Set-Valued Var. Anal (2017) 25:427–439
DOI 10.1007/s11228-016-0391-6

Stability of Weakly Pareto-Nash Equilibria and Pareto-Nash Equilibria for Multiobjective Population Games

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Received: 2 December 2015 / Accepted: 27 September 2016 / Published online: 12 October 2016

Abstract Using the method of generic continuity of set-valued mappings, this paper studies the stability of weakly Pareto-Nash and Pareto-Nash equilibria for multiobjective population games, when payoff functions are perturbed. More precisely, the paper investigates the continuity properties of the set of weakly Pareto-Nash equilibria and that of the set of Pareto-Nash equilibria under sufficiently small perturbations of payoff functions. Firstly, the set of weakly Pareto-Nash equilibria is proven to be upper semicontinuous and further generically continuous with the perturbed payoff functions. Secondly, examples are illustrated to show that the set of Pareto-Nash equilibria is neither upper semicontinuous nor lower semicontinuous. By seeking an upper semicontinuous sub-mapping, it is shown that the set of Pareto-Nash equilibria is partly upper semicontinuous and almost lower semicontinuous.

Keywords Stability · Generic continuity · Multiobjective population games · Pareto-Nash equilibrium · Sub-mapping

Mathematics Subject Classifications (2010) 18B30 · 90C29 · 91A10 · 91A44

This research is supported by the National Science Foundation of China (grant 11271098), the Guizhou Provincial Science and Technology Foundation (grant 20132116) and the Postgraduate Innovative Foundation of Guizhou University (grant 2014065).

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1 Introduction

Multiobjective population games (MPGs) are population games with vector-valued payoffs. The theory of population games [1] originates from Nash's "mass-action" interpretation of equilibrium points in his dissertation [2] and his related literatures [3, 4]. Population games serve as a general model for studying strategic interactions among large numbers of agents, hence they are widely applied to modelling many economic, social and technological environments in which large collections of small agents make strategically interdependent decisions, such as network congestion, public goods and externalities, cultural integration and assimilation, markets and bargaining, etc.

Recently, population games and their applications have attracted increasing attention [5–8]. However, it is worth noting that all the payoffs in the current researches still remain in the scalar case. In other words, these models of population games were only considered as single-objective ones. As we know, in real world the criteria for choosing a strategy usually vary from population to population, even within one population, their criteria are often more than one, such as individual payoff, social position and life satisfaction, etc. Hence, a generalization of the scalar case to multiple criteria is of both theoretical and practical significance for population games.

Weakly Pareto-Nash and Pareto-Nash equilibria, corresponding to weakly efficient and efficient solutions in vector optimization problems, have been constant topics in multiobjective games in recent years. Some scholars explored the existence of weakly Pareto-Nash and Pareto-Nash equilibria [9–11], others focused on their stability [12–14]. In essence, the stability is to study the continuity properties of the set of equilibria. This ideas has been widely used in various fields, such as optimization problems [15–18], linear systems [19], and Nash equilibrium problems [12–14, 20, 21].

In this paper, we are mainly interested in the continuity properties of the set of weakly Pareto-Nash equilibria and that of the set of Pareto-Nash equilibria for (MPGs). In particualr, about the issue of continuity, we focus on upper/lower semicontinuity of the set of weakly Pareto-Nash equilibria and that of the set of Pareto-Nash equilibria with the perturbation of payoff functions. Firstly, the set of weakly Pareto-Nash equilibria is proven to be upper semicontinuous further continuous on a subset of the space of continuous payoff functions equipped with uniform converge norm for (MPGs). However, the set of Pareto-Nash equilibria does not have the same satisfactory continuity properties as that of weakly Pareto-Nash equilibria for (MPGs). In fact, we illustrate two examples to show that the set-valued mapping of Pareto-Nash equilibria is neither upper semicontinuous nor lower semicontinuous. Therefore, to some extent, the continuity of the set of Pareto-Nash equilibria is to be discounted.

Inspired by [14, 16], we show that the Pareto-Nash equilibrium mapping is partly upper semicontinuous although it is not upper semicontinuous by seeking an upper semicontinuous sub-mapping. To achieve this result, the continuity of the set of weighted Nash equilibria and the relationship between weighted Nash equilibria and Pareto-Nash equilibria play crucial role.

The outline of the paper is as follows: in Section 2, the concepts of weakly Pareto-Nash and Pareto-Nash equilibrium are proposed for (MPGs). And some preliminaries on set-valued mappings are reviewed. Section 3 is devoted to the stability of weakly Pareto-Nash equilibria and Pareto-Nash equilibria for (MPGs). In Section 4, a concise conclusion is made for this paper.

2 Preliminaries

Throughout this paper, $\mathbb{N}_+$ denotes the set of all positive integers, $\mathbb{R}$ denotes the set of real numbers. For each $k \in \mathbb{N}_+$, $\mathbb{R}^k$ is a $k-$dimensional Euclidean space, its nonnegative orthant

$$\mathbb{R}^k_+ = \{a = (a_1, a_2, \cdots, a_k) \in \mathbb{R}^k : a_j \geq 0, j = 1, \ldots, k\},$$

and the interior of $\mathbb{R}^k_+$

$$\text{int}\mathbb{R}^k_+ = \{a = (a_1, a_2, \cdots, a_k) \in \mathbb{R}^k : a_j > 0, j = 1, \ldots, k\},$$

respectively. And denoted by $\mathbb{T}^k_+$ and int$\mathbb{T}^k_+$ the simplex of $\mathbb{R}^k_+$ and its interior

$$\mathbb{T}^k_+ = \left\{a = (a_1, a_2, \cdots, a_k) \in \mathbb{R}^k_+ : \sum_{j=1}^{k} a_j = 1\right\},$$

$$\text{int}\mathbb{T}^k_+ = \left\{a = (a_1, a_2, \cdots, a_k) \in \text{int}\mathbb{R}^k_+ : \sum_{j=1}^{k} a_j = 1\right\},$$

respectively.

