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Well-posedness and Subdifferentials of Optimal Value and Infimal Convolution

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Abstract
We show that well-posedness (namely approximative well-posedness) properties of optimization problems are very efficient tools in subdifferential calculus of optimal value (marginal) function and in particular of infimal convolution. Under well-posedness conditions we establish an inclusion for the Mordukhovich limiting subdifferential of the marginal function and obtain new properties and descriptions of the Fréchet, proximal and Mordukhovich limiting subdifferentials of the infimal convolution. We also formulate sufficient conditions for well-posedness properties under consideration.

Keywords Marginal function · Optimal value function · Infimal convolution · Well-posedness · Fréchet subdifferential · Mordukhovich subdifferential · Ekeland variational principle

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1 Introduction

The optimal value function (or marginal function in other terminology) reveals dependence of optimal value on some parameters. Quite often, the parameterized optimization problem

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is a perturbation of some original optimization problem. Such problems arise in optimization (e.g., in the method of Lagrange multipliers) and regularization methods (such as Moreau and Lasry-Lions regularizations). In particular, the Moreau-type infimal convolution problem is a very important case of parameterized optimization problems and includes such significant examples as the best approximation problem in a Banach space, the optimal control problem with constant dynamics, Moreau regularization etc.

Differential properties of the optimal value function are crucial in both theory and numerical methods. Since the marginal functions are nonsmooth in general, the proper terms to describe their differential properties are subdifferentials such as Fréchet, Clarke, limiting and proximal subdifferentials.

A lot of investigations are devoted to study and describe differentials of the optimal value function in a general setting and in particular cases (see [4–11, 13–22] and references therein). We propose in this work an approach to investigate differential properties of the optimal value function based on some special well-posedness (WP) conditions. Hadamard and Tykhonov WP and WP under perturbations [2, 5, 12, 23, 24] are well-known and very useful concepts in stability and sensitivity analysis, numerical optimization and optimal control methods, variational analysis and so on. We introduce new WP conditions, namely approximative WP and stronger Lipschitz approximative WP, which are related to famous WP conditions but differ from them. We also compare approximative WP condition with docility condition introduced in [19].

Let us focus on various ideas and results of the present paper. Applying the Ekeland variational principle we obtain, in the general Banach space framework, an inclusion for the Fréchet ε-subdifferential of the optimal value function into the Fréchet ε-subdifferential of the objective function (Theorem 3.2), which improves the result of Ngai and Penot [18, Theorem 3] obtained for Asplund space. Then we use this inclusion to prove an inclusion for the Mordukhovich limiting subdifferential under approximative WP conditions (Theorem 3.5). The latter result is akin to the result [18, Corollary 5], but neither of them is a consequence of the other even in Asplund space. As a consequence of Theorem 3.5 we obtain some known results of Thibault [22] and Ngai, Luc and Théra [17, Theorem 2.5]. Another consequence of Theorem 3.5 is the inclusion of the Mordukhovich limiting subdifferential of infimal convolution of two functions into the intersection of the Mordukhovich limiting subdifferentials of these functions (Theorem 4.6(a)) under approximative WP conditions. Though the reverse inclusion fails in general, it holds under Lipschitz approximative WP conditions for lower regular functions (Theorem 4.6(b)). Using sufficient conditions for Lipschitz approximative WP we obtain sufficient conditions for coincidence of the Mordukhovich limiting subdifferential of infimal convolution of two functions and the intersection of the Mordukhovich limiting subdifferentials of these functions (Theorem 4.8).

2 Approximative Well-posedness

In the present paper we continue research started in [6, 7] and [8]. Let (U, d) and (X, d) be metric spaces. For a real $\varepsilon > 0$ and a point $x \in X$ we will denote the open (resp. closed) *ball* centered at x with radius $\varepsilon > 0$ by $B(x, \varepsilon)$ (resp. $B[x, \varepsilon]$). The *effective domain* of an extended real-valued function $f : X \to \mathbb{R} \cup \{+\infty\}$ is $\operatorname{dom} f := \{x \in X : f(x) \in \mathbb{R}\}$.

Throughout this section, we keep U and X as *metric spaces* as stated above. Let $h : U \times X \to \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function. Consider the problem

$$\mathcal{P}_h: \quad \text{Minimize } h(u, x) \text{ over } x \in X \tag{2.1}$$

with parameter $u \in U$. The *optimal value* of $\mathcal{P}_h$ at $u \in U$ is

$$h_{\inf}(u) := \inf_{x \in X} h(u, x).$$

If $x \in X$ satisfies the equality $h(u, x) = h_{\inf}(u) \in \mathbb{R}$, it is called a *solution* of $\mathcal{P}_h$ at $u \in U$. A sequence (x_k) in X is called *minimizing* for $\mathcal{P}_h$ at $u \in U$ if

$$\lim_{k\to\infty} h(u, x_k) = h_{\inf}(u).$$

The problem $\mathcal{P}_h$ is called *Tykhonov well-posed* at $u_0 \in U$ if it admits a unique solution x_0 and every minimizing sequence for $\mathcal{P}_h$ at u_0 converges to x_0 (see [2, 12, 23, 24]). In the case when $h(u_0, \cdot)$ is lower semicontinuous, $\mathcal{P}_h$ is Tykhonov well-posed at u_0 if and only if every minimizing sequence for $\mathcal{P}_h$ at u_0 converges.

Let $x_0 \in X$ be a unique solution of $\mathcal{P}_h$ at $u_0 \in U$. Denote by $\mathcal{M}(h, u)$ the set of all minimizing sequences for $\mathcal{P}_h$ at $u \in U$ and define the function $\Delta_{h,u_0} : U \to [0, +\infty[$ as

$$\Delta_{h,u_0}(u) := \inf_{(x_k)\in\mathcal{M}(h,u)} \liminf_{k\to\infty} d(x_k, x_0), \qquad u \in U. \tag{2.2}$$

The problem $\mathcal{P}_h$ is called *approximately well-posed (AWP)* at $u_0 \in U$ if it admits a unique solution at u_0 and

$$\lim_{u \xrightarrow{h_{\inf}} u_0} \Delta_{h,u_0}(u) = 0, \tag{2.3}$$

i.e. $\Delta_{h,u_0}(u_k) \to 0$ for any sequence (u_k) in U such that $u_k \to u_0$ and $h_{\inf}(u_k) \to h_{\inf}(u_0)$.

If, in addition, there exist positive reals λ_1, λ_2 and δ such that

$$\Delta_{h,u_0}(u) \le \lambda_1 d(u, u_0) + \lambda_2 |h_{\inf}(u) - h_{\inf}(u_0)|$$

for all $u \in B(u_0, \delta)$ such that $|h_{\inf}(u) - h_{\inf}(u_0)| < \delta$, then the problem $\mathcal{P}_h$ is called *Lipschitz approximately well-posed (LAWP)* at u_0.

Clearly, if the problem $\mathcal{P}_h$ is LAWP at u_0, it is AWP at u_0.

Remark 2.1 Suppose that $x_0 \in X$ is a unique solution of $\mathcal{P}_h$ at $u_0 \in \operatorname{dom} h_{\inf}$ and that for any $u \in \operatorname{dom} h_{\inf}$ around u_0 there exists a solution $x(u)$ of $\mathcal{P}_h$ at u such that $x(u) \to x_0$ as $u \to u_0$. Then $\mathcal{P}_h$ is AWP at u_0. If, in addition, there exists $\lambda_1, \lambda_2 > 0$ such that $d(x(u), x_0) \le \lambda_1 d(u, u_0) + \lambda_2 |h_{\inf}(u) - h_{\inf}(u_0)|$ for all $u \in \operatorname{dom} h_{\inf}$ around u_0, then $\mathcal{P}_h$ is LAWP at u_0.

Remark 2.2 If $\mathcal{P}_h$ at u_0 is well-posed under perturbations (see, e.g., Zolezzi [24]), then it is AWP at u_0.

Lemmas 2.1 and 2.3 offer sufficient conditions for the problem $\mathcal{P}_h$ to be AWP. The first lemma is a variant of Berge's maximum theorem (see Example 6, Section 1, Chapter I and Propositions 1, 2, Section 1, Chapter IX in [2] and also Proposition 5.1 in [12]).

Lemma 2.1 *Let (U, d) be a metric space, (X, d) be a compact metric space and $h : U \times X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. Suppose that $\mathcal{P}_h$ admits a unique solution x_0 at $u_0 \in \operatorname{dom} h_{\inf}$. Then $\mathcal{P}_h$ is AWP at u_0.*

Proof Fix any sequence (u_k) in U such that $u_k \to u_0$ and $h_{\inf}(u_k) \to h_{\inf}(u_0)$. As $h_{\inf}(u_0)$ is finite, for sufficiently large k the value $h_{\inf}(u_k)$ is finite too. In view of compactness of X and lower semicontinuity of h for sufficiently large k there exists a solution x_k of $\mathcal{P}_h$ at u_k. Let us prove that $x_k \to x_0$. Suppose the contrary. Then extracting a subsequence due to compactness of X we may suppose that $x_k \to \widehat{x} \neq x_0$. Using the lower semicontinuity of h and the relations $h(u_k, x_k) = h_{\inf}(u_k) \to h_{\inf}(u_0)$, we arrive at $h(u_0, \widehat{x}) \leq h_{\inf}(u_0)$. This means that $\widehat{x}$ and x_0 are two different solutions of $\mathcal{P}_h$ at u_0. This contradicts the uniqueness assumption of the lemma and proves that $x_k \to x_0$. For any $k \in \mathbb{N}$ the constant sequence $(x_k, x_k, \ldots)$ belongs to $\mathcal{M}(h, u_k)$ and hence $\Delta_{h,u_0}(u_k) \leq d(x_k, x_0)$. Consequently, $\Delta_{h,u_0}(u_k) \to 0$ and $\mathcal{P}_h$ is AWP at u_0. □

Lemma 2.2 *Let $x_0 \in X$ be a unique solution of $\mathcal{P}_h$ at $u_0 \in U$. Then for any $u \in U$ there exists $(x_k) \in \mathcal{M}(h, u)$ such that*

$$\lim_{k\to\infty} d(x_k, x_0) = \Delta_{h,u_0}(u). \tag{2.4}$$

Proof Fix any $u \in U$. If $\Delta_{h,u_0}(u) = +\infty$, then by (2.2) for any $(x_k) \in \mathcal{M}(h, u)$ we have $\liminf_{k\to\infty} d(x_k, x_0) = +\infty$ and the statement of the lemma holds true. If $h_{\inf}(u) = +\infty$, then $h(u, x) = +\infty$ for any $x \in X$ and the constant sequence $(x_0, x_0, \ldots)$ is a minimizing one for $\mathcal{P}_h$ at u. In this case $\Delta_{h,u_0}(u) = 0$ and the desired statement holds true as well. Further, we suppose that $\Delta_{h,u_0}(u) < +\infty$ and $h_{\inf}(u) < +\infty$. Fix any numbers $\lambda > \Delta_{h,u_0}(u)$ and $\mu > h_{\inf}(u)$. According to (2.2) one can find a sequence $(z_i) \in \mathcal{M}(h, u)$ such that $\liminf_{i\to\infty} d(z_i, x_0) < \lambda$. Consequently, there exists $i \in \mathbb{N}$ which satisfies the inequalities $d(z_i, x_0) < \lambda$ and $h(u, z_i) < \mu$. Now fix any sequences (λ_k) and (μ_k) such that $\lambda_k \downarrow \Delta_{h,u_0}(u)$ and $\mu_k \downarrow h_{\inf}(u)$. As it was shown above, for any $k \in \mathbb{N}$ there exists $x_k \in X$ which satisfies the inequalities $d(x_k, x_0) < \lambda_k$ and $h(u, x_k) < \mu_k$. Hence, $(x_k) \in \mathcal{M}(h, u)$ and

$$\limsup_{k\to\infty} d(x_k, x_0) \leq \limsup_{k\to\infty} \lambda_k = \Delta_{h,u_0}(u) \leq \liminf_{k\to\infty} d(x_k, x_0).$$

