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On the Stability of the Directional Regularity

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Abstract
In this paper we select two tools of investigation of the classical metric regularity of set-valued mappings, namely the Ioffe criterion and the Ekeland Variational Principle, which we adapt to the study of the directional setting. In this way, we obtain in a unitary manner new necessary and/or sufficient conditions for directional metric regularity. As an application, we establish stability of this property at composition and sum of set-valued mappings. In this process, we introduce directional tangent cones and the associated generalized primal differentiation objects and concepts. Moreover, we underline several links between our main assertions by providing alternative proofs for several results.

Keywords Directional regularity · Ioffe criterion · Directional openness stability · Primal conditions

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1 Introduction

In this paper we continue the study of the directional regularity concepts introduced in [8], by getting adapted metric regularity criteria of Ioffe-type inspired by [2], which allows to derive, in a unitary way, several sufficient conditions for the stability of these properties. In our investigation, the directional character of the regularities is formulated by the use of a special type of the minimal time function introduced and studied in [7]. In the proof of the main criterion for the directional metric regularity we use, as a main tool, a directional Ekeland Variational Principle obtained in [8] by means of this minimal time function. Applying this criterion, we derive conditions for the directional regularities of a composition and a sum of set-valued maps. Furthermore, we give sufficient conditions for directional regularities in both cases in terms of directional derivatives of set-valued maps. The novelties that our work propose can be structured into three categories: firstly, some of the results are completely new, secondly, many of our results generalize, to the directional setting, known results from the classical (non-directional) case and, thirdly, we present new and shorter proofs for several assertions.

As shown in [2, 16], the Ioffe criterion for regularity allows to obtain simpler proofs for metric regularity conditions in different instances. Indeed, we show that this is the case as well for the directional regularity we study here, in the sense that, for the sufficient conditions we develop, the proofs based on this adapted criterion are much shorter than the direct proofs which can be given by the use of the directional Ekeland Variational Principle. Also, a more direct, but longer proof, based on iterations, of the Ioffe criterion is possible. However, as a general procedure, we prefer to have a unitary presentation, with concise proofs, and we leave the alternative proofs for the last section of the paper.

The paper is organized as follows. Firstly, we present the directional regularity properties we deal with, the links between them and we compare our concepts with other directional metric regularity constructions met in the literature. Next, we derive a Ioffe criterion for directional regularity of single-valued mappings, and then a criterion of the same type for set-valued maps. On this basis, we consider the stability of directional openness of set-valued compositions and, as a particular case, that of the sum of multifunctions. Moreover, we present sufficient conditions for the directional regularity of a set-valued map in terms of a special type of the directional Bouligand-Severi tangent cone with implications in the cases of composition and sum. At the end we collect, in Appendix, in two separate subsections, the direct and constructive proofs of some results which were not fitted for the unity of the presentation in the previous sections. A comparison between the proofs given in the main part of the paper and the corresponding ones from the Appendix emphasizes the usefulness of the Ioffe criterion.

2 Directional Regularity: Preliminaries and Comparison with Other Concepts

Throughout the paper, we use the convention that $\inf \emptyset = +\infty$ and, as we work with non-negative quantities, that $\sup \emptyset = 0$. $B(x, r)$ and $B[x, r]$ are the open and the closed ball in a metric space X with a center $x \in X$ and a radius $r \in [0, +\infty]$, respectively, with the convention that

$$B[x, r] = B(x, r) = \begin{cases} \{x\}, & \text{if } r = 0, \\ X, & \text{if } r = +\infty. \end{cases}$$

We denote by $B_X := B(0, 1)$, $\mathbb{B}_X := B[0, 1]$, and $S_X := B[0, 1] \setminus B(0, 1)$ the open unit ball, the closed unit ball, and the unit sphere in X, respectively. For a set $A \subset X$, we denote by $\operatorname{int} A$, and $\operatorname{cl} A$ its topological interior, and closure, respectively. The distance from a point x to a set A in the metric space (X, ϱ) is $d(x, A) := \inf\{\varrho(x, a) \mid a \in A\}$. If X is a normed vector space, the cone generated by A is designated by $\operatorname{cone} A$. For $x, y \in X$, we denote by (x, y) and $[x, y]$ the open and the closed line segment joining the points x and y, respectively.

Let $F : X \rightrightarrows Y$ be a multifunction. The domain and the graph of F are denoted respectively by $\operatorname{Dom} F := \{x \in X \mid F(x) \neq \emptyset\}$ and $\operatorname{Gr} F = \{(x, y) \in X \times Y \mid y \in F(x)\}$. If $A \subset X$, then $F(A) := \bigcup_{x \in A} F(x)$. The inverse of F, which always exists, is the set-valued mapping $F^{-1} : Y \rightrightarrows X$ given by $F^{-1}(y) = \{x \in X \mid y \in F(x)\}$, $y \in Y$.

In literature, there are several concepts dealing with regularity of mappings in various directional settings. We mention here the following works: [5, 12–14]. In this paper we are concerned with the approach proposed in [8].

Let $\emptyset \neq \Omega \subset X$ and $\emptyset \neq L \subset S_X$. Then the function

$$X \ni x \longmapsto T_L(x, \Omega) := \inf\{t \geq 0 \mid \exists \ell \in L : x + t\ell \in \Omega\} \tag{2.1}$$

is called the directional minimal time function with respect to L. Many properties of this function were systematically analyzed in [7]. Remark that

$$T_L(x, \Omega) < +\infty \text{ if and only if } x \in \Omega - \operatorname{cone} L$$

and

$$d(x, \Omega) \leq T_L(x, \Omega) \text{ for all } x \in X.$$

Moreover, if $L = S_X$, then $T_L(\cdot, \Omega) = d(\cdot, \Omega)$. If $\Omega = \{u\}$ for a point $u \in X$, we denote in what follows $T_L(\cdot, \{u\})$ by $T_L(\cdot, u)$. Clearly, for each $x, u \in X$, if $T_L(x, u) < +\infty$ (which is equivalent to $u - x \in \operatorname{cone} L$), then

$$T_{-L}(u, x) = T_L(x, u) = \|u - x\|.$$

Moreover, if $\operatorname{cone} L$ is convex, then

(i) $T_L(x, u) = 0$ if and only if $x = u$;
(ii) $T_L(x, u) \leq T_L(x, v) + T_L(v, u)$, for all $x, u, v \in X$.

We recall next the directional regularity notions introduced and studied in [8], as well as the link between them, which mimic the classical case of around-point regularities.

For two sets $A, B \subset X$, we consider the directional excess from A to B with respect to L as

$$e_L(A, B) := \sup_{x \in A} T_L(x, B).$$

Of course, $e_L(A, B) = +\infty$ if $A \not\subset B - \operatorname{cone} L$. Obviously, if $L = S_X$, then $e_L(A, B)$ becomes the usual excess from A to B defined by

$$e(A, B) := \sup_{x \in A} d(x, B).$$

Definition 1 Let $F : X \rightrightarrows Y$ be a set-valued mapping between normed spaces $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ with $(\bar{x}, \bar{y}) \in \operatorname{Gr} F$ and sets $L \subset S_X$ and $M \subset S_Y$ be nonempty.

(i) One says that F is *directionally metrically regular around* $(\bar{x}, \bar{y})$ *with respect to* L *and* M *with a constant* $c > 0$ if there are $\varepsilon > 0$ and neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that, for every $(x, y) \in U \times V$ such that $T_M(y, F(x)) < \varepsilon$,

$$T_L(x, F^{-1}(y)) \leq c \cdot T_M(y, F(x)). \tag{2.2}$$

The *modulus of directional regularity of* F *around* $(\bar{x}, \bar{y})$ *with respect to* L *and* M, denoted by $\operatorname{dirreg}_{L\times M} F(\bar{x}, \bar{y})$, is the infimum of $c > 0$ such that F is directionally metrically regular around $(\bar{x}, \bar{y})$ with respect to L and M with the constant c.

(ii) One says that F is *directionally linearly open around* $(\bar{x}, \bar{y})$ *with respect to* L *and* M *with a constant* $c > 0$ if there are $\varepsilon > 0$ and neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that, for every $r \in (0, \varepsilon)$ and every $(x, y) \in (U \times V) \cap \operatorname{Gr} F$,

$$B(y, cr) \cap (y - \operatorname{cone} M) \subset F(B(x, r) \cap (x + \operatorname{cone} L)). \tag{2.3}$$

The *modulus of directional openness of* F *around* $(\bar{x}, \bar{y})$ *with respect to* L *and* M, denoted by $\operatorname{dirsur}_{L\times M} F(\bar{x}, \bar{y})$, is the supremum of $c > 0$ such that F is directionally linearly open around $(\bar{x}, \bar{y})$ with respect to L and M with the constant c.

(iii) One says that F has the *directional Aubin property around* $(\bar{x}, \bar{y})$ *with respect to* L *and* M *with a constant* $c > 0$ if there are neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that, for every $x, u \in U$,

$$e_M(F(x) \cap V, F(u)) \leq c \cdot T_L(u, x). \tag{2.4}$$

The *modulus of the directional Aubin property of* F *around* $(\bar{x}, \bar{y})$ *with respect to* L *and* M, denoted by $\operatorname{dirlip}_{L\times M} F(\bar{x}, \bar{y})$, is the infimum of $c > 0$ such that F has the directional Aubin property around $(\bar{x}, \bar{y})$ with respect to L and M with the constant c.

Of course, when $L := S_X$ and $M := S_Y$, the previous concepts reduce to the usual metric regularity, linear openness, and Aubin property around the reference point (see, e.g., [4] for more details). Moreover, directional metric regularity and Aubin property correspond to the cases where the regularity moduli are finite, respectively, while the directional linear openness holds if and only if the modulus of directional openness is strictly positive.

Remark 2 Note that a similar concept of directional openness can be defined if one considers, instead of (2.3), that

$$B[y, cr] \cap (y - \operatorname{cone} M) \subset F(B[x, r] \cap (x + \operatorname{cone} L)) \tag{2.5}$$

holds. Indeed, if according to the definition, (2.3) holds, then for arbitrary $c' < c$,

$$\begin{aligned} B[y, c'r] \cap (y - \operatorname{cone} M) &\subset B(y, cr) \cap (y - \operatorname{cone} M) \\ &\subset F(B(x, r) \cap (x + \operatorname{cone} L)) \subset F(B[x, r] \cap (x + \operatorname{cone} L)) \end{aligned}$$

for every corresponding (x, y) and $r \in (0, \varepsilon)$, hence the inclusion holds for closed balls instead of open ones and for any $c' < c$. Also, if (2.5) holds for some $c > 0$, then for any $c' < c$,

$$\begin{aligned} B(y, c'r) \cap (y - \operatorname{cone} M) &= B(y, c(c'c^{-1}r)) \cap (y - \operatorname{cone} M) \\ &\subset B[y, c(c'c^{-1}r)] \cap (y - \operatorname{cone} M) \subset F(B[x, c'c^{-1}r] \cap (x + \operatorname{cone} L)) \\ &\subset F(B(x, r) \cap (x + \operatorname{cone} L)). \end{aligned}$$

Hence, the two notions are equivalent. Moreover, observe that the value of $\operatorname{dirsur}_{L\times M} F(\overline{x}, \overline{y})$ remains unchanged if one defines it as the supremum of positive constants c involved in (2.3) or in (2.5).

The next result contains the announced link between the notions given before (see [8, Proposition 2.3]). The convention $1/0 = +\infty$ applies here.

Proposition 3 *Let $F : X \rightrightarrows Y$ be a set-valued mapping between normed spaces $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ with $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ and sets $L \subset S_X$ and $M \subset S_Y$ be nonempty. Then*

$$\operatorname{dirreg}_{L\times M} F(\overline{x}, \overline{y}) = (\operatorname{dirsur}_{L\times M} F(\overline{x}, \overline{y}))^{-1} = \operatorname{dirlip}_{M\times L} F^{-1}(\overline{y}, \overline{x}).$$

For a set-valued mapping $F : X \times Y \rightrightarrows Z$, one may speak about the directional regularities with respect to one variable, uniformly for the other. More precisely, in this case, we use the notation $F_y := F(\cdot, y)$, we consider nonempty sets $L \subset S_X$, $M \subset S_Z$, and we say that F is directionally metrically regular relative to x uniformly in y around $(\overline{x}, \overline{y}, \overline{z}) \in \operatorname{Gr} F$ with respect to L and M with a constant $c > 0$ if there are $\varepsilon > 0$ and neighborhoods U of $\overline{x}$, V of $\overline{y}$, and W of $\overline{z}$ such that, for every $y \in V$, and every $(x, z) \in U \times W$ such that $T_M(z, F_y(x)) < \varepsilon$,

$$T_L(x, F_y^{-1}(z)) \le c \cdot T_M(z, F_y(x)). \tag{2.6}$$

The modulus of directional regularity of F relative to x uniformly in y around $(\overline{x}, \overline{y}, \overline{z})$ with respect to L and M, denoted by $\widehat{\operatorname{dirreg}}^{x}_{L\times M} F(\overline{x}, \overline{y}, \overline{z})$, is defined as the infimum of $c > 0$ such that the above property holds.

Analogously, one may define the other two regularity properties relative to one variable, uniformly for the other, and the corresponding regularity moduli are denoted by $\widehat{\operatorname{dirsur}}^{x}_{L\times M} F(\overline{x}, \overline{y}, \overline{z})$ and $\widehat{\operatorname{dirlip}}^{x}_{L\times M} F(\overline{x}, \overline{y}, \overline{z})$.

In what follows, we compare the concepts from Definition 1 with other directional regularity notions met in literature. Firstly, recall the notion of the directional metric regularity in a given direction from [14].

Definition 4 A set-valued mapping $F : X \rightrightarrows Y$ from a metric space (X, ϱ) to a normed space $(Y, \|\cdot\|)$ is said to be *directionally metrically regular at* $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ *in a direction* $w \in Y$ *with a constant* $\kappa > 0$ if there exist $\varepsilon > 0$ and $\delta > 0$ such that, for every $(x, y) \in B[\overline{x}, \varepsilon] \times B[\overline{y}, \varepsilon]$ satisfying $d(y, F(x)) < \varepsilon$ and $y \in F(x) + \operatorname{cone} B(w, \delta)$,

$$d(x, F^{-1}(y)) \le \kappa\, d(y, F(x)).$$

Proposition 5 *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be two normed spaces, $F : X \rightrightarrows Y$ be a set-valued mapping, and $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$. If F is directionally metrically regular at $(\overline{x}, \overline{y})$ in a direction $w \in Y$ with a constant $\kappa > 0$, then there is a nonempty closed $M \subset S_Y$ with* cone M *being convex such that F is directionally metrically regular around $(\overline{x}, \overline{y})$ with respect to S_X and M with the constant κ.*

Proof Find $\varepsilon > 0$ and $\delta > 0$ such that for all $(x, y) \in B[\overline{x}, \varepsilon] \times B[\overline{y}, \varepsilon]$ satisfying $d(y, F(x)) < \varepsilon$ and $y \in F(x) + \operatorname{cone} B(w, \delta)$ we have $d(x, F^{-1}(y)) \le \kappa d(y, F(x))$. Denote $M := -[S_Y \cap \operatorname{cone} B(w, \delta)]$. Fix any $(x, y) \in B[\overline{x}, \varepsilon] \times B[\overline{y}, \varepsilon]$ satisfying $T_M(y, F(x)) < \varepsilon$. Then $d(y, F(x)) \le T_M(y, F(x)) < \varepsilon$, hence there is $m \in M$ and $\varepsilon' \in (0, \varepsilon)$ such that $y + \varepsilon' m \in F(x)$. So

$$y \in F(x) - \varepsilon' m \subset F(x) + \varepsilon' [S_Y \cap \operatorname{cone} B(w, \delta)] \subset F(x) + \operatorname{cone} B(w, \delta).$$

Consequently, $T_{S_X}(x, F^{-1}(y)) = d(x, F^{-1}(y)) \le \kappa d(y, F(x)) \le \kappa T_M(y, F(x))$. □

The converse of the previous result does not hold in general, as the next example shows.

