

Necessary Conditions for Adverse Control Problems Expressed by Relaxed Derivatives

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Received: 12 February 2015 / Accepted: 25 January 2016 / Published online: 5 February 2016
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Abstract This paper provides a framework for deriving a new set of necessary conditions for adverse control problems among two players. The distinguish feature of such problems is that the first player has a priori knowledge on the second player strategy. A subclass of adverse control problems is the one of minimax control problems, which frequently arise in robust dynamic optimization. The conditions derived in this manuscript are expressed in terms of relaxed derivatives: the dual variables and the related functions are limits of computable sequences, obtained by considering a regularized version of the original problem and applying well known necessary conditions. This topic was initially treated by J. Warga.

Keywords Necessary conditions · Optimal control · Ordinary differential equations · Relaxation · Generalized derivatives · Conflicting controls

Mathematics Subject Classification (2010) MSC 49K15 · MSC 49K35

1 Introduction

In this paper, we consider adverse control problems between two players described by differential equations

$$y(t) = b + \int_{t_0}^t f(s, y(s), u(s)) ds$$

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$$\tilde{y}(t) = \tilde{b} + \int_{t_0}^t \tilde{f}(s, \hat{y}(s), u(s), v(s)) ds.$$

In this model, $u(\cdot)$, b , and $y(\cdot)$ are, respectively, the control, the initial state and the state trajectory of the first player, while $v(\cdot)$, \tilde{b} and $\tilde{y}(\cdot)$ are, respectively, the control, the initial state and the state trajectory of the second player; $\hat{y}(\cdot) = (y(\cdot), \tilde{y}(\cdot))$ is the state trajectory of the control system and $\hat{b} = (b, \tilde{b})$ is its initial condition at time t_0 , which can be varying on a given set (details will be provided in the following section). Adverse control problems concern the choice of $(u(\cdot), b, \tilde{b})$ that minimizes a given function $h_0(y(u)(t_1))$ and satisfies constraints expressed in terms of

$$h_1(y(u)(t_1)) = 0, \quad \hat{h}(\hat{y}(u, v)(t_1)) \leq 0$$

for every admissible strategy $v(\cdot)$.

Adverse control problems have some formal analogy with differential games, but yield a priori information to the second player about the first player strategy. The model formulation that we deal with emphasizes this different feature by decoupling the player one trajectory $y(\cdot)$ from the player two trajectory $\tilde{y}(\cdot)$, instead of considering a joint differential equation

$$\hat{y}(t) = \hat{b} + \int_{t_0}^t \hat{f}(s, \hat{y}(s), u(s), v(s)) ds,$$

as it is usually taken into account in the differential game framework. A particular case of adverse control problems are minimax problems, in which

$$h_0(y(u)(t_1)) = \sup_v \hat{h}_0(\hat{y}(u, v)(t_1))$$

and the minimization process is carried out following some “worst case” criteria. A deeper exposition on minimax optimal control is presented in the monograph [1].

Adverse control problems were extensively studied by Warga in his monograph [8], in which he proposed two extensions of the original problem, aimed to guarantee the existence of a solution. He denotes such enlarged problems as relaxed and hyperrelaxed, respectively. The relaxed extension can be properly used to model the case in which the function \hat{f} is additively coupled with respect to the control strategies of players one and two respectively, that is:

$$\tilde{f}(t, \hat{y}, u, v) = \tilde{f}_1(t, \hat{y}, u) + \tilde{f}_2(t, \hat{y}, v);$$

another case in which the relaxed extension can be successfully applied is when the second player does not have perfect means to detect the value $u(t)$, but can just detect an average of values of $u(\cdot)$ over short intervals of time. In all the other cases (which means, when \tilde{f} assumes a general form and when the second player can acquire information on the value $u(t)$), the relaxed problem can fail to provide the “right” value of the adverse control problem: in other words, it can occur that the value of the relaxed extension is lower than the value of the original problem, even for smooth dynamics (as it is showed in [8] and in the last section of the paper). This lack of properness motivates the attention for the hyperrelaxed extension: in this setting, the second player gains more freedom in the choice of the control strategy, making the hyperrelaxed problem formulation “fair”, in the sense that the value of the hyperrelaxed extension does not change with respect to the original one. In [8] and [7], Warga proves the properness of the hyperrelaxed problem and the existence of a sequence of original controls which approximates the hyperrelaxed problem solution.

In the same monograph, necessary conditions both for relaxed and hyperrelaxed problems are derived in the case of smooth data. The nonsmooth setting is considered in [11], where necessary conditions are derived using the notion of Warga derivative container (see [9], [10]). However, as it is explained in ([11], Section 3), there are some technical difficulties (related to the measurability of the relaxed and hyperrelaxed hamiltonians) which prevent to obtain a “pointwise” maximum principle strong as much as in the smooth setting.

A different approach to the particular case in which minimax problems are considered is provided in [6], where the set of “adverse trajectories” is identified with a compact metric space. Necessary conditions are expressed in terms of a nonsmooth Pontryagin maximum principle in which the adjoint equation and the transversality condition are modified in order to gain good compactness properties in the proofs. Main ideas and tools used in the proofs of [6] can be found in [4] and [5]

The aim of this paper is to provide a new set of necessary conditions for adverse control problems which have a stronger resemblance with the necessary conditions established in [8] for the smooth setting. Indeed, the main results, collected in theorems 31 and 32, provide a form of Pontryagin maximum principle in which the pointwise maximum condition is still preserved and the adjoint equations have limit representations.

The key idea of the proofs is to define a sequence of perturbed smooth problems for which well-known necessary conditions apply and the couples solutions/multipliers converge to a couple solution/multiplier of the hyperrelaxed (or relaxed) problem. We do not make use of variational principles, but we regularize the data by convolution techniques. We do not provide direct substitute to the non-existing derivative of nonsmooth data, but we cope with the convolution integrals, using the concept of “relaxed derivative” established in [12].

We stress that the sequence solutions/multipliers generated in the main proofs could be computed in many cases of interest and no a priori information on the minimizers of the adverse control problem is required.

The paper is organized as follows: in Section 2 we describe notations, a precise statement of the problem, an overview on relaxation and hyperrelaxation schemes and the assumptions that we refer to through all the paper; furthermore, we provide some tools, lemmas and convergence results for convolution approximations and relaxed derivatives; in particular in Section 2.7, we derive necessary conditions in the smooth case for the problem formulation we will deal with. In Section 3, the main theorems and their proofs can be found, while in Section 4 an illustrative example is discussed.

2 Preliminaries

2.1 Notation

In this section, we introduce notations and basic concepts which we will use through all the paper.

Given a compact set K , we denote as $C(K)$ the set of all continuous functions defined on K . It is well known that the set $C^*(K)$ (the set of linear and continuous functionals defined on $C(K)$) can be identified with the set of finite Radon measures on K , which we denote as $\text{f.r.m.}(K)$. Further we denote as $\text{f.r.m.}^+(K)$ the set of finite positive Radon measures and by $\text{r.p.m.}(K)$ the set of Radon probability measures on K . We denote also as $\mathcal{B}(K)$ the Borel σ -field on K and for every $\mu \in C^*(K)$, we denote as $\mu(K)$ the norm in total variation of μ .

Given a measure space (X, μ, \mathcal{F}) and a metric space (Y, d) , we denote as $L^1(X, \mu, Y) := \mathcal{L}(X, \mu, Y) / \sim$, where $\mathcal{L}(X, \mu, Y)$ is the set of the μ -integrable functions $f : X \rightarrow Y$ defined at every point $x \in X$ such that $\int |f(x)|\mu(dx) < \infty$ and \sim is the equivalence relation $f \sim g$ iff $f = g$ μ -a.e. In the paper, we use the notation $L^1(X, \mu)$ or $L^1(\mu, Y)$ when there is no disambiguation in the codomain or the domain, respectively.

Given a set A , we denote as $\text{co } A$ the convex hull of A and as χ_A the indicator function related to A . Finally, we denote as \mathbb{B} the closed unit ball in the euclidean space with suitable dimension, as $\mathcal{P}(K)$ the power set of K , as $M_{r \times k}$ the set of matrices with r rows and k columns and as I_k the identity matrix of dimension k .

2.2 Original Problem Statement

Consider the adverse control problem

$$(OP) \begin{cases} \text{Minimize}_{u \in \mathcal{U}} h_0(y(u)(t_1)) \\ \text{over measurable functions } u(\cdot), v(\cdot) \text{ such that} \\ u(t) \in U(t), \quad v(t) \in V(t) \quad \text{a.e. } t \in [t_0, t_1] \\ \dot{y}(t) = f(t, y(t), u(t)) \quad \text{a.e. } t \in [t_0, t_1] \\ \dot{\tilde{y}}(t) = \tilde{f}(t, \hat{y}(t), u(t), v(t)) \quad \text{a.e. } t \in [t_0, t_1] \\ y(t_0) = b \in B, \quad \tilde{y}(t_0) = \tilde{b} \in \tilde{B} \quad \text{and} \quad h_1(y(u)(t_1)) = 0 \\ \hat{h}(\hat{y}(u, v)(t_1)) \leq 0 \quad \text{for each } v \in \mathcal{V} \end{cases},$$

where $f : [t_0, t_1] \times \mathbb{R}^n \times \mathbf{U} \rightarrow \mathbb{R}^n$, $\tilde{f} : [t_0, t_1] \times \mathbb{R}^{n+m} \times \mathbf{U} \times \mathbf{V} \rightarrow \mathbb{R}^m$, $h_0 : \mathbb{R}^n \rightarrow \mathbb{R}$, $h_1 : \mathbb{R}^n \rightarrow \mathbb{R}$ and $\hat{h} : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ are given functions, \mathbf{U} and \mathbf{V} compact metric spaces and $[t_0, t_1] \subset \mathbb{R}$ a given interval. The initial condition $\hat{b} := (b, \tilde{b})$ takes values on the compact and convex set $\hat{B} := B \times \tilde{B} \subset \mathbb{R}^n \times \mathbb{R}^m$. It turns out that the initial condition can be regarded as a choice of control parameters for problem (OP) (cfr. [8]). We denote as $\hat{y} = (y, \tilde{y})$, as $\hat{f} = (f, \tilde{f})$ and we sometimes emphasize the dependence on the controls writing $y(u)(t)$, $\hat{y}(u, v)(t)$. The mappings $U(\cdot) : [t_0, t_1] \rightarrow \mathcal{P}(\mathbf{U})$ and $V(\cdot) : [t_0, t_1] \rightarrow \mathcal{P}(\mathbf{V})$ are given Borel measurable multifunctions with compact values and we denote as \mathcal{U} (resp. \mathcal{V}) the set of all measurable functions $u(\cdot) : [t_0, t_1] \rightarrow \mathbf{U}$ such that $u(t) \in U(t)$ a.e. $t \in [t_0, t_1]$ (resp. $v(\cdot) : [t_0, t_1] \rightarrow \mathbf{V}$ such that $v(t) \in V(t)$ a.e. $t \in [t_0, t_1]$).

2.3 Assumptions

In this paper, we assume the following hypotheses on the data: let $\Omega \subset \mathbb{R}^n$, $\tilde{\Omega} \subset \mathbb{R}^m$ be open sets, $\hat{\Omega} := \Omega \times \tilde{\Omega}$. We consider functions

$$\hat{f} = (f, \tilde{f}) : [t_0, t_1] \times \hat{\Omega} \times \mathbf{U} \times \mathbf{V} \rightarrow \mathbb{R}^{n+m}, \quad h_i : \Omega \rightarrow \mathbb{R}, \quad i = 0, 1, \\ \hat{h} : \hat{\Omega} \rightarrow \mathbb{R}$$

such that

- H1) $\hat{f}(\cdot, \hat{y}, u, v)$ is Lebesgue measurable for each (\hat{y}, u, v) and $\hat{f}(t, \dots, \dots)$ is continuous a.e. $t \in [t_0, t_1]$;
- H2) there exist a constant $L_{\hat{f}}$ and an integrable function $\chi(\cdot)$ such that

$$|\hat{f}(t, \hat{y}, u, v) - \hat{f}(t, \hat{y}', u, v)| \leq L_{\hat{f}}|\hat{y} - \hat{y}'|$$

and

$$|\hat{f}(t, \hat{y}, u, v)| \leq \chi(t)$$

for every $\hat{y}, \hat{y}' \in \hat{\Omega}, u \in \mathbf{U}, v \in \mathbf{V}$, a.e. $t \in [t_0, t_1]$;

H3) there exist positive numbers $L_{h_0}, L_{h_1}, L_{\hat{h}} \geq 0$ such that

$$|h_i(y) - h_i(y')| \leq L_{h_i}|y - y'|, \quad i = 0, 1, \quad |\hat{h}(\hat{y}) - \hat{h}(\hat{y}')| \leq L_{\hat{h}}|\hat{y} - \hat{y}'|,$$

for every $y, y' \in \Omega$ and $\hat{y}, \hat{y}' \in \hat{\Omega}$.