Consider (MPGs), we state it as follows: let $\mathscr{P} = \{1, 2, \ldots, P\}$ ($P \in \mathbb{N}_+$) be a society consisting of P populations of agents. In each population $p \in \mathscr{P}$, there are a large but finite number of agents and they are capable of independently choosing pure strategies from a finite set $S^p = \{1, 2, \cdots, n^p\}$, where $n^p \in \mathbb{N}_+$ means the total number of the pure strategies in population $p \in \mathscr{P}$, and the number n^p possibly varies from population to population. For the convenience of discussion, throughout this paper, we all assume that the mass of agents in every population is one unit. Thus, for each $p \in \mathscr{P}$, denoted by $X^p = \{x^p = (x^p_1, x^p_2, \cdots, x^p_{n^p}) \in \mathbb{R}^{n^p}_+ : \sum_{l=1}^{n^p} x^p_l = 1\}$, the set of population states, is an $n^p - 1$ dimensional simplex, and where the nonnegative scalar x^p_l represents the share distribution of members playing strategy $l \in S^p$ in population p, and the element $x^p = (x^p_1, x^p_2, \cdots, x^p_{n^p})$ is the state (vector) of population p. Let $m = \sum_{p\in\mathscr{P}} n^p \in \mathbb{N}_+$ be the total number of pure strategies in all populations, and $X = \prod_{p\in\mathscr{P}} X^p = \{x = (x^1; x^2; \cdots; x^P) \in \mathbb{R}^m : x^p \in X^p, p \in \mathscr{P}\}$ denotes the set of social states, in which $x^1, x^2, \cdots, x^P$ are vectors in different spaces and the element $x = (x^1; x^2; \cdots; x^P) \in X$ describes the all populations' behavior at once. And we assume that in each population $p \in \mathscr{P}$ agents all have $k^p \in \mathbb{N}_+$ objectives whenever they play a strategy. $F^p_l : X \to \mathbb{R}^{k^p}$ defines a vector-valued payoff function to a strategy $l \in S^p$, where the element $F^p_{lj} \in \mathbb{R}$ represents the jth objective real-valued payoff to the strategy $l \in S^p$, $j = 1, 2, \cdots, k^p$. $F^p = (F^p_1; F^p_2; \cdots; F^p_{n^p})^T : X \to \mathbb{R}^{n^p k^p}$ describes population p's payoff functions for all strategies in S^p. Now let $N = \sum_{p\in\mathscr{P}} n^p k^p \in \mathbb{N}_+$, the payoff functions $F : X \to \mathbb{R}^N$ is a map that assigns each social state a vector of payoffs, one for each criterion corresponding to each strategy in each population. Since the sets of populations and strategies are generally taken as fixed, a (MPG) is identified with its payoff functions F in the context.

The notions of weakly Pareto-Nash and Pareto-Nash equilibrium of (MPGs) are defined as follows:

Definition 2.1 For a (MPG) F,

(1) a social state $\bar{x} = (\bar{x}^1; \bar{x}^2; \cdots; \bar{x}^P) \in X$ is called a weakly Pareto-Nash equilibrium of F if for each $p \in \mathscr{P}$,

$$\bar{x}_i^p > 0 \Rightarrow F_i^p(\bar{x}) - F_l^p(\bar{x}) \notin -\text{int}\mathbb{R}_+^{k^p}, \ \forall i, l \in S^p.$$

(2) a social state $\bar{x} = (\bar{x}^1; \bar{x}^2; \cdots; \bar{x}^P) \in X$ is called a Pareto-Nash equilibrium of F if for each $p \in \mathscr{P}$,

$$\bar{x}_i^p > 0 \Rightarrow F_i^p(\bar{x}) - F_l^p(\bar{x}) \notin -\mathbb{R}_+^{k^p}\backslash\{0\}, \ \forall i, l \in S^p.$$

Denote the set of weakly Pareto-Nash equilibria and that of Pareto-Nash equilibria by $PE_w(F)$ and $PE(F)$, respectively.

Clearly, $PE(F) \subseteq PE_w(F)$. And if $k^p = 1$ for each $p \in \mathscr{P}$, a Pareto-Nash equilibrium and a weakly Pareto-Nash equilibrium reduce to a Nash equilibrium of population games [1].

Definition 2.2 A social state $\bar{x} = (\bar{x}^1; \bar{x}^2; \cdots; \bar{x}^P) \in X$ is called a weighted Nash equilibrium of F with respect to a given weight combination $\lambda = (\lambda^1; \lambda^2; \cdots; \lambda^P)$ satisfying $\lambda^p \in \mathbb{T}_+^{k^p} (\forall p \in \mathscr{P})$ if for each $p \in \mathscr{P}$,

$$\bar{x}_i^p > 0 \Rightarrow F_{\lambda,i}^p(\bar{x}) \geq F_{\lambda,l}^p(\bar{x}), \quad \forall i, l \in S^p,$$

where $F_{\lambda,i}^p(x) = \sum_{j=1}^{k^p} \lambda_j^p F_{ij}^p(x)$ is the additive weight payoff to a strategy $i \in S^p$. And denoted by $E_\lambda(F)$ the set of all weighted Nash equilibria of F with respect to a given weight combination λ.

Let us recall some necessary definitions and results, which are helpful to main results of this paper.

Let H be a topological vector space. A subset C of H is called a cone if $tc \in C$ for any $c \in C$ and any nonnegative number t. A cone C is closed if it is a closed set. A cone C is convex if it is a convex set. Further, it is pointed if $C \cap (-C) = \{\theta\}$, where θ denotes the zero element of H. In particular, $\mathbb{R}_+^k$ is a closed convex and pointed cone of $\mathbb{R}^k$. The following two definitions can be found in [22].

Definition 2.3 Let E and H be two topological vector spaces, Y a nonempty convex subset of E and C a closed, convex and pointed cone of H with int$C \neq \emptyset$. Let $f : Y \to H$ be a vector-valued function. f is said to be $C-$continuous at $y_0 \in Y$ if, for any open neighborhood V of zero element θ in H, there exists an open neighborhood U of $y_0 \in Y$ such that, for all $y \in U$,

$$f(y) \in f(y_0) + V + C,$$

and $C-$continuous on Y if it is $C-$continuous at any point of Y.

