|(\tilde{\eta} - \eta)\|)(1 + K_0) \\ &\quad \leq \theta(\|(\tilde{\xi} - \xi, \tilde{\eta} - \eta)\| + K_0\|(\tilde{\xi} - \xi, \tilde{\eta} - \eta)\|)(1 + K_0) = \tilde{\theta}(\|(\tilde{\xi},\tilde{\eta}) - (\xi,\eta)\|),\end{aligned}$$

where $\tilde{\theta} : (0,\infty) \to (0,\infty)$ is given by

$$\tilde{\theta}(s) = (1 + K_0)\theta(s + K_0 s).$$

It is easy to see that $\tilde{\theta}$ is monotone increasing and $\tilde{\theta}(s) \downarrow 0$ as $s \downarrow 0$. Therefore, assumption (ii) in Corollary 1 is satisfied. Finally, checking (iii), since

$$\nabla_\eta\mu(0,0,t) = \nabla H(\bar{z}(t),t)\Upsilon(t)' = \Upsilon(t)\Upsilon(t)',$$

it follows from (H4) that $\nabla_\eta\mu(0,0,t)$ is nonsingular for all $t \in A$. By (11), $\|\Upsilon(t)\Upsilon(t)'\| \leq K_0^2$ a.e. $t \in [0,T]$. Thus, by using (H4), from Lemma 1, there exists $M > 0$ such that

$$\|[\nabla_\eta\mu(0,0,t)]^{-1}\| = \|[\Upsilon(t)\Upsilon(t)']^{-1}\| \leq M \text{ a.e. } t \in [0,T].$$

By Corollary 1, there exist $\delta \geq 0$ and an implicit map $d : \delta B \times [0,T] \to \alpha B$ such that $d(\xi,\cdot)$ is measurable for fixed ξ, $d(\cdot,t)$ is Lipschitz continuous for every $t \in A$ with a common Lipschitz constant, $d(\cdot,t)$ is continuously differentiable for every $t \in A$,

$$d(0,t) = 0, \ \ t \in A, \tag{12}$$

$$\mu(\xi, d(\xi,t), t) = 0, \ \ \xi \in \delta B, \ \ t \in A, \tag{13}$$

$$\nabla d(0,t) = -[\Upsilon(t)\Upsilon(t)']^{-1}\Upsilon(t), \ \ t \in A. \tag{14}$$

Let us choose $\delta_1 > 0$ and α_1 satisfying

$$\delta_1 < \min\{\delta, \varepsilon/2\}, \quad \alpha_1 < \min\{\alpha, \varepsilon/2\}, \quad \delta_1 + K_0\alpha_1 < \min\{\sigma, \varepsilon\}. \tag{15}$$

In the following steps, without loss of generality, we consider the implicit function $d : \delta_1 B \times [0,T] \to \alpha_1 B$.

STEP 3: (Auxiliary problem) Consider the auxiliary problem below:

$$\begin{aligned} &\text{maximize } \tilde{P}(z) = \int_0^T \varphi(z(t), t)\, dt \\ &\text{subject to } z \in L^\infty([0, T]; \mathbb{R}^n), \end{aligned} \tag{AP}$$

where $\varphi : \mathbb{R}^n \times [0, T] \to \mathbb{R}$ is defined as $\varphi(z, t) = \phi(z + \Upsilon(t)' d(z - \bar{z}(t), t), t)$. The local optimal solution of (PEC) is also a local optimal solution of (AP). Indeed, it is clear that $\bar{z}$ is a feasible solution of (AP). Suppose that there exists $\tilde{z} \in L^\infty([0, T]; \mathbb{R}^n)$ such that $\tilde{z}(t) \in \bar{z}(t) + \delta_2 B$, $0 < \delta_2 < \delta_1$, and $\tilde{P}(\tilde{z}) > \tilde{P}(\bar{z})$. Let $\hat{z} \in L^\infty([0, T]; \mathbb{R}^n)$ given by

$$\hat{z}(t) = \tilde{z}(t) + \Upsilon(t)' d(\tilde{z}(t) - \bar{z}(t), t) \text{ a.e. } t \in [0, T].$$