2.4 Relaxed and Hyperrelaxed Problems

The adverse control problem (OP) does not always admit a solution, when we restrict the choice of controls $u(\cdot)$ and $v(\cdot)$ to be elements of \mathcal{U} and \mathcal{V} , respectively. As it is showed in [8] and [11], there are two ways to guarantee the existence of a solution for adverse control problems.

The first method concerns the symmetric relaxation of both players. More precisely, we introduce the set of Borel measurable mappings

$$\mathcal{S} := \{\sigma(\cdot) : [t_0, t_1] \rightarrow \text{r.p.m.}(\mathbf{U}) : \sigma(t)(U(t)) = 1 \text{ a.e. } t \in [t_0, t_1]\},$$

and we symmetrically extend the choice for the second player to the set of Borel measurable mappings

$$\mathcal{S}_P := \{\sigma_P(\cdot) : [t_0, t_1] \rightarrow \text{r.p.m.}(\mathbf{V}) : \sigma_P(t)(V(t)) = 1 \text{ a.e. } t \in [t_0, t_1]\}.$$

Then we consider the new relaxed problem (RP), which has the same data of (OP), but where the dynamic equations are replaced by

$$y(\sigma)(t) = b + \int_{t_0}^t ds \int f(s, y(s), u)\sigma(s)(du)$$

$$\tilde{y}(\sigma \times \sigma_P)(t) = \tilde{b} + \int_{t_0}^t ds \int \tilde{f}(s, \hat{y}(s), u, v)\sigma(s)(du) \times \sigma_P(s)(dv),$$

where $\sigma \times \sigma_P$ is the product measure between $\sigma \in \mathcal{S}$ and $\sigma_P \in \mathcal{S}_P$. The control strategies for players one and two are now elements of \mathcal{S} and \mathcal{S}_P , respectively. It is showed in ([8], Example IX.2.2, pp 453–456) that the problems (OP) and (RP) can have different values if we do not assume some special hypotheses on the structure of the dynamics.

We now move our attention to the hyperrelaxed extension, which does not modify the value of the problem, let alone special assumptions on the structure of \hat{f} . The problem is modified as follows: the first player can still choose the control strategy in the set of relaxed controls \mathcal{S} while the second player, in order to not modify the cost of the problem, has to pick controls up from a larger set than \mathcal{S} . These controls are mentioned as hyperrelaxed and lie in the set

$$\tilde{\mathcal{P}} := \{\pi(\cdot, \cdot) : [t_0, t_1] \times \mathbf{U} \rightarrow \text{r.p.m.}(\mathbf{V}) : \pi(t, u)(V(t)) = 1, \text{ a.e. } t \in [t_0, t_1], \forall u \in \mathbf{U}\}.$$

Roughly speaking, if we consider the set of Borel measurable mappings

$$\mathcal{Q} := \{\alpha : [t_0, t_1] \rightarrow \text{r.p.m.}(\mathbf{U} \times \mathbf{V}) : \alpha(U(t) \times V(t)) = 1, \text{ a.e. } t \in [t_0, t_1]\},$$

then \mathcal{S} can be considered as the set of the Borel measurable mappings from $[t_0, t_1]$ to the set of marginal probabilities on \mathbf{U} , while $\tilde{\mathcal{P}}$ can be regarded as the set of the mappings from $[t_0, t_1]$ to the set of the conditional probabilities on \mathbf{V} with respect to the information $u \in \mathbf{U}$.

The hyperrelaxed problem has the same data of (OP) , but the choice of controls is $\sigma \in \mathcal{S}$ for the first player and π belonging to a modification of $\tilde{\mathcal{P}}$ (details will be given later in the subsection) for the second one. This changes the dynamic equations as follows:

$$y(\sigma)(t) = b + \int_{t_0}^t ds \int f(s, y(\sigma)(s), u)\sigma(s)(du),$$

$$\tilde{y}(\sigma \otimes \pi)(t) = \tilde{b} + \int_{t_0}^t ds \int \tilde{f}(s, \hat{y}(\sigma \otimes \pi)(s), u, v)\sigma(s)(du) \otimes \pi(s, u)(dv),$$

where the symbol $\sigma \otimes \pi$ denotes the unique element in \mathcal{Q} such that

$$\int_{t_0}^t ds \int \varphi(s, u, v)\sigma(s)(du) \otimes \pi(s, u)(dv) = \int_{t_0}^t ds \int \sigma(s)(du) \int \varphi(s, u, v)\pi(s, u)(dv)$$

for every $\varphi(., ., .) \in L^1(dt, C(\mathbf{U} \times \mathbf{V}))$, (for more details, see [8], Lemma X.1.3, pp. 485). We denote as (HP) the hyperrelaxed version of the problem (OP) .

As it is pointed out in ([8], Remark, pp. 489), there appears not useful way to define a compact metric topology on $\tilde{\mathcal{P}}$ such that the function $\pi \mapsto \sigma \otimes \pi$ is continuous for every $\sigma \in \mathcal{S}$. However, we can overcome this difficulty proceeding as follows:

1. Restrict our attention to some fixed denumerable subset $\mathcal{S}' \subset \mathcal{S}$;
2. Introduce on $\tilde{\mathcal{P}}$ the following relation \sim : $\pi_1 \sim \pi_2$ if and only if

$$\sigma \otimes \pi_1(t) = \sigma \otimes \pi_2(t) \quad a.e. t \in [t_0, t_1], \quad \forall \sigma \in \text{co } \mathcal{S}'.$$

We denote as \mathcal{P} the set of equivalence classes on $\tilde{\mathcal{P}}$;

3. We introduce a compact metric topology on \mathcal{P} , which makes continuous the mapping $\pi \mapsto \sigma \otimes \pi$ for every $\sigma \in \text{co } \mathcal{S}'$. By ([8], Lemma X.I.I, pp. 482), for every $\tilde{\sigma} \in \text{co } \mathcal{S}'$ there exists a unique nonatomic measure $\tilde{\zeta}$ such that

$$\int_{t_0}^{t_1} dt \int h(t, u)\tilde{\sigma}(t)(du) = \int h(t, u)\tilde{\zeta}(d(t, u)) \quad \forall h \in L^1([t_0, t_1] \times \mathbf{U}, \tilde{\zeta}).$$

\mathcal{P} can be regarded as the set of the $\tilde{\zeta}$ -measurable mappings $\pi : [t_0, t_1] \times \mathbf{U} \rightarrow \text{r.p.m.}(\mathbf{V})$ such that $\pi(t, u)(V(t)) = 1$ $\tilde{\zeta}$ -a.e. which elements are also Borel measurable on the set $([t_0, t_1] \times \mathbf{U})$. (for more details, see ([8], Definition X.2.1, pp. 496) and following discussion).

We now state a lemma that provides a link between the measure $\tilde{\zeta}$ and the elements $\sigma \in \mathcal{S}'$.

Lemma 21 *Let \mathcal{S}' , \mathcal{P} and $\tilde{\zeta}$ be defined as above. Then $\lim_i \sigma \otimes \pi_i = \sigma \otimes \pi$ for every $\sigma \in \mathcal{S}'$ if $\lim_i \pi_i = \pi$ in \mathcal{P} . Furthermore, if E is a $\tilde{\zeta}$ -null set, then $\chi_E(t, u) = 0$ $\sigma(t)$ -a.e. r , a.e. $t \in [t_0, t_1]$, for every $\sigma \in \mathcal{S}'$.*

Proof See ([8], Lemma X.2.2, pp. 497). □

Using lemma 21 and the formal construction of the hyperrelaxed control set described in points (1) – (3), it is easy to check that the definition of hyperrelaxed controls does not depend on the choice of $\tilde{\zeta}$. Indeed, taking any other $\sigma \in \text{co } \mathcal{S}'$ and the associated measure

ζ , it turns out that ζ and $\tilde{\zeta}$ are equivalent and null sets of $\tilde{\zeta}$ are also null sets of ζ (and vice versa).

It is proven in ([8], Theorem VII.I, pp. 348) that the functions $\mathcal{S} \ni \sigma \mapsto y(\sigma)$ and $\mathcal{Q} \ni \sigma \otimes \pi \rightarrow \hat{y}(\sigma \otimes \pi)$ are continuous. Furthermore, since ([8], Lemma X.3.3, pp. 504), the function $\mathcal{P} \ni \pi \mapsto \hat{y}(\sigma \otimes \pi)$ is continuous for every $\sigma \in \text{co}\mathcal{S}'$, and also the function $\mathcal{S}_P \ni \sigma_P \mapsto \sigma \otimes \sigma_P$ is continuous for every $\sigma \in \mathcal{S}$.

To summarize, the new relaxed and hyperrelaxed adverse control problems can be written as follows:

$$(RP) \begin{cases} \text{Minimize}_{\sigma \in \mathcal{S}} h_0(y(\sigma)(t_1)) \\ \text{over } \sigma \times \sigma_P \in \mathcal{S} \times \mathcal{S}_P, \text{ s.t.} \\ \dot{y}(t) = f(t, y(t), \sigma(t)) \quad \text{a.e. } t \in [t_0, t_1] \\ \dot{\tilde{y}}(t) = \tilde{f}(t, \hat{y}(t), \sigma \times \sigma_P(t)) \quad \text{a.e. } t \in [t_0, t_1] \\ y(t_0) = b \in B, \quad \tilde{y}(t_0) = \tilde{b} \in \tilde{B} \quad \text{and} \quad h_1(y(\sigma)(t_1)) = 0 \\ \hat{h}(\hat{y}(\sigma \times \sigma_P)(t_1)) \leq 0 \quad \text{for each } \sigma_P \in \mathcal{S}_P \end{cases},$$

and

$$(HP) \begin{cases} \text{Minimize}_{\sigma \in \mathcal{S}} h_0(y(\sigma)(t_1)) \\ \text{over } \sigma \in \mathcal{S}, \quad \pi \in \mathcal{P}, \text{ s.t.} \\ \dot{y}(t) = f(t, y(t), \sigma(t)) \quad \text{a.e. } t \in [t_0, t_1] \\ \dot{\tilde{y}}(t) = \tilde{f}(t, \hat{y}(t), \sigma \otimes \pi(t)) \quad \text{a.e. } t \in [t_0, t_1] \\ y(t_0) = b \in B, \quad \tilde{y}(t_0) = \tilde{b} \in \tilde{B} \quad \text{and} \quad h_1(y(\sigma)(t_1)) = 0 \\ \hat{h}(\hat{y}(\sigma \otimes \pi)(t_1)) \leq 0 \quad \text{for each } \pi \in \mathcal{P} \end{cases},$$

where, to the sake of shortness, we have used the notation

$$f(t, y(t), \sigma(t)) := \int_{\mathbf{U}} f(t, y(\sigma)(t), u)\sigma(t)(du),$$

$$\tilde{f}(t, \hat{y}(t), \sigma \otimes \pi(t)) := \int_{\mathbf{U} \times \mathbf{V}} \tilde{f}(t, \hat{y}(\sigma \otimes \pi)(t), u, v)\sigma(t)(du) \otimes \pi(u, t)(dv)$$

and

$$\tilde{f}(t, \hat{y}(t), \sigma \times \sigma_P(t)) := \int_{\mathbf{U} \times \mathbf{V}} \tilde{f}(t, \hat{y}(\sigma \times \sigma_P)(t), u, v)\sigma(du) \times \sigma_P(dv)(t).$$

2.5 A Special Choice of \mathcal{S}'

Let $\bar{\sigma} \in \mathcal{S}$ be a given relaxed control. We define a special denumerable set \mathcal{S}' which will be used at a succeeding stage. By ([8] Condition IV.3.I, pp 280), there exists a denumerable subset \mathcal{U}_∞ of \mathcal{U} such that the set $\{u(t) \mid u \in \mathcal{U}_\infty\}$ is dense in $U(t)$ a.e. $t \in [t_0, t_1]$. We stress that the compactness of \mathbf{U} is a sufficient condition by which such a condition is satisfied. If we denote as I_∞ the set of all the subintervals $[a, b]$ of $[t_0, t_1]$ with rational endpoints, then the set $\mathcal{U}_\infty \times I_\infty$ is still denumerable and takes the form $\{(u^j, [a^j, b^j]) : j \in \mathbb{N}\}$. We denote as δ_r the Dirac measure at r ; we set $\sigma_0 := \bar{\sigma}$ and, for all $j \in \mathbb{N}$,

$$\sigma_j(t) := \begin{cases} \delta_{u^j(t)} & \text{if } t \in [a^j, b^j] \\ \bar{\sigma}(t) & \text{otherwise} \end{cases}.$$

We finally set $\mathcal{S}' := \{\sigma_0, \sigma_1, \sigma_2, \dots\}$. This special construction for the denumerable set \mathcal{S}' will be helpful in the proofs of theorem 31.