This implies (2.4). □

Lemma 2.3 *Let (U, d) and (X, d) be metric spaces. Suppose that $u_0 \in \operatorname{dom} h_{\inf}$ and the function $h(\cdot, x)$ is continuous at u_0 uniformly with respect to $x \in X$, i.e. there exists a function $\varepsilon : U \to [0, +\infty[$ such that*

$$h(u_0, x) - \varepsilon(u) \leq h(u, x) \leq h(u_0, x) + \varepsilon(u) \qquad \forall u \in U, \quad \forall x \in X$$

and

$$\lim_{u\to u_0} \varepsilon(u) = 0.$$

Then the following hold:

(a) The function $h_{\inf}(\cdot)$ is continuous at u_0.
(b) If additionally $\mathcal{P}_h$ is Tykhonov well-posed at u_0, then $\mathcal{P}_h$ is AWP at u_0.

Proof (a). By the assumption, we have

$$h_{\inf}(u_0) - \varepsilon(u) \leq h_{\inf}(u) \leq h_{\inf}(u_0) + \varepsilon(u) \qquad \forall u \in U,$$

and hence $\lim_{u\to u_0} h_{\inf}(u) = h_{\inf}(u_0)$.

(b). Assume additionally that $\mathcal{P}_h$ is Tykhonov well-posed at u_0. Let x_0 be the solution of $\mathcal{P}_h$ at u_0. Fix a sequence (u_k) in U that converges to u_0. Lemma 2.2 implies that for any $k \in \mathbb{N}$ there exists $x_k \in X$ such that $h(u_k, x_k) < h_{\inf}(u_k) + \frac{1}{k}$ and $d(x_k, x_0) > \Delta_{h,u_0}(u_k) - \frac{1}{k}$. Since

$$h(u_0, x_k) \leq h(u_k, x_k) + \varepsilon(u_k) \leq h_{\inf}(u_k) + \frac{1}{k} + \varepsilon(u_k) \to h_{\inf}(u_0) \quad \text{as} \quad k \to \infty,$$

it follows that (x_k) is a minimizing sequence for $\mathcal{P}_h$ at u_0. Then by Tykhonov well-posedness we have $x_k \to x_0$, and consequently $\Delta_{h,u_0}(u_k) < d(x_k, x_0) + \frac{1}{k} \to 0$. Therefore, $\mathcal{P}_h$ is AWP at u_0.

□

The following proposition provides a sufficient condition for the problem $\mathcal{P}_h$ to be AWP and LAWP.

Proposition 2.4 *Let* (U, d) *and* (X, d) *be metric spaces. Assume that* $(u_0, x_0) \in \operatorname{dom} h$ *and*

$$h(u, x) \geq h(u_0, x_0) + \varphi(d(x, x_0)) - \xi(d(u, u_0)) \qquad \forall u \in U, \quad \forall x \in X, \tag{2.5}$$

where $\varphi : [0, +\infty[\to [0, +\infty[$ *is a nondecreasing function such that* $\varphi(t) > \varphi(0) = 0$ *for all* $t > 0$ *and* $\xi : [0, +\infty[\to [0, +\infty[$ *is such that* $\lim_{t\downarrow 0} \xi(t) = \xi(0) = 0$. *Then* $\mathcal{P}_h$ *admits* x_0 *as unique solution at* u_0, $h_{\inf}$ *is lower semicontinuous at* u_0 *and* $\mathcal{P}_h$ *is AWP at* u_0.

If additionally there exist positive constants λ, μ *such that*

$$\varphi(t) \geq \lambda t, \qquad \xi(t) \leq \mu t \qquad \forall t \geq 0, \tag{2.6}$$

then $\mathcal{P}_h$ *is LAWP at* u_0.

Proof Putting $u = u_0$ in (2.5), we get

$$h(u_0, x) \geq h(u_0, x_0) + \varphi(d(x, x_0)) > h(u_0, x_0) \qquad \forall x \in X \setminus \{x_0\}.$$

Consequently, $h_{\inf}(u_0) = h(u_0, x_0)$ and $\mathcal{P}_h$ admits x_0 as unique solution at u_0. Inequality (2.5) also implies that

$$h_{\inf}(u) \geq h(u_0, x_0) - \xi(d(u, u_0)) \overset{u\to u_0}{\longrightarrow} h(u_0, x_0) = h_{\inf}(u_0)$$

and hence $h_{\inf}$ is lower semicontinuous at u_0.

Fix any $\tau_1 \in]0, \varphi(1)[$ and consider the function $\varphi^{-1} : [0, \tau_1] \to [0, 1]$ defined as

$$\varphi^{-1}(\tau) := \inf\{t > 0 : \varphi(t) > \tau\}, \qquad \tau \in [0, \tau_1]. \tag{2.7}$$

Since φ is nondecreasing and $\varphi(t) > \varphi(0) = 0$ for all $t > 0$, it follows that

$$\lim_{\tau\downarrow 0} \varphi^{-1}(\tau) = \varphi^{-1}(0) = 0. \tag{2.8}$$

Let us prove that

$$\Delta_{h,u_0}(u) \leq \varphi^{-1}(\tau_u) \tag{2.9}$$

for any $u \in \operatorname{dom} h_{\inf}$ such that

$$\xi(d(u, u_0)) + |h_{\inf}(u) - h_{\inf}(u_0)| =: \tau_u < \tau_1. \tag{2.10}$$

Assume the contrary: there exists $u \in \operatorname{dom} h_{\inf}$ which satisfies (2.10) and $\Delta_{h,u_0}(u) > \varphi^{-1}(\tau_u)$. Then one can find t_u such that $\varphi^{-1}(\tau_u) < t_u < \Delta_{h,u_0}(u)$. Fix any sequence $(x_k) \in \mathcal{M}(h, u)$. Using (2.2) one can find $k_0 \in \mathbb{N}$ such that $d(x_k, x_0) > t_u$ (and hence $\varphi(d(x_k, x_0)) \geq \varphi(t_u)$) for all $k \geq k_0$. According to (2.5) we get for all $k \geq k_0$

$$h(u, x_k) \geq h(u_0, x_0) + \varphi(d(x_k, x_0)) - \xi(d(u, u_0)) \geq h_{\inf}(u_0) + \varphi(t_u) - \xi(d(u, u_0)).$$

Passing to the limit as $k \to \infty$ and taking into account that (x_k) is a minimizing sequence for $\mathcal{P}_h$ at u, we have by (2.10)

$$\varphi(t_u) \le \xi(d(u, u_0)) + h_{\inf}(u) - h_{\inf}(u_0) \le \tau_u.$$

In view of (2.7) this contradicts the inequality $\varphi^{-1}(\tau_u) < t_u$. So, (2.9) is proved for any $u \in \operatorname{dom} h_{\inf}$ satisfying (2.10).

In order to prove that

$$\lim_{u \xrightarrow{h_{\inf}} u_0} \Delta_{h,u_0}(u) = 0, \tag{2.11}$$

we fix any sequence (u_k) in $\operatorname{dom} h_{\inf}$ such that $u_k \xrightarrow{h_{\inf}} u_0$. We also fix any $\varepsilon > 0$. In view of (2.8) one can find $\tau \in]0, \tau_1[$ such that $\varphi^{-1}(\tau) < \varepsilon$. Since $\xi(d(u_k, u_0)) + |h_{\inf}(u_k) - h_{\inf}(u_0)| \to 0$ (here we use the assumption $\lim_{t\downarrow 0} \xi(t) = 0$), we have $\xi(d(u_k, u_0)) + |h_{\inf}(u_k) - h_{\inf}(u_0)| < \tau$ for sufficiently large k. Then by (2.9) we have $\Delta_{h,u_0}(u_k) \le \varphi^{-1}(\tau) < \varepsilon$ for sufficiently large k. Consequently, $\Delta_{h,u_0}(u_k) \to 0$ as $k \to \infty$ and (2.11) is proved. From (2.11) we see that $\mathcal{P}_h$ is AWP at u_0.

Now assume that (2.6) holds for some $\lambda, \mu > 0$. If $t > \frac{\tau_u}{\lambda}$, then $\varphi(t) \ge \lambda t > \tau_u$. Hence,

$$\varphi^{-1}(\tau_u) = \inf\{t > 0 : \varphi(t) > \tau_u\} \le \inf\left\{t > 0 : t > \frac{\tau_u}{\lambda}\right\} = \frac{\tau_u}{\lambda}. \tag{2.12}$$

Define $\delta := \frac{\tau_1}{1+\mu}$. For any $u \in B(u_0, \delta)$ such that $|h_{\inf}(u) - h_{\inf}(u_0)| < \delta$ (if any) we have

$$\begin{aligned} \tau_u &= \xi(d(u, u_0)) + |h_{\inf}(u) - h_{\inf}(u_0)| \\ &\le \mu d(u, u_0) + |h_{\inf}(u) - h_{\inf}(u_0)| < (\mu + 1)\delta = \tau_1 \end{aligned}$$

and by (2.9) and (2.12) we get

$$\Delta_{h,u_0}(u) \le \varphi^{-1}(\tau_u) \le \frac{\tau_u}{\lambda} \le \frac{\mu}{\lambda} d(u, u_0) + \frac{1}{\lambda}|h_{\inf}(u) - h_{\inf}(u_0)|.$$

Consequently, $\mathcal{P}_h$ is LAWP at u_0. □

Compliance, docility and meekness conditions introduced in [19, 20] are shown to be useful tools in subdifferential calculus of optimal value functions. Let us compare AWP condition with docility one, which is the closest to AWP among above mentioned conditions.