Using (11) and (15), one has that

$$\|\hat{z}(t) - \bar{z}(t)\| \leq \|\tilde{z}(t) - \bar{z}(t)\| + \|\Upsilon(t)'\| \|d(\tilde{z}(t) - \bar{z}(t), t)\| < \delta_1 + K_0 \alpha_1 < \varepsilon$$

and then $\hat{z}(t) \in \bar{z}(t) + \varepsilon B$ a.e. $t \in [0, T]$. As $\tilde{z}(t) - \bar{z}(t) \in \delta_1 B$, by (13),

$$\mu(\tilde{z}(t) - \bar{z}(t), d(\tilde{z}(t) - \bar{z}(t), t), t) = 0 \Rightarrow H(\tilde{z}(t) + \Upsilon(t)' d(\tilde{z}(t) - \bar{z}(t), t), t) = 0$$

so that

$$h_{i_j}(\hat{z}(t), t) = 0, \text{ a.e. } t \in [0, T], \; j = 1, \ldots, r.$$

For any $i_0 \in \{1, \ldots, p\} \setminus \{i_1, \ldots, i_r\}$, by (10),

$$\begin{aligned} h_{i_0}(\hat{z}(t), t) &= \chi^t(h_{i_1}(\hat{z}(t), t), \ldots, h_{i_r}(\hat{z}(t), t)) = \chi^t(0, \ldots, 0) \\ &= \chi^t(h_{i_1}(\bar{z}(t), t), \ldots, h_{i_r}(\bar{z}(t), t)) = h_{i_0}(\bar{z}(t), t) = 0 \end{aligned}$$

for almost every $t \in [0, T]$. Thereby, $\hat{z}$ is feasible for (PEC) and, by definition of $\tilde{P}$ and (12),

$$P(\hat{z}) = \tilde{P}(\tilde{z}) > \tilde{P}(\bar{z}) = P(\bar{z}),$$

which contradicts the fact that $\bar{z}$ is a local optimal solution for (PEC).

STEP 4: (First order necessary optimality conditions) By Proposition 1,

$$0 = \nabla\varphi(\bar{z}(t), t) = [I_n + \nabla d(0, t)' \Upsilon(t)] \nabla\phi(\bar{z}(t) + \Upsilon(t)' d(0, t), t)$$

for almost every $t \in [0, T]$. By (12) and (14),

$$\begin{aligned} 0 &= \nabla\phi(\bar{z}(t), t) + \nabla d(0, t)' \Upsilon(t) \nabla\phi(\bar{z}(t), t) \\ &= \nabla\phi(\bar{z}(t), t) + \Upsilon(t)' [\Upsilon(t)\Upsilon(t)']^{-1} \Upsilon(t) \nabla\phi(\bar{z}(t), t) \\ &= \nabla\phi(\bar{z}(t), t) + \sum_{j=1}^{r} u_j(t) \nabla h_{i_j}(\bar{z}(t), t) = 0 \text{ a.e. } t \in [0, T], \end{aligned} \tag{16}$$

where $u \in L^\infty([0, T]; \mathbb{R}^r)$ is given by

$$u(t) = -[\Upsilon(t)\Upsilon(t)']^{-1} \Upsilon(t) \nabla\phi(\bar{z}(t), t) \text{ a.e. } t \in [0, T]. \tag{17}$$

Taking $u_i \equiv 0$ for $i \in \{1, \ldots, p\} \setminus \{i_1, \ldots, i_r\}$, one obtains (8).

STEP 5: (Second order necessary optimality conditions) By Proposition 1, one knows that

$$\int_0^T \gamma(t)' \nabla^2 \varphi(\bar{z}(t), t) \gamma(t)\, dt \leq 0 \; \forall \gamma \in L^\infty([0, T]; \mathbb{R}^n).$$

From the definition of φ in Step 3, one has that

$$\begin{aligned}\nabla^2\varphi(\bar{z}(t),t) &= [I_n + \nabla d(0,t)'\Upsilon(t)]\nabla^2\phi(\bar{z}(t),t)\\ &\quad +\nabla d(0,t)'\Upsilon(t)[I_n + \nabla d(0,t)'\Upsilon(t)]\nabla^2\phi(\bar{z}(t),t)\\ &\quad +\sum_{j=1}^{r}\nabla h_{i_j}(\bar{z}(t),t)'\nabla\phi(\bar{z}(t),t)\nabla^2 d_j(0,t) \text{ a.e. } t\in[0,T].\end{aligned}$$