2.6 Relaxed Derivatives

Consider an open set $\Omega' \subset \mathbb{R}^n$ and a set $\Omega \subset \Omega'$ which has compact closure into Ω . We use the notation $\Omega \subset\subset \Omega'$ and we can always suppose that there exists an $\varepsilon > 0$ small enough such that $\Omega + \varepsilon\mathbb{B} \subset \Omega'$.

We construct a C^∞ function as follows: define

$$\bar{\varrho}(x) := \begin{cases} \exp\left(\frac{-1}{1-|x|^2}\right) & \text{if } x \in \mathbb{B} \\ 0 & \text{otherwise} \end{cases}$$

and consider its normalization $\varrho(x) = \frac{\bar{\varrho}(x)}{\int_{\mathbb{B}} \bar{\varrho}(x') dx'}$. It follows that $\varrho(\cdot)$ is C^∞ , has compact support in \mathbb{B} and

$$\int_{\mathbb{R}^n} \varrho(x) dx = \int_{\mathbb{B}} \varrho(x) = 1.$$

Furthermore we can define a function $\varrho^j(x) := j^n \varrho(jx)$ which has compact support in $\frac{1}{j}\mathbb{B}$ and such that $\int \varrho^j(x) dx = 1$ for each $j \in \mathbb{N}$. We say that the function $\varrho(\cdot)$ is a C^∞ mollifier and that $\{\varrho^j\}$ is a sequence of mollifiers.

Let $\phi : \Omega' \subset \mathbb{R}^n \rightarrow \mathbb{R}$ be a locally Lipschitz function with constant L_ϕ and take a point $x \in \Omega$. We next consider the convolution between the sequence of mollifiers and the function $\phi(\cdot)$ defining

$$\phi^j(x) := (\phi * \varrho^j)(x) := \int_{j^{-1}\mathbb{B}} \varrho^j(y) \phi(x - y) dy = \int_{x + j^{-1}\mathbb{B}} \varrho^j(x - y) \phi(y) dy.$$

The last equality is well defined for j sufficiently small because $x \in \Omega$. The sequence $\{\phi^j(\cdot)\}_{j \in \mathbb{N}}$ is called convolution approximation of the function $\phi(\cdot)$. It turns out that the functions $\phi^j(\cdot)$ are C^∞ and their partial derivatives $\partial_x \phi^j(\cdot)$ are uniformly continuous on $\bar{\Omega}$. Furthermore, by the Rademacher theorem, the function $\partial_x \phi(\cdot)$ exists a.e. and, for $x \in \Omega$, we set

$$\partial_x \phi^j(x) := \int_{j^{-1}\mathbb{B}} \varrho^j(y) \partial_x \phi(x - y) dy.$$

If $\eta : [t_0, t_1] \rightarrow \Omega$ is a continuous function, then $t \mapsto \partial_x^j \phi(\eta(t))$ is Lebesgue measurable and dominated by $|\partial_x^j \phi(\eta(t))| \leq L_\phi$. It follows that the integral

$$\Phi^j := \int_{t_0}^{t_1} \partial_x^j \phi(\eta(t)) dt$$

is well defined and, by the dominated convergence theorem, there exists $\Phi := \lim_j \Phi^j$. We call Φ the relaxed derivative of $\phi(\cdot)$ evaluated along the continuous function $\eta(\cdot)$.

We have the following result:

Lemma 22 *Suppose that $\phi(\cdot)$ and Ω are the same objects defined in Section 2.6 and that $\phi^j(\cdot)$ are the convolution approximations of $\phi(\cdot)$. Then $\phi^j(\cdot)$ are locally Lipschitz with constant L_ϕ and, for all $x \in \Omega$,*

$$|\partial_x \phi^j(x)| \leq L_\phi, \quad |\phi^j(x) - \phi(x)| \leq L_\phi/j.$$

Proof See [2]. □

The analysis carried out in this section can be easily extended to any function $\phi : [t_0, t_1] \times \Omega \times \mathbf{U} \rightarrow \mathbb{R}^n$ such that $t \mapsto \phi(t, x, u)$ is integrable for every $(x, u) \in \Omega \times \mathbf{U}$, $x \mapsto \phi(t, x, u)$ is Lipschitz continuous a.e. $t \in [t_0, t_1]$, for every $u \in \mathbf{U}$ and $u \mapsto \phi(t, x, u)$ is continuous a.e. $t \in [t_0, t_1]$, for every $x \in \Omega$.

2.7 Preliminary Results: The Smooth Case

We now state results dealing with the case in which all the data of the adverse control problems (RP) and (HP) are smooth. Such lemmas are very similar to [8], X.3.5, X.3.7, but they differ in the typology of adverse control problems that we are dealing with. In what follows, we will invoke the following hypothesis:

H4) $\hat{f}(t, \cdot, u, v)$, $\hat{h}(\cdot)$ and $h_i(\cdot)$, $i = 0, 1$ are all continuously differentiable, a.e. $t \in [t_0, t_1]$, for all $u \in \mathbf{U}$, $v \in \mathbf{V}$.

Furthermore, for $t \in [t_0, t_1]$, $\bar{\sigma} \in \mathcal{S}$ and $\pi \in \mathcal{P}$, we will denote as

$$Z(t) := I_n + \int_t^{t_1} Z(s) \partial_x f(s, y(\bar{\sigma})(s), \bar{\sigma}(s)) ds,$$

$$\hat{Z}(\pi)(t) = I_{n+m} + \int_t^{t_1} \hat{Z}(\pi)(s) \partial_x \hat{f}(s, \hat{y}(\bar{\sigma} \otimes \pi)(s), (\bar{\sigma} \otimes \pi)(s)) ds,$$

which are well posed since the regularity of the dynamics with respect to the state variable.

Lemma 23 *Assume hypotheses H1) - H4). Then, given $(\bar{\sigma}, \bar{b}, \bar{b})$ minimizer for problem (HP), there exist $l_0 \geq 0$, $l_1 \in \mathbb{R}$ and $\omega \in \text{f.r.m.}^+(\mathcal{P})$ such that*

$$i) \quad 0 < l_0 + |l_1| + \omega(\mathcal{P}) \leq 1;$$

if we set

$$k(t) := (l_0 \partial_x h_0(y(\bar{\sigma})(t_1)) + l_1 \partial_x h_1(y(\bar{\sigma})(t_1))) Z(t),$$

$$\hat{k}(\pi)(t) := \partial_x \hat{h}(\hat{y}(\bar{\sigma} \otimes \pi)(t_1)) \hat{Z}(\pi)(t),$$

$$\mathfrak{h}(\pi, t, u) = \max_{v \in V(t)} \hat{k}(\pi)(t) \hat{f}(t, \hat{y}(\bar{\sigma} \otimes \pi)(t), u, v),$$

and

$$H(t, u) = k(t) f(t, y(\bar{\sigma})(t), u) + \int \mathfrak{h}(\pi, t, u) \omega(d\pi),$$

for each $\pi \in \mathcal{P}$, $u \in U(t)$, a.e. $t \in [t_0, t_1]$, then

$$ii) \quad \int H(t, u) \bar{\sigma}(t)(du) = \min_{u \in U(t)} H(t, u) \quad \text{a.e. } t \in [t_0, t_1],$$

$$iii) \quad \mathfrak{h}(\pi, t, u) = \int \hat{k}(\pi)(t) \hat{f}(t, \hat{y}(\bar{\sigma} \otimes \pi)(t), u, v) \pi(t, u)(dv),$$

for ω -a.a. $\pi \in \mathcal{P}$, a.e. $t \in [t_0, t_1]$, $\bar{\sigma}(t)$ -a.a. $u \in \mathbf{U}$,

$$iv) \quad \hat{h}(\hat{y}(\bar{\sigma} \otimes \pi^*)(t_1)) = \max_{\pi \in \mathcal{P}} \hat{h}(\hat{y}(\bar{\sigma} \otimes \pi)(t_1)) = 0$$

for ω -a.a. $\pi^* \in \mathcal{P}$, and

$$v) \quad k(t_0)\bar{b} + \lambda(\bar{b}, \bar{b}) = \min_{(b, \tilde{b}) \in \hat{B}} k(t_0)b + \lambda(b, \tilde{b}),$$

where $\lambda := \int \hat{k}(\pi)(t_0)\omega(d\pi)$.

Proof We apply theorem [8], X.2.4 to the data of the problem (HP). From condition [8], X.2.4 (1), it follows that there exist $l_0 \geq 0, l_1 \in \mathbb{R}, \omega \in \text{f.r.m.}^+(\mathcal{P})$ and an $\tilde{\omega}(\cdot)$ function which is $L^1(\omega, \mathcal{P})$ and such that $|\tilde{\omega}(\pi)| = 1$ for ω -a.a. $\pi \in \mathcal{P}$. In particular, from condition [8], X.2.4 (3), we obtain that $\tilde{\omega} \equiv 1$ and that

$$\hat{h}(\hat{y}(\bar{\sigma} \otimes \pi^*)(t_1)) = 0 \quad \omega - \text{a.a. } \pi^* \in \mathcal{P}.$$

Conditions *i*) and *iv*) are then satisfied.

Now applying the result of [8], X.3.2 to the function $(\sigma \otimes \pi) \mapsto \hat{h}(\hat{y}(\sigma \otimes \pi)(t_1))$ and combining with [8], X.1.4, we get the relation

$$\mathfrak{h}(\pi, t, u) = \int \partial_x \hat{h}(\hat{y}(\bar{\sigma} \otimes \pi)(t_1)) \hat{Z}(\pi)(t) \cdot \hat{f}(t, \hat{y}(\bar{\sigma} \otimes \pi)(t), u, v) \pi(t, u)(dv),$$

for ω -a.a. $\pi \in \mathcal{P}$, a.e. $t \in [t_0, t_1], \bar{\sigma}(t)$ -a.a. $u \in \mathbf{U}$, which is condition *iii*).

From [8], X.2.4 (2), arguing as in the proof of [8], X.3.5, Step 3, relations *ii*) and *v*) follow. This completes the proof. □

We now state a similar result for relaxed adverse control problems. We do not perform the proof since it is based on the same arguments of the proof of lemma 23. We just point out that instead of applying theorem X.2.4, [8], we invoke theorem IX.1.2, [8], which is specific for the relaxed extension of problem (OP). It is worth mentioning that the minimization principle *ii*) is performed on the set of relaxed controls \mathcal{S} and involves an ‘averaged maximized’ hamiltonian $\mathfrak{h}(\sigma_P, t, s), s \in \mathcal{S}$. For given $t \in [t_0, t_1], \bar{\sigma} \in \mathcal{S}$ and $\bar{\sigma}_P \in \mathcal{S}$, the function $Z(\cdot)$ remains unchanged, while we define

$$\hat{Z}(\sigma_P)(t) = I_{n+m} + \int_t^{t_1} \hat{Z}(\sigma_P)(s) \partial_x \hat{f}(s, \hat{y}(\bar{\sigma} \times \sigma_P)(s), (\bar{\sigma} \times \sigma_P)(s)) ds.$$

Lemma 24 *Assume hypotheses H1) - H4). Then, given $(\bar{\sigma}, \bar{b}, \bar{b})$ minimizer for problem (RP), then there exist $l_0 \geq 0, l_1 \in \mathbb{R}$ and $\omega \in \text{f.r.m.}^+(\mathcal{P})$ such that*

$$i) \quad 0 < l_0 + |l_1| + \omega(\mathcal{P}) \leq 1;$$

if we set

$$\begin{aligned} k(t) &:= (l_0 \partial_x h_0(y(\bar{\sigma})(t_1)) + l_1 \partial_x h_1(y(\bar{\sigma})(t_1))) Z(t), \\ \hat{k}(\sigma_P)(t) &:= \partial_x \hat{h}(\hat{y}(\bar{\sigma} \times \sigma_P)(t_1)) \hat{Z}(\sigma_P)(t) \\ \mathfrak{h}(\sigma_P, t, s) &:= \max_{v \in V(t)} \hat{k}(\sigma_P)(t) \cdot \int \hat{f}(t, \hat{y}(\bar{\sigma} \times \sigma_P)(t), u, v) s(du), \end{aligned}$$

and

$$H(t, s) = k(t) f(t, y(\bar{\sigma})(t), s) + \int \mathfrak{h}(\sigma_P, t, s) \omega(d\sigma_P), \quad (s \in \mathcal{S})$$

then

$$ii) \quad \int H(t, u) \bar{\sigma}(t)(du) = \min_{s \in \mathcal{S}} H(t, s) \quad \text{a.e. } t \in [t_0, t_1],$$

$$iii) \quad h(\sigma_P, t, \bar{\sigma}(t)) = \hat{k}(\sigma_P)(t) \int \hat{f}(t, \hat{y}(\bar{\sigma} \times \sigma_P)(t), \bar{\sigma}(t), v) \sigma_P(dv),$$

for ω -a.a. $\sigma_P \in \mathcal{S}_P$, a.e. $t \in [t_0, t_1]$,

$$iv) \quad \hat{h}(\hat{y}(\bar{\sigma} \times \sigma_P^*)(t_1)) = \max_{\sigma \in \mathcal{S}_P} \hat{h}(\hat{y}(\bar{\sigma} \times \sigma_P)(t_1)) = 0$$

for ω -a.a. $\sigma_P^* \in \mathcal{S}_P$, and

$$v) \quad k(t_0)\bar{b} + \lambda(\bar{b}, \bar{b}) = \min_{(b, \tilde{b}) \in \hat{B}} k(t_0)b + \lambda(b, \tilde{b})$$

where $\lambda := \int \hat{k}(\sigma_P)(t_0)\omega(d\sigma_P)$.