Given $\varepsilon \ge 0$, the ε-solution of $\mathcal{P}_h$ at $u \in U$ is

$$S_\varepsilon(u) = \{x \in X : h(u, x) \le h_{\inf}(u) + \varepsilon\}.$$

The function $h : U \times X \to \mathbb{R} \cup \{+\infty\}$ is called *docile* at $u_0 \in U$ if

$$\forall \alpha > 0\, \exists \eta > 0,\ \forall u \in B(u_0, \eta),\ \forall \beta > 0, \quad S_\beta(u) \cap S_\alpha(u_0) \ne \emptyset. \tag{2.13}$$

Lemma 2.5 *Let (U, d) and (X, d) be metric spaces. Assume that $\mathcal{P}_h$ is Tykhonov well-posed at $u_0 \in U$ and h is docile at u_0. Then $\mathcal{P}_h$ is AWP at u_0.*

Proof Fix any (u_k) in U such that $u_k \to u_0$ and $h_{\inf}(u_k) \to h_{\inf}(u_0)$ and fix any $\varepsilon > 0$. It suffices to prove that $\Delta_{h,u_0}(u_k) \le \varepsilon$ for sufficiently large $k \in \mathbb{N}$. Since $\mathcal{P}_h$ is Tykhonov well-posed at u_0 it admits a unique solution $x_0 \in X$ ($S_0(u_0) = \{x_0\}$) and there exists $\alpha_\varepsilon > 0$ such that $S_{\alpha_\varepsilon}(u_0) \subset B(x_0, \varepsilon)$. Using docility condition one can find $\eta_\varepsilon > 0$ such that for any $u \in B(u_0, \eta_\varepsilon)$ and any $\beta > 0$ we have $S_\beta(u) \cap S_{\alpha_\varepsilon}(u_0) \ne \emptyset$. Since $u_k \to u_0$ there exists K_ε such that $u_k \in B(u_0, \eta_\varepsilon)$ for all $k \ge K_\varepsilon$. Fix any $k \ge K_\varepsilon$ and let us show that $\Delta_{h,u_0}(u_k) \le \varepsilon$. Since $S_\beta(u_k) \cap S_{\alpha_\varepsilon}(u_0) \ne \emptyset$ for any $\beta > 0$, one can find a sequence (x_n) in $\mathcal{M}(h, u_k)$ such that $x_n \in S_{\alpha_\varepsilon}(u_0)$ for all $n \in \mathbb{N}$. In view of the inclusions $x_n \in S_{\alpha_\varepsilon}(u_0) \subset B(x_0, \varepsilon)$ we obtain the desired inequality $\Delta_{h,u_0}(u_k) \le \varepsilon$. □

Lemma 2.6 *Let (U, d) and (X, d) be metric spaces. Assume that $\mathcal{P}_h$ is AWP at $u_0 \in U$ and $x_0 \in X$ is the solution of $\mathcal{P}_h$ at u_0. Suppose that $h(u_0, \cdot)$ is continuous at x_0 and $h_{\inf}$ is continuous at u_0. Then h is docile at u_0.*

Proof Assume the contrary. Then there exists $\alpha > 0$ such that for any $k \in \mathbb{N}$ one can find $u_k \in B(u_0, \frac{1}{k})$ and $\beta_k > 0$ with $S_{\beta_k}(u_k) \cap S_\alpha(u_0) = \emptyset$. Since $h(u_0, \cdot)$ is continuous at x_0 one can find $\delta > 0$ such that $h(u_0, x) < h(u_0, x_0) + \alpha$ for all $x \in B(x_0, \delta)$ and hence $B(x_0, \delta) \subset S_\alpha(u_0)$. Let us show that $\Delta_{h,u_0}(u_k) \ge \delta$ for all $k \in \mathbb{N}$. Fix any $k \in \mathbb{N}$. According to Lemma 2.2 there exists a sequence (x_n) in $\mathcal{M}(h, u_k)$ such that $\lim_{n\to\infty} d(x_n, x_0) = \Delta_{h,u_0}(u_k)$. The inclusion $(x_n) \in \mathcal{M}(h, u_k)$ also gives $x_n \in S_{\beta_k}(u_k)$ for sufficiently large n. In view of $S_{\beta_k}(u_k) \cap B(x_0, \delta) = \emptyset$ we obtain $d(x_n, x_0) \ge \delta$ for sufficiently large n, and thus $\Delta_{h,u_0}(u_k) \ge \delta > 0$. Since $u_k \in B(u_0, \frac{1}{k})$ for all $k \in \mathbb{N}$ it follows that $u_k \to u_0$. By continuity of $h_{\inf}$ at u_0 we get $h_{\inf}(u_k) \to h_{\inf}(u_0)$. So, (2.3) is violated. This contradicts the assumption that $\mathcal{P}_h$ is AWP at u_0. □

Remark 2.3 In general AWP condition does not imply docility condition. For example, let

$$U = X = \mathbb{R}, \qquad h(u, x) = \begin{cases} 0, & u = x, \\ +\infty, & u \ne x. \end{cases}$$

Then $\mathcal{P}_h$ is AWP (and even LAWP) at 0 but h is not docile at 0.

On the other hand, docility condition does not imply AWP condition. This can be seen from the example $h(u, x) = \frac{x^2+u^2}{1+x^4}$ with $U = X = \mathbb{R}$. In the later example h is docile at 0 while $\mathcal{P}_h$ is not AWP at 0.

3 Subdifferentials of the Optimal Value Function

From now on let X be a normed linear space. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a given function and let $x \in \operatorname{dom} f$.

For any $\varepsilon \ge 0$, the *Fréchet ε-subdifferential* of f at x is

$$\partial^{F,\varepsilon} f(x) = \{x^* \in X^* : \forall \eta > 0\, \exists \delta > 0,\ \forall x' \in B(x, \delta), \\ \langle x^*, x' - x\rangle \le f(x') - f(x) + (\varepsilon + \eta)\|x' - x\|\}.$$

If $\varepsilon = 0$, then $\partial^F f := \partial^{F,\varepsilon} f$ is called the *Fréchet subdifferential* of f.

The *proximal subdifferential* of f at x is

$$\partial^P f(x) = \{x^* \in X^* : \exists \delta > 0,\ \exists r > 0,\ \forall x' \in B(x, \delta), \\ \langle x^*, x' - x\rangle \le f(x') - f(x) + r\|x' - x\|^2\}.$$

The *Mordukhovich limiting subdifferential* $\partial^L f(x)$ at x is the set of all $x^* \in X^*$ such that there exist $\varepsilon_k \downarrow 0$, $x_k \to x_0$ with $f(x_k) \to f(x)$, and $x_k^* \to x^*$ weak*, $x_k^* \in \partial^{F,\varepsilon_k} f(x_k)$ for all $k \in \mathbb{N}$ (see [14]).

Hereinafter let U be also a normed linear space. Consider the linear space $U \times X$ endowed with the norm

$$\|(u, x)\| = \max\{\|u\|, \|x\|\}, \qquad (u, x) \in U \times X. \tag{3.1}$$

As usual, we identify $(U \times X)^*$ with $U^* \times X^*$ through the pairing

$$\langle (u^*, x^*), (u, x)\rangle = \langle u^*, u\rangle + \langle x^*, x\rangle \qquad \forall u \in U, \quad \forall x \in X.$$

As earlier we consider an extended real-valued function $h : U \times X \to \mathbb{R} \cup \{+\infty\}$.

Given a point $(u, x) \in \operatorname{dom} h$, we shall use $\partial^{F,\varepsilon} h(u, x)$, $\partial^P h(u, x)$ and $\partial^L h(u, x)$ to denote correspondingly the Fréchet ε-subdifferential, the proximal subdifferential and the limiting subdifferential of h at (u, x) with respect to the norm (3.1) in $U \times X$.

We denote by $\partial_u^{F,\varepsilon} h(u, x)$ (respectively by $\partial_u^P h(u, x)$) the Fréchet ε-subdifferential (respectively proximal subdifferential) of the function $h(\cdot, x)$ at the point u. We shall use $\partial_u^L h(u, x)$ to denote the set of all $u^* \in U^*$ such that there exist $\varepsilon_k \downarrow 0$, $(u_k, x_k) \to (u, x)$ with $h(u_k, x_k) \to h(u, x)$, and $u_k^* \to u^*$ weak*, $u_k^* \in \partial_u^{F,\varepsilon_k} h(u_k, x_k)$ for all $k \in \mathbb{N}$. Similarly, we define $\partial_x^{F,\varepsilon} h(u, x)$, $\partial_x^P h(u, x)$ and $\partial_x^L h(u, x)$.

Remark 3.1 It follows immediately from the definitions that for any $(u, x) \in \operatorname{dom} h$

$$\partial^{F,\varepsilon} h(u, x) \subset \partial_u^{F,\varepsilon} h(u, x) \times \partial_x^{F,\varepsilon} h(u, x), \qquad \varepsilon \geq 0,$$

$$\partial^P h(u, x) \subset \partial_u^P h(u, x) \times \partial_x^P h(u, x),$$

$$\partial^L h(u, x) \subset \partial_u^L h(u, x) \times \partial_x^L h(u, x).$$

The properties in the following lemma are easily verified and we omit their proofs.

Lemma 3.1 *Let U and X be normed spaces and let $x_0 \in X$ be a solution of $\mathcal{P}_h$ at $u_0 \in \operatorname{dom} h_{\inf}$. Then for any $\varepsilon \geq 0$*

$$\partial^{F,\varepsilon} h_{\inf}(u_0) \times \{0\} \subset \partial^{F,\varepsilon} h(u_0, x_0),$$

$$\partial^{F,\varepsilon} h_{\inf}(u_0) \subset \partial_u^{F,\varepsilon} h(u_0, x_0),$$

$$\partial^P h_{\inf}(u_0) \times \{0\} \subset \partial^P h(u_0, x_0),$$

$$\partial^P h_{\inf}(u_0) \subset \partial_u^P h(u_0, x_0).$$

Remark 3.2 The inclusion $\partial_u^{F,\varepsilon} h(u_0, x_0) \subset \partial^{F,\varepsilon} h_{\inf}(u_0)$ generally fails. Consider, for example, $U = X = \mathbb{R}$, $h(u, x) = |x - u|$. Then $h_{\inf}(u) = 0$ for all $u \in \mathbb{R}$, $\partial^F h_{\inf}(0) = \{0\}$, while $\partial_u^F h(0, 0) = [-1, 1]$. However an inclusion in the form

$$\liminf_{h(u_0,x)\to h_{\inf}(u_0)} \partial_u^F h(u_0, x) \subset \partial^F h_{\inf}(u_0)$$

was obtained by Penot under docility assumption (2.13) and some additional assumptions (see [19, Proposition 3.6] and related results in [19] and [20]).

The following theorem is an analogue of the first inclusion of Lemma 3.1 in the case when the infimum is not achieved. It provides a sharp inclusion for the Fréchet ε-subdifferential of the optimal value function.