Definition 2.4 Let E and H be two topological vector spaces, Y be a nonempty convex subset of E and C be a closed, convex and pointed cone of H with int$C \neq \emptyset$. Let $f : Y \to$

H be a vector-valued function. f is called $C-$concave if for each $y_1, y_2 \in Y$ and each $t \in [0, 1]$,

$$tf(y_1) + (1-t)f(y_2) - f\big(ty_1 + (1-t)y_2\big) \in -C,$$

and $C-$convex if $-f$ is $C-$concave.

If $H = \mathbb{R}^k$, $C = \mathbb{R}^k_+$, the following two lemmas reveal the equivalent property of $\mathbb{R}^k_+-$continuity and that of $\mathbb{R}^k_+-$concavity of a vector-valued function [12], respectively.

Lemma 2.5 *Let X be a nonempty subset of a normed space and $f : X \to \mathbb{R}^k$ be a vector-valued function, where $f = (f_1, f_2, \cdots, f_k)$. Then f is $\mathbb{R}^k_+-$continuous if and only if f_j is lower semicontinuous for every $j = 1, 2, \cdots, k$.*

Lemma 2.6 *Let X be a nonempty convex subset of a normed space and $f : X \to \mathbb{R}^k$ be a vector-valued function, where $f = (f_1, f_2, \cdots, f_k)$. Then f is $\mathbb{R}^k_+-$concave if and only if f_j is concave for every $j = 1, 2, \cdots, k$.*

The following lemma is referred to Theorem 1.1 of [12].

Lemma 2.7 *Let X be a nonempty convex compact subset of a normed space and H be a topological vector space with a closed, convex and pointed cone C and $intC \neq \emptyset$. Suppose that $\phi : X \times X \to H$ satisfies the following conditions:*

(i) *for each fixed $y \in X$, $x \mapsto \phi(x, y)$ is $C-$continuous;*
(ii) *for each fixed $x \in X$, $y \mapsto \phi(x, y)$ is $C-$concave; and*
(iii) *for each $x \in X$, $\phi(x, x) \notin intC$.*

Then there exists $x^* \in X$ such that $\phi(x^*, y) \notin intC$ for all $y \in X$.

Definition 2.8 ([23]) Let X and Y be two Hausdorff topological spaces, and let $B : Y \to 2^X$ be a set-valued mapping.

(1) B is upper semicontinuous at $y \in Y$ if for any open set U in X with $U \supset B(y)$, there is an open neighborhood $\mathscr{O}(y)$ of y such that $U \supset B(y^{'})$ for each $y^{'} \in \mathscr{O}(y)$.
   B is upper semicontinuous on Y if it is upper semicontinuous at every point $y \in Y$. Further, B is an usco mapping if B is upper semicontinuous on Y and $B(y)$ is compact for every $y \in Y$.
(2) B is lower semicontinuous at $y \in Y$ if for any open set U with $U \cap B(y) \neq \emptyset$, there is an open neighborhood $\mathscr{O}(y)$ of y such that $U \cap B(y^{'}) \neq \emptyset$ for each $y^{'} \in \mathscr{O}(y)$.
   B is lower semicontinuous on Y if it is lower semicontinuous at every point $y \in Y$.
(3) B is continuous at $y \in Y$ if it is both upper and lower semicontinuous at $y \in Y$. B is continuous on Y if it is continuous at every point $y \in Y$.
(4) B is almost lower semicontinuous at $y \in Y$ if there exists at least one $x \in B(y)$ such that, for each open neighborhood $N(x)$ of x, there exists an open neighborhood $\mathscr{O}(y)$ of y such that $\mathscr{N}(x) \cap B(y^{'}) \neq \emptyset$ for any $y^{'} \in \mathscr{O}(y)$.

The following example provides a set-valued mapping that is almost lower semicontinuous but not lower semicontinuous.

Example 2.9 Let $X = [0, 1]$ and define the set-valued mapping $B : X \to 2^X$ by

$$B(x) = \begin{cases} \{0\}, & x \in [0, 1), \\ [0, 1], & x = 1. \end{cases}$$

It is easy to check that B is almost lower semicontinuous but not lower semicontinuous at $x = 1$.

The following lemma is referred to Corollary 9 in Chapter 3 of [23] or Theorem 7.1.16 of [24].

Lemma 2.10 *Let X, Y be two Hausdorff topological spaces with X compact. If the graph $Graph(B)$ of the set-valued mapping $B : Y \to 2^X$ is closed, then B is upper semicontinuous on Y, where*

$$Graph(B) = \{(y, x) \in Y \times X : x \in B(y)\}.$$

Lemma 2.11 ([25], Theorem 2) *Let X be a metric space, Y be a Baire space, and $B : Y \to 2^X$ be an usco mapping. Then there is a dense G_δ subset Q of Y such that B is lower semicontinuous on Q.*

Remark 2.12 Lemma 2.11 indicates the set-valued mapping $B : Y \to 2^X$ is continuous on a dense G_δ subset Q of Y, so B is usually called generic continuity on Y.

3 Stability of Weakly Pareto-Nash and Pareto-Nash Equilibrium for (MPGs)

In this section, we begin with the existence of weakly Pareto-Nash equilibrium for (MPGs).