From (14), one sees that, given $\gamma \in N$,

$$\gamma(t)'\nabla d(0,t)' = -\gamma(t)'\Upsilon(t)'[\Upsilon(t)\Upsilon(t)']^{-1} = 0 \text{ a.e. } t\in[0,T].$$

Hence, for $\gamma \in N$,

$$\int_0^T \gamma(t)'\Big[\nabla^2\phi(\bar{z}(t),t) + \sum_{j=1}^{r}\nabla h_{i_j}(\bar{z}(t),t)'\nabla\phi(\bar{z}(t),t)\nabla^2 d_j(0,t)\Big]\gamma(t)\,dt \le 0. \quad (18)$$

On the other hand, deriving the expression in (13) twice with respect to ξ and then evaluating at $\xi = 0$, one gets, for $j = 1\ldots,r$, that

$$\begin{aligned}0 &= [I_n + \nabla d(0,t)'\Upsilon(t)]\nabla^2 h_{i_j}(\bar{z}(t),t)\\ &\quad +\nabla d(0,t)'\Upsilon(t)[I_n + \nabla d(0,t)'\Upsilon(t)]\nabla^2 h_{i_j}(\bar{z}(t),t)\\ &\quad +\sum_{l=1}^{r}\nabla h_{i_l}(\bar{z}(t),t)'\nabla h_{i_j}(\bar{z}(t),t)\nabla^2 d_l(0,t) \text{ a.e. } t\in[0,T].\end{aligned}$$

Thus, for $\gamma \in N$,

$$\begin{aligned}0 &= \int_0^T \gamma(t)'\sum_{j=1}^{r}u_j(t)\nabla^2 h_{i_j}(\bar{z}(t),t)\gamma(t)\,dt\\ &\quad +\int_0^T \gamma(t)'\sum_{j=1}^{r}\sum_{l=1}^{r}u_j(t)\nabla h_{i_l}(\bar{z}(t),t)'\nabla h_{i_j}(\bar{z}(t),t)\nabla^2 d_l(0,t)\gamma(t)\,dt. \quad (19)\end{aligned}$$

Adding (18) and (19), and using (16), one obtains

$$
\begin{aligned}
0 \geq & \int_0^T \gamma(t)' \Big[\nabla^2 \phi(\bar{z}(t), t) + \sum_{l=1}^r \nabla h_{i_l}(\bar{z}(t), t)' \nabla \phi(\bar{z}(t), t) \nabla^2 d_j(0, t) \Big] \gamma(t)\, dt \\
& + \int_0^T \gamma(t)' \sum_{j=1}^r u_j(t) \nabla^2 h_{i_j}(\bar{z}(t), t) \gamma(t)\, dt \\
& + \int_0^T \gamma(t)' \sum_{j=1}^r \sum_{l=1}^r u_j(t) \nabla h_{i_l}(\bar{z}(t), t)' \nabla h_{i_j}(\bar{z}(t), t) \nabla^2 d_l(0, t) \gamma(t)\, dt \\
= & \int_0^T \gamma(t)' \Big[\nabla^2 \phi(\bar{z}(t), t) + \sum_{j=1}^r u_j(t) \nabla^2 h_{i_j}(\bar{z}(t), t) \Big] \gamma(t)\, dt \\
& + \int_0^T \gamma(t)' \sum_{l=1}^r \nabla h_{i_l}(\bar{z}(t), t)' \\
& \quad \Big[\nabla \phi(\bar{z}(t), t) + \sum_{j=1}^r u_j(t) \nabla h_{i_j}(\bar{z}(t), t) \Big] \nabla^2 d_l(0, t) \gamma(t)\, dt \\
= & \int_0^T \gamma(t)' \Big[\nabla^2 \phi(\bar{z}(t), t) + \sum_{j=1}^r u_j(t) \nabla^2 h_{i_j}(\bar{z}(t), t) \Big] \gamma(t)\, dt
\end{aligned}
$$

for all $\gamma \in N$, so that (9) is verified. □

Remark 3 In the proof of Theorem 2, Equation (17) uniquely determines the multipliers u_j in terms of the data of the problem. However, they depend on the matrix Υ. Observe in (H4) that the choice of Υ may not be unique. Indeed, since $\operatorname{rank}(\nabla h(\bar{z}(t), t)) = r$, there are up to $C(p, r)$ ways to pick r linearly independent rows of $\nabla h(\bar{z}(t), t)$. Therefore, the set of multipliers for which KKT conditions are valid for (PEC) under the constant rank condition may not be a singleton. When there are several matrices Υ so that (H4) is satisfied, an interesting way to select one of them is choosing the one in such a way the multipliers have minimum norm.