Theorem 3.2 *Let U and X be Banach spaces and $h : U \times X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. Let $u^* \in \partial^{F,\varepsilon} h_{\inf}(u)$ for some $u \in \operatorname{dom} h_{\inf}$ and $\varepsilon \geq 0$. Let $\delta > 0$. Then there exists $\beta > 0$ such that for any $x \in X$ with $h(u, x) < h_{\inf}(u) + \beta$ there exist $\widehat{u} \in B(u, \delta)$, $\widehat{x} \in B(x, \delta)$ with $(u^*, 0) \in \partial^{F,\varepsilon+\delta} h(\widehat{u}, \widehat{x})$ and $|h(\widehat{u}, \widehat{x}) - h_{\inf}(u)| < \delta$.*

Proof Fix any $\eta > 0$. By definition of the Fréchet ε-subdifferential there exists $\theta > 0$ such that

$$\langle u^*, u' - u\rangle \le h_{\inf}(u') - h_{\inf}(u) + (\varepsilon + \eta)\|u' - u\| \qquad \forall u' \in B[u, 2\theta]. \tag{3.2}$$

Define

$$\beta_1 := \min\left\{\theta,\ \delta,\ \frac{\delta}{\delta + \|u^*\|_*},\ \frac{\delta}{\varepsilon + 2\eta + \|u^*\|_*}\right\} \quad \text{and} \quad \beta := \beta_1^2.$$

Fix any $x \in X$ with $h(u, x) < h_{\inf}(u) + \beta = h_{\inf}(u) + \beta_1^2$. Taking into account that $h_{\inf}(u') \le h(u', x')$ for any $u' \in U$, $x' \in X$, we have

$$\langle u^*, u' - u\rangle \le h(u', x') - h(u, x) + (\varepsilon + \eta)\|u' - u\| + \beta_1^2 \qquad \forall u' \in B[u, 2\theta], \quad \forall x' \in X.$$

In terms of the function

$$g(u', x') := h(u', x') + (\varepsilon + \eta)\|u' - u\| - \langle u^*, u'\rangle, \qquad u' \in U,\ x' \in X$$

it means that

$$g(u', x') \ge g(u, x) - \beta_1^2 \qquad \forall u' \in B[u, 2\theta], \quad \forall x' \in X.$$

Applying the Ekeland variational principle (see [3]) for the complete metric space $B[u, 2\theta]$ gives some $\widehat{u} \in B[u, 2\theta]$ and $\widehat{x} \in X$ with

$$\|\widehat{u} - u\| + \|\widehat{x} - x\| \le \beta_1, \tag{3.3}$$

$$g(\widehat{u}, \widehat{x}) \le g(u, x), \tag{3.4}$$

$$g(\widehat{u}, \widehat{x}) \le g(u', x') + \beta_1 \max\{\|u' - \widehat{u}\|, \|x' - \widehat{x}\|\} \qquad \forall u' \in B[u, 2\theta], \quad \forall x' \in X.$$

The latter inequality and the definition of g entail for all $u' \in B[\widehat{u}, \beta_1] \subset B[u, 2\theta]$ and all $x' \in X$

$$\begin{aligned}\langle u^*, u' - \widehat{u}\rangle &\le h(u', x') - h(\widehat{u}, \widehat{x}) + \beta_1 \max\{\|u' - \widehat{u}\|, \|x' - \widehat{x}\|\} \\ &\quad + (\varepsilon + \eta)(\|u' - u\| - \|\widehat{u} - u\|) \\ &\le h(u', x') - h(\widehat{u}, \widehat{x}) + (\varepsilon + \delta + \eta) \max\{\|u' - \widehat{u}\|, \|x' - \widehat{x}\|\},\end{aligned}$$

and hence $(u^*, 0) \in \partial^{F, \varepsilon+\delta} h(\widehat{u}, \widehat{x})$. Inequality (3.4) implies that

$$\begin{aligned}h(\widehat{u}, \widehat{x}) &\le h(u, x) + \langle u^*, \widehat{u} - u\rangle \le h(u, x) + \beta_1\|u^*\|_* \\ &< h_{\inf}(u) + \beta_1^2 + \beta_1\|u^*\|_* \le h_{\inf}(u) + \beta_1(\delta + \|u^*\|_*) \le h_{\inf}(u) + \delta.\end{aligned}$$

Using (3.2), (3.3), we have

$$\begin{aligned}h_{\inf}(u) - h(\widehat{u}, \widehat{x}) &\le h_{\inf}(u) - h_{\inf}(\widehat{u}) \le -\langle u^*, \widehat{u} - u\rangle + (\varepsilon + \eta)\|\widehat{u} - u\| \\ &\le (\|u^*\|_* + \varepsilon + \eta)\beta_1 < \delta.\end{aligned}$$

So, $|h(\widehat{u}, \widehat{x}) - h_{\inf}(u)| < \delta$, which finishes the proof. □

Lemma 3.3 *Let X be an Asplund space, $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function, $x_0 \in \operatorname{dom} f$, $\varepsilon \ge 0$, $\delta \ge 0$ and $x_0^* \in \partial^{F, \varepsilon+\delta} f(x_0)$. Then for any $\eta > 0$ there exist $x \in \operatorname{dom} f$ and $x^* \in X^*$ such that*

$$\|x - x_0\| < \eta, \quad |f(x) - f(x_0)| < \eta, \quad \|x^* - x_0^*\|_* < \delta + \eta, \quad x^* \in \partial^{F, \varepsilon} f(x).$$

Proof According to the definition of the Fréchet ε-subdifferential we have $\partial^{F,t} f(x_0) = \partial^F f_{t,x_0}(x_0)$ for any $t \ge 0$, where $f_{t,x_0}(x) := f(x) + t\|x - x_0\|$ for any $x \in X$. Applying the fuzzy sum rule [4, Theorem 3] (valid in Asplund spaces) for the sum $f_{\varepsilon+\delta,x_0}(x) = f_{\varepsilon,x_0}(x) + \delta\|x - x_0\|$, we complete the proof. □

As a consequence of Theorem 3.2 we obtain the result of Ngai and Penot [18, Theorem 3] for Asplund spaces.

Corollary 3.4 *Let U and X be Asplund spaces and $h : U \times X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. Then for any $u \in \operatorname{dom} h_{\inf}$, $\varepsilon \geq 0$ and $u^* \in \partial^{F,\varepsilon} h_{\inf}(u)$ there exist sequences (u_k) in U, (u_k^*) in U^*, (x_k) in X and (x_k^*) in X^* with $(u_k^*, x_k^*) \in \partial^{F,\varepsilon} h(u_k, x_k)$ for all $k \in \mathbb{N}$ such that $\|u_k - u\| \to 0$, $\|u_k^* - u^*\|_* \to 0$, $\|x_k^*\|_* \to 0$ and $h(u_k, x_k) \to h_{\inf}(u)$.*

Proof Fix any sequence $\delta_k \downarrow 0$. By Theorem 3.2 there exist sequences $(\widehat{u}_k)$ in U and $(\widehat{x}_k)$ in X with $(u^*, 0) \in \partial^{F,\varepsilon+\delta_k} h(\widehat{u}_k, \widehat{x}_k)$, $\|\widehat{u}_k - u\| < \delta_k$ and $|h(\widehat{u}_k, \widehat{x}_k) - h_{\inf}(u)| < \delta_k$ for all $k \in \mathbb{N}$. For each $k \in \mathbb{N}$ the inclusion $(u^*, 0) \in \partial^{F,\varepsilon+\delta_k} h(\widehat{u}_k, \widehat{x}_k)$ and Lemma 3.3 (where we put $\eta = \delta = \delta_k$) give $u_k \in U$, $u_k^* \in U^*$, $x_k \in X$ and $x_k^* \in X^*$ such that $(u_k^*, x_k^*) \in \partial^{F,\varepsilon} h(u_k, x_k)$, $\|u_k - \widehat{u}_k\| < \delta_k$, $\|x_k^*\|_* < 2\delta_k$, $\|u_k^* - u^*\|_* < 2\delta_k$, $|h(u_k, x_k) - h(\widehat{u}_k, \widehat{x}_k)| < \delta_k$. This completes the proof. □

Via Theorem 3.2 we have, under the AWP property for $\mathcal{P}_h$, inclusions for the Mordukhovich limiting subdifferential similar to those in Lemma 3.1.

Theorem 3.5 *Let U and X be Banach spaces and $h : U \times X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. Assume that $\mathcal{P}_h$ is AWP at $u_0 \in U$ and $x_0 \in X$ is the solution of $\mathcal{P}_h$ at u_0. Then*

$$\partial^L h_{\inf}(u_0) \times \{0\} \subset \partial^L h(u_0, x_0), \tag{3.5}$$

$$\partial^L h_{\inf}(u_0) \subset \partial_u^L h(u_0, x_0). \tag{3.6}$$

Proof Fix $u^* \in \partial^L h_{\inf}(u_0)$. By definition of the limiting subdifferential there exist $\varepsilon_k \downarrow 0$, $u_k \to u_0$ with $h_{\inf}(u_k) \to h_{\inf}(u_0)$, and $u_k^* \to u^*$ weak*, such that $u_k^* \in \partial^{F,\varepsilon_k} h_{\inf}(u_k)$ for all $k \in \mathbb{N}$. Since $\mathcal{P}_h$ is AWP at u_0 it follows that $\Delta_{h,u_0}(u_k) \to 0$, where $\Delta_{h,u_0}(\cdot)$ is defined by (2.2). Applying Theorem 3.2 for $u = u_k$, $u^* = u_k^*$, $\varepsilon = \delta = \varepsilon_k$ gives $\beta_k > 0$ such that for any $x \in X$ with $h(u_k, x) < h_{\inf}(u_k) + \beta_k$ there exist $\widehat{u}_k = \widehat{u}_k(x) \in B(u_k, \varepsilon_k)$, $\widehat{x}_k = \widehat{x}_k(x) \in B(x, \varepsilon_k)$ with $(u_k^*, 0) \in \partial^{F,2\varepsilon_k} h(\widehat{u}_k, \widehat{x}_k)$ and $|h(\widehat{u}_k, \widehat{x}_k) - h_{\inf}(u_k)| < \varepsilon_k$. Using Lemma 2.2 one can choose $x_k \in B(x_0, \Delta_{h,u_0}(u_k) + \varepsilon_k)$ such that $h(u_k, x_k) < h_{\inf}(u_k) + \beta_k$. Denoting $x_k' = \widehat{x}_k(x_k)$, $u_k' = \widehat{u}_k(x_k)$, we have for any $k \in \mathbb{N}$

$$u_k' \in B(u_k, \varepsilon_k), \quad x_k' \in B(x_k, \varepsilon_k),$$

$$(u_k^*, 0) \in \partial^{F,2\varepsilon_k} h(u_k', x_k'), \quad |h(u_k', x_k') - h_{\inf}(u_k)| < \varepsilon_k.$$

Observing that

$$\|x_k' - x_0\| \leq \|x_k' - x_k\| + \|x_k - x_0\| \leq \varepsilon_k + \Delta_{h,u_0}(u_k) + \varepsilon_k \longrightarrow 0,$$

$$\|u_k' - u_0\| \leq \|u_k' - u_k\| + \|u_k - u_0\| \leq \varepsilon_k + \|u_k - u_0\| \longrightarrow 0,$$

we conclude that $(u_k', x_k') \to (u_0, x_0)$. Further, taking into account that $h_{\inf}(u_k) \to h_{\inf}(u_0) = h(u_0, x_0)$ and $|h(u_k', x_k') - h_{\inf}(u_k)| < \varepsilon_k \to 0$, we see that $h(u_k', x_k') \to h(u_0, x_0)$. Consequently, $(u^*, 0) \in \partial^L h(u_0, x_0)$, which proves (3.5). The other inclusion (3.6) follows from (3.5) and Remark 3.1. □

Remark 3.3 The assumption of Theorem 3.5 that $\mathcal{P}_h$ is AWP at u_0 is essential. Indeed, consider the following continuous function

$$h(u,x) = \begin{cases} |x| + u, & u \le 0 \text{ or } |xu| \le 1, \\ \max\left\{u + \frac{2}{u} - |x|, 0\right\}, & u > 0 \text{ and } |xu| > 1. \end{cases}$$

One can easily see that $h_{\inf}(u) = \min\{u, 0\}$ and $\mathcal{P}_h$ at $u_0 = 0$ admits a unique solution $x_0 = 0$. So, $0 \in \partial^L h_{\inf}(0)$, while $\partial_u^L h(0,0) = \{1\}$. Consequently, $\partial^L h_{\inf}(u_0) \not\subset \partial_u^L h(u_0, x_0)$. According to Theorem 3.5 the problem $\mathcal{P}_h$ is not AWP at u_0. We suggest the reader to check it directly.