Theorem 3.1 *Let $F : X \to \mathbb{R}^N$ be a (MPG), where $X = \prod_{p \in \mathscr{P}} X^p$, X^p is a simplex in $\mathbb{R}^{n^p}$ for each $p \in \mathscr{P}$, and $N = \sum_{p \in \mathscr{P}} n^p k^p \in \mathbb{N}_+$. If F is continuous on X, then it admits at least one weakly Pareto-Nash equilibrium.*

Proof Define a vector-valued function $\phi : X \times X \to \mathbb{R}^k$ by

$$\phi(x, y) = \sum_{p \in \mathscr{P}} \sum_{l \in S^p} \left(y_l^p - x_l^p\right) \widehat{F}_l^p(x),$$

where $k = \max\limits_{p \in \mathscr{P}} k^p$, and for any $p \in \mathscr{P}$ and $l \in S^p$,

$$\widehat{F_l^p}(x) = (\underbrace{F_l^p(x)}_{k^p components}; \underbrace{F_{l1}^p(x), \cdots, F_{l1}^p(x)}_{k - k^p \ times}) \in \mathbb{R}^k.$$

It is easy to check that

(i) for each fixed $y \in X$, $x \mapsto \phi(x, y)$ is $\mathbb{R}_+^k$−continuous (by Lemma 2.5);
(ii) for each fixed $x \in X$, $y \mapsto \phi(x, y)$ is $\mathbb{R}_+^k$−concave (by Lemma 2.6); and
(iii) for each $x \in X$, $\phi(x, x) = 0 \notin \text{int}\mathbb{R}_+^k$.

Therefore, by Lemma 2.7, there exists $\bar{x} \in X$ such that

$$\phi(\bar{x}, y) \notin \text{int}\mathbb{R}^k_+, \quad \forall y \in X.$$

Now for any $p \in \mathscr{P}$, denotes $I^p(\bar{x}) = \{l \in S^p : \bar{x}^p_l > 0\}$, obviously $I^p(\bar{x}) \neq \emptyset$ as X^p is a simplex of $\mathbb{R}^{n^p}$ for each $p \in \mathscr{P}$. For each fixed $i \in I^p(\bar{x})$, for any $l \in S^p$, setting $y = (\bar{x}^1; \cdots; \bar{x}^{p-1}; y^p; \bar{x}^{p+1}; \cdots; \bar{x}^P) \in X$ and

$$y^p = (\bar{x}^p_1, \cdots, \bar{x}^p_{i-1}, \underbrace{0}_{i}, \bar{x}^p_{i+1}, \cdots, \bar{x}^p_{l-1}, \underbrace{\bar{x}^p_i + \bar{x}^p_l}_{l}, \bar{x}^p_{l+1}, \cdots, \bar{x}^p_{n^p}) \in X^P.$$

Furthermore,

$$\bar{x}^p_i \big(\underbrace{F^p_i(\bar{x}) - F^p_l(\bar{x})}_{k^p components}; \underbrace{F^p_{i1}(\bar{x}) - F^p_{l1}(\bar{x}), \cdots, F^p_{i1}(\bar{x}) - F^p_{l1}(\bar{x})}_{k-k^p\ times}\big) = \phi(\bar{x}, y) \notin \text{int}\mathbb{R}^k_+.$$

Because of $\bar{x}^p_i > 0$, then

$$\big(\underbrace{F^p_i(\bar{x}) - F^p_l(\bar{x})}_{k^p components}; \underbrace{F^p_{i1}(\bar{x}) - F^p_{l1}(\bar{x}), \cdots, F^p_{i1}(\bar{x}) - F^p_{l1}(\bar{x})}_{k-k^p\ times}\big) \notin \text{int}\mathbb{R}^k_+. \tag{1}$$

If $F^p_i(\bar{x}) - F^p_l(\bar{x}) \in \text{int}\mathbb{R}^{k^p}_+$, then $F^p_{ij}(\bar{x}) - F^p_{lj}(\bar{x}) \in \text{int}\mathbb{R}^{k^p}_+$ for each $j = 1, 2, \cdots, k^p$ and thus

$$\big(\underbrace{F^p_i(\bar{x}) - F^p_l(\bar{x})}_{k^p components}; \underbrace{F^p_{i1}(\bar{x}) - F^p_{l1}(\bar{x}), \cdots, F^p_{i1}(\bar{x}) - F^p_{l1}(\bar{x})}_{k-k^p\ times}\big) \in \text{int}\mathbb{R}^k_+,$$

which contradicts that expression (1). Consequently, for all $p \in \mathscr{P}$, $i \in I^p(\bar{x})$, it holds true that

$$\bar{x}^p_i > 0 \Rightarrow F^p_i(\bar{x}) - F^p_l(\bar{x}) \notin \text{int}\mathbb{R}^{k^p}_+, \forall l \in S^p.$$

Hence, $\bar{x}$ is a weakly Pareto-Nash equilibrium of F. The proof is complete. □

Let $\mathscr{F}$ be the collection of (MPGs) satisfying that: (i) X^p is a simplex in $\mathbb{R}^{n^p}$ for all $p \in \mathscr{P}$; (ii)$F : X = \prod_{p\in\mathscr{P}} X^p \to \mathbb{R}^N$ is continuous.

For any $F, G \in \mathscr{F}$, define

$$\rho(F, G) = \max_{x\in X} \sum_{p\in\mathscr{P}} \|F^p(x) - G^p(x)\|,$$

where $\|F^p(x)-G^p(x)\| = \big(\sum_{l\in S^p} |F^p_l(x)-G^p_l(x)|^2\big)^{\frac{1}{2}}$, the classical Euclidean norm in $\mathbb{R}^{n^p}$.

Clearly, ρ is a metric on $\mathscr{F}$. Indeed, $(\mathscr{F}, \rho)$ is complete. From Theorem 3.1, $PE_w(F) \neq \emptyset$ for each $F \in \mathscr{F}$. And PE_w, PE are both set-valued mappings from $\mathscr{F}$ to X, which are called weakly Pareto-Nash and Pareto-Nash equilibrium mapping in the remainder of this paper, respectively.

Definition 3.2 (i) For each $F \in \mathscr{F}$, let $x \in PE_w(F)$ (resp. $x \in PE(F)$). Then x is called an essential weakly Pareto-Nash (resp. Pareto-Nash) equilibrium of F provided that for any open neighborhood U of x in X, there exists an open neighborhood $\mathscr{N}(F)$ of $F \in \mathscr{F}$ such that $U \cap PE_w(F') \neq \emptyset$ (resp. $U \cap PE(F') \neq \emptyset$) for any $F' \in \mathscr{N}(F)$.