2.8 Perturbed Problems

Convolution approximations can be used to define a sequence of problems whose limit approximates the behavior of (OP) . If we consider the function $\hat{f}(t, y, u, v)$, we can construct its convolution approximation with respect to y as

$$\hat{f}^j(t, \hat{y}, u, v) = \int_{j^{-1}\mathbb{B}} \varrho^j(x) \hat{f}(t, \hat{y} - x, u, v) dx.$$

The same procedure can be carried out on the functions

$$h_i^j(y) = \int_{j^{-1}\mathbb{B}} \varrho^j(x) h_i(y - x) dx, \quad i = 0, 1,$$

and

$$\hat{h}^j(\hat{y}) = \int_{j^{-1}\mathbb{B}} \varrho^j(x) \hat{h}(\hat{y} - x) dx.$$

The next properties is helpful for the pursuance of the discussion:

Lemma 25 *Fix*

$$\alpha := \int_{t_0}^{t_1} \chi(\tau) d\tau, \quad c_{\hat{y}} := L_{\hat{f}} + \alpha e^\alpha, \quad c_{h_i} := L_{h_i}(c_y + 1) \quad \forall i = 0, 1, \quad c_{\hat{h}} := L_{\hat{h}}(c_y + 1).$$

Then, for each $\sigma \in \mathcal{S}$ and $\pi \in \mathcal{P}$ (or $\pi \in \mathcal{S}_P$), we have:

- (i) $w^j(t) := |\hat{y}^j(\sigma \otimes \pi)(t) - \hat{y}(\sigma \otimes \pi)(t)| \leq c_{\hat{y}}/j$;
- (ii) $|\hat{f}^j(t, \hat{y}^j(\sigma \otimes \pi)(t), (\sigma \otimes \pi)(t)) - \hat{f}(t, \hat{y}(\sigma \otimes \pi)(t), (\sigma \otimes \pi)(t))| \leq (c_{\hat{y}} + 1)/j$;
- (iii) $|h_i^j(y^j(\sigma)(t)) - h_i(y(\sigma)(t))| \leq c_{h_i}/j \quad \forall i = 0, 1$;
- (iv) $|\hat{h}^j(\hat{y}^j(\sigma \otimes \pi)(t)) - \hat{h}(\hat{y}(\sigma \otimes \pi)(t))| \leq c_{\hat{h}}/j$.

for every $t \in [t_0, t_1]$.

Proof To prove relation (i), we fix $\sigma \in \mathcal{S}$ and $\pi \in \mathcal{P}$ (or $\pi \in \mathcal{S}_P$) and from the definition of $\hat{y}^j(\sigma \otimes \pi)(\cdot)$, $\hat{y}(\sigma \otimes \pi)(\cdot)$ respectively, it follows

$$w^j(t) := |\hat{y}^j(\sigma \otimes \pi)(t) - \hat{y}(\sigma \otimes \pi)(t)|$$

$$\begin{aligned} &\leq \int [|\hat{f}^j(s, \hat{y}^j(\sigma \otimes \pi)(s), (\sigma \otimes \pi)(s)) - \hat{f}(s, \hat{y}^j(\sigma \otimes \pi)(s), (\sigma \otimes \pi)(s))| \\ &\quad + |\hat{f}(s, \hat{y}^j(\sigma \otimes \pi)(s), (\sigma \otimes \pi)(s)) - \hat{f}(s, \hat{y}(\sigma \otimes \pi)(s), (\sigma \otimes \pi)(s))|] ds. \end{aligned} \tag{2.1}$$

By lemma 22, the first term of the integrand is bounded by $j^{-1}L_{\hat{f}}$, while, in view of the Lipschitz continuity of $\hat{f}(t, \cdot, u, v)$, the second term of the integrand is bounded by $L_{\hat{f}}w^j(t)$. From the Gronwall inequality, relation (i) follows.

The proof of relations (ii) – (iv) is consequence of relation (i) and of the Lipschitz continuity of the functions $\hat{f}(t, \cdot, u, v)$, $h_0(\cdot)$, $h_1(\cdot)$ and $\hat{h}(\cdot)$. □

It follows from lemma 25 that the sequences $\{\hat{y}^j(\sigma \otimes \pi)(\cdot)\}_{j \in J}$ and $\{\hat{f}^j(t, \hat{y}^j(\sigma \otimes \pi)(\cdot), (\sigma \otimes \pi)(\cdot))\}_{j \in J}$ converge uniformly with respect to $\sigma \otimes \pi \in \mathcal{Q}$, a.e. $t \in [t_0, t_1]$.

In the following, we define problems that will be helpful in the proofs of theorems 31 and 32.

Suppose that (RP) has a solution $(\bar{\sigma}, \bar{b}, \bar{b}) \in \mathcal{S} \times B \times \bar{B}$. Then we consider the problem of seeking the optimal strategy $\sigma^j \in \mathcal{S}$ which minimizes the cost $h_0^j(y(\sigma)(t_1))$ and such that

$$H_1^j(y(\sigma)(t_1)) = 0, \quad \hat{H}^j(\hat{y}(\sigma \times \sigma_P)(t_1)) \leq 0, \quad \forall \sigma_P \in \mathcal{S}_P,$$

where $y(\sigma)(\cdot)$ and $\hat{y}(\sigma \times \sigma_P)(\cdot)$ are the solutions of the equations

$$y(\sigma)(t) = \bar{b} + \int_{t_0}^t ds \int f^j(s, y(\sigma)(s), u)\sigma(s)(du), \tag{2.2}$$

$$\hat{y}(\sigma \times \sigma_P)(t) = (\bar{b}, \bar{b}) + \int_{t_0}^t ds \int \hat{f}^j(s, \hat{y}(\sigma \times \sigma_P)(s), u, v)\sigma(s)(du) \times \sigma_P(s)(dv), \tag{2.3}$$

and the functions $H_1^j(\cdot)$, $\hat{H}^j(\cdot)$ are defined as

$$H_1^j(y^j(\sigma)(t_1)) := h_1^j(y^j(\sigma)(t_1)) - a^j,$$

for some suitable choice of $a^j \in \frac{c_{h_1}}{j}\mathbb{B}$, and as

$$\hat{H}^j(\hat{y}^j(\sigma \times \sigma_P)(t_1)) := \hat{h}^j(\hat{y}^j(\sigma \times \sigma_P)(t_1)) + \frac{c_{\hat{h}}}{j},$$

respectively. At the light of the previous discussion, we denote as (RP^j) the following problem

$$(RP^j) \begin{cases} \text{Minimize}_{\sigma \in \mathcal{S}} h_0^j(y(\sigma)(t_1)) \\ \text{over } \sigma \times \sigma_P \in \mathcal{S} \times \mathcal{S}_P, \text{ s.t.} \\ \dot{y}(t) = f^j(t, y(t), \sigma(t)) \quad \text{a.e. } t \in [t_0, t_1] \\ \dot{\hat{y}}(t) = \hat{f}^j(t, \hat{y}(t), \sigma \times \sigma_P(t)) \quad \text{a.e. } t \in [t_0, t_1] \\ y(t_0) = \bar{b}, \quad \hat{y}(t_0) = \bar{b} \quad \text{and} \quad H_1^j(y(\sigma)(t_1)) = 0 \\ \hat{H}^j(\hat{y}(\sigma \times \sigma_P)(t_1)) \leq 0 \quad \text{for each } \sigma_P \in \mathcal{S}_P \end{cases},$$

If $(\bar{\sigma}, \bar{b}, \bar{b})$ is a solution for the problem (RP) stated in Section 2.4, it is easy to check that, using lemma 25, we can choose the parameter a^j in such a manner that $\bar{\sigma}$ is also an admissible strategy for the perturbed problem (RP^j) , for every j sufficiently large. From

general compactness arguments (see [8], Theorem IX.1.1, pp 445), it follows that there exists a minimizing control $\sigma^j \in \mathcal{S}$ that solves the problem (RP^j) .

The same procedure can be carried out when the second player chooses control strategies in \mathcal{P} . In this case, (2.2) is not modified, while (2.3) becomes

$$\hat{y}(\sigma \otimes \pi)(t) = (\bar{b}, \bar{\tilde{b}}) + \int_{t_0}^t ds \int \sigma(s)(du) \int \hat{f}^j(s, \hat{y}(\sigma \otimes \pi)(s), u, v)\pi(s, u)(dv). \tag{2.4}$$

The functions $h_0^j(y(\sigma)(t_1))$, $H_1^j(y(\sigma)(t_1))$ and $\hat{H}^j(\hat{y}(\sigma \otimes \pi)(t_1))$ remain unchanged (we have just replaced $\sigma \times \sigma_P$ with $\sigma \otimes \pi$).

We define the problem of finding a control $\sigma \in \mathcal{S}$ which minimizes $h_0^j(y(\sigma)(t_1))$, such that $y(\sigma)(\cdot)$ and $\hat{y}(\sigma \otimes \pi)(\cdot)$ are solutions of (2.2), (2.4) and

$$H_1^j(y(\sigma)(t_1)) = 0, \quad \hat{H}^j(\hat{y}(\sigma \otimes \pi)(t_1)) \leq 0 \quad \forall \pi \in \mathcal{P}.$$

We denote as (HP^j) the hyperrelaxed perturbed problem

$$(HP^j) \begin{cases} \text{Minimize}_{\sigma \in \mathcal{S}} h_0^j(y(\sigma)(t_1)) \\ \text{over } \sigma \in \mathcal{S}, \pi \in \mathcal{P}, \text{ s.t.} \\ \dot{y}(t) = f^j(t, y(t), \sigma(t)) \quad \text{a.e. } t \in [t_0, t_1] \\ \dot{\tilde{y}}(t) = \tilde{f}^j(t, \hat{y}(t), \sigma \otimes \pi(t)) \quad \text{a.e. } t \in [t_0, t_1] \\ y(t_0) = \bar{b}, \quad \tilde{y}(t_0) = \bar{\tilde{b}} \quad \text{and} \quad H_1^j(y(\sigma)(t_1)) = 0 \\ \hat{H}^j(\hat{y}(\sigma \otimes \pi)(t_1)) \leq 0 \quad \text{for each } \pi \in \mathcal{P} \end{cases},$$

and as $y^j(\sigma)(\cdot)$ and $\hat{y}^j(\sigma \otimes \pi)(\cdot)$ the solutions of (2.2), (2.4) respectively.

The following remark is helpful in the proof of theorem 31. Suppose we deal with the set of hyperrelaxed controls \mathcal{P} with the particular choice of \mathcal{S}' in Section 2.5 and assume that $(\bar{\sigma}, \bar{b}, \bar{\tilde{b}})$ is a solution for the problem (HP) . From the particular choice of \mathcal{S}' , it follows that $\bar{\sigma}$ is also an admissible strategy for (HP^j) when we restrict our attention to controls in $\text{co}\mathcal{S}'$ for the first player. Furthermore, since $\text{co}\mathcal{S}'$ has the same properties of \mathcal{S} (which means $\text{co}\mathcal{S}'$ is convex and sequentially compact), it follows that (HP^j) has also a solution $\sigma^j \in \text{co}\mathcal{S}'$ for every j (again, see [8], Theorem IX.1.1, pp. 445).