Inspired by the famous Palais-Smale property, Ngai and Penot [18] introduced the following condition (C_p) at $u_0 \in \operatorname{dom} h_{\inf}$:

(C_p): if sequences (u_k) in U, (u_k^*) in U^*, (x_k) in X and (x_k^*) in X^* are such that $(x_k^*, u_k^*) \in \partial^F h(u_k, x_k)$ for all $k \in \mathbb{N}$, $\|u_k - u_0\| \to 0$, $u_k^* \to u^*$ weak*, $\|x_k^*\|_* \to 0$ and $h(u_k, x_k) \to h_{\inf}(u_0)$, then there exists a convergent subsequence of (x_k).

The following result of Ngai and Penot is akin to Theorem 3.5.

Proposition 3.6 *([18, Corollary 5]). Assume that U and X are Asplund spaces, a function $h : U \times X \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous, $u_0 \in \operatorname{dom} h_{\inf}$, $u^* \in \partial^L h_{\inf}(u_0)$ and condition (C_p) is satisfied. Then there exists a solution x_0 of $\mathcal{P}_h$ such that $(u^*, 0) \in \partial^L h(u_0, x_0)$.*

Example 3.1 Let $U = X = \mathbb{R}$, $u_0 = 0$, $u^* = 0$ and

$$h(u,x) = |x|\Big((1 - xu)^2 + u^4\Big), \qquad u, x \in \mathbb{R}.$$

For any $u \in \mathbb{R}$ one can easily see that $h_{\inf}(u) = 0$, $x_k \to 0$ for any minimizing sequence $(x_k) \in \mathcal{M}(h, u)$ and $x_0 = 0$ is a unique solution for $\mathcal{P}_h$ at u. Hence, $\Delta_{h,u_0}(u) = 0$ for all $u \in \mathbb{R}$ and $\mathcal{P}_h$ is AWP at u_0. However condition (C_p) is not satisfied in this example. Indeed, consider sequences $x_k = k$, $u_k = \frac{1}{k}$, $x_k^* = \frac{1}{k^4}$, $u_k^* = \frac{4}{k^2}$. Observe that $(x_k^*, u_k^*) \in \partial^F h(u_k, x_k)$ for all $k \in \mathbb{N}$, $\|u_k - u_0\| \to 0$, $\|x_k^*\|_* \to 0$, $u_k^* \to u^*$ and $h(u_k, x_k) \to h_{\inf}(u_0)$, but the sequence (x_k) has no convergent subsequence. This example shows that Theorem 3.5 does not follow from Proposition 3.6 even in Asplund space.

In the next corollary we shall need the following proposition (see [14, Theorem 2.33]) on the Mordukhovich limiting subdifferential of a sum.

Proposition 3.7 *Let X be an Asplund space. Let $f, g : X \to \mathbb{R} \cup \{+\infty\}$ be two proper lower semicontinuous functions, one of them being Lipschitz continuous around $x \in \operatorname{dom} f \cap \operatorname{dom} g$. Then*

$$\partial^L (f + g)(x) \subset \partial^L f(x) + \partial^L g(x).$$

For a set $C \subset X$ we denote by d_C and ψ_C the *distance* function and the *indicator* function, that is,

$$d_C(x) := \inf_{y \in C} \|x - y\|, \tag{3.7}$$

$$\psi_C(x) := \begin{cases} 0, & x \in C, \\ +\infty, & x \in X \setminus C. \end{cases}$$

The *Mordukhovich limiting normal cone* of the set $C \subset X$ at $x \in C$ is the limiting subdifferential of the function ψ_C:

$$N_C^L(x) := \partial^L \psi_C(x). \tag{3.8}$$

Given a function $f : U \times X \to \mathbb{R} \cup \{+\infty\}$ and a multifunction $G : U \rightrightarrows X$, consider for $u \in U$ the problem:

$$\text{Minimize } f(u, x) \text{ over } x \in G(u). \tag{3.9}$$

Problem (3.9) is equivalent to the problem $\mathcal{P}_h$ with

$$h(u, x) := f(u, x) + \psi_{\operatorname{gph} G}(u, x), \qquad u \in U, \ x \in X,$$

where $\operatorname{gph} G = \{(u, x) \in U \times X : \ x \in G(u)\}$ is the graph of G.

The optimal value of (3.9) is

$$h_{\inf}(u) = \inf_{x \in X} h(u, x) = \inf_{x \in G(u)} f(u, x). \tag{3.10}$$

We shall say that problem (3.9) is AWP at $u_0 \in U$ if $\mathcal{P}_h$ is AWP at u_0.

Corollary 3.8 *Let U and X be Asplund spaces, $G : U \rightrightarrows X$ be a multifunction with closed graph and $f : U \times X \to \mathbb{R}$ be a lower semicontinuous function. Let problem (3.9) be AWP at $u_0 \in U$ and let $x_0 \in X$ be the solution of this problem at u_0. Assume that f is Lipschitz continuous around (u_0, x_0).*

Then the limiting subdifferential of the optimal value function (3.10) satisfies the following inclusion:

$$\partial^L h_{\inf}(u_0) \times \{0\} \subset \partial^L f(u_0, x_0) + N_{\operatorname{gph} G}^L(u_0, x_0). \tag{3.11}$$

If additionally for any u sufficiently close to u_0 the function $f(u, \cdot)$ is Lipschitz on X with some Lipschitz constant $\kappa'(u) < \kappa$, where κ doesn't depend on u, then

$$\partial^L h_{\inf}(u_0) \times \{0\} \subset \partial^L f(u_0, x_0) + \kappa \partial^L \varrho_G(u_0, x_0) \tag{3.12}$$

where

$$\varrho_G(u, x) := d_{G(u)}(x) = \inf_{y \in G(u)} \|x - y\|, \qquad u \in U, \ x \in X.$$

Proof Using Theorem 3.5 and Proposition 3.7 we obtain

$$\begin{aligned}\partial^L h_{\inf}(u_0) \times \{0\} &\subset \partial^L h(u_0, x_0) = \partial^L (f + \psi_{\operatorname{gph} G})(u_0, x_0) \\ &\subset \partial^L f(u_0, x_0) + \partial^L \psi_{\operatorname{gph} G}(u_0, x_0) \\ &= \partial^L f(u_0, x_0) + N_{\operatorname{gph} G}^L(u_0, x_0).\end{aligned}$$

So, (3.11) is proved. Let us prove (3.12).

Let a neighborhood U_0 of u_0 be such that for any $u \in U_0$ the function $f(u, \cdot)$ is Lipschitz continuous on X with Lipschitz constant $\kappa'(u) < \kappa$. Consider the function

$$\tilde{h}(u, x) = f(u, x) + \kappa d_{G(u)}(x) = f(u, x) + \kappa \varrho_G(u, x), \qquad u \in U, \ x \in X.$$

Observe that for each $u \in U_0$ one has $\tilde{h}_{\inf}(u) = h_{\inf}(u)$ and any minimizing sequence for $\mathcal{P}_h$ at u is a minimizing sequence for $\mathcal{P}_{\tilde{h}}$ at u. Consequently, $\Delta_{\tilde{h}, u_0}(u) \le \Delta_{h, u_0}(u)$ for all $u \in U_0$. Since x_0 is a unique solution of problem (3.9) at u_0, it follows that x_0 is a unique solution for $\mathcal{P}_{\tilde{h}}$ at u_0. So, $\mathcal{P}_{\tilde{h}}$ is AWP at u_0. According to Proposition 3.7 we have

$$\partial^L \tilde{h}(u_0, x_0) \subset \partial^L f(u_0, x_0) + \kappa \partial^L \varrho_G(u_0, x_0).$$

Using Theorem 3.5 for $\tilde{h}$ in place of h, we obtain

$$\partial^L h_{\inf}(u_0) \times \{0\} = \partial^L \tilde{h}_{\inf}(u_0) \times \{0\} \subset \partial^L \tilde{h}(u_0, x_0) \subset \partial^L f(u_0, x_0) + \kappa \partial^L \varrho_G(u_0, x_0).$$

□

In view of the equality $N_C^L(u_0, x_0) = \bigcup_{s\geq 0} s\partial^L d_C(u_0, x_0)$ (see (4.21) and [22]) inclusion (3.11) may be rewritten in the form

$$\partial^L h_{\inf}(u_0) \times \{0\} \subset \partial^L f(u_0, x_0) + \bigcup_{s\geq 0} s\partial^L d_{\operatorname{gph} G}(u_0, x_0).$$

So, inclusion (3.11) correlates with [22, Proposition 3.1] and other results of Thibault. Inclusion (3.12) is akin to the result of Ngai, Luc and Théra [17, Theorem 2.5].

4 Subdifferentials of the Infimal Convolution

The Moreau-type *infimal convolution* of two functions $f, g : X \to \mathbb{R} \cup \{+\infty\}$ is

$$(f \,\square\, g)(u) = \inf_{x\in X} \Big(f(x) + g(u - x) \Big), \qquad u \in X.$$

The *infimal convolution problem* $\mathcal{P}_{f,g}$ at a point $u \in X$ is the problem $\mathcal{P}_h$ with

$$h(u, x) = f(x) + g(u - x), \qquad u, x \in X.$$

Applying Lemma 2.3(b) to the function $h(u, x) = f(x) + g(u - x)$, we obtain the following lemma.