(ii) For each $F \in \mathscr{F}$, let $e(F)$ be a nonempty closed subset of $PE_w(F)$ (resp.$PE(F)$), Then $e(F)$ is called an essential set of weakly Pareto-Nash (resp. Pareto-Nash) equilibria of F provided that for any open set $U \supset e(F)$, there exists an open neighborhood

$\mathcal{N}(F)$ of $F \in \mathscr{F}$ such that $U \cap PE_w(F^{'}) \neq \emptyset$ (resp. $U \cap PE(F^{'}) \neq \emptyset$) for any $F^{'} \in \mathcal{N}(F)$.

Remark 3.3 (i) x is an essential weakly Pareto-Nash (resp. Pareto-Nash) equilibrium of $F \in \mathscr{F}$, namely, the mapping PE_w (resp. PE) is almost lower semicontinuous at F.
(ii) An essential set $e(F)$ of weakly Pareto-Nash (resp. Pareto-Nash) equilibria of F means that near the game F there exists at least one weakly Pareto-Nash (resp. Pareto-Nash) equilibrium near $e(F)$. In particular, if $e(F)$ is a singleton set, i.e., $e(F) = \{\bar{x}\}$, then $\bar{x}$ is an essential weakly Pareto-Nash (resp. Pareto-Nash) equilibrium of F.

Theorem 3.4 *The weakly Pareto-Nash equilibrium mapping $PE_w : \mathscr{F} \to 2^X$ is usco on $\mathscr{F}$.*

Proof Since $X = \prod_{p\in\mathscr{P}} X^p$ is compact, from Lemma 2.10, it suffices to verify that the graph $Graph(PE_w)$ of the set-valued mapping PE_w is closed, where

$$Graph(PE_w) = \{(F, x) \in \mathscr{F} \times X : x \in PE_w(F)\}.$$

Let $\{(F^n, x^n)\}_{n\in\mathbb{N}_+}$ be a sequence in $Graph(PE_w)$ with $(F^n, x^n) \to (\bar{F}, \bar{x}) \in \mathscr{F} \times X$, then $F^n \in \mathscr{F}, x^n \in PE_w(F^n)$. It needs to prove $\bar{x} \in PE_w(\bar{F})$. Argue by contradiction. Suppose that $\bar{x} \notin PE_w(\bar{F})$, then there are some $p_0 \in \mathscr{P}$ and $i_0, l_0 \in S^{p_0}$, though $\bar{x}_{i_0}^{p_0} > 0$,

$$\bar{F}_{i_0}^{p_0}(\bar{x}) - \bar{F}_{l_0}^{p_0}(\bar{x}) \in -\text{int}\mathbb{R}_+^{k^{p_0}}.$$

As $(F^n, x^n) \to (\bar{F}, \bar{x})$, then $F^n \to \bar{F}, x^n \to \bar{x}$, there exists $N_1 \in \mathbb{N}_+$, such that $F^n(\bar{x}) \to \bar{F}(\bar{x})$ and $(x^n)_{i_0}^{p_0} \to \bar{x}_{i_0}^{p_0}$ with $(x^n)_{i_0}^{p_0} > 0$ whenever $n > N_1$. Further by combining the continuity of F^n on X with the fact $F^n \to \bar{F}, x^n \to \bar{x}$, then there exists $N_2 \in \mathbb{N}_+$ and $N_2 > N_1$, such that

$$(F^n)_{i_0}^{p_0}(x^n) - (F^n)_{l_0}^{p_0}(x^n) \in -\text{int}\mathbb{R}_+^{k^{p_0}},$$

whenever $n > N_2$.

In a word, although $(x^n)_{i_0}^{p_0} > 0$, $(F^n)_{i_0}^{p_0}(x^n) - (F^n)_{l_0}^{p_0}(x^n) \in -\text{int}\mathbb{R}_+^{k^{p_0}}$ whenever $n > N_2$. By Definition 2.1, thus $x^n \notin PE_w(F^n)$, which contradicts the fact $x^n \in PE_w(F^n)$. Therefore, $\bar{x} \in PE_w(\bar{F})$, and hence $Graph(PE_w)$ is closed. Following from Lemma 2.10, then PE_w is upper semicontinuous on $\mathscr{F}$.

Next, we need prove $PE_w(F)$ is compact for all $F \in \mathscr{F}$. Observing that $X = \prod_{p\in\mathscr{P}} X^p$ is compact, so we have just to check $PE_w(F)$ is closed for all $F \in \mathscr{F}$. To do this, we merely repeat the first part of the proof with $F \in \mathscr{F}$. The proof is complete. □

From Theorem 3.4 and Lemma 2.11, it is easy to obtain the generic continuity of the set of weakly Pateto-Nash equilibria for (MPGs) as follows:

Theorem 3.5 *There exists a dense G_δ subset Q of $\mathscr{F}$ such that the weakly Pareto-Nash equilibrium mapping $PE_w : \mathscr{F} \to 2^X$ is lower semicontinuous further continuous on Q. That is, every weakly Pareto-Nash equilibrium of each $F \in Q$ is essential.*

Proof From Theorem 3.4 and Lemma 2.11, there exists a dense G_δ subset Q of $\mathscr{F}$ such that PE_w is lower semicontinuous further continuous on Q. Let $x^* \in PE_w(F)$ for $F \in Q$. Since PE_w is lower semicontinuous at F, then x^* is an essential weakly Pareto-Nash equilibrium from Definition 3.2 and Remark 3.3. The proof is complete. □

One can observe that such generic continuity on the set of weakly Pareto-Nash equilibria is due to its upper semicontinuity property. On involving the set of Pareto-Nash equilibria of (MPGs), however, it is not a satisfactory one. The following example shows this point.