3 Main Theorems

In the statement and the proof of theorem 31, we use the following notation. We denote as $y^j(\sigma)(\cdot)$ the unique solution of (2.2) for $\sigma \in \text{co}\mathcal{S}'$ and as $y^j(\sigma \otimes \pi)(\cdot)$ the unique solution of (2.4) for $\sigma \in \text{co}\mathcal{S}'$ and $\pi \in \mathcal{P}$ (we suppose that $\bar{b} := (\bar{b}, \bar{\tilde{b}})$ is fixed for the perturbed problem). From the discussion in Section 2.8, it follows that the problem (HP^j) has a solution $\sigma^j \in \text{co}\mathcal{S}'$ for every j . We define the adjoint backward equations

$$Z^j(t) := I_n + \int_t^{t_1} Z^j(s) \partial_x f^j(s, y^j(\sigma^j)(s), \sigma^j(s)) ds,$$

$$\hat{Z}^j(\pi)(t) = I_{n+m} + \int_t^{t_1} \hat{Z}^j(\pi)(s) \partial_x \hat{f}^j(s, \hat{y}^j(\sigma^j \otimes \pi)(s), (\sigma^j \otimes \pi)(s)) ds.$$

Since the function $x \mapsto \hat{f}^j(t, x, u, v)$ is C^∞ for every j , the functions above are uniquely defined.

In the convergence analysis of theorem 31, we deal with the derivatives of the functions $h_1^j(\cdot)$ and $\hat{h}^j(\cdot)$ instead of considering the derivatives of the functions $H_1^j(\cdot)$ and $\hat{H}^j(\cdot)$. It is easy to check that this simplification does not affect the statements *i) – vii)* of the following theorem.

Theorem 31 *Let $(\bar{\sigma}, \bar{b}, \bar{\tilde{b}})$ be an optimal solution to the problem (HP). Then there exist a set of index $J \subset \mathbb{N}$, limiting multipliers $l_0 \geq 0, l_1 \in \mathbb{R}$, limiting initial directions $\mathcal{H}_0, \mathcal{H}_1 \in \mathbb{R}^n$, a $\omega \in \text{f.r.m.}^+(\mathcal{P})$, a ω -integrable function $\hat{\mathcal{H}} : \mathcal{P} \rightarrow \mathbb{R}^{n+m}$, and, for each $\pi \in \mathcal{P}$, continuous functions $Z : [t_0, t_1] \rightarrow M_{n \times n}, \hat{Z}(\pi) : [t_0, t_1] \rightarrow M_{(n+m) \times (n+m)}$, such that:*

$$i) \quad Z(t) = \lim_{j \in J} Z^j(t), \quad \hat{Z}(\pi)(t) = \lim_{j \in J} \hat{Z}^j(\pi)(t) \quad \text{uniformly } t \in [t_0, t_1], \quad \omega\text{-a.a. } \pi \in \mathcal{P};$$

$$(l_0, l_1) = \lim_{j \in J} (l_0^j, l_1^j), \quad \mathcal{H}_0 = \lim_{j \in J} \partial_x h_0^j(y^j(t_1)), \quad \mathcal{H}_1 = \lim_{j \in J} \partial_x h_1^j(y^j(t_1)),$$

$$\hat{\mathcal{H}}(\pi) = \lim_{j \in J} \partial_x \hat{h}^j(\hat{y}(\sigma^j \otimes \pi)(t_1)), \quad \omega - a.a. \quad \pi \in \mathcal{P};$$

$$ii) \quad 0 < l_0 + |l_1| + \omega(\mathcal{P}) \leq 1$$

Define:

$$k(t) := (l_0 \mathcal{H}_0 + l_1 \mathcal{H}_1) Z(t), \quad \hat{k}(\pi)(t) := \hat{\mathcal{H}}(\pi) \hat{Z}(\pi)(t),$$

$$\mathfrak{h}(\pi, t, u) := \max_{v \in V(t)} \hat{k}(\pi)(t) \cdot \hat{f}(t, \hat{y}(\bar{\sigma} \otimes \pi)(t), u, v),$$

$$H(t, u) := k(t) f(t, y(\bar{\sigma})(t), u) + \int \mathfrak{h}(\pi, t, u) \omega(d\pi),$$

Then

$$iii) \quad \mathfrak{h}(\pi, t, u) = \int \hat{k}(\pi)(t) \cdot \hat{f}(t, \hat{y}(\bar{\sigma} \otimes \pi)(t), u, v) \pi(t, u)(dv),$$

ω -a.a. $\pi \in \mathcal{P}$, a.e. $t \in [t_0, t_1]$, $\bar{\sigma}(t)$ -a.a. $u \in \mathbf{U}$.

$$iv) \quad \int_{t_0}^{t_1} dt \int H(t, u) (\sigma - \bar{\sigma})(t)(du) \geq 0 \quad \forall \sigma \in \mathcal{S}',$$

$$v) \quad \hat{h}(\hat{y}(\bar{\sigma} \otimes \pi^*)(t_1)) = \max_{\pi \in \mathcal{P}} \hat{h}(\hat{y}(\bar{\sigma} \otimes \pi)(t_1)) = 0 \quad \omega - a.a. \quad \pi^* \in \mathcal{P}.$$

$$vi) \quad k(t_0) \bar{b} + \lambda(\bar{b}, \bar{\tilde{b}}) = \min_{(b, \tilde{b}) \in \hat{B}} k(t_0) b + \lambda(b, \tilde{b}),$$

where $\lambda := \int \hat{k}(\pi)(t_0) \omega(d\pi)$. Furthermore, since the choice of \mathcal{S}' as in Section 2.5, condition *iv)* can be strengthened, obtaining

$$vii) \quad \int H(t, u) \bar{\sigma}(t)(du) = \min_{u \in U(t)} H(t, u) \quad \text{a.e. } t \in [t_0, t_1].$$

Proof Step 1: We first show that it suffices to prove the theorem in the special case where $B = \{\bar{b}\}$ and $\tilde{B} = \{\bar{\tilde{b}}\}$. Indeed, suppose that B^* and \tilde{B}^* are two arbitrary convex and compact neighbourhood of \bar{b} and $\bar{\tilde{b}}$, respectively and that condition *vii)* of theorem 31 has been

already proved for a problem formulation of (HP) with fixed initial conditions. Denote as $\hat{B}^* := B^* \times \tilde{B}^*$. We now consider a new problem related to (HP)

$$(MHP) \left\{ \begin{array}{l} \text{Minimize}_{\sigma \in \mathcal{S}} h_0(y(\sigma)(t_1)) \\ \text{over } \sigma \in \mathcal{S}, \pi \in \mathcal{P} \text{ and } (b(t), \tilde{b}(t)) \in \hat{B}^* \text{ a.e. } t \in [t_0 - 1, t_1] \text{ s.t.} \\ \dot{y}(t) = b(t) - \bar{b} \quad \text{a.e. } t \in [t_0 - 1, t_0] \\ \dot{y}(t) = f(t, y(t), \sigma(t)) \quad \text{a.e. } t \in [t_0, t_1] \\ \dot{\tilde{y}}(t) = \tilde{b}(t) - \bar{\tilde{b}} \quad \text{a.e. } t \in [t_0 - 1, t_0] \\ \dot{\tilde{y}}(t) = \tilde{f}(t, \hat{y}(t), \sigma \otimes \pi(t)) \quad \text{a.e. } t \in [t_0, t_1] \\ y(t_0) = \bar{b}, \quad \tilde{y}(t_0) = \bar{\tilde{b}} \quad \text{and} \quad h_1(y(\sigma)(t_1)) = 0 \\ \hat{h}(\hat{y}(\sigma \otimes \pi)(t_1)) \leq 0 \quad \text{for each } \pi \in \mathcal{P} \end{array} \right. ,$$

It is a straightforward matter to check that if $(\bar{\sigma}, \bar{b}, \bar{\tilde{b}})$ is a minimizer for problem (HP) , then $((\bar{\sigma}, \bar{b}, \bar{\tilde{b}}), \bar{b}, \bar{\tilde{b}})$ is a minimizer for problem (MHP) . This implies that theorem 31 can be applied. In particular, notice that in the interval $[1 - t_0, t_0]$ the dynamic equation derivatives are vanishing and the adjoint equations are $Z(t) \equiv Z(t_0), \hat{Z}(\pi)(t) \equiv \hat{Z}(\pi)(t_0)$ for all $t \in [t_0 - 1, t_0]$, for all $\pi \in \mathcal{P}$ (notice that the convolution does not change the the dynamic equations in $t \in [t_0 - 1, t_1]$, since they are not state dependent). Thus, applying condition *vii*) of theorem 31, we obtain

$$\min_{(b, \tilde{b}) \in \hat{B}^*} (k(t_0)(b - \bar{b}) + \lambda((b, \tilde{b}) - (\bar{b}, \bar{\tilde{b}}))) = 0$$

which gives condition *vi*) of theorem 31 when we pose $\hat{B}^* = \hat{B}$. In particular, it follows that we can prove the theorem in the special case $B = \{\bar{b}\}$ and $\tilde{B} = \{\bar{\tilde{b}}\}$.

Step 2: Consider the sequences of functions

$$\left\{ Z^j(t) = I_n + \int_t^{t_1} Z^j(s) \partial_x f^j(s, y^j(\sigma^j)(s), \sigma^j(s)) ds \right\}_{j \in \mathbb{N}}$$

and

$$\left\{ \hat{Z}^j(t)(\pi) = I_{n+m} + \int_t^{t_1} \hat{Z}^j(\pi)(s) \partial_x \hat{f}^j(s, \hat{y}^j(\sigma^j \otimes \pi)(s), (\sigma^j \otimes \pi)(s)) ds \right\}_{j \in \mathbb{N}}$$

for every $\pi \in \mathcal{P}$. It is easy to check that both sequences are equibounded and equicontinuous. The first property follows because

$$|\partial_x f^j(t, y^j(\sigma^j)(t), \sigma^j(t))| \leq L_f, \quad |\partial_x \hat{f}^j(t, \hat{y}^j(\sigma^j \otimes \pi)(t), (\sigma^j \otimes \pi)(t))| \leq L_{\hat{f}}$$

for every $j \in \mathbb{N}$, a.e. $t \in [t_0, t_1]$, and therefore

$$|Z^j(t)| \leq L_f \int_{t_0}^{t_1} |Z^j(s)| ds + 1, \quad |\hat{Z}^j(\pi)(t)| \leq L_{\hat{f}} \int_{t_0}^{t_1} |\hat{Z}^j(\pi)(s)| ds + 1.$$

In view of the previous inequalities, we can use the Gronwall lemma which yields

$$|Z^j(s)| \leq 1 + (t_1 - t_0)L_f e^{(t_1 - t_0)}, \quad |\hat{Z}^j(\pi)(s)| \leq 1 + (t_1 - t_0)L_{\hat{f}} e^{(t_1 - t_0)} =: \hat{C}.$$

Therefore both sequences are uniformly bounded. Furthermore, by an easy calculation, it follows that, for each $\pi \in \mathcal{P}$,

$$|(Z^j)'(t)| \leq |Z^j(t)|, \quad |(\hat{Z}^j)'(\pi)(t)| \leq |\hat{Z}^j(\pi)(t)|,$$

a.e. $t \in [t_0, t_1]$. These arguments show that we can apply the Ascoli-Arzelà theorem and that there exist $J_1 \subset \mathbb{N}$ and continuous functions $Z(\cdot), \hat{Z}(\pi)(\cdot)$ such that, for each $\pi \in \mathcal{P}$,

$$\lim_{j \in J_1} Z^j(t) = Z(t), \quad \lim_{j \in J_1} \hat{Z}^j(\pi)(t) = \hat{Z}(\pi)(t),$$

uniformly with respect to $t \in [t_0, t_1]$.