Example 6 Consider $X := Y := \mathbb{R}$, $M := \{1\}$, $F : X \rightrightarrows Y$ given by

$$F(x) := \begin{cases} \mathbb{R}, & \text{for } x \le 0 \text{ or } x \ge 1 \\ \left(-\infty, x^2\right] \cup \left[\sqrt{x}, +\infty\right), & \text{for } x \in (0, 1), \end{cases}$$

and $(\bar{x}, \bar{y}) := (0, 0)$. Let us prove first that F is directionally metrically regular around $(\bar{x}, \bar{y})$ with respect to S_X and M with the constant $c = 1$. For this, take $\varepsilon > 0$ such that $\sqrt{\varepsilon} + \varepsilon \le 1$, and $x, y \in (-\varepsilon, \varepsilon)$ such that $T_M(y, F(x)) < \varepsilon$. If $y \le 0$, then $F^{-1}(y) = \mathbb{R}$, hence the inequality

$$d(x, F^{-1}(y)) \le T_M(y, F(x)) \tag{2.7}$$

trivially holds. Suppose now that $y \in (0, \varepsilon)$. Then $F^{-1}(y) = \left(-\infty, y^2\right] \cup \left[\sqrt{y}, +\infty\right)$. If $x \le 0$, then $d(x, F^{-1}(y)) = 0$, hence (2.7) holds again. Consider next that $x \in (0, \varepsilon)$. If $y \le x^2$, then $\sqrt{y} \le x$, so $d(x, F^{-1}(y)) = T_M(y, F(x)) = 0$. If $y \ge \sqrt{x}$, then $y^2 \ge x$, and again $d(x, F^{-1}(y)) = T_M(y, F(x)) = 0$. Finally, consider the case that $x^2 < y < \sqrt{x}$. Then $T_M(y, F(x)) = \sqrt{x} - y$, and $d(x, F^{-1}(y)) = \min\left\{x - y^2, \sqrt{y} - x\right\}$. But we have that

$$d(x, F^{-1}(y)) \le x - y^2 = \left(\sqrt{x} - y\right)\left(\sqrt{x} + y\right) \le \sqrt{x} - y = T_M(y, F(x)).$$

We conclude that the claim is proved. Let us show next that for any direction $w \in \mathbb{R}$, F is not directionally metrically regular at $(\bar{x}, \bar{y})$ in the direction w. Take arbitrary $\varepsilon \in (0, 1)$ and pick $x, y \in (0, \varepsilon)$ such that $x^2 < y < \sqrt{x}$ and $d(y, F(x)) < \varepsilon$. Observe that it is enough to consider that $w \in \{-1, 0, 1\}$. In any of these cases, and for any $\delta > 0$, $y \in F(x) + \operatorname{cone} B(w, \delta)$ means that $y \in \mathbb{R}$. Moreover, if $y \downarrow x^2$, then $\sqrt{y} \downarrow x$, hence in this situation $d(y, F(x)) = y - x^2$, and

$$d(x, F^{-1}(y)) = \min\left\{x - y^2, \sqrt{y} - x\right\} = \sqrt{y} - x.$$

So, in order to have that F is directionally metrically regular at $(\bar{x}, \bar{y})$ in the direction w, we should find $\kappa > 0$ such that for any small x, and for $y > x^2$ arbitrarily close to x^2, we have

$$d(x, F^{-1}(y)) = \sqrt{y} - x \le \kappa \cdot \left(y - x^2\right) = \kappa \cdot d(y, F(x)).$$

But this should mean that

$$1 \le \kappa \cdot \left(\sqrt{y} + x\right)$$

for any $x, y \in (0, \varepsilon)$ with $y > x^2$ sufficiently close to x^2 such that $d(y, F(x)) < \varepsilon$ and $\sqrt{y} - x < x - y^2$, which obviously cannot hold. In conclusion, the claim is proved.

Another related concept is the regularity along a subspace from [5, 13].

Definition 7 A set-valued mapping $F : X \rightrightarrows Y$ from a normed space $(X, \|\cdot\|)$ to a metric space (Y, ϱ) is called *metrically regular along a (closed) subspace H of X around* $(\bar{x}, \bar{y}) \in \operatorname{Gr} F$ *with a constant* $\kappa > 0$ if there exists $\varepsilon > 0$ such that, for every $(x, y) \in B[\bar{x}, \varepsilon] \times B[\bar{y}, \varepsilon]$,

$$\inf\{\|h\| : h \in H \text{ and } x + h \in F^{-1}(y)\} \le \kappa\, d(y, F(x)). \tag{2.8}$$

Lemma 8 *Let H be a (closed) subspace of a normed space $(X, \|\cdot\|)$. Then, for every $\Omega \subset X$ and every $x \in X$,*

$$d_H(x, \Omega) := \inf\{\|h\| : \ h \in H \text{ and } x + h \in \Omega\} = T_{S_H}(x, \Omega).$$

Proof Let $\lambda > d_H(x, \Omega)$ be arbitrary (if there is any). Find $h \in H$ such that $\lambda > \|h\|$ and $x + h \in \Omega$. If $h = 0$ then $x \in \Omega$ and thus $T_{S_H}(x, \Omega) = 0 \ (< \lambda)$. If not, then $h/\|h\| \in S_H$ and $x + \|h\|(h/\|h\|) \in \Omega$. Hence $T_{S_H}(x, \Omega) \leq \|h\| < \lambda$. Letting $\lambda \downarrow d_H(x, \Omega)$ we get $d_H(x, \Omega) \geq T_{S_H}(x, \Omega)$. On the other hand, let $\lambda > T_{S_H}(x, \Omega)$ be arbitrary (if there is any). Find $h \in S_H$ and $t \in [0, \lambda)$ such that $x + th \in \Omega$. So $d_H(x, \Omega) \leq \|th\| \leq t < \lambda$. Letting $\lambda \downarrow T_{S_H}(x, \Omega)$ we get $d_H(x, \Omega) \leq T_{S_H}(x, \Omega)$. □

Corollary 9 *Let $F : X \rightrightarrows Y$ be a set-valued mapping between normed spaces $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ with $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$, H be a (closed) subspace of X, and $\kappa > 0$. If F is metrically regular along H around $(\overline{x}, \overline{y})$ with the constant κ, then F is directionally metrically regular around $(\overline{x}, \overline{y})$ with respect to S_H and S_Y with the constant κ. Conversely, if F is directionally metrically regular around $(\overline{x}, \overline{y})$ with respect to S_H and S_Y with the constant κ, then there is $\varepsilon > 0$ such that for all $(x, y) \in B[\overline{x}, \varepsilon] \times B[\overline{y}, \varepsilon]$ with $d(y, F(x)) < \varepsilon$ inequality (2.8) holds.*

Proof As $T_{S_Y}(y, \Gamma) = d(y, \Gamma)$ for any $y \in Y$ and $\Gamma \subset Y$, Lemma 8 implies the result. □

3 Criteria for Directional Regularity

In what follows, we provide some criteria for directional regularity in the lines of those given, e.g., in [2] (see also, [3, 10, 15]). We analyze first the case of single-valued mappings, which we see as particular instances of set-valued mappings. It is important to emphasize that the domain of the function g in the sequel must be understood as the set where g is defined, hence we think g as a multifunction having the cardinality of the set $\{g(x)\}$ at most one at every $x \in X$.

For the first proposition, we use the directional Ekeland Variational Principle (EVP, for short) given in [8, Corollary 3.2].

Theorem 10 *Let X be a Banach space and $A \subset X$ be a closed set. Let $M \subset S_X$ be a closed set such that* cone M *is convex, and $f : A \to \mathbb{R} \cup \{+\infty\}$ be a bounded from below lower semicontinuous function. Then, for every $x_0 \in A$ with $f(x_0) < +\infty$, and for every $\varepsilon > 0$, there exists $x_\varepsilon \in A$ such that*

$$f(x_\varepsilon) \leq f(x_0) - \varepsilon T_M(x_\varepsilon, x_0)$$

and for any $x \in A \setminus \{x_\varepsilon\}$,

$$f(x_\varepsilon) < f(x) + \varepsilon T_M(x, x_\varepsilon).$$

As usual, the application of Ekeland Variational Principle allows to avoid the explicit use of iterations. Another, more constructive, but longer proof of the next result is given in the Appendix.

Proposition 11 (general criterion for single-valued maps) *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be Banach spaces. Consider a nonempty closed subset L of S_X such that* cone L *is convex, a* *nonempty closed subset M of S_Y, a point $\overline{x} \in X$, and a mapping $g : X \to Y$ such that there is a neighborhood U of $\bar{x}$ such that the set $D := U \cap \operatorname{Dom} g$ is closed and g is continuous on D. Then $\operatorname{dirsur}_{L\times M} g(\overline{x})$ equals to the supremum of $c > 0$ for which there is $r > 0$ such that for all $(x, y) \in (B[\overline{x}, r] \cap \operatorname{Dom} g) \times B[g(\overline{x}), r]$, with $0 < T_M(y, g(x)) < +\infty$, there is a point $x' \in \operatorname{Dom} g$ satisfying*

$$cT_L(x, x') < T_M(y, g(x)) - T_M(y, g(x')).$$

Proof Let $\lambda := \operatorname{dirsur}_{L\times M} g(\overline{x})$ and s be the supremum from the statement. To show that $s \le \lambda$, fix an arbitrary $c \in (0, s)$ (if there is any). Find $\delta > 0$ such that the set $A := B[\overline{x}, 2\delta] \cap \operatorname{Dom} g$ is a subset of D and for all $(u, y) \in (B[\overline{x}, 2\delta] \cap \operatorname{Dom} g) \times B[g(\overline{x}), 2\delta]$, with $0 \ne g(u) - y \in \operatorname{cone} M$, there is a point $x' \in \operatorname{Dom} g$ such that

$$cT_L(u, x') < T_M(y, g(u)) - T_M(y, g(x')). \tag{3.1}$$

By the continuity of g on A, there is $\varepsilon \in (0, \delta \min\{1/2, 1/c\})$ such that for every $x \in B[\overline{x}, \varepsilon] \cap \operatorname{Dom} g$, one has that $\|g(x) - g(\overline{x})\| < \delta$.

Fix any $r \in (0, \varepsilon)$ and any $x \in B[\overline{x}, \varepsilon] \cap \operatorname{Dom} g$. We show that

$$B(g(x), cr) \cap (g(x) - \operatorname{cone} M) \subset g\left(B(x, r) \cap (x + \operatorname{cone} L) \cap \operatorname{Dom} g\right). \tag{3.2}$$

Pick an arbitrary $y \in B(g(x), cr) \cap (g(x) - \operatorname{cone} M)$. If $y = g(x)$ then (3.2) holds trivially. Suppose that $y \ne g(x)$. Let $f : A \to [0, +\infty]$ be defined by $f(z) := T_M(y, g(z))$, $z \in A$. Then f is lower semicontinuous on A, since g is continuous on A and $T_M(y, \cdot)$ is lower semicontinuous. Moreover, as $g(x) - y \in \operatorname{cone} M$, we have $f(x) = T_M(y, g(x)) = \|g(x) - y\| < cr < +\infty$. Since A is closed, applying the directional Ekeland Variational Principle (Theorem 10), we find $u \in A$ such that

$$f(u) \le f(x) - cT_{-L}(u, x) \tag{3.3}$$

and

$$f(u) \le f(z) + cT_{-L}(z, u), \quad \text{for all} \quad z \in A. \tag{3.4}$$

By (3.3), we have $cT_L(x, u) = cT_{-L}(u, x) \le f(x) - f(u) \le f(x) < cr < +\infty$, hence $u - x \in \operatorname{cone} L$ and $c\|u - x\| = cT_L(x, u) < cr$. Consequently, we have $u \in B(x, r) \cap (x + \operatorname{cone} L) \cap \operatorname{Dom} g$. As $y \in B(g(x), cr) \cap (g(x) - \operatorname{cone} M)$ is arbitrary, (3.2) will follow once we show that $y = g(u)$. Suppose on the contrary that $y \ne g(u)$. Note that

$$\|u - \overline{x}\| \le \|u - x\| + \|x - \overline{x}\| < r + \varepsilon < 2\varepsilon < \delta$$

and

$$\|y - g(\overline{x})\| \le \|y - g(x)\| + \|g(x) - g(\overline{x})\| < cr + \delta < c\varepsilon + \delta < \delta + \delta = 2\delta.$$

By (3.3), we have $T_M(y, g(u)) = f(u) \le f(x) < cr < +\infty$, hence $0 \ne g(u) - y \in \operatorname{cone} M$. Find $x' \in \operatorname{Dom} g$ such that (3.1) holds. In particular, we have $cT_L(u, x') < T_M(y, g(u)) < cr < +\infty$, and therefore

$$\left\|x' - \overline{x}\right\| \le \left\|x' - u\right\| + \|u - \overline{x}\| = T_L(u, x') + \|u - \overline{x}\| < r + \delta < \varepsilon + \delta < 2\delta.$$

Thus $x' \in A$ and $f(x')$ is well defined. Combing (3.1) and (3.4) with $z := x'$, we get

$$cT_L(u, x') < T_M(y, g(u)) - T_M(y, g(x')) = f(u) - f(x') \le cT_{-L}(x', u) = cT_L(u, x'),$$

a contradiction. Hence $y = g(u)$.

By (3.2), $\lambda \geq c$, for any $c \in (0, s)$. Thus $s \leq \lambda$. Assume that $s < \lambda$. Fix any $c \in (s, \lambda)$. Find $\varepsilon > 0$ and $c' \in (c, \lambda)$ such that $B[\overline{x}, \varepsilon] \cap \operatorname{Dom} g$ is a subset of D and for each $x \in B[\overline{x}, \varepsilon] \cap \operatorname{Dom} g$ and each $t \in (0, \varepsilon)$ we have

$$g(B(x,t) \cap (x + \operatorname{cone} L) \cap \operatorname{Dom} g) \supset B(g(x), c't) \cap (g(x) - \operatorname{cone} M).$$

By the continuity of g, we find $r \in (0, \varepsilon)$ such that $\|g(x) - y\| < c\varepsilon$ for each $(x, y) \in (B[\overline{x}, r] \cap \operatorname{Dom} g) \times B[g(\overline{x}), r]$. Fix any $y \in B[g(\overline{x}), r]$ and any $x \in B[\overline{x}, r] \cap \operatorname{Dom} g$ with $0 \neq g(x) - y \in \operatorname{cone} M$. Let $t := T_M(y, g(x))/c = \|g(x) - y\|/c$. Then $t \in (0, \varepsilon)$ and $y \in B[g(x), ct] \subset B(g(x), c't)$. Therefore there is $x' \in B(x, t) \cap \operatorname{Dom} g$ such that $y = g(x')$ and $x' - x \in \operatorname{cone} L$. Noting that $x' \neq x$, because $t > 0$, we get

$$0 < c\, T_L(x, x') = c\,\|x' - x\| < ct = T_M(y, g(x)) = T_M(y, g(x)) - T_M(y, g(x')).$$

Hence $s \geq c > s$, a contradiction. □

As in the case of the classical regularity properties, the directional openness of a set-valued mapping $F : X \rightrightarrows Y$ can be deduced from the directional openness of a simple single-valued mapping, namely, the restriction of the canonical projection from $X \times Y$ onto Y, that is the assignment $\operatorname{Gr} F \ni (x, y) \longmapsto y \in Y$. In order to do this, we give a technical lemma, which shows the possibility to see the minimal time function on product spaces as maximum of corresponding minimal time functions defined on coordinate spaces.

Lemma 12 *Let $(X_1, \|\cdot\|), \ldots, (X_n, \|\cdot\|)$ be normed spaces and positive constants $\alpha_1, \ldots, \alpha_n$ be given. Consider nonempty closed subsets L_i of S_{X_i} for $i = 1, \ldots, n$. Define the equivalent norm $\|\cdot\|_{\widetilde{X}}$ on $\widetilde{X} := X_1 \times \ldots \times X_n$ for each $(u_1, \ldots, u_n) \in \tilde{X}$ by $\|(u_1, \ldots, u_n)\|_{\widetilde{X}} := \max\{\alpha_1\|u_1\|, \ldots, \alpha_n\|u_n\|\}$. Then there exists $\widetilde{L} \subset S_{\widetilde{X}}$ such that* $\operatorname{cone} \widetilde{L} = \operatorname{cone} L_1 \times \ldots \times \operatorname{cone} L_n$ *and, for each $(u_1, \ldots, u_n), (u'_1, \ldots, u'_n) \in \widetilde{X}$,*

$$T_{\widetilde{L}}((u_1, \ldots, u_n), (u'_1, \ldots, u'_n)) = \max\{\alpha_1 T_{L_1}(u_1, u'_1), \ldots, \alpha_n T_{L_n}(u_n, u'_n)\}. \tag{3.5}$$

Proof We will proceed inductively. On $X_1 \times X_2$, consider the norm defined, for each $(u, v) \in X_1 \times X_2$, by $\|(u, v)\|_{X_1 \times X_2} := \max\{\alpha_1\|u\|, \alpha_2\|v\|\}$. Take

$$\widetilde{L_1} := \left(\alpha_1^{-1} L_1 \times [\alpha_2^{-1}\mathbb{B}_{X_2} \cap \operatorname{cone} L_2]\right) \cup \left([\alpha_1^{-1}\mathbb{B}_{X_1} \cap \operatorname{cone} L_1] \times \alpha_2^{-1} L_2\right).$$

Clearly, $\widetilde{L_1} \subset S_{X_1 \times X_2}$. We show next that $\operatorname{cone} \widetilde{L_1} = \operatorname{cone} L_1 \times \operatorname{cone} L_2$. Indeed,

$$\begin{aligned} \operatorname{cone}\left(\alpha_1^{-1} L_1 \times [\alpha_2^{-1}\mathbb{B}_{X_2} \cap \operatorname{cone} L_2]\right) &\subset \operatorname{cone}\left(\alpha_1^{-1} L_1\right) \times \operatorname{cone}\left(\alpha_2^{-1}\mathbb{B}_{X_2} \cap \operatorname{cone} L_2\right) \\ &= \operatorname{cone} L_1 \times \operatorname{cone} L_2, \end{aligned}$$

and similarly

$$\begin{aligned} \operatorname{cone}\left([\alpha_1^{-1}\mathbb{B}_{X_1} \cap \operatorname{cone} L_1] \times \alpha_2^{-1} L_2\right) &\subset \operatorname{cone}\left(\alpha_1^{-1}\mathbb{B}_{X_1} \cap \operatorname{cone} L_1\right) \times \operatorname{cone}(\alpha_2^{-1} L_2) \\ &= \operatorname{cone} L_1 \times \operatorname{cone} L_2. \end{aligned}$$

Thus $\operatorname{cone} \widetilde{L_1} \subset \operatorname{cone} L_1 \times \operatorname{cone} L_2$.