Remark 4 Apart from Step 1, the proof of Theorem 2 is strongly inspired by techniques in de Pinho and Ilchmann [23].

5 Optimality Conditions for Problems with Equality and Inequality Constraints

The continuous-time problem with equality and inequality constraints (CTP) is studied here. Optimality conditions are obtained under the constant condition stated below.

Definition 2 The constant rank constraint qualification (CRCQ) for the problem (CTP) is said to be satisfied at $\bar{z} \in \Omega$ if there exists $\varepsilon > 0$ such that the matrix

$$
M(z, w, t) = \begin{bmatrix} \nabla h(z, t) & 0 \\ \nabla g(z, t) & \operatorname{diag}\{-2w_j\}_{j=1}^m \end{bmatrix}
$$

has constant rank on $(\bar{z}(t), \bar{w}(t)) + \varepsilon B$ a.e. $t \in [0, T]$, where $\bar{w}_j(t) = \sqrt{g_j(\bar{z}(t), t)}$ a.e. in $[0, T]$, $j \in J$.

Also, we will need assumption (H5) below.

(H5) Let $\bar{z} \in \Omega$ and consider $\bar{w}$ and M given as in Definition 2. If $r = \text{rank}(M(\bar{z}(t), \bar{w}(t), t))$ a.e. $t \in [0, T]$, there exist an index subset, say $\{i_1, \ldots, i_r\}$, and a constant $C > 0$ such that

$$\det\{\Upsilon(t)\Upsilon(t)'\} \geq C \text{ a.e. } t \in [0, T],$$

where $\Upsilon(t)$ is the matrix obtained after removing from $M(\bar{z}(t), \bar{w}(t), t)$ the rows of index $i \notin \{i_1, \ldots, i_r\}$.

A full rank assumption on $M(\bar{z}(t), \bar{w}(t), t)$ implies the validity of (CRCQ) and (H5), see Proposition 2. Next we show an illustrative example, where (CRCQ) and (H5) are satisfied but the full rank condition does not hold.

Example 2 Consider the system

$$\begin{cases} tz_1(t) + t^2 z_2(t) - t^3 z_3(t) + 3t^4 = 0 \text{ a.e. } t \in [0, 1], \\ -z_1(t) - tz_2(t) + t^2 z_3(t) - 3t^3 = 0 \text{ a.e. } t \in [0, 1], \\ z_1(t)z_2(t) + z_3(t) - 3t \geq 0 \text{ a.e. } t \in [0, 1], \end{cases}$$

and $\bar{z}(t) = (0, 0, 3t)$ a.e. in $[0, 1]$. One has that $\bar{w}(t) = 0$ a.e. $[0, 1]$ and

$$M(z, w, t) = \begin{bmatrix} t & t^2 & -t^3 & 0 \\ -1 & -t & t^2 & 0 \\ z_2 & z_1 & 1 & -2w \end{bmatrix} \text{ a.e. } t \in [0, 1].$$

Then, $\text{rank}(M(z, w, t)) = 2$ for (z, w) in $(\bar{z}(t), \bar{w}(t)) + B$ a.e. $t \in [0, 1]$. Indeed, it is easy to see that $\text{rank}(M(z, w, t)) \leq 2$ a.e. $t \in [0, 1]$ for all (z, w). If $w \neq 0$, it is clear that $\text{rank}(M(z, w, t)) = 2$ a.e. $t \in [0, 1]$. If $w = 0$, assume that $\text{rank}(M(z, w, t)) = 1$ in $A \subset [0, 1]$, where A has positive measure. After some simple calculations one sees that $z_1 = -1/t$ and $z_2 = -1/t^2$, so that $(z, w) \notin (\bar{z}, \bar{w}) + B$ a.e. $t \in A$, since $|z_1| = 1/t \geq 1$ and $|z_2| = 1/t^2 \geq 1$ a.e. $t \in A \subset [0, 1]$. By choosing $i_1 = 2$ and $i_2 = 3$,

$$\Upsilon(t) = \begin{bmatrix} -1 & -t & t^2 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ a.e. } t \in [0, 1]$$

and $\det\{\Upsilon(t)\Upsilon(t)'\} = 1 + t^2 \geq 1$ a.e. $t \in [0, 1]$. Therefore, both (CRCQ) and (H5) are satisfied at $\bar{z}$. Note that $M(\bar{z}(t), \bar{w}(t), t)$ is not a full rank matrix.