The perturbed problems (HP^j) have C^∞ data and solutions $\sigma^j \in \text{co}\mathcal{S}'$. Therefore lemma 23 can be applied for the minimizer $(\sigma^j, \bar{b}, \bar{b})$. In particular, from lemma 23, condition *i*), it follows that there exist $l_0^j \geq 0, l_1^j \in \mathbb{R}$ and $\omega^j \in \text{f.r.m.}^+(\mathcal{P})$ such that,

$$0 < l_0^j + |l_1^j| + \omega^j(\mathcal{P}) \leq 1.$$

By standard compactness arguments, we can find a subset $J_2 \subset J_1, l_0 \geq 0, l_1 \in \mathbb{R}$ and $\omega \in \text{f.r.m.}^+(\mathcal{P})$ such that

$$\lim_{j \in J_2} l_0^j = l_0, \quad \lim_{j \in J_2} l_1^j = l_1, \quad \lim_{j \in J_2} \omega^j = \omega \quad \text{weakly} - *$$

and

$$0 < l_0 + |l_1| + \omega(\mathcal{P}) \leq 1.$$

By lemma 25 (*i*) and using the result of ([8], Theorem VI.I.6, pp. 348), there exists $J_3 \subset J_2$ such that

$$\lim_{j \in J_3} y^j(\sigma^j)(t) = \lim_{j \in J_3} y(\sigma^j)(t) = y(\bar{\sigma})(t) \tag{3.1}$$

and, since the continuity of the functions $\pi \mapsto \hat{y}^j(\sigma \otimes \pi)(\cdot)$ for every $\sigma \in \text{co}\mathcal{S}'$,

$$\lim_{j \in J_3} \hat{y}^j(\sigma^j \otimes \pi)(t) = \lim_{j \in J_3} \hat{y}(\sigma^j \otimes \pi)(t) = \hat{y}(\bar{\sigma} \otimes \pi)(t) \tag{3.2}$$

uniformly with respect to $\pi \in \mathcal{P}$ and $t \in [t_0, t_1]$. (The fact that $\sigma^j \rightarrow \bar{\sigma}$ follows from the optimality of σ^j . Indeed

$$\lim_{j \in J_3} h_0^j(y^j(\sigma^j)(t_1)) = h_0(y(\bar{\sigma})(t_1)) \leq \lim_{j \in J_3} h_0^j(y^j(\sigma)(t_1)) = h_0(y(\sigma)(t_1))$$

for all $\sigma \in \text{co}\mathcal{S}'$).

From lemma 22, it follows that $|\partial_x h_0^j(y^j(\sigma^j)(t_1))| \leq L_{h_0}, |\partial_x h_1^j(y^j(\sigma^j)(t_1))| \leq L_{h_1}$ and $|\partial_x \hat{h}(\hat{y}^j(\sigma^j \otimes \pi)(t_1))| \leq L_{\hat{h}}$ for every $\pi \in \mathcal{P}$. Then, using a similar convergence analysis, we can suppose that there exist $J \subset J_3, \mathcal{H}_0, \mathcal{H}_1 \in \mathbb{R}^n$ such that

$$\lim_{j \in J} \partial_x h_0^j(y^j(\sigma^j)(t_1)) = \mathcal{H}_0, \quad \lim_{j \in J} \partial_x h_1^j(y^j(\sigma^j)(t_1)) = \mathcal{H}_1,$$

and, using ([8], Lemma X.I.6 pp. 489) for each fixed $\pi \in \mathcal{P}$, there exist a vector $\hat{\mathcal{H}}(\pi)$ and a sequence $\{\pi^j\}_{j \in J} \subset \mathcal{P}$, such that

$$\lim_{j \in J} \partial_x \hat{h}^j(\hat{y}^j(\sigma^j \otimes \pi^j)(t_1)) = \hat{\mathcal{H}}(\pi).$$

This completes the proof of points *i*) and *ii*).

Step 3: We observe that the functions $\hat{\mathcal{H}}(\cdot) : \mathcal{P} \rightarrow \mathbb{R}^n$ and $\hat{Z}(\cdot)(t) : \mathcal{P} \rightarrow \mathbb{R}^{n+m}$ are pointwise limits of sequences of continuous functions $\pi \mapsto \{\partial_x \hat{h}^j(\hat{y}^j(\sigma^j \otimes \pi)(t_1))\}_{j \in \mathbb{N}}$ and $\pi \mapsto \{\hat{Z}^j(\pi)(t)\}$. Since the set \mathcal{P} is equipped with the Borel $\mathcal{B}(\mathcal{P})$ -field, the continuous functions $\pi \mapsto \{\partial_x \hat{h}^j(\hat{y}^j(\sigma^j \otimes \pi)(t_1))\}_{j \in \mathbb{N}}$ and $\pi \mapsto \{\hat{Z}^j(\pi)(t)\}$ are ω -measurable and their limit functions $\mathcal{H}(\cdot)$ and $\hat{Z}(\cdot)(t)$ are also ω -measurable (and bounded). From lemma 23, condition *iv*), it follows that

$$\hat{h}^j(\hat{y}^j(\sigma^j \otimes \pi^*)(t_1)) = \max_{\pi \in \mathcal{P}} \hat{h}^j(\hat{y}^j(\sigma^j \otimes \pi)(t_1)), \quad \omega^j - \text{a.a. } \pi^*$$

which is equivalent to

$$\hat{h}^j(\hat{y}^j(\sigma^j \otimes \pi)(t_1)) \leq \hat{h}^j(\hat{y}^j(\sigma^j \otimes \pi^*)(t_1)) \quad \omega^j - \text{a.a. } \pi^*, \quad \forall \pi \in \mathcal{P}.$$

Since ω^j is a positive measure for each j , it follows that

$$\int \left[\hat{h}^j(\hat{y}^j(\sigma^j \otimes \pi) - \hat{h}^j(\hat{y}^j(\sigma^j \otimes \pi^*))) \right] \omega^j(d\pi^*) \leq 0, \quad \forall \pi \in \mathcal{P}. \quad (3.3)$$

We recall that, for $\sigma \in \text{co}S'$ and $\pi \in \mathcal{P}$, the function $\sigma \otimes \pi \mapsto \hat{h}^j(\hat{y}^j(\sigma \otimes \pi)(t_1))$ is continuous for every j , $\pi \mapsto \hat{h}^j(\hat{y}^j(\sigma^j \otimes \pi)(t_1))$ is continuous for every j and $\omega^j \rightarrow \omega$, $\sigma^j \rightarrow \bar{\sigma}$ weakly-*.

Adding and subtracting the term $\int \hat{h}^j(\hat{y}^j(\sigma^j \otimes \pi^*)(t_1))\omega(d\pi^*)$ (which is well-defined since the continuity of $\hat{h}(\hat{y}(\sigma \otimes \cdot)(t_1))$), we can estimate

$$\left| \int \hat{h}^j(\hat{y}^j(\sigma^j \otimes \pi^*)(t_1))(\omega - \omega^j)(d\pi^*) \right| \leq \sup_{\sigma \otimes \pi} |\hat{h}(\hat{y}(\sigma \otimes \pi)(t_1))| \times \left| \int (\omega - \omega^j)(d\pi^*) \right| \rightarrow 0,$$

where the right hand side converges to 0 since $\omega^j \rightarrow \omega$ weakly-*. Then, with the help of the dominated convergence theorem, we can pass to the limit in (3.3), obtaining

$$\int \left[\hat{h}(\hat{y}(\bar{\sigma} \otimes \pi)) - \hat{h}(\hat{y}(\bar{\sigma} \otimes \pi^*)) \right] \omega(d\pi^*) \leq 0, \quad \forall \pi \in \mathcal{P}.$$

Define the sequences of functions $\hat{k}^j : \mathcal{P} \times [t_0, t_1] \rightarrow \mathbb{R}^{n+m}$ and $\mathfrak{h}^j : \mathcal{P} \times [t_0, t_1] \times \mathbf{U} \rightarrow \mathbb{R}$ such that

$$\hat{k}^j(\pi)(t) := \partial_x \hat{h}^j(\hat{y}^j(\sigma^j \otimes \pi)(t_1)) \hat{Z}^j(\pi)(t)$$

and

$$\mathfrak{h}^j(\pi, t, u) := \max_{v \in V(t)} \hat{k}^j(\pi)(t) \cdot \hat{f}^j(t, \hat{y}^j(\sigma^j \otimes \pi)(t), u, v).$$

With the help of lemma 25, it is easy to check that, for every $\pi \in \mathcal{P}$, $\hat{k}^j(\cdot)(\pi) \rightarrow \hat{k}(\cdot)(\pi)$ in $L^1([t_0, t_1], dt)$. Using again lemma 25, it follows that $\mathfrak{h}^j(\pi, \cdot, u) \rightarrow \mathfrak{h}(\pi, \cdot, u)$ in L^1 for every $\pi \in \mathcal{P}$, $u \in \mathbf{U}$, and $\mathfrak{h}^j(\pi, t, \cdot) \rightarrow \mathfrak{h}(\pi, t, \cdot)$ uniformly a.e. $t \in [t_0, t_1]$, for every $\pi \in \mathcal{P}$ and $\mathfrak{h}^j(\cdot, t, u) \rightarrow \mathfrak{h}(\cdot, t, u)$ pointwise, for every $u \in \mathbf{U}$, a.e. $t \in [t_0, t_1]$.

By lemma 23, condition *iii*), the function $\pi \rightarrow \mathfrak{h}^j(\pi, t, u)$ is ω^j -integrable for each j , $u \in \mathbf{U}$, a.e. $t \in [t_0, t_1]$ and can be expressed by the relation

$$\mathfrak{h}^j(\pi, t, u) = \int \hat{k}^j(\pi)(t) \cdot \hat{f}^j(t, \hat{y}^j(\sigma^j \otimes \pi)(t), u, v) \pi(t, u)(dv),$$

for ω^j -a.a. $\pi \in \mathcal{P}$ and $\tilde{\zeta}$ -a.a. $(t, u) \in [t_0, t_1] \times \mathbf{U}$, where $\tilde{\zeta}$ is the positive Radon measure introduced in Section 2.4.

It follows that

$$\int \omega^j(d\pi) \int [\mathfrak{h}^j(\pi, t, u) - \int \hat{k}^j(\pi)(t) \cdot \hat{f}^j(t, \hat{y}^j(\sigma^j \otimes \pi)(t), u, v) \pi(t, u)(dv)] \tilde{\zeta}(dt, du) = 0.$$

Define the function

$$I^j(\pi) := \int \left[\mathfrak{h}^j(\pi, t, u) - \int \hat{k}^j(\pi)(t) \cdot \hat{f}^j(t, \hat{y}^j(\sigma^j \otimes \pi)(t), u, v) \pi(t, u)(dv) \right] \tilde{\zeta}(dt, du)$$

which is ω^j -integrable and the function

$$I(\pi) := \int \left[\mathfrak{h}(\pi, t, u) - \int \hat{k}(\pi)(t) \cdot \hat{f}(t, \hat{y}(\bar{\sigma} \otimes \pi)(t), u, v) \pi(t, u)(dv) \right] \tilde{\zeta}(dt, du)$$

which is the pointwise limit of $I^j(\pi)$.

With a similar analysis used above, we can add and subtract $\int I^j(\pi) \omega(d\pi)$ (still well defined by the continuity of $I^j(\cdot)$) and

$$\left| \int I^j(\pi) (\omega - \omega^j)(d\pi) \right| \leq K \int (\omega - \omega^j)(d\pi) \rightarrow 0$$

since $\omega^j \rightarrow \omega$ weakly-*, where $K := 2L_{\hat{h}} \hat{C} \|\chi(\cdot)\|_{L^1}$.

Then, with the help of the dominated convergence theorem, it follows that

$$\int \omega(d\pi) \int \left[\mathfrak{h}(\pi, t, u) - \int \hat{k}(\pi)(t) \cdot \hat{f}(t, \hat{y}(\bar{\sigma} \otimes \pi)(t), u, v) \pi(t, u)(dv) \right] \tilde{\zeta}(dt, du) = 0.$$

From the definition of $\mathfrak{h}^j(\cdot, \cdot, \cdot)$, it follows that $I^j(\pi) \geq 0$ ω^j -a.a. $\pi \in \mathcal{P}$, as well as its limit function $I(\pi) \geq 0$ ω -a.a. $\pi \in \mathcal{P}$. Using lemma 21, we obtain

$$\int \omega(d\pi) \int_{t_0}^{t_1} dt \int \left[\mathfrak{h}(\pi, t, u) - \int \hat{k}(\pi)(t) \cdot \hat{f}(t, \hat{y}(\bar{\sigma} \otimes \pi)(t), u, v) \pi(t, u)(dv) \right] \bar{\sigma}(t)(du) = 0.$$

Since all the measures involved in the integrals are positive definite, it follows that

$$\mathfrak{h}(\pi, t, u) = \int \hat{k}(\pi)(t) \cdot \hat{f}(t, \hat{y}(\bar{\sigma} \otimes \pi)(t), u, v) \pi(t, u)(dv),$$

ω -a.a. $\pi \in \mathcal{P}$, a.e. $t \in [t_0, t_1]$, $\bar{\sigma}(t)$ -a.a. $u \in \mathbf{U}$, which is condition *iii*).