On the other hand, pick an arbitrary non-zero $(u, v) \in \operatorname{cone} L_1 \times \operatorname{cone} L_2$. Then there are non-negative t_1 and t_2, which are not both zero, such that $u \in t_1(\alpha_1^{-1} L_1)$ and $v \in t_2(\alpha_2^{-1} L_2)$. If $t_1 \geq t_2$, then $v \in t_1(t_2 t_1^{-1})(\alpha_2^{-1} L_2) \subset t_1(\alpha_2^{-1}\mathbb{B}_{X_2} \cap \operatorname{cone} L_2)$. Hence $(u, v) \in t_1\widetilde{L_1} \subset \operatorname{cone} \widetilde{L_1}$. If $t_2 > t_1$, then $u \in t_2(t_1 t_2^{-1})(\alpha_1^{-1} L_1) \subset t_2(\alpha_1^{-1}\mathbb{B}_{X_1} \cap \operatorname{cone} L_1)$. Hence $(u, v) \in t_2\widetilde{L_1} \subset \operatorname{cone} \widetilde{L_1}$. Thus $\operatorname{cone} \widetilde{L_1} \supset \operatorname{cone} L_1 \times \operatorname{cone} L_2$.

We want to prove next that

$$T_{\widetilde{L_1}}((u,v),(u',v')) = \max\{\alpha_1 T_{L_1}(u,u'), \alpha_2 T_{L_2}(v,v')\}, \quad \text{for all } (u,v),(u',v') \in X_1 \times X_2.$$

For this, fix any $(u,v),(u',v') \in X_1 \times X_2$. Since cone $\widetilde{L_1}$ = cone $L_1 \times$ cone L_2, we conclude that $T_{\widetilde{L_1}}((u,v),(u',v'))$ is finite if and only if both $T_{L_1}(u,u')$ and $T_{L_2}(v,v')$ are finite. In this case, we have

$$\begin{aligned}\max\{\alpha_1 T_{L_1}(u,u'), \alpha_2 T_{L_2}(v,v')\} &= \max\{\alpha_1\|u'-u\|, \alpha_2\|v'-v\|\}\\ &= \|(u',v')-(u,v)\|_{X_1\times X_2} = T_{\widetilde{L_1}}((u,v),(u',v')),\end{aligned}$$

as claimed.

Next, observe that if one considers on $X_1 \times X_2 \times X_3$ the norm defined, for each $(u,v,w) \in X_1 \times X_2 \times X_3$, by $\|(u,v,w)\|_{X_1\times X_2\times X_3} := \max\left\{\|(u,v)\|_{X_1\times X_2}, \alpha_3\|w\|\right\}$, and repeats the previous proof for

$$\widetilde{L_2} := \left(\widetilde{L_1} \times [\alpha_3^{-1}\mathbb{B}_{X_3} \cap \operatorname{cone} L_3]\right) \cup \left([\mathbb{B}_{X_1\times X_2} \cap \operatorname{cone}\widetilde{L_1}] \times \alpha_3^{-1} L_3\right),$$

it follows that cone $\widetilde{L_2}$ = cone $\widetilde{L_1}$ × cone L_3 = cone L_1 × cone L_2 × cone L_3, and for all $(u,v,w),(u',v',w') \in X_1 \times X_2 \times X_3$,

$$\begin{aligned}T_{\widetilde{L_2}}((u,v,w),(u',v',w')) &= \max\{T_{\widetilde{L_1}}((u,v),(u',v')), \alpha_3 T_{L_3}(w,w')\}\\ &= \max\{\max\{\alpha_1 T_{L_1}(u,u'), \alpha_2 T_{L_2}(v,v')\}, \alpha_3 T_{L_3}(w,w')\}\\ &= \max\{\alpha_1 T_{L_1}(u,u'), \alpha_2 T_{L_2}(v,v'), \alpha_3 T_{L_3}(w,w')\}.\end{aligned}$$

After a finite number of similar steps, the result follows. □

Proposition 13 (**general criterion for set-valued maps**) *Let* $(X, \|\cdot\|)$ *and* $(Y, \|\cdot\|)$ *be Banach spaces. Consider nonempty closed subsets* L *of* S_X *and* M *of* S_Y *such that* cone L *is convex, a point* $(\bar{x}, \bar{y}) \in X \times Y$*, and a set-valued mapping* $F : X \rightrightarrows Y$ *the graph of which is locally closed near* $(\bar{x}, \bar{y}) \in \operatorname{Gr} F$. *Then* $\operatorname{dirsur}_{L\times M} F(\bar{x},\bar{y})$ *equals to the supremum of all* $c > 0$ *for which there are* $r > 0$ *and* $\alpha \in (0, 1/c)$ *such that for any* $(x, v) \in (B[\bar{x}, r] \times B[\bar{y}, r]) \cap \operatorname{Gr} F$ *and any* $y \in B[\bar{y}, r]$*, with* $0 < T_M(y, v) < +\infty$*, there is a pair* $(x', v') \in \operatorname{Gr} F$ *such that*

$$c\max\{T_L(x,x'), \alpha\|v-v'\|\} < T_M(y,v) - T_M(y,v'). \tag{3.6}$$

Proof Let $\lambda := \operatorname{dirsur}_{L\times M} F(\bar{x},\bar{y})$ and denote by s the supremum from the statement. First, we show that $\lambda \geq s$. Fix an arbitrary $c \in (0, s)$ (if there is any). Find $\alpha \in (0, 1/c)$ and $r > 0$ such that the property involving (3.6) holds. Let $\widetilde{X} := X \times Y$ be equipped with the norm $\|(u,v)\|_{\widetilde{X}} := \max\{\|u\|, \alpha\|v\|\}$, $(u,v) \in \widetilde{X}$. Let $\widetilde{L}$ be the cone from the conclusion of Lemma 12 with $n := 2$, $X_1 := X$, $X_2 := Y$, $L_1 := L$, $L_2 := S_Y$, $\alpha_1 := 1$, and $\alpha_2 := \alpha$.

Let $g := p_Y|_{\operatorname{Gr} F}$, where p_Y is the canonical projection from $X \times Y$ onto Y. Then there is a neighborhood $\widetilde{U}$ of $(\bar{x},\bar{y})$ in $\widetilde{X}$ such that the set $\widetilde{D} := \widetilde{U} \cap \operatorname{Gr} F$ is closed and g is continuous on $\widetilde{D}$. Let $\tilde{r} \in (0, r\min\{1,\alpha\})$. Fix any $(x,v) \in B_{\widetilde{X}}[(\bar{x},\bar{y}),\tilde{r}] \cap \operatorname{Dom} g = (B[\bar{x},\tilde{r}] \times B[\bar{y},\tilde{r}/\alpha]) \cap \operatorname{Gr} F \subset (B[\bar{x},r] \times B[\bar{y},r]) \cap \operatorname{Gr} F$ and any $y \in B[\bar{y},\tilde{r}]$ with $0 < T_M(y,v) < +\infty$. Find a pair $(x',v') \in \operatorname{Gr} F = \operatorname{Dom} g$ satisfying (3.6). This and (3.5) imply that

$$c\, T_{\widetilde{L}}((x,v),(x',v')) = c\,\max\{T_L(x,x'), \alpha T_{S_Y}(v,v')\} < T_M(y,v) - T_M(y,v'),$$

because $T_{S_Y}(v,v') = \|v - v'\|$. Proposition 11 says that $\operatorname{dirsur}_{\widetilde{L}\times M} g(\bar{x},\bar{y}) \geq s$ (note that $c \in (0,s)$ is arbitrary). Keeping the same c, since $\operatorname{dirsur}_{\widetilde{L}\times M} g(\bar{x},\bar{y}) \geq s > c$, by the definition of the modulus of directional openness, there is $\varepsilon > 0$ such that for each $(x, y) \in (B[\overline{x}, \varepsilon] \times B[\overline{y}, \varepsilon/\alpha]) \cap \operatorname{Gr} F$ and each $t \in (0, \varepsilon)$ we have

$$\begin{aligned} B(y, ct) \cap (y - \operatorname{cone} M) &\subset g\left(B_{\widetilde{X}}((x, y), t) \cap ((x, y) + \operatorname{cone} \widetilde{L})\right) \\ &= g\left(\operatorname{Gr} F \cap B_{\widetilde{X}}((x, y), t) \cap ((x, y) + \operatorname{cone} \widetilde{L})\right) \\ &= g\left(\operatorname{Gr} F \cap ([B(x, t) \cap (x + \operatorname{cone} L)] \times B(y, t/\alpha))\right), \end{aligned}$$

where we used that $\operatorname{cone} \widetilde{L} = \operatorname{cone} L \times \operatorname{cone} S_Y = \operatorname{cone} L \times Y$. Note that $B(y, ct) \subset B(y, t/\alpha)$. Pick an arbitrary $\varepsilon' \in (0, \varepsilon \min\{1, 1/\alpha\})$. Fix any $(x, y) \in (B[\overline{x}, \varepsilon'] \times B[\overline{y}, \varepsilon']) \cap \operatorname{Gr} F$ and any $t \in (0, \varepsilon')$. For any $w \in B(y, ct) \cap (y - \operatorname{cone} M)$, there is $u \in B(x, t) \cap [x + \operatorname{cone} L]$ and $w' \in B(y, t/\alpha) \cap F(u)$ such that $g(u, w') = w$, hence $w = w' \in F(u)$. Thus

$$B(y, ct) \cap (y - \operatorname{cone} M) \subset F(B(x, t) \cap [x + \operatorname{cone} L]).$$

Letting $c \uparrow s$, we get $\lambda \geq s$ as desired. Suppose that $\lambda > s$. Then there are $c' > s$ and $\varepsilon' > 0$ such that for all $(x, y) \in (B[\overline{x}, \varepsilon'] \times B[\overline{y}, \varepsilon']) \cap \operatorname{Gr} F$ and all $t \in (0, \varepsilon')$ we have

$$B(y, c't) \cap (y - \operatorname{cone} M) \subset F(B(x, t) \cap [x + \operatorname{cone} L]).$$

Let $\alpha := 1/c'$. Define $\widetilde{X}$, $\| \cdot \|_{\widetilde{X}}$, $\widetilde{L}$, and g as before. Then

$$\begin{aligned} B(y, c't) \cap (y - \operatorname{cone} M) &\subset g\left(\operatorname{Gr} F \cap ([B(x, t) \cap (x + \operatorname{cone} L)] \times B(y, t/\alpha))\right) \\ &= g\left(\operatorname{Gr} F \cap B_{\widetilde{X}}((x, y), t) \cap ((x, y) + \operatorname{cone} \widetilde{L})\right). \end{aligned}$$

Pick any $c \in (s, c')$. Then $\operatorname{dirsur}_{\widetilde{L} \times M} g(\overline{x}, \overline{y}) > c$. Proposition 11 implies that there is $r' > 0$ such that for any $(x, v) \in (B[\overline{x}, r'] \times B[\overline{y}, c'r']) \cap \operatorname{Gr} F$ and any $y \in B[\overline{y}, r']$ with $0 < T_M(y, v) < +\infty$ there is a pair $(x', v') \in \operatorname{Dom} g = \operatorname{Gr} F$ such that

$$c\, T_{\widetilde{L}}((x, v), (x', v')) < T_M(y, v) - T_M(y, v').$$

As $T_{S_Y}(v, v') = \|v - v'\|$, using (3.5), we get (3.6). Picking $r \in (0, \min\{r', c'r'\})$ and noting that $\alpha = 1/c' < 1/c$, we conclude that $s \geq c$, a contradiction. □

4 Directional Openness Stability

In what follows, we speak about the local stability at composition of a pair of multifunctions, which essentially says that a point from the graph of the composed multifunction, close to the reference one, can be written by the use of points from the graphs of the involved set-valued maps, which are also close to the corresponding reference ones. For more details on this notion, as well as for links to the preservation of Aubin-type properties of set-valued mappings, and also for the involvement in the stability under compositions, see [6, 9]. Given metric spaces (X, ϱ), (Y, ϱ), and (Z, ϱ), a composition of set-valued mappings $F : X \rightrightarrows Y$ and $G : Y \rightrightarrows Z$ is the mapping $G \circ F : X \rightrightarrows Z$ defined by

$$(G \circ F)(x) := \bigcup_{y \in F(x)} G(y), \quad x \in X;$$

and a product of set-valued mappings $F_1 : X \rightrightarrows Y$ and $F_2 : X \rightrightarrows Z$ is the mapping $(F_1, F_2) : X \rightrightarrows Y \times Z$ defined by

$$(F_1, F_2)(x) := F_1(x) \times F_2(x), \quad x \in X.$$

Definition 14 Let (X, ϱ), (Y, ϱ), and (Z, ϱ) be metric spaces and $(\overline{x}, \overline{y}, \overline{z}) \in X \times Y \times Z$ be fixed. Consider set-valued mappings $F : X \rightrightarrows Y$ and $G : Y \rightrightarrows Z$ such that $\overline{y} \in F(\overline{x})$

and $\overline{z} \in G(\overline{y})$. We say that the pair F, G is *composition-stable around* $(\overline{x}, \overline{y}, \overline{z})$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for every $x \in B(\overline{x}, \delta)$ and every $z \in (G \circ F)(x) \cap B(\overline{z}, \delta)$, there exists $y \in F(x) \cap B(\overline{y}, \varepsilon)$ such that $z \in G(y)$.

Remark 15 It is important to note that the composition stability, coupled with a weaker directional openness property, implies genuine directional linear openness (cf. [6, Proposition 3.6]).

To see this, suppose that F and G are as in the previous definition with X, Y, and Z being normed vector spaces, and $L \subset S_X$ and $M \subset S_Z$ are nonempty sets for which there are positive constants ε, r, and c such that, for every $(x, y, z) \in B(\overline{x}, r) \times B(\overline{y}, r) \times B(\overline{z}, r)$ with $y \in F(x)$ and $z \in G(y)$, and for every $t \in (0, \varepsilon)$,

$$B(z, ct) \cap [z - \operatorname{cone} M] \subset (G \circ F)(B(x, t) \cap [x + \operatorname{cone} L]). \tag{4.1}$$

Since F, G are composition-stable around $(\overline{x}, \overline{y}, \overline{z})$, one gets the existence of $\delta \in (0, r)$ such that, for every $x \in B(\overline{x}, \delta)$ and every $z \in (G \circ F)(x) \cap B(\overline{z}, \delta)$, there exists $y \in F(x) \cap B(\overline{y}, r)$ with $z \in G(y)$.

Take arbitrary $(x, z) \in \operatorname{Gr}(G \circ F) \cap (B(\overline{x}, \delta) \times B(\overline{z}, \delta))$ and $t \in (0, \varepsilon)$. By the composition stability of F, G, there is $y \in F(x) \cap B(\overline{y}, r)$ such that $z \in G(y)$, hence (4.1) holds. This means that, indeed, $G \circ F$ is directionally linearly open with respect to L and M around $(\overline{x}, \overline{z})$.

We present next the main result of this section, which asserts the stability of directional regularity under composition. Note that, in what follows, we write $-A \times B$ for $(-A) \times B$.