Example 3.6 Let F be a bi-objective population game played by two unit mass populations with two strategies for each ($n^1 = n^2 = 2$). Let the corresponding sets of population states be $Y = \{y = (y_1, y_2) \in \mathbb{R}^2_+ : y_1 + y_2 = 1\}$, $Z = \{z = (z_1, z_2) \in \mathbb{R}^2_+ : z_1 + z_2 = 1\}$, respectively. Let $X = Y \times Z = \{x = (y, z) : y \in Y, z \in Z\}$ and $b \in \mathbb{R}$ be a constant. Let $F(x) = (F^1(x); F^2(x))$, where for population 1, $F^1(x) = (F^1_1(x); F^1_2(x))^T$,

$$\begin{aligned} F^1_1(x) &= (F^1_{11}(x), F^1_{12}(x)) = (y_1, y_2), \\ F^1_2(x) &= (F^1_{21}(x), F^1_{22}(x)) = (1 + y_1, 1 + y_2). \end{aligned}$$

And for population 2, $F^2(x) = (F^2_1(x); F^2_2(x))^T$,

$$\begin{aligned} F^2_1(x) &= (F^2_{11}(x), F^2_{12}(x)) = (z_1, b), \\ F^2_2(x) &= (F^2_{21}(x), F^2_{22}(x)) = (1 + z_1, b). \end{aligned}$$

It is easy to see that the payoff to the 2nd strategy dominates that of the 1st strategy in each population, thus $PE(F)$ contains only one state $\bar{x} = \{(0, 1), (0, 1)\}$, i.e., $PE(F) = \{\bar{x}\}$.

Let $\{F^n(x)\}_{n\in\mathbb{N}_+} = \{((F^n)^1(x); (F^n)^2(x))\}_{n\in\mathbb{N}_+}$ be a perturbed sequence of F, where $(F^n)^1(x) = F^1(x)$;

$$\begin{aligned} (F^n)^2_1(x) &= ((F^n)^2_{11}(x), (F^n)^2_{12}(x)) = (z_1, b + 1/n), \\ (F^n)^2_2(x) &= ((F^n)^2_{21}(x), (F^n)^2_{22}(x)) = (1 + z_1, b). \end{aligned}$$

Clearly, $F^n \to F (n \to +\infty)$, and we find that the Pareto-Nash equilibria of F^n is $PE(F^n) = \{(0, 1)\} \times Z$ since for all $x \in X$, $(F^n)^1(x) = F^1(x)$; yet

$$\begin{aligned} (F^n)^2_{11}(x) = z_1 &< (F^n)^2_{21}(x) = 1 + z_1, \\ (F^n)^2_{12}(x) = b + 1/n &> (F^n)^2_{22}(x) = b. \end{aligned}$$

It is easy to check that all the other states $x = (y, z) \in (Y \backslash \{(0, 1)\}) \times Z$ are not Pareto-Nash equilibrium of F^n. Therefore, $PE(F^n) = \{(0, 1)\} \times Z$ is the Pareto-Nash equilibria of F^n.

So the Pareto-Nash equilibrium mapping PE is not upper semicontinuous at F, because however close F^n approaches F, the Pareto-Nash equilibria of F^n cannot be covered in any small neighborhood of $\bar{x} = \{(0, 1), (0, 1)\}$, which is the unique Pareto-Nash equilibrium of F. However, it is not difficult to examine that PE_w is upper semicontinuous at F, because all $x = (y, z) \in \{(0, 1)\} \times Z$ are weakly Pareto-Nash equilibria of F and of F^n as well.

To obtain the stability of Pareto-Nash equilibria for (MPGs), we define a sub-mapping and partly upper semicontinuity of the Pareto-Nash equilibrium mapping PE below:

Definition 3.7 A mapping $E_0 : \mathscr{F} \to 2^X$ is said to be a sub-mapping of the Pareto-Nash equilibrium mapping PE if $E_0(F) \subset PE(F)$ holds for each $F \in \mathscr{F}$. Furthermore, if the sub-mapping $E_0 : \mathscr{F} \to 2^X$ is upper semicontinuous, then the Pareto-Nash equilibrium mapping PE is said to be partly upper semicontinuous.

Example 3.8 Consider a bi-objective single-population game $\tilde{F} = (\tilde{F}_1; \tilde{F}_2)^T$ with two strategies. The state set is denoted by $X = \{x = (x_1, x_2) \in \mathbb{R}^2_+ : x_1 + x_2 = 1\}$.

For each $x = (x_1, x_2) \in X$,

$$\tilde{F}_1(x) = (\tilde{F}_{11}(x), \tilde{F}_{12}(x)) = (x_1, 2),$$
$$\tilde{F}_2(x) = (\tilde{F}_{21}(x), \tilde{F}_{22}(x)) = (1 + x_1, x_2).$$

Clearly, each $x = (x_1, x_2) \in X$ is a Pareto-Nash equilibrium of $\tilde{F}$, i.e., $PE(\tilde{F}) = X$.

For a given weight combination $\lambda = (\lambda_1, \lambda_2) = (1/3, 2/3) \in \text{int}\mathbb{T}^2_+$, the resulting additive weight payoffs to strategy 1 and 2:

$$\tilde{F}_{\lambda,1}(x) = \lambda_1 \tilde{F}_{11}(x) + \lambda_2 \tilde{F}_{12}(x) = x_1/3 + 4/3,$$
$$\tilde{F}_{\lambda,2}(x) = \lambda_1 \tilde{F}_{21}(x) + \lambda_2 \tilde{F}_{22}(x) = (1 + x_1)/3 + 2x_2/3 = -x_1/3 + 1,$$

respectively. By Definition 2.2, obviously, $\tilde{F}$ has only one weighted Nash equilibrium $(1, 0) \in X$ with respect to the given weight combination $\lambda = (\lambda_1, \lambda_2) = (1/3, 2/3) \in \text{int}\mathbb{T}^2_+$, since $\tilde{F}_{\lambda,1}(x) > \tilde{F}_{\lambda,2}(x)$ holds for any $x = (x_1, x_2) \in X$ no matter what value $x_2 > 0$ takes except for the state $(1, 0) \in X$. Consequently, for $\tilde{F}$, $E_\lambda(\tilde{F}) = \{(1, 0)\} \subset X = PE(\tilde{F})$. By Definition 3.7, $E_\lambda(\tilde{F})$ is a sub-mapping of $PE(\tilde{F})$.