Step 4: We now derive relation *iv*) from lemma 23, condition *ii*). Define the functions

$$k^j(t) := \left(l_0^j \partial_x h_0^j(y^j(\sigma^j)(t_1)) + l_1^j \partial_x h_1^j(y^j(\sigma^j)(t_1)) \right) Z^j(t)$$

and

$$H^j(t, u) := k^j(t) \cdot f^j(t, y^j(\sigma^j)(t), u) + \int \mathfrak{h}^j(\pi, t, u) \omega^j(d\pi).$$

From lemma 23, condition *ii*), it follows that

$$\int H^j(t, u) (\sigma - \sigma^j)(t)(du) \geq 0, \quad \text{a.e. } t \in [t_0, t_1],$$

which can be explicitly written as

$$\int k^j(t) \cdot f^j(t, y^j(\sigma^j)(t), u)(\sigma - \sigma^j)(t)(du) + \int (\sigma - \sigma^j)(t)(du) \int \mathfrak{h}^j(\pi, t, u)\omega^j(d\pi) \geq 0,$$

for every $\sigma \in \text{co}\mathcal{S}'$, a.e. $t \in [t_0, t_1]$. We now observe that the function $u \mapsto \int \mathfrak{h}^j(\pi, t, u)\omega^j(d\pi)$ is continuous for every $j \in \mathbb{N}$, a.e. $t \in [t_0, t_1]$. Furthermore, adding and subtracting the term $\int \mathfrak{h}^j(\pi, t, u)\omega(d\pi)$, we can prove that $u \mapsto \int \mathfrak{h}^j(\pi, t, u)\omega^j(d\pi)$ converges uniformly to $u \mapsto \int \mathfrak{h}(\pi, t, u)\omega(d\pi)$, a.e. $t \in [t_0, t_1]$ (in turns, we have used the weakly-* convergence of the sequence $\{\omega^j\}$ and the continuity of the function $u \mapsto \mathfrak{h}(\pi, t, u)$, for every $\pi \in \mathcal{P}$ and a.e. $t \in [t_0, t_1]$). This in particular implies that the functions $u \mapsto \int \mathfrak{h}(\pi, t, u)\omega(d\pi)$ and $u \mapsto H(t, u)$ are continuous, a.e. $t \in [t_0, t_1]$. A use of the dominated convergence theorem and the convergence of $\sigma^j \rightarrow \bar{\sigma}$ weakly-* permits us to pass to the limit in the relation above, yielding

$$\int k(t) \cdot f(t, y(\bar{\sigma})(t), u)(\sigma - \bar{\sigma})(t)(du) + \int (\sigma - \bar{\sigma})(t)(du) \int \mathfrak{h}(\pi, t, u)\omega(d\pi) \geq 0,$$

which is exactly the relation

$$\int H(t, u)(\sigma - \bar{\sigma})(t)(du) \geq 0$$

for every $\sigma \in \text{co}\mathcal{S}'$, a.e. $t \in [t_0, t_1]$. By integrating with respect to t on $[t_0, t_1]$, we obtain relation v).

Step 5: In this last step, we derive a pointwise condition for the function $H(\cdot, \cdot)$. We preliminary observe that the $H(\cdot, \cdot)$ is integrable with respect to t for every $u \in \mathbf{U}$ and continuous with respect to u a.e. $t \in [t_0, t_1]$, since the functions $(t, u) \mapsto \mathfrak{h}(\pi, t, u)$ and $(t, u) \mapsto k(t)f(t, y(\bar{\sigma})(t), u)$ satisfy the same property.

We recall that \mathcal{S}' is defined as in Section 2.5. It follows that, for every $u(\cdot) \in \mathcal{R}_\infty$, there exists a null set $Z_u \subset [t_0, t_1]$ such that

$$H(t, u(t)) - \int H(t, u)\bar{\sigma}(t)(du) \geq 0 \quad \forall t \in [t_0, t_1] \setminus Z_u.$$

Since the set \mathcal{U}_∞ is denumerable, then $Z := \cup_{u \in \mathcal{U}_\infty} Z_u$ is still a null set. The set $\{u(t) : u \in \mathcal{U}_\infty\}$ is dense in $U(t)$ a.e. $t \in [t_0, t_1]$ and this implies

$$\int H(t, u)\bar{\sigma}(t)(du) = \inf_{u \in \mathcal{U}_\infty} H(t, u(t)) = \min_{u \in U(t)} H(t, u).$$

This completes the proof. □

The result proved for hyperrelaxed controls can be similarly derived also for relaxed adverse control problems. In this case, we do not need to choose any denumerable subset \mathcal{S}' of \mathcal{S} and the convergence analysis is more straightforward. Now we denote by $y^j(\sigma)(\cdot)$ the unique solution of (2.2) for $\sigma \in \mathcal{S}$ and by $y^j(\sigma \times \sigma_P)(\cdot)$ the unique solution of (2.3) for $\sigma \in \mathcal{S}$ and $\sigma_P \in \mathcal{S}_P$. The problem (RP^j) has a solution $\sigma^j \in \mathcal{S}$ for every j . The function $Z^j(t)$ is the same of theorem 31, while

$$\hat{Z}^j(\sigma_P)(t) = I_{n+m} + \int_t^{t_1} \hat{Z}^j(\sigma_P)(s)\partial_x \hat{f}^j(s, \hat{y}^j(\sigma^j \times \sigma_P)(s), (\sigma^j \times \sigma_P)(s))ds$$

We deal with the derivatives of the functions $h_1^j(\cdot)$ and $\hat{h}^j(\cdot)$ instead that considering the derivatives of $H_1^j(\cdot)$ and $\hat{H}^j(\cdot)$. In what follows, we propose the proof in the relaxed case, pointing out the main differences with the analysis carried out in theorem 31.

Theorem 32 *Let $(\bar{\sigma}, \bar{b}, \bar{b})$ be an optimal solution to the problem (RP). Then there exist a set of index $J \subset \mathbb{N}$, limiting multipliers $l_0 \geq 0, l_1 \in \mathbb{R}$, limiting initial directions $\mathcal{H}_0, \mathcal{H}_1 \in \mathbb{R}^n, a \omega \in \text{f.r.m.}^+(\mathcal{S}_P), a \omega$ -integrable function $\hat{\mathcal{H}} : \mathcal{S}_P \rightarrow \mathbb{R}^{n+m}$, and the continuous functions $Z : [t_0, t_1] \rightarrow M_{n \times n}, \hat{Z}(\mathcal{S}_P) : [t_0, t_1] \rightarrow M_{(n+m) \times (n+m)}$, such that:*

$$i) \quad Z(t) = \lim_{j \in J} Z^j(t), \quad \hat{Z}(\mathcal{S}_P)(t) = \lim_{j \in J} \hat{Z}^j(\mathcal{S}_P)(t) \quad \text{uniformly } t \in [t_0, t_1], \quad \omega - a.a. \sigma_P \in \mathcal{S}_P,$$

$$(l_0, l_1) = \lim_{j \in J} (l_0^j, l_1^j), \quad \mathcal{H}_0 = \lim_{j \in J} \partial_x h_0^j(y^j(t_1)), \quad \mathcal{H}_1 = \lim_{j \in J} \partial_x h_1^j(y^j(t_1)),$$

$$\hat{\mathcal{H}}(\mathcal{S}_P) = \lim_{j \in J} \partial_x \hat{h}^j(\hat{y}(\sigma^j \times \mathcal{S}_P)(t_1)), \quad \omega - a.a. \sigma_P \in \mathcal{S}_P;$$

$$ii) \quad 0 < l_0 + |l_1| + \omega(\mathcal{S}_P) \leq 1$$

Define:

$$k(t) := (l_0 \mathcal{H}_0 + l_1 \mathcal{H}_1) Z(t), \quad \hat{k}(\mathcal{S}_P)(t) := \hat{\mathcal{H}}(\mathcal{S}_P) \hat{Z}(\mathcal{S}_P)(t),$$

$$\mathfrak{h}(\mathcal{S}_P, t, s) := \max_{v \in V(t)} \hat{k}(\mathcal{S}_P)(t) \cdot \int \hat{f}(t, \hat{y}(\bar{\sigma} \times \mathcal{S}_P)(t), u, v) s(du),$$

$$H(t, s) := k(t) f(t, y(\bar{\sigma})(t), s) + \int \mathfrak{h}(\mathcal{S}_P, t, s) \omega(d\mathcal{S}_P), \quad (s \in \mathcal{S}).$$

Then

$$iii) \quad \mathfrak{h}(\mathcal{S}_P, t, \bar{\sigma}(t)) = \int \hat{k}(\mathcal{S}_P)(t) \hat{f}(t, \hat{y}(\bar{\sigma} \times \mathcal{S}_P)(t), \bar{\sigma}(t), v) \sigma_P(t)(dv),$$

ω -a.a. $\sigma_P \in \mathcal{S}_P, a.e. t \in [t_0, t_1]$,

$$iv) \quad \int H(t, u) \bar{\sigma}(t)(du) = \min_{s \in \mathcal{S}} H(t, s) \quad a.e. t \in [t_0, t_1],$$

$$v) \quad \hat{h}(\hat{y}(\bar{\sigma} \times \sigma_P^*)(t_1)) = \max_{\sigma_P \in \mathcal{S}_P} \hat{h}(\hat{y}(\bar{\sigma} \times \mathcal{S}_P)(t_1)) = 0 \quad \omega - a.a. \sigma_P^* \in \mathcal{S}_P.$$

and

$$vi) \quad k(t_0) \bar{b} + \lambda(\bar{b}, \bar{b}) = \min_{(b, \tilde{b}) \in \hat{B}} k(t_0) b + \lambda(b, \tilde{b})$$

where $\lambda := \int \hat{k}(\mathcal{S}_P)(t_0) \omega(d\mathcal{S}_P)$.

Proof Steps 1–2: The hypothesis reduction argument showed in theorem 31, Step 1, can be carried out in identical fashion, obtaining in this way condition *vi*). Also the compactness argument showed in 31, Step 2 can be reproduce in the relaxed framework. We just recall that, if $\sigma^j \rightarrow \sigma$ weakly-* in \mathcal{S} , then also $\sigma^j \times \sigma_P \rightarrow \sigma \times \sigma_P$ weakly-* in \mathcal{Q} . Furthermore the mapping $\sigma_P \mapsto \sigma \times \sigma_P$ is continuous as well as the function $\sigma_P \mapsto \hat{y}^j(\sigma \times \sigma_P)$ for every j . This remark justifies the use of theorem 31, Step 2 arguments, regarding functions $Z(\cdot)$ and $\hat{Z}(\mathcal{S}_P)(\cdot)$. The perturbed problems (RP^j) have C^∞ data and solutions $\sigma^j \in \mathcal{S}$. It

follows that theorem lemma 24 can be applied. In particular from lemma 24, condition *i*), it follows that there exist $l_0^j \geq 0, l_1^j \in \mathbb{R}$ and $\omega^j \in \text{f.r.m.}^+(\mathcal{S}_P)$ such that,

$$0 < l_0^j + |l_1^j| + \omega^j(\mathcal{S}_P) \leq 1$$

and, considering a subsequence of index $J_2 \subset J_1$, we obtain $l_0 \geq 0, l_1 \in \mathbb{R}$ and $\omega \in \text{f.r.m.}^+(\mathcal{S}_P)$ such that

$$\lim_{j \in J_2} l_0^j = l_0, \quad \lim_{j \in J_2} l_1^j = l_1, \quad \lim_{j \in J_2} \omega^j = \omega \quad \text{weakly} - *$$

and such that

$$0 < l_0 + |l_1| + \omega(\mathcal{S}_P) \leq 1.$$

From lemma 25, *i*) and using the result of ([8], Theorem VI.I.6, pp. 348), there exists $J_3 \subset J_2$ such that

$$\lim_{j \in J_3} y^j(\sigma^j)(t) = \lim_{j \in J_3} y(\sigma^j)(t) = y(\bar{\sigma})(t)$$

and, since $\sigma^j \times \sigma_P \rightarrow \bar{\sigma} \times \sigma_P$ in \mathcal{Q} , we have

$$\lim_{j \in J_3} \hat{y}^j(\sigma^j \times \sigma_P)(t) = \lim_{j \in J_3} \hat{y}(\sigma^j \times \sigma_P)(t) = \hat{y}(\bar{\sigma} \times \sigma_P)(t)$$

uniformly with respect to $t \in [t_0, t_1]$, $\sigma_P \in \mathcal{S}_P$. (Again, the fact that $\sigma^j \rightarrow \bar{\sigma}$ follows from the optimality of σ^j . Indeed

$$\lim_{j \in J_3} h_0^j(y^j(\sigma^j)(t_1)) = h_0(y(\bar{\sigma})(t_1)) \leq \lim_{j \in J_3} h_0^j(y^j(\sigma)(t_1)) = h_0(y(\sigma)(t_1))$$

for all $\sigma \in \mathcal{S}$).