Theorem 16 *Let* $(X, \|\cdot\|)$, $(Y, \|\cdot\|)$, $(Z, \|\cdot\|)$, *and* $(W, \|\cdot\|)$ *be Banach spaces and* $(\overline{x}, \overline{y}, \overline{z}, \overline{w}) \in X \times Y \times Z \times W$ *be fixed. Consider nonempty closed subsets L of S_X, M of S_Y, N of S_Z, and P of S_W such that* cone L, cone M, cone N, *and* cone P *are convex, set-valued mappings* $F_1 : X \rightrightarrows Y$, $F_2 : X \rightrightarrows Z$, *and* $G : Y \times Z \rightrightarrows W$ *such that F_1 has a locally closed graph near* $(\overline{x}, \overline{y}) \in \operatorname{Gr} F_1$, *$F_2$ has a locally closed graph near* $(\overline{x}, \overline{z}) \in \operatorname{Gr} F_2$, *and G has a locally closed graph near* $(\overline{y}, \overline{z}, \overline{w}) \in \operatorname{Gr} G$. *Define the mapping* $\mathcal{E}_{G,(F_1,F_2)} : X \times Y \times Z \rightrightarrows W$ *by*

$$\mathcal{E}_{G,(F_1,F_2)}(x, y, z) := \begin{cases} G(y, z), & \textit{if } (y, z) \in (F_1, F_2)(x), \\ \emptyset, & \textit{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \operatorname{dirsur}_{L\times -M\times N\times P} \mathcal{E}_{G,(F_1,F_2)}(\overline{x}, \overline{y}, \overline{z}, \overline{w}) \geq\ & \operatorname{dirsur}_{L\times M} F_1(\overline{x}, \overline{y}) \cdot \widehat{\operatorname{dirsur}}^{\,y}_{-M\times P} G(\overline{y}, \overline{z}, \overline{w}) \\ & - \operatorname{dirlip}_{-L\times N} F_2(\overline{x}, \overline{z}) \cdot \widehat{\operatorname{dirlip}}^{\,z}_{-N\times P} G(\overline{y}, \overline{z}, \overline{w}). \end{aligned} \tag{4.2}$$

If, in addition, the pair (F_1, F_2), G *is composition-stable around* $(\overline{x}, (\overline{y}, \overline{z}), \overline{w})$, *then*

$$\begin{aligned} \operatorname{dirsur}_{L\times P} (G \circ (F_1, F_2))(\overline{x}, \overline{w}) \geq\ & \operatorname{dirsur}_{L\times M} F_1(\overline{x}, \overline{y}) \cdot \widehat{\operatorname{dirsur}}^{\,y}_{-M\times P} G(\overline{y}, \overline{z}, \overline{w}) \\ & - \operatorname{dirlip}_{-L\times N} F_2(\overline{x}, \overline{z}) \cdot \widehat{\operatorname{dirlip}}^{\,z}_{-N\times P} G(\overline{y}, \overline{z}, \overline{w}). \end{aligned} \tag{4.3}$$

Proof If the quantity from the right-hand side of the inequality (4.2) is negative, then there is nothing to prove. If it is positive, then there are positive constants α, β, β', γ, and δ such that

$$\operatorname{dirsur}_{L\times M} F_1(\overline{x},\overline{y}) > \alpha, \quad \widehat{\operatorname{dirsur}}^{y}_{-M\times P} G(\overline{y},\overline{z},\overline{w}) > \gamma, \tag{4.4}$$

and

$$\beta > \beta' > \operatorname{dirlip}_{-L\times N} F_2(\overline{x},\overline{z}), \quad \delta > \widehat{\operatorname{dirlip}}^{z}_{-N\times P} G(\overline{y},\overline{z},\overline{w}) \tag{4.5}$$

and also $c := \alpha\gamma - \beta\delta > 0$.

Then, thanks to (4.4), there exists $\varepsilon > 0$ such that, for each $t \in (0,\varepsilon)$, and each $(x,y) \in (B[\overline{x},\varepsilon] \times B[\overline{y},\varepsilon]) \cap \operatorname{Gr} F_1$, we have

$$B[y,\alpha t] \cap (y - \operatorname{cone} M) \subset F_1(B[x,t] \cap (x + \operatorname{cone} L)) \tag{4.6}$$

and, for each $t \in (0,\varepsilon)$, each $z \in B[\overline{z},\varepsilon]$, and each $(y,w) \in (B[\overline{y},\varepsilon] \times B[\overline{w},\varepsilon]) \cap \operatorname{Gr} G_z$, we have

$$B[w,\gamma t] \cap (w - \operatorname{cone} P) \subset G_z(B[y,t] \cap (y - \operatorname{cone} M)), \tag{4.7}$$

and also, thanks to (4.5), that

$$e_N(F_2(x) \cap B[\overline{z},\varepsilon], F_2(x')) \le \beta' T_{-L}(x',x) = \beta' T_L(x,x'), \quad \text{for all} \quad x,x' \in B[\overline{x},\varepsilon], \tag{4.8}$$

and, similarly,

$$e_P(G_y(z) \cap B[\overline{w},2\varepsilon], G_y(z')) \le \delta T_N(z,z'), \quad \text{whenever} \quad z,z' \in B[\overline{z},\varepsilon] \text{ and } y \in B[\overline{y},\varepsilon]. \tag{4.9}$$

Observe that, since $\operatorname{Gr} F_1$, $\operatorname{Gr} F_2$ and $\operatorname{Gr} G$ are closed near the corresponding reference points, $\operatorname{Gr} \mathcal{E}_{G,(F_1,F_2)}$ is closed near $(\overline{x},\overline{y},\overline{z},\overline{w})$. Let $r := \min\left\{(2\alpha)^{-1}\varepsilon, (2\beta)^{-1}\varepsilon, 2^{-1}\varepsilon\right\}$ and $\lambda := (\alpha\gamma + \beta\delta)^{-1} \in (0, 1/c)$. Define the norm on $X \times Y \times Z$ by $\|(x,y,z)\| := \max\left\{\|x\|, \alpha^{-1}\|y\|, \beta^{-1}\|z\|\right\}$, $(x,y,z) \in X \times Y \times Z$. Use Lemma 12 to find a cone $\widetilde{L} \subset S_{X\times Y\times Z}$ such that $\operatorname{cone}\widetilde{L} = \operatorname{cone} L \times \operatorname{cone}(-M) \times \operatorname{cone} N$ and, for each (x,y,z), $(x',y',z') \in X \times Y \times Z$,

$$T_{\widetilde{L}}((x,y,z),(x',y',z')) = \max\left\{T_L(x,x'), \alpha^{-1}T_{-M}(y,y'), \beta^{-1}T_N(z,z')\right\}.$$

Fix an arbitrary $(x,y,z,w) \in (B[\overline{x},r] \times B[\overline{y},r] \times B[\overline{z},r] \times B[\overline{w},r]) \cap \operatorname{Gr} \mathcal{E}_{G,(F_1,F_2)}$ and $u \in B[\overline{w},r]$ such that $0 \ne w - u \in \operatorname{cone} P$.

In order to apply Proposition 13, we must find a point $(x',y',z',w') \in X \times Y \times Z \times W$ such that

$$c \max\left\{T_{\widetilde{L}}((x,y,z),(x',y',z')), \lambda\|w - w'\|\right\} < T_P(u,w) - T_P(u,w').$$

To do so, take $t \in \left(0, \min\left\{r, \alpha^{-1}\gamma^{-1}\|u - w\|\right\}\right)$, and define

$$h := \alpha\gamma t \cdot \frac{u - w}{\|u - w\|} \ne 0.$$

Then $w + h \in B[w,\alpha\gamma t] \cap (w - \operatorname{cone} P)$ and, moreover, $\alpha t \in (0,\varepsilon)$, $z \in B[\overline{z},\varepsilon]$, and $(y,w) \in (B[\overline{y},\varepsilon] \times B[\overline{w},\varepsilon]) \cap \operatorname{Gr} G_z$, which means by (4.7), with αt instead of t, that there exists $y' \in B[y,\alpha t] \cap (y - \operatorname{cone} M)$ such that $w + h \in G(y',z)$. Since $t \in (0,\varepsilon)$, and $(x,y) \in (B[\overline{x},\varepsilon] \times B[\overline{y},\varepsilon]) \cap \operatorname{Gr} F_1$, it follows by (4.6) that there is $x' \in B[x,t] \cap (x + \operatorname{cone} L)$ such that $y' \in F_1(x')$. Observing that

$$\|x' - \overline{x}\| \le \|x' - x\| + \|x - \overline{x}\| \le t + 2^{-1}\varepsilon < \varepsilon,$$

one has by (4.8) that

$$T_N(z, F_2(x')) \le e_N(F_2(x) \cap B[\overline{z},\varepsilon], F_2(x')) \le \beta' T_L(x,x') = \beta'\|x - x'\| \le \beta' t < \beta t,$$

hence there exists $z' \in F_2(x') \cap (z + \operatorname{cone} N)$ such that $\|z' - z\| < \beta t$. Then

$$\|z' - \overline{z}\| \le \|z' - z\| + \|z - \overline{z}\| < \beta t + 2^{-1}\varepsilon \; < \beta r + 2^{-1}\varepsilon \le \varepsilon,$$

and

$$\|y' - \overline{y}\| \le \|y' - y\| + \|y - \overline{y}\| \le \alpha t + 2^{-1}\varepsilon < \alpha r + 2^{-1}\varepsilon \le \varepsilon,$$

and also

$$\|w + h - \overline{w}\| \le \|h\| + \|w - \overline{w}\| < \|u - w\| + \|w - \overline{w}\| \le \|u - \overline{w}\| + 2\|w - \overline{w}\| \le 3r < 2\varepsilon.$$

It follows by (4.9), with y' instead of y, that

$$T_P(w + h, G_{y'}(z')) \le e_P(G_{y'}(z) \cap B[\overline{w}, 2\varepsilon], G_{y'}(z')) \le \delta\, T_N(z, z') = \delta\, \|z - z'\| < \beta\delta t,$$

hence there exists $w' \in G(y', z') \cap (w + h + \operatorname{cone} P)$ such that $\|w + h - w'\| < \beta\delta t$. Moreover, since $(y', z') \in (F_1, F_2)(x')$, we have $(x', y', z', w') \in \operatorname{Gr} \mathcal{E}_{G,(F_1,F_2)}$.

Observe that, due to the choice of t,

$$w + h - u = \left(1 - \frac{\alpha\gamma t}{\|u - w\|}\right)(w - u) \in \operatorname{cone} P.$$

Then

$$\begin{aligned} T_P(u, w') &\le T_P(u, w + h) + T_P(w + h, w') < \|w + h - u\| + \beta\delta t \\ &= \|u - w\| - \alpha\gamma t + \beta\delta t = T_P(u, w) - ct. \end{aligned} \tag{4.10}$$

Moreover,

$$T_L(x, x') \le t, \quad T_{-M}(y, y') \le \alpha t, \quad T_N(z, z') < \beta t,$$

hence

$$T_{\widetilde{L}}((x, y, z), (x', y', z')) \le t.$$

Also,

$$\|w - w'\| \le \|h\| + \|w + h - w'\| < \alpha\gamma t + \beta\delta t = \lambda^{-1} t.$$

But the final relations, combined with (4.10), mean that

$$c \max\left\{T_{\widetilde{L}}((x, y, z), (x', y', z')), \lambda\, \|w - w'\|\right\} \le ct < T_P(u, w) - T_P(u, w').$$

Proposition 13 shows that

$$\operatorname{dirsur}_{L\times -M\times N\times P} \mathcal{E}_{G,(F_1,F_2)}(\overline{x}, \overline{y}, \overline{z}, \overline{w}) \ge c.$$

Therefore letting $\alpha \uparrow \operatorname{dirsur}_{L\times M} F_1(\overline{x}, \overline{y})$, $\gamma \uparrow \widehat{\operatorname{dirsur}}^{y}_{-M\times P} G(\overline{y}, \overline{z}, \overline{w})$, $\beta \downarrow \operatorname{dirlip}_{-L\times N} F_2(\overline{x}, \overline{z})$, and also $\delta \downarrow \widehat{\operatorname{dirlip}}^{z}_{-N\times P} G(\overline{y}, \overline{z}, \overline{w})$, we get (4.2).

For the second part of the conclusion, assume that $X \times Y \times Z \times W$ is equipped with the product (box) topology. Denote

$$\lambda := \operatorname{dirsur}_{L\times M\times N\times P} \mathcal{E}_{G,(F_1,F_2)}(\overline{x}, \overline{y}, \overline{z}, \overline{w}).$$

If $\lambda = 0$,we have that the right-hand side of (4.2) is nonpositive, hence (4.3) is trivial. Suppose that $\lambda > 0$, and fix any $c \in (0, \lambda)$. Then there exists $r > 0$ such that, for every $t \in (0, r)$, and every $(x, y, z, w) \in \operatorname{Gr} \mathcal{E}_{G,(F_1,F_2)} \cap (B[\overline{x}, r] \times B[\overline{y}, r] \times B[\overline{z}, r] \times B[\overline{w}, r])$, one has

$$B(w, ct) \cap [w - \operatorname{cone} P] \subset \mathcal{E}_{G,(F_1,F_2)}\left(B((x, y, z), t) \cap ((x, y, z) + \operatorname{cone}(L \times M \times N))\right). \tag{4.11}$$

Since $(F_1, F_2), G$ is composition-stable around $(\overline{x}, (\overline{y}, \overline{z}), \overline{w})$, there exists $\rho \in (0, r)$ such that, for every $x \in B(\overline{x}, \rho)$ and every $w \in (G \circ (F_1, F_2))(x) \cap B(\overline{w}, \rho)$, there exists $(y, z) \in (F_1, F_2)(x) \cap (B(\overline{y}, r) \times B(\overline{z}, r))$ such that $w \in G(y, z)$.

Take now arbitrary $(x, w) \in \mathrm{Gr}(G \circ (F_1, F_2)) \cap (B(\overline{x}, \rho) \times B(\overline{w}, \rho))$ and arbitrary $t \in (0, r)$. Then there is $(y, z) \in (F_1, F_2)(x) \cap (B(\overline{y}, r) \times B(\overline{z}, r))$ such that $w \in G(y, z)$. By inclusion (4.11), for every $w' \in B(w, ct) \cap [w - \mathrm{cone}\, P]$, there is

$$\begin{aligned}(x', y', z') &\in B((x, y, z), t) \cap ((x, y, z) + \mathrm{cone}(L \times M \times N)) \\ &\subset (B(x, t) \cap (x + \mathrm{cone}\, L)) \times (B(y, t) \cap (y + \mathrm{cone}\, M)) \\ &\quad \times (B(z, t) \cap (z + \mathrm{cone}\, N))\end{aligned}$$

such that $(y', z') \in (F_1, F_2)(x')$ and $w' \in G(y', z')$, that is, $w' \in (G \circ (F_1, F_2))(x')$. Consequently,

$$B(w, ct) \cap [w - \mathrm{cone}\, P] \subset (G \circ (F_1, F_2))(B(x, t) \cap (x + \mathrm{cone}\, L)).$$

As $(x, w) \in \mathrm{Gr}(G \circ (F_1, F_2)) \cap (B(\overline{x}, \rho) \times B(\overline{w}, \rho))$ and $t \in (0, r)$ were arbitrary, we conclude that $\mathrm{dirsur}_{L \times P}\, (G \circ (F_1, F_2))\, (\overline{x}, \overline{w}) \geq c$. Letting $c \uparrow \lambda$, we finish the proof. □

As a consequence of the result above, we obtain a stability of directional openness under summation, which represents, in fact, a directional Lyusternik-Graves type assertion. As in the previous case, to get genuine openness, we need to impose some local sum-stability property. For more details about the origin of this notion and its links to local composition stability, see [9].

Definition 17 Let (X, ϱ) and (Y, ϱ) be metric spaces and $(\overline{x}, \overline{y}, \overline{z}) \in X \times Y \times Y$ be fixed. Consider set-valued mappings $F : X \rightrightarrows Y$ and $G : X \rightrightarrows Y$ such that $\overline{y} \in F(\overline{x})$ and $\overline{z} \in G(\overline{x})$. We say that the pair F, G is *sum-stable around* $(\overline{x}, \overline{y}, \overline{z})$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for every $x \in B(\overline{x}, \delta)$ and every $w \in (F + G)(x) \cap B(\overline{y} + \overline{z}, \delta)$, there exist $y \in F(x) \cap B(\overline{y}, \varepsilon)$ and $z \in G(x) \cap B(\overline{z}, \varepsilon)$ such that $w = y + z$.

Remark 18 Observe that, if one takes in Definition 14 $F : X \rightrightarrows Y \times Y$, $F := (F_1, F_2)$, where $F_1 : X \rightrightarrows Y$, $F_2 : X \rightrightarrows Y$ are two multifunctions, $G := g$, where $g : Y \times Y \to Y$ is given by $g(y, z) := y + z$, for each $(y, z) \in Y \times Y$, and $(\overline{x}, \overline{y}, \overline{z}) \in X \times Y \times Y$ such that $(\overline{y}, \overline{z}) \in F_1(\overline{x}) \times F_2(\overline{x})$, then the composition-stability of the pair F, G around $(\overline{x}, (\overline{y}, \overline{z}), \overline{y} + \overline{z})$ is just the sum-stability of F_1, F_2 around $(\overline{x}, \overline{y}, \overline{z})$.