Theorem 3 *Let $\bar{z}$ be local optimal solution of (CTP). Suppose that the assumptions (H1)-(H3), (H5) and (CRCQ) are satisfied. Then, there exist $u \in L^\infty([0, T]; \mathbb{R}^p)$ and $v \in L^\infty([0, T]; \mathbb{R}^m)$ such that*

$$\nabla\phi(\bar{z}(t), t) + \sum_{i=1}^{p} u_i(t)\nabla h_i(\bar{z}(t), t) + \sum_{j=1}^{m} v_j(t)\nabla g_j(\bar{z}(t), t) = 0, \tag{20}$$

$$v_j(t) \geq 0, \ j \in J, \tag{21}$$

$$v_j(t) g_j(\bar{z}(t), t) = 0, \ j \in J, \tag{22}$$

for almost every $t \in [0, T]$, *and*

$$\int_0^T \gamma(t)' \Big[\nabla^2 \phi(\bar{z}(t), t) + \sum_{i=1}^{p} u_i(t) \nabla^2 h_i(\bar{z}(t), t) + \sum_{j=1}^{m} v_j(t) \nabla^2 g_j(\bar{z}(t), t) \Big] \gamma(t)\, dt \leq 0, \tag{23}$$

for all $\gamma \in N$, *where*

$$N = \{\gamma \in L^\infty([0, T]; \mathbb{R}^n) : \nabla h_i(\bar{z}(t), t)' \gamma(t) = 0 \text{ a.e. } t \in [0, T],\ i \in I, \nabla g_j(\bar{z}(t), t)' \gamma(t) = 0 \text{ a.e. } t \in [0, T],\ j \in I_a(t)\}$$

and

$$I_a(t) = \{j \in J : g_j(\bar{z}(t), t) = 0\} \text{ a.e. } t \in [0, T].$$

Proof We will proceed in several steps.

STEP 1: (Auxiliary problem) Consider the following auxiliary problem

$$\begin{aligned} &\text{maximize } \tilde{P}(z, w) = \int_0^T \phi(z(t), t)\, dt \\ &\text{subject to } h(z(t), t) = 0 \text{ a.e. } t \in [0, T], \\ &\qquad g(z(t), t) - \text{diag}\{w(t)\} w(t) = 0 \text{ a.e. } t \in [0, T], \\ &\qquad z \in L^\infty([0, T]; \mathbb{R}^n),\ w \in L^\infty([0, T]; \mathbb{R}^m). \end{aligned} \tag{AUX}$$

Let $\bar{z} \in \Omega$ be an optimal solution of (CTP) on $\bar{z}(t) + \varepsilon \bar{B}$ and let $\bar{w}_j(t) = \sqrt{g_j(\bar{z}(t), t)}$ a.e. $t \in [0, T]$, $j \in J$. Then $(\bar{z}, \bar{w})$ is an optimal solution for (AUX). Indeed, it is clear that $\bar{z}$ is feasible for (AUX). Let $0 < \bar{\varepsilon} < \varepsilon$ and suppose that there exists a feasible solution $(\tilde{z}, \tilde{w})$ of (AUX) with $(\tilde{z}(t), \tilde{w}(t)) \in (\bar{z}(t), \bar{w}(t)) + \bar{\varepsilon} \bar{B}$ and $\tilde{P}(\tilde{z}, \tilde{w}) > \tilde{P}(\bar{z}, \bar{w})$. Then, $h(\tilde{z}(t), t) = 0$ a.e. $t \in [0, T]$ and

$$g(\tilde{z}(t), t) = \text{diag}\{\tilde{w}(t)\} \tilde{w}(t) = (\tilde{w}_1(t)^2, \ldots, \tilde{w}_m(t)^2) \geq 0 \text{ a.e. } t \in [0, T].$$

Therefore, $\tilde{z}$ is feasible point for (CTP) with $P(\tilde{z}) = \tilde{P}(\tilde{z}, \tilde{w}) > \tilde{P}(\bar{z}, \bar{w}) = P(\bar{z})$. This is a contradiction.