Lemma 4.1 *Let X be a normed space. Assume that $g : X \to \mathbb{R}$ is uniformly continuous and $\mathcal{P}_{f,g}$ is Tykhonov well-posed at $u_0 \in$* dom $(f \,\square\, g)$*. Then $\mathcal{P}_{f,g}$ is AWP at u_0.*

Lemma 4.2 *Let X be a normed space and let any $\varepsilon \geq 0$ be given. Assume that $x_0 \in X$ is a solution of $\mathcal{P}_{f,g}$ at $u_0 \in$* dom $(f \,\square\, g)$*. Then*

$$\partial^{F,\varepsilon}(f \,\square\, g)(u_0) \subset \partial^{F,\varepsilon} f(x_0) \bigcap \partial^{F,\varepsilon} g(u_0 - x_0), \tag{4.1}$$

$$\partial^{P}(f \,\square\, g)(u_0) \subset \partial^{P} f(x_0) \bigcap \partial^{P} g(u_0 - x_0). \tag{4.2}$$

Proof Using the second inclusion of Lemma 3.1 for $h(u, x) = f(x) + g(u - x)$, we obtain

$$\partial^{F,\varepsilon}(f \,\square\, g)(u_0) = \partial^{F,\varepsilon} h_{\inf}(u_0) \subset \partial_u^{F,\varepsilon} h(u_0, x_0) = \partial^{F,\varepsilon} g(u_0 - x_0).$$

Similarly, using the second inclusion of Lemma 3.1 for $\tilde{h}(u, \tilde{x}) = f(u - \tilde{x}) + g(\tilde{x})$ and $\tilde{x}_0 = u_0 - x_0$, we have

$$\partial^{F,\varepsilon}(f \,\square\, g)(u_0) = \partial^{F,\varepsilon} \tilde{h}_{\inf}(u_0) \subset \partial_u^{F,\varepsilon} \tilde{h}(u_0, \tilde{x}_0) = \partial^{F,\varepsilon} f(u_0 - \tilde{x}_0) = \partial^{F,\varepsilon} f(x_0).$$

So, (4.1) is proved. The proof of (4.2) is similar. □

Inclusion (4.1) was previously obtained in [1, Lemma 3.6] for $\varepsilon = 0$ and in [15, Proposition 2.1] for $\varepsilon \geq 0$. This inclusion correlates with the result of Kecis and Thibault [10, Theorem 3.1]. In a particular case when g is a Minkowski functional, Lemma 4.2 was proved in [6, Theorem 3.1].

In the case of infimal convolution, Theorem 3.2 can be translated as follows.

Corollary 4.3 *Let X be a Banach space, $f, g : X \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous functions, $u \in \operatorname{dom}(f \,\Box\, g)$, $\varepsilon \geq 0$ and $u^* \in \partial^{F,\varepsilon}(f \,\Box\, g)(u)$. Let $\delta > 0$. Then there exists $\beta > 0$ such that for any $x, z \in X$ with $x + z = u$ and $f(x) + g(z) < (f \,\Box\, g)(u) + \beta$ there exist $\widehat{x} \in B(x, \delta)$ and $\widehat{z} \in B(z, \delta)$ with*

$$u^* \in \partial^{F,\varepsilon+\delta} f(\widehat{x}) \bigcap \partial^{F,\varepsilon+\delta} g(\widehat{z}), \tag{4.3}$$

$$|f(\widehat{x}) + g(\widehat{z}) - (f \,\Box\, g)(u)| < \delta. \tag{4.4}$$

Proof Applying Theorem 3.2 for the function

$$h(u', x') = f(x') + g(u' - x'), \qquad u', x' \in X$$

and $\frac{\delta}{2}$ in place of δ gives $\beta > 0$ such that for any $x \in X$ with $f(x) + g(u - x) < (f \,\Box\, g)(u) + \beta$ there exist $\widehat{u} \in B\left(u, \frac{\delta}{2}\right)$, $\widehat{x} \in B\left(x, \frac{\delta}{2}\right)$ such that $(u^*, 0) \in \partial^{F,\varepsilon+\delta/2} h(\widehat{u}, \widehat{x})$ and (4.4) holds true with $\widehat{z} = \widehat{u} - \widehat{x}$. Due to the inclusion $(u^*, 0) \in \partial^{F,\varepsilon+\delta/2} h(\widehat{u}, \widehat{x})$ there exists $\theta > 0$ such that for all $u' \in B(\widehat{u}, \theta)$, $x' \in B(\widehat{x}, \theta)$

$$\begin{aligned} \langle u^*, u' - \widehat{u} \rangle \;&\leq\; h(u', x') - h(\widehat{u}, \widehat{x}) + (\varepsilon + \delta) \max\{\|u' - \widehat{u}\|, \|x' - \widehat{x}\|\} \\ &=\; f(x') - f(\widehat{x}) + g(u' - x') - g(\widehat{u} - \widehat{x}) \\ &\quad + (\varepsilon + \delta) \max\{\|u' - \widehat{u}\|, \|x' - \widehat{x}\|\}. \end{aligned}$$

Putting $u' = \widehat{u} - \widehat{x} + x'$ we get

$$\langle u^*, x' - \widehat{x} \rangle \leq f(x') - f(\widehat{x}) + (\varepsilon + \delta)\|x' - \widehat{x}\| \qquad \forall x' \in B(\widehat{x}, \theta).$$

Consequently, $u^* \in \partial^{F,\varepsilon+\delta} f(\widehat{x})$.

Similarly, setting $x' = \widehat{x}$, $u' = \widehat{u} - \widehat{z} + z'$ we obtain

$$\langle u^*, z' - \widehat{z} \rangle \leq g(z') - g(\widehat{z}) + (\varepsilon + \delta)\|z' - \widehat{z}\| \qquad \forall z' \in B(\widehat{z}, \theta)$$

and hence $u^* \in \partial^{F,\varepsilon+\delta} g(\widehat{z})$. □

Similarly to the proof of Corollary 3.4 one can easily see that for Asplund spaces Corollary 6 of Ngai and Penot [18] follows directly from Corollary 4.3.

The next theorem gives the description of the Fréchet and proximal subdifferentials of the infimal convolution under LAWP conditions.

Theorem 4.4 *Let X be a normed space. Assume that $x_0 \in X$ is a solution of $\mathcal{P}_{f,g}$ at $u_0 \in \operatorname{dom}(f \,\Box\, g)$, $f \,\Box\, g$ is lower semicontinuous at u_0 and $\mathcal{P}_{f,g}$ is LAWP at u_0. Then for any $R > 0$ there exist positive reals λ and ε_0 such that for any $\varepsilon \in [0, \varepsilon_0]$ and $z_0 = u_0 - x_0$ one has*

$$\partial^{F,\varepsilon} f(x_0) \bigcap \partial^{F,\varepsilon} g(z_0) \bigcap B(0, R) \subset \partial^{F,\lambda\varepsilon}(f \,\Box\, g)(u_0). \tag{4.5}$$

Furthermore,

$$\partial^F (f \,\Box\, g)(u_0) = \partial^F f(x_0) \bigcap \partial^F g(z_0), \tag{4.6}$$

$$\partial^P (f \,\Box\, g)(u_0) = \partial^P f(x_0) \bigcap \partial^P g(z_0). \tag{4.7}$$

Proof Consider the functions $h(u, x) = f(x) + g(u - x)$ with $u, x \in X$ and $h_{\inf} = f \,\Box\, g$. Since $\mathcal{P}_{f,g}$ is LAWP at u_0, there exist positive reals $\lambda_1, \lambda_2, \delta_1$ such that

$$\begin{aligned} \Delta_{h,u_0}(u) \;&\leq\; \lambda_1 \|u - u_0\| + \lambda_2\, |h_{\inf}(u) - h_{\inf}(u_0)| \\ &\qquad \text{for all } u \in B(u_0, \delta_1) \;\text{ with }\; |h_{\inf}(u) - h_{\inf}(u_0)| < \delta_1. \end{aligned} \tag{4.8}$$

Fix any $R > 0$ and denote

$$\lambda = 4(1 + \lambda_1 + \lambda_2 R), \qquad \varepsilon_0 = \frac{1}{8\lambda_2}.$$

Consider any $\varepsilon \in [0, \varepsilon_0]$ and let us prove (4.5).

Fix any $u^* \in \partial^{F,\varepsilon} f(x_0) \bigcap \partial^{F,\varepsilon} g(z_0) \bigcap B(0, R)$ and $\eta \in]0, \varepsilon_0]$. By definition of the Fréchet ε-subdifferential there exists a positive real $\delta_2 < \min\{\delta_1, 4\lambda_2\delta_1\}$ such that

$$\langle u^*, x - x_0\rangle \le f(x) - f(x_0) + (\varepsilon + \eta)\|x - x_0\| \qquad \forall x \in B(x_0, \delta_2), \tag{4.9}$$

$$\langle u^*, z - z_0\rangle \le g(z) - g(z_0) + (\varepsilon + \eta)\|z - z_0\| \qquad \forall z \in B(z_0, \delta_2). \tag{4.10}$$

As $h_{\inf}$ is lower semicontinuous at u_0, one can find a positive real $\delta_3 \le \delta_2/\lambda$ such that

$$h_{\inf}(u_0) \le h_{\inf}(u) + \frac{\delta_2}{4\lambda_2} \qquad \forall u \in B(u_0, \delta_3). \tag{4.11}$$

Fix any $u \in B(u_0, \delta_3)$ and let us prove that

$$\langle u^*, u - u_0\rangle \le h_{\inf}(u) - h_{\inf}(u_0) + \lambda(\varepsilon + \eta)\|u - u_0\|. \tag{4.12}$$

If $h_{\inf}(u) - h_{\inf}(u_0) > \delta_2/(4\lambda_2)$, then

$$\langle u^*, u - u_0\rangle \le \|u^*\| \cdot \|u - u_0\| \le R\delta_3 \le \frac{R\delta_2}{\lambda} < \frac{\delta_2}{4\lambda_2} < h_{\inf}(u) - h_{\inf}(u_0)$$

and (4.12) holds true. Further, we suppose that $h_{\inf}(u) - h_{\inf}(u_0) \le \delta_2/(4\lambda_2)$. Taking into account (4.11), we get

$$|h_{\inf}(u_0) - h_{\inf}(u)| \le \frac{\delta_2}{4\lambda_2} < \delta_1.$$

Consequently, (4.8) implies that

$$\Delta_{h,u_0}(u) \le \lambda_1\delta_3 + \lambda_2\frac{\delta_2}{4\lambda_2} \le \frac{\lambda_1\delta_2}{\lambda} + \frac{\delta_2}{4} < \frac{\delta_2}{2}.$$

According to Lemma 2.2 there exists a sequence $(x_k) \in \mathcal{M}(h, u)$ such that $\lim_{k\to\infty} \|x_k - x_0\| = \Delta_{h,u_0}(u) < \frac{\delta_2}{2}$. Denoting $z_k = u - x_k$, we see that $\|z_k - z_0\| \le \|x_k - x_0\| + \|u - u_0\| < \|x_k - x_0\| + \delta_3$, and hence $\limsup_{k\to\infty} \|z_k - z_0\| < \delta_2$. Using (4.9), (4.10), we have for sufficiently large k

$$\langle u^*, x_k - x_0\rangle \le f(x_k) - f(x_0) + (\varepsilon + \eta)\|x_k - x_0\|,$$

$$\langle u^*, z_k - z_0\rangle \le g(z_k) - g(z_0) + (\varepsilon + \eta)\|z_k - z_0\|.$$