Corollary 19 *Let* $(X, \|\cdot\|)$ *and* $(Y, \|\cdot\|)$ *be Banach spaces and* $(\overline{x}, \overline{y}, \overline{z}) \in X \times Y \times Y$ *be fixed. Consider nonempty closed subsets* L *of* S_X *and* M *of* S_Y *such that* cone L *and* cone M *are convex, set-valued mappings* $F_1, F_2 : X \rightrightarrows Y$ *such that* F_1 *has a locally closed graph near* $(\overline{x}, \overline{y}) \in \mathrm{Gr}\, F_1$ *and* F_2 *has a locally closed graph near* $(\overline{x}, \overline{z}) \in \mathrm{Gr}\, F_2$. *Define the mapping* $\mathcal{E}_{F_1, F_2} : X \times Y \times Y \rightrightarrows Y$ *by*

$$\mathcal{E}_{F_1, F_2}(x, y, z) := \begin{cases} y + z, & \text{if } (y, z) \in (F_1, F_2)(x), \\ \emptyset, & \text{otherwise.} \end{cases}$$

Then

$$\mathrm{dirsur}_{L \times -M \times M \times M}\, \mathcal{E}_{F_1, F_2}(\overline{x}, \overline{y}, \overline{z}, \overline{y} + \overline{z}) \geq \mathrm{dirsur}_{L \times M}\, F_1(\overline{x}, \overline{y}) - \mathrm{dirlip}_{-L \times M}\, F_2(\overline{x}, \overline{z}). \tag{4.12}$$

If, in addition, the pair F_1, F_2 *is sum-stable around* $(\overline{x}, \overline{y}, \overline{z})$*, then*

$$\mathrm{dirsur}_{L \times M}(F_1 + F_2)(\overline{x}, \overline{y} + \overline{z}) \geq \mathrm{dirsur}_{L \times M}\, F_1(\overline{x}, \overline{y}) - \mathrm{dirlip}_{-L \times M}\, F_2(\overline{x}, \overline{z}). \tag{4.13}$$

Proof Take in Theorem 16 $W = Z = Y$, $\overline{w} := \overline{y} + \overline{z}$, $P := N := M$, and $G := g$, where $g : Y \times Y \to Y$ is given by $g(y, z) := y + z$, for all $(y, z) \in Y \times Y$.

Observe that G is directionally linearly open with respect to y uniformly in z around $((\overline{y}, \overline{z}), \overline{w}) \in \operatorname{Gr} G$ with respect to $-M$ and M with modulus 1, and also that G is directionally Aubin continuous with respect to z uniformly in y around $((\overline{y}, \overline{z}), \overline{w})$ with respect to $-M$ and M with modulus 1.

Hence,

$$\widehat{\mathrm{dirsur}}^{y}_{-M\times M} G(\overline{y}, \overline{z}, \overline{w}) = 1 \quad \text{and} \quad \widehat{\mathrm{dirlip}}^{z}_{-M\times M} G(\overline{y}, \overline{z}, \overline{w}) = 1.$$

Moreover, $\mathcal{E}_{F_1,F_2} = \mathcal{E}_{G,(F_1,F_2)}$. Consequently, we get (4.12).

The second part of the conclusion also follows from Theorem 16 by Remark 18. □

Corollary 20 *Let* $(X, \|\cdot\|)$ *and* $(Y, \|\cdot\|)$ *be Banach spaces and* $(\overline{x}, \overline{y}) \in X \times Y$ *be fixed. Consider nonempty closed subsets* L *of* S_X *and* M *of* S_Y *such that* cone L *and* cone M *are convex, a set-valued mapping* $F : X \rightrightarrows Y$ *the graph of which is locally closed near* $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$, *and a single-valued mapping* $f : X \to Y$ *which is continuous at* $\overline{x}$. *Then*

$$\mathrm{dirsur}_{L\times M}(f + F)(\overline{x}, f(\overline{x}) + \overline{y}) \geq \mathrm{dirsur}_{L\times M} F(\overline{x}, \overline{y}) - \mathrm{dirlip}_{-L\times M} f(\overline{x}).$$

Proof Since $f : X \to Y$ is continuous at $\overline{x}$, it follows that the pair F, f is sum-stable around $(\overline{x}, \overline{y}, f(\overline{x}))$. Since all the assumptions in Corollary 19 are satisfied, the conclusion follows. □

5 Primal Conditions for Directional Regularity

In this section, we employ a directional version of the Bouligand (graphical) derivative to obtain sufficient conditions for the directional regularity, the idea coined by J.-P. Aubin in [1]. Let us start with a directional version of the Bouligand-Severi tangent cone to a set.

Definition 21 Let Ω be a nonempty subset of a normed space $(X, \|\cdot\|)$, $M \subset S_X$ be a nonempty set, and $\overline{x} \in \Omega$. The *Bouligand-Severi tangent cone to* Ω *at* $\overline{x}$ *with respect to* M is the set

$$T(\Omega, \overline{x}, M) = \left\{ u \in X \mid \liminf_{t\downarrow 0}\, t^{-1} T_M(\overline{x} + tu, \Omega) = 0 \right\}. \tag{5.1}$$

Observe that $T(\Omega, \overline{x}, M)$ is a cone and contains all points $u \in X$ such that there are sequences (u_n) in $u + \operatorname{cone} M$ converging to u and (t_n) in $(0, +\infty)$ converging to 0 such that, for each $n \in \mathbb{N}$, we have $\overline{x} + t_n u_n \in \Omega$. Symbolically,

$$T(\Omega, \overline{x}, M) = \{u \in X \mid \exists (t_n) \downarrow 0,\ \exists (u_n) \subset u + \operatorname{cone} M,\ (u_n) \to u,\ \forall n \in \mathbb{N},\ \overline{x} + t_n u_n \in \Omega\}.$$

Moreover, note that this notion is naturally obtained by replacing the distance function in the usual definition of Bouligand-Severi tangent cone (denoted $T(\Omega, \overline{x})$) by the minimal time function. Clearly, $T(\Omega, \overline{x}, M) \subset T(\Omega, \overline{x})$, and if $M = S_X$, the equality holds. However, if $M \neq S_X$, the Bouligand-Severi tangent cone with respect to M does not enjoy the usual properties of the classical contingent cone, as the next example shows.

Example 22 Take $X = \mathbb{R}^2$, $\Omega = \left\{(a, b) \in \mathbb{R}^2 \mid (a-1)^2 + b^2 \leq 1\right\}$ and $\overline{x} = (0, 0)$. Now, for $M = [0, +\infty)^2 \cap S_X$, we have that $T(\Omega, \overline{x}, M) = T(\Omega, \overline{x}) = [0, +\infty) \times \mathbb{R}$. On the other hand, for $M = (-\infty, 0]^2 \cap S_X$, we have that $T(\Omega, \overline{x}, M) = (0, +\infty) \times \mathbb{R} \subset T(\Omega, \overline{x}) =$

$[0, +\infty) \times \mathbb{R}$. In particular, this shows that $T(\Omega, \overline{x}, M)$ may not be closed. Furthermore, if Ω is convex, it is easy to see that, for any M,

$$\operatorname{cone}(\Omega - \overline{x}) \subset T(\Omega, \overline{x}, M) \subset \operatorname{cl}\operatorname{cone}(\Omega - \overline{x}) = T(\Omega, \overline{x}). \tag{5.2}$$

The first choice of M from above shows that the first inclusion in (5.2) can be strict, while the second choice shows that the second inclusion in (5.2) does not hold as equality, in general.

Similar to the classical case, one can introduce the directional adjacent cone, as follows.

Definition 23 Let Ω be a nonempty subset of a normed space $(X, \|\cdot\|)$, $M \subset S_X$ be a nonempty set, and $\overline{x} \in \Omega$. The *adjacent cone to* Ω *at* $\overline{x}$ *with respect to* M is the set

$$T^\flat(\Omega, \overline{x}, M) = \left\{u \in X \mid \lim_{t \downarrow 0} t^{-1} T_M(\overline{x} + tu, \Omega) = 0\right\}, \tag{5.3}$$

that is,

$$T^\flat(\Omega, \overline{x}, M) = \{u \in X \mid \forall (t_n) \downarrow 0, \exists (u_n) \subset u + \operatorname{cone} M,\ (u_n) \to u,\ \forall n \in \mathbb{N},\ \overline{x} + t_n u_n \in \Omega\}.$$

It is clear that, in general,

$$T^\flat(\Omega, \overline{x}, M) \subset T(\Omega, \overline{x}, M).$$

Definition 24 Let $F : X \rightrightarrows Y$ be a set-valued mapping between normed spaces $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ with $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$, $L \subset S_X$ and $M \subset S_Y$ be nonempty sets.

(i) The *Bouligand derivative of* F *at* $(\overline{x}, \overline{y})$ *with respect to* L *and* M is the set-valued mapping $D_{L,M}F(\overline{x}, \overline{y})$ from X into Y defined, for each $u \in X$, by

$$D_{L,M}F(\overline{x}, \overline{y})(u) = \{v \in Y \mid \exists (t_n) \downarrow 0,\ \exists (u_n) \subset u + \operatorname{cone} L,\ (u_n) \to u,\ \exists (v_n) \subset v + \operatorname{cone} M,\ (v_n) \to v, \forall n \in \mathbb{N},\ \overline{y} + t_n v_n \in F(\overline{x} + t_n u_n)\}.$$

(ii) The *adjacent derivative of* F *at* $(\overline{x}, \overline{y})$ *with respect to* L *and* M is the set-valued mapping denoted $D^\flat_{L,M}F(\overline{x}, \overline{y})$ from X into Y defined, for each $u \in X$, by

$$D^\flat_{L,M}F(\overline{x}, \overline{y})(u) = \{v \in Y \mid \forall (t_n) \downarrow 0,\ \exists (u_n) \subset u + \operatorname{cone} L,\ (u_n) \to u,\ \exists (v_n) \subset v + \operatorname{cone} M,\ (v_n) \to v, \forall n \in \mathbb{N},\ \overline{y} + t_n v_n \in F(\overline{x} + t_n u_n)\}.$$

(iii) One says that F is *directionally proto-differentiable with respect to* $L \times M$ *at* $\overline{x}$ *relative to* $\overline{y}$ if $D_{L,M}F(\overline{x}, \overline{y}) = D^\flat_{L,M}F(\overline{x}, \overline{y})$.

Observe that if $\operatorname{cone} \widetilde{L} = \operatorname{cone} L \times \operatorname{cone} M$, with an appropriate choice of $\widetilde{L} \subset S_{X \times Y}$, then

$$\begin{aligned} \operatorname{Gr} D_{L,M}F(\overline{x}, \overline{y}) &= T(\operatorname{Gr} F, (\overline{x}, \overline{y}), \widetilde{L}) \text{ and} \\ \operatorname{Gr} D^\flat_{L,M}F(\overline{x}, \overline{y}) &= T^\flat(\operatorname{Gr} F, (\overline{x}, \overline{y}), \widetilde{L}). \end{aligned}$$

The following statement is a directional version of [2, Theorem 3.2].

Theorem 25 *Let* $(X, \|\cdot\|)$ *and* $(Y, \|\cdot\|)$ *be Banach spaces. Consider nonempty closed subsets* L *of* S_X *and* M *of* S_Y *such that* $\operatorname{cone} L$ *and* $\operatorname{cone} M$ *are convex and a mapping* $F : X \rightrightarrows Y$ *the graph of which is locally closed near* $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$. *Assume that there are* *positive constants* β, ϱ, *and* r *such that for every* $(x, v) \in (B[\bar{x}, r] \times B[\bar{y}, r]) \cap \operatorname{Gr} F$ *we have*

$$D_{L,M}F(x, v)(\mathbb{B}_X \cap \operatorname{cone} L) + B[0, \beta] \cap (-\operatorname{cone} M) \supset -(\beta + \varrho)M.$$

Then $\operatorname{dirsur}_{L\times M} F(\bar{x}, \bar{y}) \geq \varrho$.

Proof Fix any $c \in (0, \varrho)$. Pick $\gamma > 1$ such that $c\gamma < \varrho$. Fix any $(x, v) \in \operatorname{Gr} F \cap (B[\bar{x}, r] \times B[\bar{y}, r])$ and any $y \in Y$ with $0 \neq v - y \in \operatorname{cone} M$. Let

$$z := (\beta + \varrho)\frac{y - v}{\|y - v\|} \in (\beta + \varrho)(-M).$$

By the assumption, there is a pair $(h, w) \in (\mathbb{B}_X \cap \operatorname{cone} L) \times Y$ such that $w \in D_{L,M}F(x, v)(h)$ with $\|w - z\| \leq \beta$ and $w - z \in \operatorname{cone} M$. Hence $\|w\| \leq 2\beta + \varrho$. The definition of $D_{L,M}F(x, v)$ yields a triple $(t, h', w') \in (0, +\infty) \times X \times Y$ such that $v + tw' \in F(x + th')$ with $(h' - h, w' - w) \in \operatorname{cone} L \times \operatorname{cone} M$ and satisfying

$$(\beta+\varrho)t < \|y-v\|, \quad \|w-w'\| < \varrho-c\gamma, \quad \|w'\| < \gamma(2\beta+\varrho), \quad \text{and} \quad \|h'\| < \gamma. \tag{5.4}$$

Let $x' := x + th'$ and $v' := v + tw'$. Then $(x', v') \in \operatorname{Gr} F$. As $v - y \in \operatorname{cone} M$, the first inequality in (5.4) implies that we have

$$v + tz - y = (1 - (\beta + \varrho)t/\|y - v\|)\,(v - y) \in \operatorname{cone} M,$$

which means that $T_M(y, v + tz) = \|y - v - tz\| = \|y - v\| - t(\beta + \varrho)$. Since $w - z$ and $w' - w$ are in $\operatorname{cone} M$, we have $v' - (v + tz) = tw' - tz = t(w' - w) + t(w - z) \in \operatorname{cone} M$. So, by the second inequality in (5.4), we get

$$T_M(v + tz, v') \leq t(\|w' - w\| + \|w - z\|) < t(\varrho - c\gamma + \beta).$$

Remembering that $v - y \in \operatorname{cone} M$, we conclude that

$$\begin{aligned} T_M(y, v') &\leq T_M(y, v + tz) + T_M(v + tz, v') < \|y - v\| - t(\beta + \varrho) + t(\varrho - c\gamma + \beta) \\ &= T_M(y, v) - c\,(t\gamma). \end{aligned}$$

The last two inequalities in (5.4) reveal that $\|v'-v\| = t\|w'\| < t\gamma(2\beta+\varrho)$ and $T_L(x, x') = \|x' - x\| = t\|h'\| < t\gamma$ because $x' - x = th' = t(h' - h) + th \in \operatorname{cone} L$. Therefore

$$T_M(y, v') < T_M(y, v) - c \max\left\{T_L(x, x'), \|v' - v\|/(2\beta + \varrho)\right\}.$$

Using Proposition 13 with $\alpha := 1/(2\beta + \varrho)$ and then letting $c \uparrow \varrho$ we conclude the proof. □

We present now a necessary and sufficient condition for the directional regularity based on a directional version of the contingent variation, a concept coined by H. Frankowska in [11].

Definition 26 Let $F : X \rightrightarrows Y$ be a set-valued mapping between normed spaces $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ with $(\bar{x}, \bar{y}) \in \operatorname{Gr} F$, $L \subset S_X$ and $M \subset S_Y$ be nonempty sets. The *contingent variation of* F *at* $(\bar{x}, \bar{y})$ *with respect to* L *and* M is the set $F^{(1)}_{L,M}(\bar{x}, \bar{y})$ of all vectors $v \in Y$ such that there are sequences (t_n) in $(0, +\infty)$ converging to 0 and (v_n) in $v + \operatorname{cone} M$ converging to v such that, for each $n \in \mathbb{N}$,

$$\bar{y} + t_n v_n \in F(B[\bar{x}, t_n] \cap [\bar{x} + \operatorname{cone} L]). \tag{5.5}$$

Let us present a directional version of [11, Theorem 6.1 and Corollary 6.2].

Theorem 27 *Let* $(X, \|\cdot\|)$ *and* $(Y, \|\cdot\|)$ *be Banach spaces. Consider nonempty closed subsets* L *of* S_X *and* M *of* S_Y *such that* cone L *and* cone M *are convex and a mapping* $F : X \rightrightarrows Y$ *the graph of which is locally closed near* $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$. *Then* $\operatorname{dirsur}_{L\times M} F(\overline{x}, \overline{y})$ *is equal to the supremum of all* $\varrho > 0$ *for which there is* $r > 0$ *such that*

$$F^{(1)}_{L,M}(x, v) \supset -\varrho M \quad \textit{for every} \quad (x, v) \in (B[\overline{x}, r] \times B[\overline{y}, r]) \cap \operatorname{Gr} F. \tag{5.6}$$

Proof Let $\lambda := \operatorname{dirsur}_{L\times M} F(\overline{x}, \overline{y})$ and denote by s the supremum from the statement. First, we show that $\lambda \geq s$. Fix an arbitrary $\varrho \in (0, s)$ (if there is any). Find $r > 0$ such that (5.6) holds. Pick any $c \in (0, \varrho)$ and then find $\gamma > 1$ such that $c\gamma < \varrho$. Let $(x, v) \in (B[\overline{x}, r] \times B[\overline{y}, r]) \cap \operatorname{Gr} F$ and $y \in Y$ with $0 \neq v - y \in \operatorname{cone} M$ be arbitrary. Set

$$z := \varrho \frac{y - v}{\|y - v\|} \in \varrho(-M).$$

By (5.6), $F^{(1)}_{L,M}(x, v) \ni z \neq 0$. Hence there is $(t, x', z') \in (0, +\infty) \times (x + \operatorname{cone} L) \times (z + \operatorname{cone} M)$ such that $v + tz' \in F(x')$ with

$$\varrho t < \|y - v\|, \quad \|z - z'\| < \varrho - \gamma c, \quad \|z'\| < \gamma\varrho, \quad \text{and} \quad \|x' - x\| \leq t. \tag{5.7}$$

Let $v' := v + tz'$. Then $(x', v') \in \operatorname{Gr} F$. As $v - y \in \operatorname{cone} M$, the first inequality in (5.7) implies that

$$v + tz - y = (1 - \varrho t/\|y - v\|)(v - y) \in \operatorname{cone} M,$$

which means that $T_M(y, v + tz) = \|y - v - tz\| = \|y - v\| - t\varrho$. Since $v' - (v + tz) = t(z' - z) \in \operatorname{cone} M$, the second inequality in (5.7) implies that

$$T_M(v + tz, v') = \|v' - v - tz\| = t\|z' - z\| < t(\varrho - \gamma c).$$

We conclude that

$$T_M(y, v') \leq T_M(y, v + tz) + T_M(v + tz, v') < \|y - v\| - t\varrho + t(\varrho - \gamma c) = T_M(y, v) - c(t\gamma).$$

As $x' - x \in \operatorname{cone} L$, using the last two inequalities in (5.7), we get that $T_L(x, x') = \|x' - x\| < t\gamma$ and $\|v' - v\| = t\|z'\| < t\gamma\varrho$. Thus

$$T_M(y, v') < T_M(y, v) - c \max\left\{T_L(x, x'), \|v' - v\|/\varrho\right\}.$$

Now, Proposition 13 with $\alpha := 1/\varrho$ says that $\lambda \geq c$. Letting $c \uparrow \varrho$ and then $\varrho \uparrow s$, we conclude that $\lambda \geq s$ as claimed.