Now, by seeking an upper semicontinous sub-mapping, we obtain the following partly upper semicontinuity result of the Pareto-Nash equilibrium mapping PE.

Theorem 3.9 *There exists an upper semicontinuous sub-mapping of PE, i.e., $E_0 : \mathscr{F} \to 2^X$ such that $E_0(F) \subset PE(F)$ for each $F \in \mathscr{F}$ and upper semicontinuous on $\mathscr{F}$. That is, PE is partly upper semicontinuous.*

Proof Given a weight combination $\lambda = (\lambda^1; \lambda^2; \cdots; \lambda^P)$ with $\lambda^p \in \text{int}\mathbb{T}^{k^p}_+ (\forall p \in \mathscr{P})$, for each $F \in \mathscr{F}$, since the additive weight payoff $F^p_{\lambda,i}(x)$ to the strategy $i \in S^p$ is continuous on X for each population $p \in \mathscr{P}$, then there exists weighted Nash equilibria for each $F \in \mathscr{F}$ from Theorem 2.1.1 of [1], i.e., $E_\lambda(F) \neq \emptyset$ and E_λ is a set-valued mapping from $\mathscr{F}$ to X.

Set $E_0 = E_\lambda : \mathscr{F} \to 2^X$. Next, it is proven that the mapping $E_\lambda : \mathscr{F} \to 2^X$ is usco.

Let $\{(F^n, x^n)\}_{n \in \mathbb{N}_+}$ be a sequence in $Graph(E_\lambda)$ with $(F^n, x^n) \to (\bar{F}, \bar{x}) \in \mathscr{F} \times X$, then $F^n \in \mathscr{F}, x^n \in E_\lambda(F^n)$. Suppose that $\bar{x} \notin E_\lambda(\bar{F})$, then there exist some $p_0 \in \mathscr{P}$ and $i_0, l_0 \in S^{p_0}$ such that $\bar{x}^{p_0}_{i_0} > 0$, however,

$$\bar{F}^{p_0}_{\lambda,i_0}(\bar{x}) < \bar{F}^{p_0}_{\lambda,l_0}(\bar{x}).$$

As $F^n_\lambda \to \bar{F}_\lambda$ due to $F^n \to \bar{F}$, further,

$$(F^n)^{p_0}_{\lambda,i_0}(\bar{x}) < (F^n)^{p_0}_{\lambda,l_0}(\bar{x}).$$

Because $(F^n)^{p_0}_{\lambda,l}(x)$ is continuous on X for any $l \in S^p$ and $x^n \to \bar{x}$, then $(x^n)^{p_0}_{i_0} \to \bar{x}^{p_0}_{i_0}$ satisfying $(x^n)^{p_0}_{i_0} > 0$, and

$$(F^n)^{p_0}_{\lambda,i_0}(x^n) < (F^n)^{p_0}_{\lambda,l_0}(x^n),$$

whenever n is sufficiently large.

In a word, $(x^n)^{p_0}_{i_0} > 0$, yet $(F^n)^{p_0}_{\lambda,i_0}(x^n) < (F^n)^{p_0}_{\lambda,l_0}(x^n)$ whenever n is sufficiently large. By Definition 2.2, thus $x^n \notin E_\lambda(F^n)$, which contradicts the assumption $x^n \in E_\lambda(F^n)$. Therefore, $\bar{x} \in E_\lambda(\bar{F})$, namely, $Graph(E_\lambda)$ is closed. From Lemma 2.10, then E_λ is upper semicontinuous on $\mathscr{F}$. Furthermore, to show $E_\lambda(F)$ is compact for each $F \in \mathscr{F}$, we have just to check $PE_w(F)$ is closed for all $F \in \mathscr{F}$ as $X = \prod_{p\in\mathscr{P}} X^p$ is compact. For this purpose, we repeat the above proof on the closedness of $Graph(E_\lambda)$ with $F^n = F$. Consequently, E_λ is an usco mapping on $\mathscr{F}$.

Besides, it holds true that $E_\lambda(F) \subset PE(F)$ for $\lambda = (\lambda^1; \lambda^2; \cdots; \lambda^P)$ with $\lambda^p \in \text{int}\mathbb{T}_+^{k^p}$ for all $p \in \mathscr{P}$. If not, there is one point $\tilde{x} \in E_\lambda(F)$, nevertheless, $\tilde{x} \notin PE(F)$. Then there are some $p_0 \in \mathscr{P}$ and $i_0, l_0 \in S^{p_0}$, $\tilde{x}_{i_0}^{p_0} > 0$, but $F_{i_0}^{p_0}(\tilde{x}) - F_{l_0}^{p_0}(\tilde{x}) \in -\mathbb{R}_+^{k^{p_0}} \backslash \{0\}$, i.e.,

$$F_{l_0 j}^{p_0}(\tilde{x}) \geq F_{i_0 j}^{p_0}(\tilde{x}), \ \forall j \in \{1, 2, \cdots, k^{p_0}\}, \text{ and}$$
$$F_{l_0 j}^{p_0}(\tilde{x}) > F_{i_0 j}^{p_0}(\tilde{x}), \text{ for some } j \in \{1, 2, \cdots, k^{p_0}\}.$$

Since $\lambda^{p_0} \in \text{int}\mathbb{T}_+^{k^{p_0}}$ within $\lambda = (\lambda^1; \lambda^2; \cdots; \lambda^P)$, it immediately follows that $F_{\lambda,l_0}^{p_0}(\tilde{x}) > F_{\lambda,i_0}^{p_0}(\tilde{x})$. To sum up, $\tilde{x}_{i_0}^{p_0} > 0$, however, $F_{\lambda,l_0}^{p_0}(\tilde{x}) > F_{\lambda,i_0}^{p_0}(\tilde{x})$. From Definition 2.2, this means that $\tilde{x} \notin E_\lambda(F)$, which is a contradiction. The proof is complete. □

Remark 3.10 From the proof of Theorem 3.9, it is known that

(1) for each $F \in \mathscr{F}$, $\emptyset \neq E_\lambda(F) \subset PE(F)$ and $E_\lambda(F)$ is closed for a given $\lambda = (\lambda^1; \lambda^2; \cdots; \lambda^P)$ with $\lambda^p \in \text{int}\mathbb{T}_+^{k^p}$, $\forall p \in \mathscr{P}$. Further, $E_\lambda(F)$ is an essential set of $PE(F)$ by Definition 3.2(ii).
(2) The set of weighted Nash equilibria is usco on $\mathscr{F}$. From Lemma 2.11, there is a dense G_δ subset Q of $\mathscr{F}$ such that the set of weighted Nash equilibria is lower semicontinuous further continuous on Q.