Adding this two inequalities together, we arrive at

$$\langle u^*, u - u_0\rangle \le f(x_k) + g(z_k) - f(x_0) - g(z_0) + (\varepsilon + \eta)\left(\|x_k - x_0\| + \|z_k - z_0\|\right).$$

Since $(x_k) \in \mathcal{M}(h, u)$, it follows that $f(x_k) + g(z_k) = h(u, x_k) \to h_{\inf}(u)$. Passing to the limit as $k \to \infty$, we obtain

$$\langle u^*, u - u_0\rangle \le h_{\inf}(u) - h_{\inf}(u_0) + (\varepsilon + \eta)\left(2\Delta_{h,u_0}(u) + \|u - u_0\|\right). \tag{4.13}$$

Denoting $D := \langle u^*, u - u_0\rangle - h_{\inf}(u) + h_{\inf}(u_0)$, by (4.8) we get

$$\begin{aligned}\Delta_{h,u_0}(u) &\le \lambda_1\|u - u_0\| + \lambda_2\,|h_{\inf}(u) - h_{\inf}(u_0)|\\ &\le \lambda_1\|u - u_0\| + \lambda_2(|D| + |\langle u^*, u - u_0\rangle|)\\ &\le (\lambda_1 + \lambda_2 R)\|u - u_0\| + \lambda_2|D|.\end{aligned}$$

So, (4.13) implies that

$$D \le 2(\varepsilon+\eta)\Big((\lambda_1+\lambda_2 R)\|u-u_0\|+\lambda_2|D|+\|u-u_0\|\Big)$$
$$\le \frac{\lambda(\varepsilon+\eta)}{2}\|u-u_0\|+2(\varepsilon+\eta)\lambda_2|D|.$$

As $\varepsilon \le \varepsilon_0$ and $\eta \le \varepsilon_0$, we have $\varepsilon+\eta \le 2\varepsilon_0 = \frac{1}{4\lambda_2}$, and hence $D \le \frac{\lambda(\varepsilon+\eta)}{2}\|u-u_0\| + \frac{|D|}{2}$. If $D \ge 0$, the latter inequality yields

$$D \le \lambda(\varepsilon+\eta)\|u-u_0\|. \tag{4.14}$$

In the other case ($D < 0$) (4.14) is still valid. Thus (4.12) is proved, that is, for any $\eta \in]0, \varepsilon_0]$ one can find $\delta_3 > 0$ such that

$$\langle u^*, u-u_0\rangle \le h_{\inf}(u) - h_{\inf}(u_0) + \lambda(\varepsilon+\eta)\|u-u_0\| \qquad \forall u \in B(u_0, \delta_3).$$

Consequently, $u^* \in \partial^{F,\lambda\varepsilon}(f \,\square\, g)(u_0)$ and (4.5) is proved. Using (4.5) and (4.1) for $\varepsilon = 0$ we obtain (4.6).

The proof of inclusion (4.7) is similar. □

Example 4.1 Let $X = \mathbb{R}^2$,

$$f(x_1,x_2) = \begin{cases} -|x_2|^3, & x_1 = 0, \\ +\infty, & x_1 \ne 0, \end{cases} \qquad g(x_1,x_2) = \begin{cases} 0, & x_1 = x_2^3, \\ +\infty, & x_1 \ne x_2^3, \end{cases}$$

for all $(x_1,x_2) \in \mathbb{R}^2$. One can easily see that $(x_1,x_2) = \left(0, u_2 - u_1^{1/3}\right)$ is a unique solution of $\mathcal{P}_{f,g}$ at (u_1,u_2). Therefore $(f \,\square\, g)(u_1,u_2) = -\left|u_2 - u_1^{1/3}\right|^3$, in particular, $(f \,\square\, g)(u_1,0) = -|u_1|$, $(0,0) \in \partial^F f(0,0) \bigcap \partial^F g(0,0)$, but $(0,0) \notin \partial^F (f \,\square\, g)(0,0)$. According to Remark 2.1, $\mathcal{P}_{f,g}$ is AWP at $(0,0)$. This example shows that in Theorem 4.4 the assumption that $\mathcal{P}_{f,g}$ is LAWP can't be reduced to the assumption that $\mathcal{P}_{f,g}$ is AWP.

The next theorem provides sufficient conditions for $\mathcal{P}_{f,g}$ to be LAWP and as a consequence sufficient conditions for description of the subdifferentials of the infimal convolution.

Theorem 4.5 *Let X be a normed space and $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$. Suppose that functions $f : X \to \mathbb{R}\cup\{+\infty\}$, $g : X \to \mathbb{R}$ and points $x_0 \in \operatorname{dom} f$, $z_0 \in \operatorname{dom} g$ satisfy the inequalities*

$$f(x) - f(x_0) \ge -\alpha\|x-x_0\| \qquad \forall x \in X, \tag{4.15}$$

$$g(z) - g(z_0) \ge \beta\|z-z_0\| \qquad \forall z \in X. \tag{4.16}$$

Then

- *(a)* *$\mathcal{P}_{f,g}$ admits x_0 as unique solution at $u_0 = x_0 + z_0$, $f \,\square\, g$ is lower semicontinuous at u_0 and $\mathcal{P}_{f,g}$ is LAWP at u_0;*
- *(b)* *equalities (4.6), (4.7) are valid;*
- *(c)* *for any $R > 0$ there exist positive reals λ and ε_0 such that for any $\varepsilon \in [0, \varepsilon_0]$ inclusion (4.5) is valid as well.*

Proof For the function $h(u,x) = f(x) + g(u-x)$ we have for any $u, x \in X$

$$h(u,x) - h(u_0,x_0) \ge -\alpha\|x-x_0\| + \beta\|u-x-u_0+x_0\| \ge (\beta-\alpha)\|x-x_0\| - |\beta|\cdot\|u-u_0\|.$$

Using Proposition 2.4 for $\varphi(t) = \lambda t$, $\xi(t) = \mu t$ with $\lambda = \beta - \alpha$, $\mu = |\beta|$, we obtain assertion (a). Applying Theorem 4.4, we get assertions (b) and (c). □

A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is called *lower regular* at a point $x \in \operatorname{dom} f$ (see [13]) whenever

$$\partial^L f(x) = \partial^F f(x).$$

Theorem 4.6 *Let X be a Banach space, $f, g : X \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous functions and $x_0 \in X$ be the solution of $\mathcal{P}_{f,g}$ at $u_0 \in \operatorname{dom}(f \,\square\, g)$, $z_0 = u_0 - x_0$.*

(a) If $\mathcal{P}_{f,g}$ is AWP at $u_0 \in X$, then

$$\partial^L (f \,\square\, g)(u_0) \subset \partial^L f(x_0) \bigcap \partial^L g(z_0). \tag{4.17}$$

(b) If $\mathcal{P}_{f,g}$ is LAWP at u_0, f and g are lower regular at x_0 and z_0 respectively, and $f \,\square\, g$ is lower semicontinuous at u_0, then $f \,\square\, g$ is lower regular at u_0 and inclusion (4.17) holds as an equality.

Proof (a). Using Theorem 3.5 for $h(u, x) = f(x) + g(u - x)$, we obtain

$$\partial^L (f \,\square\, g)(u_0) = \partial^L h_{\inf}(u_0) \subset \partial^L_u h(u_0, x_0) = \partial^L g(z_0).$$

Similarly, using Theorem 3.5 for $\tilde{h}(u, z) = f(u - z) + g(z)$, we obtain

$$\partial^L (f \,\square\, g)(u_0) = \partial^L \tilde{h}_{\inf}(u_0) \subset \partial^L_u \tilde{h}(u_0, z_0) = \partial^L f(u_0 - z_0) = \partial^L f(x_0).$$

So, (4.17) is proved.

(b). Using inclusion (4.17) and equality (4.6) of Theorem 4.4, one has

$$\begin{aligned} \partial^L (f \,\square\, g)(u_0) &\subset \partial^L f(x_0) \bigcap \partial^L g(z_0) = \partial^F f(x_0) \bigcap \partial^F g(z_0) \\ &= \partial^F (f \,\square\, g)(u_0) \subset \partial^L (f \,\square\, g)(u_0). \end{aligned}$$ □

Theorem 4.6(b) improves Theorem 4.1 from [7], where g was supposed to be a Minkowski functional.

Using Theorem 4.6(a) and sufficient conditions for $\mathcal{P}_{f,g}$ to be AWP we obtain the following corollary.

Corollary 4.7 *Let X be a Banach space, $f, g : X \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous functions and $x_0 \in X$ be the solution of $\mathcal{P}_{f,g}$ at $u_0 \in \operatorname{dom}(f \,\square\, g)$, $z_0 = u_0 - x_0$. Assume, in addition, that at least one of the following conditions holds:*

(a) the function $g : X \to \mathbb{R}$ is uniformly continuous and the problem $\mathcal{P}_{f,g}$ is Tykhonov well-posed at u_0; or
(b) inequalities (4.15), (4.16) are satisfied with some $\alpha, \beta \in \mathbb{R}$ such that $\alpha < \beta$.

Then inclusion (4.17) holds true.

Proof In case (a) inclusion (4.17) follows from Lemma 4.1 and Theorem 4.6(a). In case (b) it suffices to apply Theorem 4.5(a) and Theorem 4.6(a). □

In the particular case when g is a Minkowski functional, Corollary 4.7 was proved in [7, Theorems 3.1, 3.2]. In [16, Theorem 5.5] Corollary 4.7(b) was proved in another case, namely when g is subaditive and coercive with some constant $\ell > 0$ and f is Lipschitz continuous on dom f with a constant $m < \ell$.

Remark 4.1 Let the problem $\mathcal{P}_{f,g}$ be LAWP (and hence AWP) at a point $u \in \mathrm{dom}\,(f \,\square\, g)$ and let $x \in X$ be the solution of $\mathcal{P}_{f,g}$ at u. The inclusion

$$\partial^L f(x) \bigcap \partial^L g(u-x) \subset \partial^L (f \,\square\, g)(u) \tag{4.18}$$

fails in general. Indeed, consider in $\mathbb{R}^2$ the functions

$$f(x_1, x_2) = |x_1| + \min\{|x_1|, |x_2|\}, \qquad g(x_1, x_2) = |x_2| + \min\{|x_1|, |x_2|\}.$$

Observing that $f(x_1, x_2) \geq 0$ and $g(x_1, x_2) \geq 0$ for all $(x_1, x_2) \in \mathbb{R}^2$, we get $(f \,\square\, g)(u_1, u_2) \geq 0$ for all $(u_1, u_2) \in \mathbb{R}^2$. On the other hand, since $f(0, u_2) = g(u_1, 0) = 0$ it follows that $(f \,\square\, g)(u_1, u_2) \leq f(0, u_2) + g(u_1, 0) = 0$. Thus, $(f \,\square\, g)(u_1, u_2) = 0$ and $(0, u_2)$ is a solution of $\mathcal{P}_{f,g}$ at any $(u_1, u_2) \in \mathbb{R}^2$. Since the function $(u_1, u_2) \mapsto (0, u_2)$ is Lipschitz continuous, it follows that $\mathcal{P}_{f,g}$ is LAWP at any point $(u_1, u_2) \in \mathbb{R}^2$.