To show that $\lambda = s$, assume on the contrary that $\lambda > s$. Pick $\varrho \in (s, \lambda)$. Find $r > 0$ such that, for each $(x, v) \in (B[\overline{x}, r] \times B[\overline{y}, r]) \cap \operatorname{Gr} F$ and each $t \in (0, r]$,

$$B[v, \varrho t] \cap (v - \operatorname{cone} M) \subset F(B[x, t] \cap (x + \operatorname{cone} L)).$$

Fix any $(x, v) \in (B[\overline{x}, r] \times B[\overline{y}, r]) \cap \operatorname{Gr} F$. Pick an arbitrary $w \in -\varrho M$. For each $n \in \mathbb{N}\setminus\{0\}$, set $t_n := r/n$ and $w_n := w$. Then $t_n \downarrow 0$ and $w_n \to w$ as $n \to +\infty$. Clearly, for each $n \in \mathbb{N}\setminus\{0\}$, we have $w_n \in w + \operatorname{cone} M$ and $v + t_n w_n \in B[v, \varrho t_n] \cap (v - \operatorname{cone} M)$. Hence $v + t_n w_n \in F(B[x, t_n] \cap (x + \operatorname{cone} L))$, that is, $w \in F^{(1)}_{L,M}(x, v)$. We showed that $-\varrho M \subset F^{(1)}_{L,M}(x, v)$ for each $(x, v) \in (B[\overline{x}, r] \times B[\overline{y}, r]) \cap \operatorname{Gr} F$. Thus $s \geq \varrho > s$, a contradiction. □

Note that Theorem 25 with $\beta := 0$ follows from the above statement, as the next remark shows.

Remark 28 Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be Banach spaces. Consider nonempty closed subsets L of S_X and M of S_Y such that cone L and cone M are convex and a mapping $F : X \rightrightarrows Y$ with $(\bar{x}, \bar{y}) \in \operatorname{Gr} F$. Then

$$(-\operatorname{cone} M) \cap D_{L,M} F(\bar{x}, \bar{y})(\mathbb{B}_X \cap \operatorname{cone} L) \subset (-\operatorname{cone} M) \cap F^{(1)}_{L,M}(\bar{x}, \bar{y}). \tag{5.8}$$

Indeed, pick an arbitrary v from the set on the left-hand side of (5.8). Find $u \in \mathbb{B}_X \cap \operatorname{cone} L$ such that $v \in D_{L,M} F(\bar{x}, \bar{y})(u)$. Find sequences (t_n) in $(0, +\infty)$ converging to 0, (u_n) in $u + \operatorname{cone} L$ converging to u, and (v_n) in $v + \operatorname{cone} M$ converging to v such that, for each $n \in \mathbb{N}$, we have $\bar{y} + t_n v_n \in F(\bar{x} + t_n u_n)$. Let $N_1 := \{n \in \mathbb{N} | \, \|u_n\| \le 1\}$ and $N_2 := \mathbb{N} \setminus N_1$. Suppose that N_1 is infinite. Then, for each $n \in N_1$, we have $t_n u_n = t_n u + t_n(u_n - u) \in \operatorname{cone} L + \operatorname{cone} L \subset \operatorname{cone} L$, that is, $\bar{x} + t_n u_n \in B[\bar{x}, t_n] \cap [\bar{x} + \operatorname{cone} L]$ and thus (5.5) holds. Using the subsequences $(v_n)_{n \in N_1}$ and $(t_n)_{n \in N_1}$, we conclude that $v \in F^{(1)}_{L,M}(\bar{x}, \bar{y})$. Second, suppose that N_2 is infinite. Then $\|u\| = 1$. Let $t'_n := t_n \|u_n\|$, $u'_n := u_n / \|u_n\|$, and $v'_n := v_n / \|u_n\|$ for each $n \in N_2$. Then $t'_n \downarrow 0$, $u'_n \to u$, and $v'_n \to v$ as $N_2 \ni n \to +\infty$. For each $n \in N_2$, we have $t'_n u'_n = t_n u_n$ and $t'_n v'_n = t_n v_n$. Similarly to the previous case, we conclude that, for each $n \in N_2$, we have $\bar{x} + t'_n u'_n \in B[\bar{x}, t'_n] \cap [\bar{x} + \operatorname{cone} L]$ which means that

$$\bar{y} + t'_n v'_n = \bar{y} + t_n v_n \in F(B[\bar{x}, t'_n] \cap [\bar{x} + \operatorname{cone} L]);$$

moreover

$$v'_n - v = \frac{1}{\|u_n\|}(v_n - v) + \frac{\|u_n\| - 1}{\|u_n\|}(-v) \in \operatorname{cone} M + \operatorname{cone} M \subset \operatorname{cone} M.$$

Hence $v \in F^{(1)}_{L,M}(\bar{x}, \bar{y})$, which proves (5.8).

On the other hand, the equality in (5.8) holds provided that X is finite dimensional and $\operatorname{cone} L - \operatorname{cone} L \subset \operatorname{cone} L$, which means that cone L is a linear subspace. Indeed, pick any v from the set on the right-hand side of (5.8). Find sequences (t_n) in $(0, +\infty)$ converging to 0 and (v_n) in $v + \operatorname{cone} M$ converging to v such that (5.5) holds for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, find $x_n \in B[\bar{x}, t_n] \cap [\bar{x} + \operatorname{cone} L]$ such that $\bar{y} + t_n v_n \in F(x_n)$, that is, for $u_n := (x_n - \bar{x})/t_n$ we have $\bar{y} + t_n v_n \in F(\bar{x} + t_n u_n)$. As (u_n) lies in $\mathbb{B}_X \cap \operatorname{cone} L$, there is an infinite set $N \in \mathbb{N}$ such that $u := \lim_{N \ni n \to +\infty} u_n$ exists and lies in $\mathbb{B}_X \cap \operatorname{cone} L$. Since $\operatorname{cone} L - \operatorname{cone} L \subset \operatorname{cone} L$, the sequence (u_n) lies in $u + \operatorname{cone} L$, hence $v \in D_{L,M} F(\bar{x}, \bar{y})(u)$.

At the end of this section, we formulate results that use Theorem 25 in order to give primal sufficient conditions for the directional metric regularity of compositions and sums. Note that the next theorem is new even for the non-directional case. For the next results, under the notation of Definition 24, we denote by $D_{L,M} F(\bar{x}, \bar{y}) \cap \mathbb{B}_Y \cap \operatorname{cone} M$ the multifunction $H : X \rightrightarrows Y$ given by

$$H(u) = D_{L,M} F(\bar{x}, \bar{y})(u) \cap \mathbb{B}_Y \cap \operatorname{cone} M, \quad u \in X.$$

Theorem 29 *Let spaces X, Y, Z, and W, a point $(\bar{x}, \bar{y}, \bar{z}, \bar{w})$, sets L, M, N, and P, and mappings F_1, F_2, G and $\mathcal{E}_{G,(F_1,F_2)}$ be as in Theorem 16. Assume that there exist positive constants β, ϱ, and r such that, for every $(x, y, z, w) \in (B[\bar{x}, r] \times B[\bar{y}, r] \times B[\bar{z}, r] \times B[\bar{w}, r]) \cap \operatorname{Gr} \mathcal{E}_{G,(F_1,F_2)}$:*

(i) *the next relation holds*

$$\begin{aligned} D_{M,N,P} G(y, z, w)\big((&D_{L,M} F_1(x, y) \cap \mathbb{B}_Y \cap \operatorname{cone} M, \\ &D_{L,N} F_2(x, z) \cap \mathbb{B}_Z \cap \operatorname{cone} N)(\mathbb{B}_X \cap \operatorname{cone} L)\big) \\ &+ B[0, \beta] \cap (-\operatorname{cone} P) \supset -(\beta + \varrho) P; \end{aligned}$$

(ii) *either* F_1 *is directionally proto-differentiable with respect to* $L \times M$ *at* x *relative to* y *or* F_2 *is directionally proto-differentiable with respect to* $L \times N$ *at* x *relative to* z;

(iii) *either* F_1 *has the directional Aubin property with respect to* S_X *and* M *around* (x, y) *or* F_2 *has the directional Aubin property with respect to* S_X *and* N *around* (x, z);

(iv) G *is directionally proto-differentiable with respect to* $M \times N \times P$ *at* (y, z) *relative to* w;

(v) G *has the directional Aubin property with respect to* $S_Y \times S_Z$ *and* P *around* (y, z, w);

(vi) *the pair* (F_1, F_2), G *is composition-stable around* $(x, (y, z), w)$.

Then $\operatorname{dirsur}_{L\times P}(G \circ (F_1, F_2))(\overline{x}, \overline{w}) \geq \varrho$.

Proof We prove that, for every $(x, y, z, w) \in (B[\overline{x}, r] \times B[\overline{y}, r] \times B[\overline{z}, r] \times B[\overline{w}, r]) \cap \operatorname{Gr} \mathcal{E}_{G,(F_1,F_2)}$,

$$D_{L,M,N,P}\mathcal{E}_{G,(F_1,F_2)}(x, y, z, w)((\mathbb{B}_X \cap \operatorname{cone} L) \times (\mathbb{B}_Y \cap \operatorname{cone} M) \times (\mathbb{B}_Z \cap \operatorname{cone} N)) + B[0, \beta] \cap (-\operatorname{cone} P) \supset -(\beta + \varrho)P.$$

Fix any such (x, y, z, w). It suffices to show that

$$D_{M,N,P}G(y, z, w)\big((D_{L,M}F_1(x, y) \cap \mathbb{B}_Y \cap \operatorname{cone} M, D_{L,N}F_2(x, z) \cap\mathbb{B}_Z \cap \operatorname{cone} N)(\mathbb{B}_X \cap \operatorname{cone} L)\big) \subset D_{L,M,N,P}\mathcal{E}_{G,(F_1,F_2)}(x, y, z, w)((\mathbb{B}_X \cap \operatorname{cone} L) \times(\mathbb{B}_Y \cap \operatorname{cone} M) \times (\mathbb{B}_Z \cap \operatorname{cone} N)). \tag{5.9}$$

To show this, consider $a \in \mathbb{B}_X \cap \operatorname{cone} L$, $b \in D_{L,M}F_1(x, y)(a) \cap \mathbb{B}_Y \cap \operatorname{cone} M$, $c \in D_{L,N}F_2(x, z)(a) \cap \mathbb{B}_Z \cap \operatorname{cone} N$, and $d \in D_{M,N,P}G(y, z, w)(b, c)$.

Suppose that F_2 is directionally proto-differentiable with respect to $L \times N$ at x relative to z, and has the directional Aubin property around (x, z) with respect to S_X and N. Since $b \in D_{L,M}F_1(x, y)(a)$, there exist $(t_n) \downarrow 0$, $(a_n) \to a$, $(b_n) \to b$, $(a_n) \subset a + \operatorname{cone} L$, $(b_n) \subset b + \operatorname{cone} M$, such that, for every $n \in \mathbb{N}$, $y + t_n b_n \in F_1(x + t_n a_n)$.

Now, since F_2 is directionally proto-differentiable with respect to $L \times N$ at x relative to z, for the sequence (t_n) chosen before, and because $c \in D_{L,N}F_2(x, z)(a)$, there exist $(a'_n) \to a$, $(c'_n) \to c$, $(a'_n) \subset a + \operatorname{cone} L$, $(c'_n) \subset c + \operatorname{cone} N$ such that, for every $n \in \mathbb{N}$, $z + t_n c'_n \in F_2(x + t_n a'_n)$. The directional Aubin property of F_2 around (x, z) with respect to S_X and N means that, for sufficiently large $n \in \mathbb{N}$,

$$T_N(z + t_n c'_n, F_2(x + t_n a_n)) \leq e_N(F_2(x + t_n a'_n) \cap U, F_2(x + t_n a_n)) \leq \ell t_n \left\|a_n - a'_n\right\|,$$

where a constant $\ell > 0$ and a neighborhood U of z are appropriately chosen. Fix any such $n \in \mathbb{N}$ for a longer while. Then, there is $z_n \in F_2(x + t_n a_n)$ such that $z + t_n c'_n \in z_n - \operatorname{cone} N$ and $\left\|z + t_n c'_n - z_n\right\| \leq \ell t_n \left\|a_n - a'_n\right\| + t_n^2$. Denote

$$c_n = \frac{z_n - z}{t_n}, \quad \text{that is,} \quad z + t_n c_n = z_n.$$

Then $\left\|c'_n - c_n\right\| \leq \ell \left\|a_n - a'_n\right\| + t_n$ and $c_n = c'_n + (c_n - c'_n) \in c + \operatorname{cone} N + \operatorname{cone} N = c + \operatorname{cone} N$. Hence,

$$z + t_n c_n \in F_2(x + t_n a_n),\ (c_n) \to c \text{ and } (c_n) \subset c + \operatorname{cone} N.$$

The other three cases described by (ii) and (iii) lead, similarly, to a relation of the type

$$(y + t_n b_n, z + t_n c_n) \in (F_1, F_2)(x + t_n a_n),$$
$$(b_n) \to b,\ (b_n) \subset b + \operatorname{cone} M \text{ and } (c_n) \to c,\ (c_n) \subset c + \operatorname{cone} N.$$

Next, using that $d \in D_{M,N,P}G(y,z,w)(b,c)$ and the fact that G is directionally proto-differentiable with respect to $M \times N \times P$ at (y,z) relative to w, for the (t_n) from before, we find $(b_n'') \to b$, $(c_n'') \to c$, $(d_n'') \to d$, $(b_n'') \subset b + \operatorname{cone} M$, $(c_n'') \subset c + \operatorname{cone} N$, $(d_n'') \subset d + \operatorname{cone} P$, such that, for every $n \in \mathbb{N}$, $w + t_n d_n'' \in G(y + t_n b_n'', z + t_n c_n'')$. The directional Aubin property of G around (y,z,w) with respect to $S_Y \times S_Z$ and P means that, for sufficiently large $n \in \mathbb{N}$,

$$\begin{aligned} T_P(w+t_n d_n'', G(y+t_n b_n, z+t_n c_n)) &\le e_P(G(y+t_n b_n'', z+t_n c_n'') \cap V, G(y+t_n b_n, z+t_n c_n)) \\ &\le m t_n \max\left\{\|b_n - b_n''\|, \|c_n - c_n''\|\right\}, \end{aligned}$$

where a constant $m > 0$ and a neighborhood V of w are appropriately chosen. Fix any such $n \in \mathbb{N}$ for a longer while. There exists $w_n \in G(y + t_n b_n, z + t_n c_n)$ such that $w + t_n d_n'' \in w_n - \operatorname{cone} P$ and $\|w_n - w - t_n d_n''\| \le m t_n \max\left\{\|b_n - b_n''\|, \|c_n - c_n''\|\right\} + t_n^2$. Denote

$$d_n = \frac{w_n - w}{t_n}, \quad \text{that is,} \quad w + t_n d_n = w_n.$$

Then $\|d_n'' - d_n\| \le m \max\left\{\|b_n - b_n''\|, \|c_n - c_n''\|\right\} + t_n$ and $d_n = d_n'' + (d_n - d_n'') \in d + \operatorname{cone} P + \operatorname{cone} P = d + \operatorname{cone} P$. In conclusion, there exist $(t_n) \downarrow 0$, $(a_n) \to a$, $(b_n) \to b$, $(c_n) \to c$, $(d_n) \to d$, $(a_n) \subset a + \operatorname{cone} L$, $(b_n) \subset b + \operatorname{cone} M$, $(c_n) \subset c + \operatorname{cone} N$, $(d_n) \subset d + \operatorname{cone} P$, such that, for every $n \in \mathbb{N}$,

$$w + t_n d_n \in G(y + t_n b_n, z + t_n c_n), \ y + t_n b_n \in F_1(x + t_n a_n), \ z + t_n c_n \in F_2(x + t_n a_n).$$

Hence,

$$d \in D_{L,M,N,P}\mathcal{E}_{G,(F_1,F_2)}(x,y,z,w)(a,b,c).$$

Since $a \in \mathbb{B}_X \cap \operatorname{cone} L$, $b \in \mathbb{B}_Y \cap \operatorname{cone} M$, $c \in \mathbb{B}_Z \cap \operatorname{cone} N$, inclusion (5.9) is proved.