STEP 2: (Verifying assumptions of Theorem 2) Let $b : \mathbb{R}^{n+m} \times [0, T] \to \mathbb{R}^{p+m}$ be given as

$$b(z, w, t) = (h(z, t), g(z(t), t) - \text{diag}\{w\} w, t).$$

Assumptions (H1) and (H2) are immediate for b. Let us check (H3). Take $(\tilde{z}, \tilde{w}),\ (z, w) \in (\bar{z}(t), \bar{w}(t)) \in \varepsilon \bar{B}$. One has, for almost every $t \in [0, T]$, that

$$\nabla b(z, w, t) = \begin{bmatrix} \nabla h(z, t) & 0 \\ \nabla g(z, t) & \text{diag}\{-2w\} \end{bmatrix} \forall z, w.$$

So, since (h, g) satisfies (H3),

$$\begin{aligned} &\|\nabla b(\tilde{z}, \tilde{w}, t) - \nabla b(z, w, t)\| \\ &\quad \leq \|\nabla(h, g)(\tilde{z}, t) - \nabla(h, g)(z, t)\| + \|\text{diag}\{-2(\tilde{w} - w)\}\| \\ &\quad \leq \theta(\|\tilde{z} - z\|) + 2\|\tilde{w} - w\| \\ &\quad \leq \theta(\|(\tilde{z} - z, \tilde{w} - w)\|) + 2\|(\tilde{z} - z, \tilde{w} - w)\| \\ &\quad = \bar{\theta}(\|(\tilde{z}, \tilde{w}) - (z, w)\|) \text{ a.e. } t \in [0, T], \end{aligned}$$

where $\bar{\theta} : (0, \infty) \to (0, \infty)$ is given by $\bar{\theta}(s) = \theta(s) + 2s$. It is clear that $\bar{\theta}$ is monotone increasing and $\bar{\theta}(s) = \theta(s) + 2s \downarrow 0$ as $s \downarrow 0$. Moreover, by (H2), $g(\bar{z}(\cdot), \cdot)$ is essentially bounded on $[0, T]$,

$$\begin{aligned}\|\nabla b(\bar{z}(t), \bar{w}(t), t)\| &\leq \|\nabla(h, g)(\bar{z}(t), t)\| + \|\mathrm{diag}\{-2\bar{w}(t)\}\| \\ &\leq K_0 + 2\|\bar{w}(t)\| = K_1 \text{ a.e. } t \in [0, T].\end{aligned}$$

Thus b verifies (H3). Finally, as $\nabla b(z, w, t) = M(z, w, t)$ for all (z, w) a.e. $t \in [0, T]$, directly from (CRCQ) according to Definition 2 and (H5), one sees that the constraints of (AUX) satisfy (CRCQ) according to Definition 1 and (H4) at $(\bar{z}, \bar{w})$.

STEP 3: (First order optimality conditions) By Theorem 2, there exists $(u, v) \in L^\infty([0, T]; \mathbb{R}^p) \times L^\infty([0, T]; \mathbb{R}^m)$ such that, for almost every $t \in [0, T]$,

$$\begin{bmatrix} \nabla\phi(\bar{z}(t), t) \\ 0 \end{bmatrix} + \begin{bmatrix} \nabla h(\bar{z}(t), t)' & \nabla g(\bar{z}(t), t)' \\ 0 & \mathrm{diag}\{-2\bar{w}(t)\} \end{bmatrix} \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Then, for almost every $t \in [0, T]$,

$$\nabla\phi(\bar{z}(t), t) + \sum_{i=1}^{p} u_i(t)\nabla h_i(\bar{z}(t), t) + \sum_{j=1}^{m} v_j(t)\nabla g_j(\bar{z}(t), t) = 0$$

and

$$\bar{w}_j(t)v_j(t) = 0,\ \ j \in J \Rightarrow g_j(\bar{z}(t), t)v_j(t) = 0,\ \ j \in J,$$

that is, (20) and (22) are proved. (21) will be postponed to Step 5.