Consider the functional $x^* = (1, 1)$, i.e., $\langle x^*, (u, x)\rangle = u + x$ for all $(u, x) \in \mathbb{R}^2$. Since $f(x_1, x_2) = x_1 + x_2$ for all (x_1, x_2) such that $0 < x_2 < x_1$, it follows that $x^* \in \partial^F f(x_1, x_2)$ whenever $0 < x_2 < x_1$ and, therefore, $x^* \in \partial^L f(0, 0)$. Similarly, $x^* \in \partial^L g(0, 0)$. On the other hand, $x^* \notin \partial^L (f \,\square\, g)(0, 0) = \{(0, 0)\}$, since $(f \,\square\, g)(u_1, u_2) = 0$ for all $(u_1, u_2) \in \mathbb{R}^2$. So, in this example inclusion (4.18) is false.

Theorem 4.8 *Let X be a Banach space, $f, g : X \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous functions. Let g be lower regular at $z_0 \in \mathrm{dom}\, g$ and satisfy (4.16) for some $\beta \in \mathbb{R}$. Suppose, in addition, that at least one of the following conditions holds:*

(a) X is finite-dimensional, f is Lipschitz continuous on dom f *with some constant $\alpha < \beta$ and $x_0 \in \mathrm{dom}\, f$; or*
(b) f is lower regular at $x_0 \in \mathrm{dom}\, f$ and satisfies the inequality (4.15) with $\alpha < \beta$.

Then

$$\partial^L (f \,\square\, g)(x_0 + z_0) = \partial^L f(x_0) \bigcap \partial^L g(z_0).$$

In case (b) the function $f \,\square\, g$ is lower regular at the point $(x_0 + z_0)$.

Proof First assume that condition (a) holds. Fix any $x_0' \in \mathrm{dom}\, f$. Since f is Lipschitz continuous on dom f with constant α, it follows that inequality (4.15) with x_0' in place of x_0 holds true. According to Theorem 4.5(a) the problem $\mathcal{P}_{f,g}$ is LAWP at $x_0' + z_0$ for any $x_0' \in \mathrm{dom}\, f$.

Fix any $u^* \in \partial^L f(x_0) \bigcap \partial^L g(z_0)$. By definition of the limiting subdifferential there exist $\varepsilon_k \downarrow 0$, $u_k^* \to u^*$ and $x_k \to x_0$ such that $f(x_k) \to f(x_0)$ and $u_k^* \in \partial^{F,\varepsilon_k} f(x_k)$ for all $k \in \mathbb{N}$. Due to the lower regularity of g one has $\partial^L g(z_0) = \partial^F g(z_0)$. Since $u_k^* \to u^* \in \partial^F g(z_0)$, there exists $\varepsilon_k' \downarrow 0$ with $\varepsilon_k' \geq \varepsilon_k$ and $u_k^* \in \partial^F g(z_0) + B(0, \varepsilon_k') \subset \partial^{F,\varepsilon_k'} g(z_0)$ for all $k \in \mathbb{N}$.

According to Theorem 4.5(c) there exists $\lambda > 0$ such that for all sufficiently large k we have $u_k^* \in \partial^{F,\lambda\varepsilon_k'}(f \,\square\, g)(x_k + z_0)$. Observe that $(f \,\square\, g)(x_k + z_0) = f(x_k) + g(z_0) \to f(x_0) + g(z_0) = (f \,\square\, g)(x_0 + z_0)$. It ensures that $u^* \in \partial^L(f \,\square\, g)(x_0 + z_0)$ and hence

$$\partial^L f(x_0) \bigcap \partial^L g(z_0) \subset \partial^L(f \,\square\, g)(x_0 + z_0).$$

The reverse inclusion follows from Corollary 4.7(b).

In the case (b) Theorem 4.5(a) and Theorem 4.6(b) imply the desired statement. □

Under additional assumptions that g is positively homogeneous and subadditive, while f is Lipschitz continuous on dom f in both cases (a) and (b), Theorem 4.8 was proved in [16, Theorem 5.5]. Case (b) of Theorem 4.8 under the assumption that g is a Minkowski functional was established in [7, Theorem 4.2].

Observe that for $g(\cdot) = \|\cdot\|$ one has $(\psi_C \,\square\, g)(\cdot) = d_C(\cdot)$ (see (3.7)). Given a closed set C in a Banach space X and $x_0 \in C$, applying Theorem 4.8 for $f(\cdot) = \psi_C(\cdot)$ and $g(\cdot) = \|\cdot\|$ either in the case when X is finite-dimensional or in the case when ψ_C is lower regular at x_0, one obtains the well known equality

$$\partial^L d_C(x_0) = N_C^L(x_0) \bigcap B[0, 1].$$

In the paper [7] we give an example of a closed set $C \subset \ell_2$ such that for some $x_0 \in C$

$$N_C^L(x_0) \bigcap B[0, 1] \not\subset \partial^L d_C(x_0).$$

This example shows that the assumption of Theorem 4.8 that either X is finite-dimensional or f is lower regular at x_0 is essential.

Theorem 4.9 *Let C be a closed subset of a Banach space X and $x_0 \in C$. Let $g : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function which satisfies (4.16) for some $\beta > 0$ and $z_0 \in \operatorname{dom} g$. Then*

$$N_C^L(x_0) = \bigcup_{s \geq 0} s\partial^L(\psi_C \,\square\, g)(x_0 + z_0).$$

Proof In view of (4.16) one has $g(z_0) = \min_{z \in X} g(z)$. Consequently, for any $x \in C$

$$g(z_0) = \min_{x' \in X} \psi_C(x') + \min_{z \in X} g(z) \leq (\psi_C \,\square\, g)(x + z_0) \leq \psi_C(x) + g(z_0) = g(z_0).$$

So,

$$(\psi_C \,\square\, g)(x + z_0) = g(z_0) \qquad \forall x \in C. \tag{4.19}$$

Fix any $u^* \in N_C^L(x_0) = \partial^L \psi_C(x_0)$. By the definition of the limiting subdifferential there exist $\varepsilon_k \downarrow 0$, $u_k^* \to u^*$ weak* and $x_k \to x_0$ such that $x_k \in C$ and $u_k^* \in \partial^{F,\varepsilon_k}\psi_C(x_k)$ for all $k \in \mathbb{N}$. Since the sequence (u_k^*) converges weak*, it is bounded and hence there exists $s > 0$ such that $u_k^* \in B(0, s\beta)$ for all $k \in \mathbb{N}$. Inequality (4.16) implies that $B(0, \beta) \subset \partial^F g(z_0)$. Therefore, $\frac{u_k^*}{s} \in \partial^F g(z_0)$ for all $k \in \mathbb{N}$. Since $u_k^* \in \partial^{F,\varepsilon_k}\psi_C(x_k)$, it follows that $\frac{u_k^*}{s} \in \partial^{F,\varepsilon_k/s}\psi_C(x_k)$. Observe that the function $f = \psi_C$ satisfies (4.15) with $\alpha = 0$ at x_0 and at x_k in place of x_0 as well. According to Theorem 4.5(c) there exists $\lambda > 0$ such that $\frac{u_k^*}{s} \in \partial^{F,\lambda\varepsilon_k}(\psi_C \,\square\, g)(x_k + z_0)$ for all sufficiently large k. It follows by (4.19) that

$(\psi_C \,\square\, g)(x_k + z_0) = g(z_0) = (\psi_C \,\square\, g)(x_0 + z_0)$. Consequently, by the definition of the limiting subdifferential $\frac{u^*}{s} \in \partial^L(\psi_C \,\square\, g)(x_0 + z_0)$. So, the inclusion

$$N_C^L(x_0) \subset \bigcup_{s \geq 0} s\partial^L(\psi_C \,\square\, g)(x_0 + z_0) \tag{4.20}$$

is proved. Corollary 4.7(b) implies that $\partial^L(\psi_C \,\square\, g)(x_0 + z_0) \subset \partial^L \psi_C(x_0) = N_C^L(x_0)$. Due to conicity of $N_C^L(x_0)$ we get the reverse inclusion to (4.20). □

Taking $g(\cdot) := \|\cdot\|$ and $z_0 := 0$, Theorem 4.9 generalizes the result of Thibault [22] that for any point x_0 of a closed subset C in a Banach space

$$N_C^L(x_0) = \bigcup_{s \geq 0} s\partial^L d_C(x_0). \tag{4.21}$$

Theorem 4.9 also includes, as a corollary, Proposition 2.7 in Thibault [22] for the Mordukhovich limiting normal cone of the graph of a multifunction $G : U \rightrightarrows X$ which is closed near $(u_0, x_0) \in \operatorname{gph} G$, that is, there is a neighborhood W of (u_0, x_0) such that $W \cap \operatorname{gph} G$ is closed in W relative to the induced topology.

Corollary 4.10 *Let U and X be Banach spaces and let $G : U \rightrightarrows X$ be a multifunction the graph of which is closed near $(u_0, x_0) \in \operatorname{gph} G$. For the function $\varrho_G : U \times X \to \mathbb{R}$ defined by*

$$\varrho_G(u, x) = \inf_{y \in G(u)} \|x - y\|, \quad x \in X,\ u \in U,$$

one has the equality

$$N_{\operatorname{gph} G}^L(u_0, x_0) = \bigcup_{s \geq 0} s\partial^L \varrho_G(u_0, x_0). \tag{4.22}$$

Proof Without loss of generality, we may and do suppose that $\operatorname{gph} G$ is closed. Consider for $(u, x) \in U \times X$ the problem:

$$\text{Minimize } \|x - y\| \text{ over } y \in G(u).$$

The optimal value function of this problem coincides with

$$\varrho_G(u, x) = \inf_{y \in G(u)} \|x - y\|, \qquad x \in X,\ u \in U,$$

and this function ϱ_G is clearly the infimal convolution

$$\varrho_G = \psi_{\operatorname{gph} G} \,\square\, g$$

of the indicator function of the graph $\operatorname{gph} G$ and of the function

$$g(u, x) = \|x\| + \psi_{\{0\}}(u) = \begin{cases} \|x\|, & u = 0, \\ +\infty, & u \neq 0, \end{cases} \qquad u \in U,\ x \in X.$$

Fix any $(u_0, x_0) \in \operatorname{gph} G$. Since g satisfies (4.16) with $z_0 = (0, 0)$ and $\beta = 1$ (as easily seen), Theorem 4.9 applied to the closed set $C = \operatorname{gph} G$ yields (4.22). □

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