Using now Theorem 25, it follows that $\operatorname{dirsur}_{L\times M\times N\times P} \mathcal{E}_{G,(F_1,F_2)}(\bar{x},\bar{y},\bar{z},\bar{w}) \ge \varrho$. Since the pair $(F_1, F_2), G$ is composition-stable around $(\bar{x},(\bar{y},\bar{z}),\bar{w})$, we have as in the final part of the proof of Theorem 16 that $\operatorname{dirsur}_{L\times P}(G \circ (F_1, F_2))(\bar{x},\bar{w}) \ge \varrho$. □

Remark 30 In fact, (5.9) is equality. Indeed, take $a \in \mathbb{B}_X \cap \operatorname{cone} L$, $b \in \mathbb{B}_Y \cap \operatorname{cone} M$, $c \in \mathbb{B}_Z \cap \operatorname{cone} N$ and

$$d \in D_{L,M,N,P}\mathcal{E}_{G,(F_1,F_2)}(x,y,z,w)(a,b,c).$$

Then there exist $(t_n) \downarrow 0$, $(a_n) \to a$, $(b_n) \to b$, $(c_n) \to c$, $(d_n) \to d$, $(a_n) \subset a + \operatorname{cone} L$, $(b_n) \subset b + \operatorname{cone} M$, $(c_n) \subset c + \operatorname{cone} N$, $(d_n) \subset d + \operatorname{cone} P$, such that, for every $n \in \mathbb{N}$,

$$w + t_n d_n \in G(y + t_n b_n, z + t_n c_n), \ y + t_n b_n \in F_1(x + t_n a_n), \ z + t_n c_n \in F_2(x + t_n a_n).$$

This means that $b \in D_{L,M}F_1(x,y)(a)$, $c \in D_{L,N}F_2(x,z)(a)$ and $d \in D_{M,N,P}G(y,z,w)(b,c)$.

Note that in (iii) one can assume that the directional Aubin property holds with respect to L and M and $\operatorname{cone} L - \operatorname{cone} L \subset \operatorname{cone} L$. Also (v) can be modified similarly.

As a consequence of Theorem 29, we present the next result for the particular case of the sum of two set-valued maps.

Corollary 31 *Let spaces X and Y, a point $(\overline{x}, \overline{y}, \overline{z})$, sets L and M, and mappings F_1, F_2 and $\mathcal{E}_{F_1,F_2}$ be as in Corollary 19. Assume that there exist positive constants β, ϱ, and r such that, for every $(x, y, z) \in (B[\overline{x}, r] \times B[\overline{y}, r] \times B[\overline{z}, r]) \cap \operatorname{Gr} \mathcal{E}_{F_1,F_2}$:*

(i) *the next relation holds*

$$(D_{L,M}F_1(x, y) \cap \mathbb{B}_Y \cap \operatorname{cone} M + D_{L,M}F_2(x, z) \cap \mathbb{B}_Y \cap \operatorname{cone} M)(\mathbb{B}_X \cap \operatorname{cone} L) \\ +B[0, \beta] \cap (-\operatorname{cone} M) \supset -(\beta + \varrho)M;$$

(ii) *either F_1 is directionally proto-differentiable with respect to $L \times M$ at x relative to y or F_2 is directionally proto-differentiable with respect to $L \times M$ at x relative to z;*

(iii) *either F_1 has the directional Aubin property with respect to S_X and M around (x, y) or F_2 has the directional Aubin property with respect to S_X and M around (x, z);*

(vi) *the pair F_1, F_2 is sum-stable around (x, y, z).*

Then $\operatorname{dirsur}_{L\times M}(F_1 + F_2)(\overline{x}, \overline{y} + \overline{z}) \geq \varrho$.

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Appendix

In this section, we illustrate some other connections between our results and several well-known tools in variational analysis. In this sense, we provide some different proofs for two of the key results in this work.

On one hand, we present a direct and constructive proof for the general criterion for the directional regularity of single-valued maps. This underlines again the fact that the use of the Ekeland Variational Principle is an alternative for explicit iterative procedures. On the other hand, we provide as well another proof for the result about the stability at composition of the directional regularity. For this, we employ now, instead of Proposition 13, a variant of the directional Ekeland Variational Principle, formulated on product spaces on the basis of Lemma 12.

A.1 Proof of the Criterion for the Directional Regularity by an Iterative Procedure

Let us present next the announced constructive proof of the criterion for the directional regularity of single-valued maps.

Proof (of Proposition 11 by Iterative Procedure) Let $\lambda := \operatorname{dirsur}_{L\times M} g(\overline{x})$ and s be the supremum from the statement. We only prove that $s \leq \lambda$, since the opposite inequality is straightforward, as shown in the proof given in Section 3.

Define a function $\varphi : X \times X \to [0, +\infty]$ by $\varphi(u, v) = T_L(u, v)$, $(u, v) \in X \times X$, and a function $\psi : Y \times Y \to [0, +\infty]$ by $\psi(y, z) = T_M(y, z)$, $(y, z) \in Y \times Y$. Observe that the convexity of cone L implies that

$$\varphi(u, v) \leq \varphi(u, w) + \varphi(w, v), \quad \text{for all} \quad u, v, w \in X. \tag{A.1}$$

To show that $s \leq \lambda$, fix an arbitrary $c \in (0, s)$ (if there is any) for which there is $r > 0$ such that for all $(x, y) \in (B[\bar{x}, r] \cap \operatorname{Dom} g) \times B[g(\bar{x}), r]$, with $0 \neq g(x) - y \in \operatorname{cone} M$, there is a point $x' \in \operatorname{Dom} g$ such that

$$c\,\varphi(x, x') < \psi(y, g(x)) - \psi(y, g(x')). \tag{A.2}$$

Make $r > 0$ smaller, if necessary, so that the set $B[\bar{x}, r] \cap \operatorname{Dom} g$ is complete and g is continuous on this set. By the continuity of g, there is $\varepsilon \in (0, r)$ such that

$$B[g(u), c\varepsilon] \subset B[g(\bar{x}), r] \text{ and } B[u, \varepsilon] \subset B[\bar{x}, r] \text{ whenever } u \in B[\bar{x}, \varepsilon] \cap \operatorname{Dom} g. \tag{A.3}$$

Fix any $t \in (0, \varepsilon)$ and any $u \in B[\bar{x}, \varepsilon] \cap \operatorname{Dom} g$. Let $\Lambda := B[u, t] \cap (u + \operatorname{cone} L) \cap \operatorname{Dom} g$. As $L \subset S_X$ is closed, so is cone L. Consequently, Λ is complete. We have to show that

$$g(\Lambda) \supset B[g(u), ct] \cap (g(u) - \operatorname{cone} M).$$

Consider any fixed $y \in B[g(u), ct] \cap (g(u) - \operatorname{cone} M)$; we will find $x \in \Lambda$ such that $y = g(x)$. If $y = g(u)$, take $x := u$ and we are done. Assume further that $y \neq g(u)$. We will construct a sequence $x_1, x_2, \ldots$ in Λ satisfying

$$c\,\varphi(u, x_m) \leq \psi(y, g(u)) - \psi(y, g(x_m)), \quad m \in \mathbb{N}. \tag{A.4}$$

As $\varphi(u, u) = 0$ and $g(u) - y \in \operatorname{cone} M$ (thus $\psi(y, g(u))$ is finite), the point $x_1 := u$ satisfies (A.4) with $m = 1$. Let $n \in \mathbb{N}$ and assume that $x_n \in \Lambda$ satisfying (A.4) with $m = n$ was already found. If $g(x_n) = y$, then take $x := x_n$, and stop the construction. Assume further that $g(x_n) \neq y$. Then (A.4), with $m := n$, implies that $\psi(y, g(x_n))$ is finite, meaning that $g(x_n) - y \in \operatorname{cone} M$. Using (A.3) and (A.2), we find $x_{n+1} \in \operatorname{Dom} g$ such that

$$c\,\varphi(x_n, x_{n+1}) < \psi(y, g(x_n)) - \psi(y, g(x_{n+1})) \quad \text{and that} \quad \varphi(x_n, x_{n+1}) \geq \tfrac{1}{2} s_n \tag{A.5}$$

where

$$s_n := \sup\left\{\varphi(x_n, x') : \ x' \in \operatorname{Dom} g \ \text{ and } \ c\,\varphi(x_n, x') < \psi(y, g(x_n)) - \psi(y, g(x'))\right\}.$$

Note that $0 \leq s_n \leq \frac{1}{c}\psi(y, g(x_n)) < +\infty$. Using (A.1), the first inequality in (A.5), and (A.4) with $m := n$, we get

$$c\,\varphi(u, x_{n+1}) \leq c\,\varphi(u, x_n) + c\,\varphi(x_n, x_{n+1}) < \psi(y, g(u)) - \psi(y, g(x_{n+1})),$$

which is (A.4) with $m := n + 1$. In particular, we have $c\,\varphi(u, x_{n+1}) \leq \psi(y, g(u)) = \|y - g(u)\| \leq ct$; thus $x_{n+1} \in u + \operatorname{cone} L$ and $\varphi(u, x_{n+1}) = \|u - x_{n+1}\|$. Consequently, $x_{n+1} \in \Lambda$. If the process stops at some $n \in \mathbb{N}$, we are done. Assume that this was not the case, that is, $g(x_n) \neq y$ for every $n \in \mathbb{N}$. From (A.5) and (A.1) we have, for all $1 \leq n < m$, that

$$\begin{aligned} 0 \leq c\,\varphi(x_n, x_m) &\leq c\,\varphi(x_n, x_{n+1}) + \cdots + c\,\varphi(x_{m-1}, x_m) \\ &< (\psi(y, g(x_n)) - \psi(y, g(x_{n+1}))) + \cdots + (\psi(y, g(x_{m-1})) - \psi(y, g(x_m))) \\ &= \psi(y, g(x_n)) - \psi(y, g(x_m)), \end{aligned} \tag{A.6}$$

and so, $\psi(y, g(x_n)) > \psi(y, g(x_m))$. Thus $\ell := \lim_{n\to+\infty} \psi(y, g(x_n))$ exists and is finite. By (A.6), for all $1 \leq n < m$, we have $\varphi(x_n, x_m) < +\infty$, and hence $\varphi(x_n, x_m) = \|x_n - x_m\|$. Consequently, (x_n) is a Cauchy sequence in Λ (which is a complete metric space). Put $x := \lim_{n\to+\infty} x_n$. Then $x \in \Lambda$ and $\psi(y, g(x)) \leq \ell < +\infty$ because $\psi(y, \cdot)$ is lower semicontinuous and g is continuous. Moreover, for any $n \in \mathbb{N}$, using the lower semicontinuity of $\varphi(x_n, \cdot)$ and (A.6) we get that

$$c\,\varphi(x_n, x) \leq c \liminf_{p\to+\infty} \varphi(x_n, x_{n+p}) \leq \psi(y, g(x_n)) - \ell.$$

Consequently, $\lim_{n\to+\infty}\varphi(x_n, x) = 0$. Suppose that $y \neq g(x)$. By (A.2), there is $x' \in \operatorname{Dom} g$ such that

$$c\,\varphi(x, x') < \psi(y, g(x)) - \psi(y, g(x')) \le \ell - \psi(y, g(x')). \tag{A.7}$$

Then (A.1) implies that $\limsup_{n\to+\infty}\varphi(x_n, x') \le \lim_{n\to+\infty}\varphi(x_n, x) + \varphi(x, x') = \varphi(x, x')$. This and (A.7) imply that, for each $n \in \mathbb{N}$ sufficiently large, we have

$$c\,\varphi(x_n, x') < \psi(y, g(x_n)) - \psi(y, g(x')).$$

As $x \neq x'$ by (A.7), we have $\varphi(x, x') > 0$. The lower semicontinuity of $\varphi(\cdot, x')$, the choice of s_n, and (A.5) yield that

$$0 < \varphi(x, x') \le \liminf_{n\to+\infty}\varphi(x_n, x') \le \limsup_{n\to+\infty} s_n \le 2\lim_{n\to+\infty}\varphi(x_n, x_{n+1}) = 0,$$

a contradiction. Therefore $y = g(x)$. We proved that $c \le \lambda$, and thus $s \le \lambda$.

A.2 Proof of Directional Openness Stability at Composition by Directional EVP

As mentioned before, in the second part of this appendix, we discuss the possibility to give an alternative proof of the main result of the paper, namely Theorem 16, by the use of the next variant of the directional Ekeland Variational Principle.

Theorem 32 *Let $(X_1, \|\cdot\|), \ldots, (X_n, \|\cdot\|)$ be Banach spaces and $A \subset X_1 \times \ldots \times X_n$ be a nonempty closed set. Consider nonempty closed sets $L_i \subset S_{X_i}$, $i = 1, \ldots, n$ such that $\operatorname{cone} L_i$ are convex. Then, for every lower semicontinuous bounded from below function $f : A \to \mathbb{R} \cup \{+\infty\}$, every $a_0 := (x_{01}, \ldots, x_{0n}) \in A$ such that $f(a_0) < +\infty$, and every $\delta, \alpha_1, \ldots, \alpha_n > 0$, there exists $a_\delta := (x_{\delta 1}, \ldots, x_{\delta n}) \in A$ such that*

$$f(a_\delta) \le f(a_0) - \delta \max\{\alpha_1 T_{L_1}(x_{\delta 1}, x_{01}), \ldots, \alpha_n T_{L_n}(x_{\delta n}, x_{0n})\}$$

and, for every $a := (x_1, \ldots, x_n) \in A \setminus \{a_\delta\}$,

$$f(a_\delta) < f(a) + \delta \max\{\alpha_1 T_{L_1}(x_1, x_{\delta 1}), \ldots, \alpha_n T_{L_n}(x_n, x_{\delta n})\}.$$

Proof Take $\widetilde{L}$ as in the Lemma 12 and observe that $\operatorname{cone}\widetilde{L} = \operatorname{cone} L_1 \times \ldots \times \operatorname{cone} L_n$ is convex. Apply Theorem 10 with $X := X_1 \times \cdots \times X_n$ and $M := \widetilde{L}$ to get the statement. □

Now, we are ready to provide the announced proof of the main (and the essential) part of Theorem 16.

Proof (of Theorem 16 by Directional EVP) Again, as in the proof of Theorem 16, we only have to consider the case where the right-hand side of the inequality (4.2) is positive. We find again positive constants $\alpha, \beta, \beta', \gamma$, and δ such that $c := \alpha\gamma - \beta\delta > 0$, and inequalities (4.4) and (4.5) hold. Moreover, keeping the notation of Theorem 16, there is $\varepsilon > 0$ such that (4.6), (4.7) and (4.9) hold. Also, taking into account Proposition 3, we may suppose that for any $z \in B(\overline{z}, \varepsilon)$, the mapping G_z^{-1} is directionally Aubin continuous around $(\overline{w}, \overline{y})$ with respect to P and $-M$ with modulus γ^{-1}, i.e.,

$$e_{-M}\left(G_z^{-1}(w) \cap B(\overline{y}, \varepsilon), G_z^{-1}(w')\right) \le \gamma^{-1} T_P(w', w) = \gamma^{-1} T_{-P}(w, w'), \tag{A.8}$$

for any $z \in B(\overline{z}, \varepsilon)$, and any $w, w' \in B(\overline{w}, \varepsilon)$.

Also, in view of the local closedness of the graphs of F_1, F_2 and G, we can consider that $\operatorname{Gr} F_1 \cap (B[x, \varepsilon] \times B[y, \alpha\varepsilon])$, $\operatorname{Gr} F_2 \cap (B[x, \varepsilon] \times B[z, \beta\varepsilon])$ and $\operatorname{Gr} G \cap (B[y, \alpha\varepsilon] \times$

$B[z, \beta\varepsilon] \times B[w, (\alpha\gamma + \beta\delta)\varepsilon])$ are closed, for any $(x, y, z, w) \in B(\overline{x}, \varepsilon) \times B(\overline{y}, \alpha\varepsilon) \times B(\overline{z}, \beta\varepsilon) \times B(\overline{w}, (\alpha\gamma + \beta\delta)\varepsilon)$ with $y \in F_1(x)$, $z \in F_2(x)$ and $w \in G(y, z)$.

Take

$$\rho := \min\left\{3^{-1}\varepsilon, (3\alpha)^{-1}\varepsilon, (3\beta)^{-1}\varepsilon, (3\alpha\gamma + 3\beta\delta)^{-1}\varepsilon\right\}.$$

Fix $t \in (0, \rho)$ and $(x, y, z, w) \in B(\overline{x}, \rho) \times B(\overline{y}, \alpha\rho) \times B(\overline{z}, \beta\rho) \times B(\overline{w}, (\alpha\gamma + \beta\delta)\rho)$ with $y \in F_1(x)$, $z \in F_2(x)$ and $w \in G(y, z)$. We want to prove that

$$B(w, ct) \cap [w - \text{cone } P] \subset \mathcal{E}_{G,(F_1,F_2)}\left(B_{X\times Y\times Z}((x, y, z), t) \cap ((x, y, z) + \text{cone } \widetilde{L})\right),$$

where the norm on $X \times Y \times Z$ and $\widetilde{L} \subset S_{X\times Y\times Z}$ are as in the proof of Theorem 16.