STEP 4: (Second order optimality conditions) Let us denote by e_j the j-th canonical vector in $\mathbb{R}^m$. By Theorem 2,

$$\int_0^T [\gamma(t)'\ \ \zeta(t)'] \left\{ \begin{bmatrix} \nabla^2\phi(\bar{z}(t), t) & 0 \\ 0 & 0 \end{bmatrix} + \sum_{i=1}^{p} u_i(t) \begin{bmatrix} \nabla^2 h_i(\bar{z}(t), t) & 0 \\ 0 & 0 \end{bmatrix} \right.$$

$$\left. + \sum_{j=1}^{m} v_j(t) \begin{bmatrix} \nabla^2 g_j(\bar{z}(t), t) & 0 \\ 0 & \mathrm{diag}\{-2e_j\} \end{bmatrix} \right\} \begin{bmatrix} \gamma(t) \\ \zeta(t) \end{bmatrix} dt \leq 0$$

for all $(\gamma, \zeta) \in L^\infty([0, T]; \mathbb{R}^{n+m})$ such that

$$\nabla h(\bar{z}(t), t)'\gamma(t) = 0 \text{ a.e. } t \in [0, T],\ \ i \in I, \tag{24}$$

$$\nabla g_j(\bar{z}(t), t)'\gamma(t) - 2\bar{w}_j(t)\zeta_j(t) = 0 \text{ a.e. } t \in [0, T],\ \ j \in J. \tag{25}$$

From the latter inequality, one obtains

$$\begin{aligned}\int_0^T \Big\{ \gamma(t)'\Big[\nabla^2\phi(\bar{z}(t), t) &+ \sum_{i=1}^{p} u_i(t)\nabla^2 h_i(\bar{z}(t), t) \\ &+ \sum_{j=1}^{m} v_j(t)\nabla^2 g_j(\bar{z}(t), t)\Big]\gamma(t) - 2\sum_{j=1}^{m} v_j(t)\zeta_j(t)^2 \Big\} dt \leq 0 \end{aligned} \tag{26}$$

for all (γ, ζ) satisfying (24)–(25). Let $\gamma \in N$ and ζ_j, $j \in J$, be given, for almost every $t \in [0, T]$, as

$$\zeta_j(t) = \begin{cases} 0, & j \in I_a(t), \\ \dfrac{\nabla g_j(\bar{z}(t), t)'\gamma(t)}{2\bar{w}_j(t)}, & j \in J \setminus I_a(t). \end{cases}$$

If $\zeta = (\zeta_1, \ldots, \zeta_m)$, note that (γ, ζ) satisfies (24)–(25) and that $v_j(t)\zeta_j(t) = 0$ a.e. $t \in [0, T]$, $j \in J$, since, by (22), $v_j(t) = 0$ for $j \notin I_a(t)$. For this particular choice of (γ, ζ), (23) follows directly from (26).

STEP 5: (Nonnegativity of the multipliers associated with the inequality constraints) Suppose that $v_l(t) < 0$ for all $t \in D \subset [0, T]$, where D has positive measure, for some $l \in J$. By (22), $l \in I_a(t)$ for all $t \in D$. Take (γ, ζ) such that $\gamma(t) \equiv 0$, $\zeta_j(t) \equiv 0$ for $j \neq l$ and

$$\zeta_l(t) = \begin{cases} 0, & t \in [0, T] \setminus D, \\ k, & t \in D, \end{cases}$$

where k is an arbitrary constant. Observe that (γ, ζ) satisfies (24)–(25). From (26), it follows that

$$\int_D v_l(t)\zeta_l(t)^2\, dt \geq 0.$$

But,

$$v_i(t) < 0 \text{ and } \zeta_l(t) = k \neq 0,\ t \in D \Rightarrow \int_D v_l(t)\zeta_l(t)^2\, dt < 0,$$

which is a contradiction. □

Proposition 2 *Let* $\bar{z} \in \Omega$ *and* $\bar{w}_j(t) = \sqrt{g_j(\bar{z}(t), t)}$ *a.e. in* $[0, T]$*,* $j \in J$*. If* $\det\{M(\bar{z}(t), \bar{w}(t), t)M(\bar{z}(t), \bar{w}(t), t)'\} \geq K$ *a.e.* $t \in [0, T]$ *for some* $K > 0$*, then (CRCQ) and (H5) hold.*