The following special case shows that the Pareto-Nash equilibrium mapping PE is not lower semicontinuous on $\mathscr{F}$.

Example 3.11 Let us consider a bi-objective single-population game $F = (F_1; F_2)^T$ with two strategies. Then the state set is denoted by $X = \{x = (x_1, x_2) \in \mathbb{R}_+^2 : x_1 + x_2 = 1\}$, the simplex in $\mathbb{R}^2$. Let $a, b \in \mathbb{R}$ be two constants.

For each $x = (x_1, x_2) \in X$,

$$F_1(x) = (F_{11}(x), F_{12}(x)) = (a, b),$$
$$F_2(x) = (F_{21}(x), F_{22}(x)) = (a, b).$$

Clearly, each $x = (x_1, x_2) \in X$ is a Pareto-Nash equilibrium of F, i.e., $PE(F) = X$.

Let $\{F^n\}_{n\in\mathbb{N}_+} = \{\big((F^n)_1; (F^n)_2\big)^T\}_{n\in\mathbb{N}_+}$ be an approximating sequence of F in which for each $x = (x_1, x_2) \in X$ we have

$$(F^n)_1(x) = \big((F^n)_{11}(x), (F^n)_{12}(x)\big) = (a, b),$$
$$(F^n)_2(x) = \big((F^n)_{21}(x), (F^n)_{22}(x)\big) = (a + 1/n, b + 1/n).$$

Obviously, the perturbed bi-objective population game F^n admits unique Pareto-Nash equilibrium $(0, 1) \in X$, namely, $PE(F^n) = \{(0, 1)\}$. Nevertheless, for a special Pareto-Nash equilibrium $(1, 0) \in PE(F)$, we can choose a small enough neighborhood $\mathscr{N}(1, 0)$; no matter how close F^n is to F, $\{(0, 1)\} \cap \mathscr{N}(1, 0) = \emptyset$. Therefore, PE is not lower semicontinuous at F. Meanwhile, it is easy to check that PE_w is also not lower semicontinuous at F, too, as $PE_w(F) = X$ and $PE_w(F^n) = \{(0, 1)\}$.

Theorem 3.12 *There exists a dense G_δ subset Q of $\mathscr{F}$ such that the Pareto-Nash equilibrium mapping PE is almost lower semicontinuous on Q.*

Proof From Lemma 2.11 and Remark 3.10(2), there exists a dense G_δ subset Q of $\mathscr{F}$ such that the sub-mapping $E_0 = E_\lambda$ is lower semicontinuous on Q, where $\lambda = (\lambda^1; \lambda^2; \cdots; \lambda^P)$ satisfying $\lambda^p \in \text{int}\mathbb{T}_+^{k^p}, \forall p \in \mathscr{P}$. Let $x^* \in E_\lambda(F)$ for $F \in Q$. Then for any open neighborhood $\mathscr{O}(x^*)$ of x^*, there exists an open neighborhood $\mathscr{N}(F)$ of F such that $\mathscr{O}(x^*) \cap E_\lambda(F') \neq \emptyset$ for all $F' \in \mathscr{N}(F)$. Since $E_0(F) \subset PE(F)$, it also holds that $\mathscr{O}(x^*) \cap PE(F) \neq \emptyset$. Therefore, PE is almost lower semicontinuous on Q by Definition 2.8(4). The proof is complete. □

Remark 3.13 (1) There exists at least one essential Pareto-Nash equilibrium $x^* \in PE(F)$ for most $F \in \mathscr{F}$ by the proof of Theorem 3.12.

(2) Combining the proof of Theorem 3.12 with Example 3.6, each $x \in PE(F)$ being an essential Pareto-Nash equilibrium is a necessary but not sufficient condition for the continuity of PE at $F \in \mathscr{F}$.

4 Conclusion

In this paper, we have proven some stability results for weakly Pareto-Nash and Pareto-Nash equilibria of (MPGs) with the perturbed payoff functions. Here, the stability is dependent on the semicontinuity property of the set-valued mapping $PE_w(F)$ (resp. $PE(F)$), which associates to (MPGs) F. The weakly Pareto-Nash equilibrium mapping PE_w is upper semicontinuous. This leads to the generic continuity of weakly Pareto-Nash equilibrium, that is, each weakly Pareto-Nash equilibrium is stable for most (MPGs) with continuous payoff functions in the sense of Baire category. However, the problem is nontrivial as the Pareto-Nash equilibrium mapping $PE(F)$ is generally neither upper semicontinuous nor lower semicontinuous (see Example 3.6 and 3.11). We prove the partly upper semicontinuity of $PE(F)$ by seeking an upper semicontinuous sub-mapping. Based on this fact, along with the generic continuity of the set of weighted Nash equilibria (Remark 3.10(2)), it is furthermore shown that $PE(F)$ is almost lower semicontinuous for most (PMGs). Therefore, most (PMGs) have at least one stable Pareto-Nash equilibrium in the sense of Baire category. Our work extends population games with single objective [1] to multiobjective cases. And our results are new for population games.

Acknowledgments The authors would like to express their deep gratitude to the anonymous reviewers for their positive comments and numerous valuable suggestions.

Compliance with Ethical Standards

Conflict of interests The authors declare that they have no conflict of interest.

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