Denote

$$\begin{aligned} A &:= B(x, 2\rho) \times B(y, 2\alpha\rho) \times B(z, 2\beta\rho) \times B(w, 2(\alpha\gamma + \beta\delta)\rho), \\ \overline{A} &:= B[x, 2\rho] \times B[y, 2\alpha\rho] \times B[z, 2\beta\rho] \times B[w, 2(\alpha\gamma + \beta\delta)\rho], \\ \Omega &:= \left\{(x', y', z', w') \in X \times Y \times Z \times W \mid (y', z') \in (F_1, F_2)(x') \text{ and } w' \in G(y', z')\right\}. \end{aligned}$$

Take an arbitrary $v \in w - [0, ct) \cdot P$. We must prove that $v \in \mathcal{E}_{G,(F_1,F_2)}((x, y, z) + [0, t) \cdot \widetilde{L})$.

We can find $\tau \in (0, 1)$ such that $\|v - w\| < \tau ct$. Remark that $\Omega \cap \overline{A}$ is closed (since $2\rho < \varepsilon$). Define

$$h : \Omega \cap \overline{A} \to [0, +\infty], \quad h(p, q, r, s) := T_P(v, s) = T_{-P}(s, v),$$

and observe that it is lower semicontinuous and bounded from below. Thus, we can apply Theorem 32, for $\tau c > 0$ instead of δ, and $a_0 = (x, y, z, w)$ and $-L$, M, $-N$, and S_W as sets in X, Y, Z and W, respectively, to find $(\widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{d}) \in \Omega \cap \overline{A}$ satisfying

$$\begin{aligned} T_P(v, \widetilde{d}) &\leq T_P(v, w) - \tau c \max\{T_{-L}(\widetilde{a}, x), \alpha^{-1} T_M(\widetilde{b}, y), \beta^{-1} T_{-N}(\widetilde{c}, z), \\ &\qquad (\alpha\gamma + \beta\delta)^{-1} \|\widetilde{d} - w\|\} \\ T_P(v, \widetilde{d}) &\leq T_P(v, s) + \tau c \max\{T_{-L}(p, \widetilde{a}), \alpha^{-1} T_M(q, \widetilde{b}), \beta^{-1} T_{-N}(r, \widetilde{c}), \\ &\qquad (\alpha\gamma + \beta\delta)^{-1} \|s - \widetilde{d}\|\}, \end{aligned}$$

for every $(p, q, r, s) \in \Omega \cap \overline{A}$. As an immediate consequence, $\widetilde{b} \in F_1(\widetilde{a})$, $\widetilde{c} \in F_2(\widetilde{a})$, $\widetilde{d} \in G(\widetilde{b}, \widetilde{c})$, and

$$\max\{T_{-L}(\widetilde{a}, x), \alpha^{-1} T_M(\widetilde{b}, y), \beta^{-1} T_{-N}(\widetilde{c}, z), (\alpha\gamma + \beta\delta)^{-1} \|\widetilde{d} - w\|\} < \infty,$$

which implies the following:

$$\begin{aligned} \widetilde{a} &\in x + \text{cone } L, \quad \widetilde{b} \in y - \text{cone } M, \quad \widetilde{c} \in z + \text{cone } N, \\ T_{-L}(\widetilde{a}, x) &= \|\widetilde{a} - x\|, \quad T_M(\widetilde{b}, y) = \|\widetilde{b} - y\|, \quad T_{-N}(\widetilde{c}, z) = \|\widetilde{c} - z\|. \end{aligned}$$

Moreover, since $v \in w - \text{cone } P$, we also have

$$\begin{aligned} &\tau c \max\{T_{-L}(\widetilde{a}, x), \alpha^{-1} T_M(\widetilde{b}, y), \beta^{-1} T_{-N}(\widetilde{c}, z), (\alpha\gamma + \beta\delta)^{-1} \|\widetilde{d} - w\|\} \\ &\leq T_P(v, w) = \|v - w\| < \tau ct, \end{aligned}$$

so

$$\begin{aligned} \widetilde{a} &\in B(x, t) \cap (x + \text{cone } L) = x + [0, t) \cdot L \subset B(x, t) \subset B(x, \rho), \\ \widetilde{b} &\in B(y, \alpha t) \cap (y - \text{cone } M) = y - [0, \alpha t) \cdot M \subset B(y, \alpha t) \subset B(y, \alpha\rho), \\ \widetilde{c} &\in B(z, \beta t) \cap (z + \text{cone } N) = z + [0, \beta t) \cdot N \subset B(z, \beta t) \subset B(z, \beta\rho), \\ \widetilde{d} &\in B(w, (\alpha\gamma + \beta\delta)t) \subset B(w, (\alpha\gamma + \beta\delta)\rho). \end{aligned}$$

Hence, $(\widetilde{a},\widetilde{b},\widetilde{c},\widetilde{d}) \in A$. Now, if $v=\widetilde{d}$, then

$$\begin{aligned} v \in \mathcal{E}_{G,(F_1,F_2)}(\widetilde{a},\widetilde{b},\widetilde{c}) &\subset \mathcal{E}_{G,(F_1,F_2)}(x+[0,t)\cdot L, y-[0,\alpha t)\cdot M, z+[0,\beta t)\cdot N) \\ &= \mathcal{E}_{G,(F_1,F_2)}((x,y,z)+[0,t)\cdot \widetilde{L}), \end{aligned}$$

which is exactly what we need. We will prove that $v=\widetilde{d}$ is the only possibility.

Assume, on the contrary, that $v\neq \widetilde{d}$. Remark that $T_P(v,\widetilde{d}) \le T_P(v,w) < \infty$, which means that $v-\widetilde{d} \in -\text{cone}\, P$. Then

$$v' := \frac{v-\widetilde{d}}{\|v-\widetilde{d}\|} \in -P,$$

since its norm equals 1 and it belongs to $-\text{cone}\, P$. Fix $\sigma \in (0,\alpha\gamma)$ such that

$$c-\sigma > \tau c, \tag{A.9}$$

and choose $\zeta \in \left(0, \min\left\{3^{-1}\rho, (\alpha\gamma-\sigma)^{-1}\left\|v-\widetilde{d}\right\|\right\}\right)$.

We have that

$$\begin{aligned} \|\widetilde{a}-\overline{x}\| &\le \|\widetilde{a}-x\| + \|x-\overline{x}\| < \rho+\rho < \varepsilon, \\ \left\|\widetilde{b}-\overline{y}\right\| &\le \left\|\widetilde{b}-y\right\| + \|y-\overline{y}\| < \alpha\rho+\alpha\rho < \varepsilon, \\ \|\widetilde{c}-\overline{z}\| &\le \|\widetilde{c}-z\| + \|z-\overline{z}\| < \beta\rho+\beta\rho < \varepsilon, \\ \left\|\widetilde{d}-\overline{w}\right\| &\le \left\|\widetilde{d}-w\right\| + \|w-\overline{w}\| < (\alpha\gamma+\beta\delta)\rho + (\alpha\gamma+\beta\delta)\rho < \varepsilon, \\ \left\|\widetilde{d}+(\alpha\gamma-\sigma)\zeta v'-\overline{w}\right\| &\le \left\|\widetilde{d}-w\right\| + \|w-\overline{w}\| + (\alpha\gamma-\sigma)\zeta < (\alpha\gamma+\beta\delta)\rho \\ &\quad +(\alpha\gamma+\beta\delta)\rho + 3^{-1}\varepsilon < \varepsilon, \end{aligned}$$

hence by (A.8),

$$\begin{aligned} T_{-M}(\widetilde{b}, G_{\widetilde{c}}^{-1}(\widetilde{d}+(\alpha\gamma-\sigma)\zeta v')) &\le e_{-M}\left(G_{\widetilde{c}}^{-1}(\widetilde{d})\cap B(\overline{y},\varepsilon), G_{\widetilde{c}}^{-1}(\widetilde{d}+(\alpha\gamma-\sigma)\zeta v')\right) \\ &\le \gamma^{-1}T_{-P}(\widetilde{d},\widetilde{d}+(\alpha\gamma-\sigma)\zeta v') = \gamma^{-1}(\alpha\gamma-\sigma)\zeta \\ &< \gamma^{-1}(\alpha\gamma-2^{-1}\sigma)\zeta, \end{aligned}$$

hence there exists $m \in \text{cone}\, M$ with $\|m\| < 1$ such that $\widetilde{b}-\gamma^{-1}(\alpha\gamma-2^{-1}\sigma)\zeta m \in G_{\widetilde{c}}^{-1}(\widetilde{d}+(\alpha\gamma-\sigma)\zeta v')$ or, equivalently,

$$\widetilde{d}+(\alpha\gamma-\sigma)\zeta v' \in G(\widetilde{b}-\gamma^{-1}(\alpha\gamma-2^{-1}\sigma)\zeta m, \widetilde{c}).$$

Now, since $\zeta < \varepsilon$ and $\widetilde{b}-\gamma^{-1}(\alpha\gamma-2^{-1}\sigma)\zeta m \in \widetilde{b}-[0,\alpha\zeta)\cdot M$, it follows using (4.6) that

$$\widetilde{b}-\gamma^{-1}(\alpha\gamma-2^{-1}\sigma)\zeta m \in \widetilde{b}-[0,\alpha\zeta)\cdot M \subset F_1(\widetilde{a}+[0,\zeta)\cdot L),$$

hence there exists $\ell \in \text{cone}\, L$ with $\|\ell\| < 1$ such that $\widetilde{b}-\gamma^{-1}(\alpha\gamma-2^{-1}\sigma)\zeta m \in F_1(\widetilde{a}+\zeta\ell)$. But we have

$$\|\widetilde{a}+\zeta\ell-\overline{x}\| < \|\widetilde{a}-x\| + \|x-\overline{x}\| + \zeta < \rho+\rho+\rho \le \varepsilon,$$

and since $\widetilde{c} \in B(\overline{z},\varepsilon)$, we can apply the directional Aubin property of F_2 (4.8) to find that

$$T_N(\widetilde{c}, F_2(\widetilde{a}+\zeta\ell)) \le e_N(F_2(\widetilde{a})\cap B(\overline{z},\varepsilon), F_2(\widetilde{a}+\zeta\ell)) \le \beta T_L(\widetilde{a},\widetilde{a}+\zeta\ell) = \beta\zeta\,\|\ell\| < \beta\zeta.$$

It follows that we can find $n \in \text{cone}\, N$ with $\|n\| < 1$ such that $\widetilde{c}+\beta\zeta n \in F_2(\widetilde{a}+\zeta\ell)$.

Finally, since

$$\begin{aligned} \left\|\widetilde{b}-\gamma^{-1}(\alpha\gamma-2^{-1}\sigma)\zeta m-\overline{y}\right\| &< \left\|\widetilde{b}-y\right\| + \|y-\overline{y}\| + \alpha\zeta < \alpha\rho+\alpha\rho+\alpha\rho \le \varepsilon, \\ \|\widetilde{c}+\beta\zeta n-\overline{z}\| &< \|\widetilde{c}-z\| + \|z-\overline{z}\| + \beta\zeta < \beta\rho+\beta\rho+\beta\rho \le \varepsilon, \end{aligned}$$

we can use the directional Aubin property of G with respect to z (4.9) to get that

$$\begin{aligned}
&T_P(\widetilde{d}+(\alpha\gamma-\sigma)\zeta v', G_{\widetilde{b}-\gamma^{-1}(\alpha\gamma-2^{-1}\sigma)\zeta m}(\widetilde{c}+\beta\zeta n))\\
&\leq e_P(G_{\widetilde{b}-\gamma^{-1}(\alpha\gamma-2^{-1}\sigma)\zeta m}(\widetilde{c})\cap B(\overline{w},\varepsilon), G_{\widetilde{b}-\gamma^{-1}(\alpha\gamma-2^{-1}\sigma)\zeta m}(\widetilde{c}+\beta\zeta n))\\
&\leq \delta T_N(\widetilde{c},\widetilde{c}+\beta\zeta n)=\beta\delta\zeta\,\|n\|<\beta\delta\zeta,
\end{aligned}$$

hence there exists $p\in\operatorname{cone}P$ with $\|p\|<1$ such that

$$\widetilde{d}+(\alpha\gamma-\sigma)\zeta v'+\beta\delta\zeta p\in G(\widetilde{b}-\gamma^{-1}(\alpha\gamma-2^{-1}\sigma)\zeta m,\widetilde{c}+\beta\zeta n).$$

Observe that

$$\begin{aligned}
\left\|\widetilde{d}+(\alpha\gamma-\sigma)\zeta v'+\beta\delta\zeta p-\overline{w}\right\| &< \left\|\widetilde{d}-w\right\|+\|w-\overline{w}\|+(\alpha\gamma-\sigma+\beta\delta)\zeta\\
&< (\alpha\gamma+\beta\delta)\rho+(\alpha\gamma+\beta\delta)\rho+(\alpha\gamma+\beta\delta)\rho\ <\varepsilon,
\end{aligned}$$

hence

$$\left(\widetilde{a}+\zeta\ell,\widetilde{b}-\gamma^{-1}(\alpha\gamma-2^{-1}\sigma)\zeta m,\widetilde{c}+\beta\zeta n,\widetilde{d}+(\alpha\gamma-\sigma)\zeta v'+\beta\delta\zeta p\right)\in\Omega\cap\overline{A},$$

and we can use the second relation in the Ekeland variational principle to find that

$$\begin{aligned}
\left\|v-\widetilde{d}\right\| \leq\ & T_P(v,\widetilde{d}+(\alpha\gamma-\sigma)\zeta v'+\beta\delta\zeta p)\\
&+\ \tau c\max\left\{\begin{array}{c}T_{-L}(\widetilde{a}+\zeta\ell,\widetilde{a}),\alpha^{-1}T_M(\widetilde{b}-\gamma^{-1}(\alpha\gamma-2^{-1}\sigma)\zeta m,\widetilde{b}),\\ \beta^{-1}T_{-N}(\widetilde{c}+\beta\zeta n,\widetilde{c}),(\alpha\gamma+\beta\delta)^{-1}\left\|(\alpha\gamma-\sigma)\zeta v'+\beta\delta\zeta p\right\|\end{array}\right\}.
\end{aligned}$$

Remark that

$$\begin{aligned}
\widetilde{d}+(\alpha\gamma-\sigma)\zeta v'+\beta\delta\zeta p &= v+(\widetilde{d}-v)-(\alpha\gamma-\sigma)\zeta\frac{\widetilde{d}-v}{\left\|v-\widetilde{d}\right\|}+\beta\delta\zeta p\\
&= v+\left(1-\zeta\frac{\alpha\gamma-\sigma}{\left\|v-\widetilde{d}\right\|}\right)(\widetilde{d}-v)+\beta\delta\zeta p\\
&\in v+\operatorname{cone}P+\operatorname{cone}P=v+\operatorname{cone}P.
\end{aligned}$$

Then the previous relation becomes

$$\begin{aligned}
\left\|v-\widetilde{d}\right\| &\leq \left\|v-\left[v+\left(1-\zeta\frac{\alpha\gamma-\sigma}{\left\|v-\widetilde{d}\right\|}\right)(\widetilde{d}-v)+\beta\delta\zeta p\right]\right\|\\
&\quad+\tau c\max\{\zeta\,\|\ell\|,\alpha^{-1}\gamma^{-1}(\alpha\gamma-2^{-1}\sigma)\zeta\,\|m\|,\beta^{-1}\beta\zeta\,\|n\|,\\
&\quad(\alpha\gamma+\beta\delta)^{-1}\zeta\left\|(\alpha\gamma-\sigma)v'+\beta\delta p\right\|\}\\
&\leq \left\|\left(1-\zeta\frac{\alpha\gamma-\sigma}{\left\|v-\widetilde{d}\right\|}\right)(\widetilde{d}-v)\right\|+\beta\delta\zeta+\tau c\zeta\\
&= \left\|v-\widetilde{d}\right\|-\zeta\,(\alpha\gamma-\sigma)+\beta\delta\zeta+\tau c\zeta.
\end{aligned}$$

Using this and (A.9), we get

$$\tau c\zeta\geq\zeta\,(\alpha\gamma-\beta\delta-\sigma)=\zeta(c-\sigma)>\zeta\tau c,$$

a contradiction. This finishes the proof. □

Taking into account that the proof of the directional EVP is based on an iterative procedure, we can summarize the implications between the assertions in this work as follows:

directional EVP	$\Rightarrow$	criterion for directional regularity of single-valued maps	$\Rightarrow$	criterion for directional regularity of set-valued maps	$\Rightarrow$	directional regularity of compositions
		$\Uparrow$				$\Uparrow$
		iterative procedure		$\Rightarrow$		directional EVP.

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