Proof By hypothesis $M(\bar{z}(t), \bar{w}(t), t)$ has full rank and it is clear that (H5) holds with $\Upsilon(t) = M(\bar{z}(t), \bar{w}(t), t)$ a.e. $t \in [0, T]$. Moreover, from Step 2 of the proof of Theorem 3, one knows that there exist a monotone increasing function $\bar{\theta} : (0, \infty) \to (0, \infty)$ with $\bar{\theta}(s) \downarrow 0$ when $s \downarrow 0$ and a positive constant K_1 such that

$$\|M(\tilde{z}, \tilde{w}, t) - M(z, w, t)\| \leq \bar{\theta}(\|(\tilde{z}, \tilde{w}) - (z, w)\|) \tag{27}$$

for all $(\tilde{w}, \tilde{w}), (z, w) \in (\bar{z}(t), \bar{w}(t)) + \varepsilon\bar{B}$ a.e. $t \in [0, T]$, for some $\varepsilon > 0$, and

$$\|M(\bar{z}(t), \bar{w}(t), t)\| \leq K_1 \text{ a.e. } t \in [0, T]. \tag{28}$$

By the continuity assumptions on the data and (28), there exists $\varepsilon_1 > 0$ such that

$$\|M(z, w, t)\| \leq K_2\ \forall (z, w) \in (\bar{z}(t), \bar{w}(t)) + \varepsilon_1\bar{B} \text{ a.e. } t \in [0, T]. \tag{29}$$

For almost every $t \in [0, T]$, denote $A(t) = M(\bar{z}(t), \bar{w}(t), t)M(\bar{z}(t), \bar{w}(t), t)'$ and $A(z, w, t) = M(z, w, t)M(z, w, t)'$. By the hypothesis and (28), it follows from Lemma 1 that there exists $K_3 > 0$ such that

$$\|[A(t)]^{-1}\| \leq K_3 \text{ a.e. } t \in [0, T]. \tag{30}$$

Let $K_4 = \max\{K_1, K_2\}$. Since $\bar{\theta}(s) \downarrow 0$ when $s \downarrow 0$, let $0 < \varepsilon_2 < \min\{\varepsilon, \varepsilon_1\}$ such that

$$\bar{\theta}(\varepsilon_2) \leq \frac{1}{4K_3K_4}. \tag{31}$$

Let $(z, w) \in (\bar{z}(t), \bar{w}(t)) + \varepsilon_2 B$. From (27)–(31), one has that

$$\begin{aligned}
\|A(z, w, t) - A(t)\| &= \|M(z, w, t)M(z, w, t)' - M(\bar{z}(t), \bar{w}(t), t)M(\bar{z}(t), \bar{w}(t), t)'\| \\
&\leq \|M(z, w, t) - M(\bar{z}(t), \bar{w}(t), t)\|\|M(z, w, t)'\| \\
&\quad + \|M(\bar{z}(t), \bar{w}(t), t)\|\|M(z, w, t)' - M(\bar{z}(t), \bar{w}(t), t)'\| \\
&\leq 2K_4\bar{\theta}(\varepsilon_2) \leq \frac{1}{2K_3} \leq \frac{1}{2\|[A(t)]^{-1}\|} \text{ a.e. in } [0, T].
\end{aligned}$$

It is known (see Noble and Daniel [19], for example) that if A and R are matrices such that A is non-singular and $\|R\| < \|A^{-1}\|^{-1}$, then $A + R$ is non-singular. It follows that $A(z, w, t) = A(t) + [A(z, w, t) - A(t)]$ has full (and consequently constant) rank for all $(z, w) \in (\bar{z}(t), \bar{w}(t)) + \varepsilon_2 B$. Therefore (CRCQ) is satisfied. □

Remark 5 Although it was showed that a full rank condition is stronger than the constant rank condition, condition (CRCQ) given in Definition 2 is, in general, difficult to be verified. This is due to presence of the diagonal matrix diag$\{-2w_j\}_{j=1}^m$. Clearly, this matrix does not maintain its rank in a neighbourhood of $\bar{w}_j(t) = \sqrt{g_j(\bar{z}(t), t)}$ a.e. in $[0, T]$, $j \in J$, so that the whole matrix $M(z, w, t)$ may not have constant rank locally around $(\bar{z}(t), \bar{w}(t)) + \varepsilon B$ a.e. $t \in [0, T]$, even when $[\nabla h(z, t)' \quad \nabla g(z, t)']'$ does. We believe that it is possible to establish Karush-Kuhn-Tucker type optimality conditions for (CTP) when (CRCQ) given in Definition 2 is replaced by merely a constant rank condition imposed on the gradients of the equality and active inequality constraints. By means of the technique employed in the proof of Theorem 3, this is not possible, however. It is going to be a topic of future work to find a more appropriate approach in which this can be carried out.

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