By the same analysis described in theorem 31, there exist $J \subset J_3, \mathcal{H}_0, \mathcal{H}_1 \in \mathbb{R}^n$ such that

$$\lim_{j \in J} \partial_x h_0^j(y^j(\sigma^j)(t_1)) = \mathcal{H}_0, \quad \lim_{j \in J} \partial_x h_1^j(y^j(\sigma^j)(t_1)) = \mathcal{H}_1,$$

and, for each fixed $\sigma_P \in \mathcal{S}_P$, a vector $\hat{\mathcal{H}}(\sigma_P)$ such that

$$\lim_{j \in J} \partial_x \hat{h}(\hat{y}^j(\sigma^j \times \sigma_P)(t_1)) = \hat{\mathcal{H}}(\sigma_P).$$

Relations *i*) and *ii*) are proved. Since the function $\hat{\mathcal{H}} : \mathcal{P} \rightarrow \mathbb{R}^n$ is the pointwise limit of the sequence of continuous functions $\{\sigma_P \mapsto \partial_x \hat{h}(\hat{y}^j(\sigma^j \times \sigma_P)(t_1))\}_{j \in J}$, it follows that $\hat{\mathcal{H}}(\cdot)$ is $\mathcal{B}(\mathcal{S}_P)$ -measurable.

Step 3: From lemma 24, condition *iv*), we have

$$\hat{h}^j(\hat{y}^j(\sigma^j \times \sigma_P^*)(t_1)) = \max_{\sigma_P \in \mathcal{S}_P} \hat{h}^j(\hat{y}^j(\sigma^j \times \sigma_P)(t_1)) \quad \omega^j - a.a. \sigma_P^* \in \mathcal{S}_P.$$

Expressing the relation above in integral form and using the same convergence analysis showed in theorem 31, Step 3, we obtain

$$\hat{h}(\hat{y}(\bar{\sigma} \times \sigma_P^*)(t_1)) = \max_{\sigma_P \in \mathcal{S}_P} \hat{h}(\hat{y}(\bar{\sigma} \times \sigma_P)(t_1)) \quad \omega - a.a. \sigma_P^* \in \mathcal{S}_P.$$

We set

$$\hat{k}^j(\sigma_P)(t) := \partial_x \hat{h}^j(\hat{y}^j(\sigma^j \times \sigma_P)(t_1)) \hat{Z}^j(\sigma_P)(t),$$

and

$$\mathfrak{h}^j(\sigma_P, t, s) := \max_{v \in V(t)} \hat{k}^j(\sigma_P)(t) \cdot \int \hat{f}^j(t, \hat{y}^j(\sigma^j \times \sigma_P)(t), u, v) s(du), \quad (s \in \mathcal{S}).$$

From lemma 24, condition *iii*), it follows that

$$\mathfrak{h}^j(\sigma_P, t, \sigma^j(t)) = \int \hat{k}^j(\sigma_P)(t) \hat{f}^j(t, \hat{y}^j(\sigma^j \times \sigma_P)(t), \sigma^j(t), v) \sigma_P(t)(dv),$$

for ω^j -a.a. $\sigma_P \in \mathcal{S}_P$, a.e. $t \in [t_0, t_1]$. Preliminary, we observe that the function $(\sigma_P, t) \rightarrow \mathfrak{h}^j(\sigma_P, t, \sigma(t))$ is $\omega^j \times dt$ -integrable for every $\sigma \in \mathcal{S}$. We can write the relation above as

$$\int \omega^j(d\sigma_P) \int_{t_0}^{t_1} [\mathfrak{h}^j(\sigma_P, t, \sigma^j(t)) - \int \hat{k}^j(\sigma_P)(t) \hat{f}^j(t, \hat{y}^j(\sigma^j \times \sigma_P)(t), \sigma^j(t), v) \sigma_P(t)(dv)] dt = 0$$

and, using the same procedure carried out in theorem 31, Step 3, where this time we deal with the functions

$$I^j(\sigma_P) := \int_{t_0}^{t_1} [\mathfrak{h}^j(\sigma_P, t, \sigma^j(t)) - \int \hat{k}^j(\sigma_P)(t) \hat{f}^j(t, \hat{y}^j(\sigma^j \times \sigma_P)(t), \sigma^j(t), v) \sigma_P(t)(dv)] dt$$

and the pointwise limit

$$I(\sigma_P) := \int_{t_0}^{t_1} [\mathfrak{h}(\sigma_P, t, \bar{\sigma}(t)) - \int \hat{k}(\sigma_P)(t) \hat{f}(t, \hat{y}(\bar{\sigma} \times \sigma_P)(t), \bar{\sigma}(t), v) \sigma_P(t)(dv)] dt,$$

it follows that

$$\mathfrak{h}(\sigma_P, t, \bar{\sigma}(t)) = \int \hat{k}(\sigma_P)(t) \hat{f}(t, \hat{y}(\bar{\sigma} \times \sigma_P)(t), \bar{\sigma}(t), v) \sigma_P(t)(dv),$$

ω -a.a. $\sigma_P \in \mathcal{S}_P$, a.e. $t \in [t_0, t_1]$. This proves relation *iii*). We also observe that, in this case, we do not make use of ζ in the convergence analysis.

Step 4: We follow the same approach used in theorem 31, Step 4.

Define the functions

$$k^j(t) := \left(l_0^j \partial_x h_0^j(y^j(\sigma^j)(t_1)) + l_1^j \partial_x h_1^j(y^j(\sigma^j)(t_1)) \right) Z^j(t)$$

and

$$H^j(t, s) := k^j(t) \cdot f^j(t, y(\sigma^j)(t), s) + \int \mathfrak{h}^j(\sigma_P, t, s) \omega^j(\sigma_P),$$

for every $s \in \mathcal{S}$, a.e. $t \in [t_0, t_1]$. (Notice that this time s is a relaxed control). It is easy to check that the sequence of functions $\{t \mapsto k^j(t)\}_{j \in \mathbb{N}}$ is continuous and $\{t \mapsto H^j(t, s)\}_{j \in \mathbb{N}}$ is integrable for every $s \in \mathcal{S}$. From lemma 24, condition *ii*), it follows that

$$\int H^j(t, u)(s - \sigma^j)(t)(du) \geq 0,$$

for every $s \in \mathcal{S}$ and a.e. $t \in [t_0, t_1]$. Recalling that $\sigma^j \rightarrow \bar{\sigma}$ weakly-*, we obtain

$$\int H(t, u)(s - \bar{\sigma})(t)(du) \geq 0$$

for every $s \in \mathcal{S}$, a.e. $t \in [t_0, t_1]$, from which relation *iv*) follows. This completes the proof. □

Comments:

1. Theorems 31 and 32 appear as a generalisation to the nonsmooth case of lemmas 23 and 24. In particular, in the case in which differentiability hypotheses are invoked (namely, hypothesis $H4$), it turns out that the limits appearing in theorems 31 and 32 converge to the unique adjoint arcs $k(\cdot)$ and $\hat{k}(\cdot)(\pi)$ of lemmas 23 and 24, respectively. In the nonsmooth case, we lose such representations and it remains just limiting relations for the adjoint arcs $k(\cdot)$ and $\hat{k}(\cdot)(\pi)$ for each $\pi \in \mathcal{P}$. However, for such limits, the pointwise version of the maximum principle *vii*) can be still derived in theorem 31. This condition was achieved in [11] just in the case in which the dynamic constraint has an additive structure and some further a priori hypotheses on the optimal trajectory were required.
2. The notion of “relaxed derivative” appears in the adjoint equations $Z(\cdot)$, $\hat{Z}(\cdot)(\cdot)$ and the related functions of the theorems 31, 32. The convergence analysis used to obtain such functions avoids some measurability issues which come out dealing with derivative containers (see [11], comments in Section 3 and following discussion). In the present theorems, the measurability is guaranteed by the limit process.
3. In this paper we have chosen the standard C^∞ mollifier

$$\bar{q}(x) := \begin{cases} \exp\left(\frac{-1}{1-|x|^2}\right) & \text{if } x \in \mathbb{B} \\ 0 & \text{otherwise} \end{cases} .$$

The limits appearing in theorems 31 and 32 depend on such a choice. A more detailed discussion on how to construct nonsmooth subgradients based on convolution argument can be found in [3]. However, as long as the compactness and the continuous differentiability of the mollified sequences appearing in the proofs of theorems 31 and 32 are preserved, the main results of the paper can be still proved in this new framework.

4. The use of the function $h(\cdot, \cdot, \cdot)$ makes a breakthrough with respect to other necessary conditions for nonsmooth problems obtained in the earlier literature. Indeed, it is sensitive to expect that theoretical and computational methods based on the present necessary conditions might take advantage of the “maximization” related to function $h(\cdot, \cdot, \cdot)$ and of the “minimization” related to function $H(\cdot, \cdot)$ (or, to be more precise, the maximization and minimization process related to the sequence of functions $h^j(\cdot, \cdot, \cdot)$ and $H^j(\cdot, \cdot)$, respectively). This will be matter of studies in following papers.

4 An Example

Consider the minimax optimal control problem

$$\begin{cases} \text{Minimize}_{u \in \mathcal{U}} \max_{v \in \mathcal{V}} y(u, v)(1) \\ \text{over measurable functions } u(\cdot), v(\cdot) \text{ such that} \\ u(t) \in \{-1, 1\}, \quad v(t) \in \{-1, 1\} \quad \text{a.e. } t \in [0, 1] \text{ .} \\ \dot{y}(u, v)(t) = |y(u, v)(t)|u(t)v(t) \quad \text{a.e. } t \in [0, 1] \\ y(0) = 1 \end{cases}$$

This problem can be reformulated as

$$(E) \begin{cases} \text{Minimize}_{u \in \mathcal{U}} \alpha \\ \text{over measurable functions } u(\cdot), v(\cdot) \text{ such that} \\ u(t) \in \{-1, 1\}, \quad v(t) \in \{-1, 1\} \quad \text{a.e. } t \in [0, 1] \\ \dot{\alpha}(t) = 0, \\ \dot{y}(u, v)(t) = |y(u, v)(t)|u(t)v(t) \quad \text{a.e. } t \in [0, 1] \\ y(0) = 1 \\ y(u, v)(1) - \alpha \leq 0 \quad \forall v \in \mathcal{V} \end{cases}$$

This is a version of (OP) , with the choice of data: $[t_0, t_1] = [0, 1]$, $\mathbf{U} = \mathbf{V} = \{-1\} \cup \{+1\}$, $B \times \tilde{B} = [-a, a] \times \{1\}$ with a large enough, $(f(t, \alpha, u), \tilde{f}(t, \alpha, y, u, v)) = (0, |y|uv)$, $h_0(\alpha) = \alpha$, $h_1(\alpha) \equiv 0$ and $\hat{h}(\alpha, y) = y - \alpha$. Notice that this kind of example cannot be treated in the framework of [8] and [11].

We aim to study the hyperrelaxed version of problem (E) , which concerns a problem with same data, but where the dynamic constraint is expressed by

$$y(\sigma \otimes \pi)(t) = 1 + \int_0^t ds \int_{\mathbf{U}} |y(s)| \sigma(s) (du) \int_{\mathbf{V}} u v \pi(s, u) (dv)$$

(notice that the role of α is not relevant in the dynamic equation) and the relation $y(u, v)(1) - \alpha \leq 0$ becomes

$$y(\sigma \otimes \pi)(1) - \alpha \leq 0 \quad \forall \pi \in \mathcal{P}.$$

From theorem 31, condition i), it follows that there exists a sequence $\hat{k}^\varepsilon(\pi)(t) \rightarrow \hat{k}(\pi)(t)$ converging uniformly w.r.t. $t \in [0, 1]$, pointwise w.r.t. $\pi \in \mathcal{P}$, to some function $\hat{k}(\pi)(t)$. The function $\mathfrak{h}(\pi, t, u)$ has the form

$$\mathfrak{h}(\pi, t, u) = \max_{v \in \{-1, 1\}} \hat{k}(\pi)(t) |y|u v = |y| |\hat{k}(\pi)(t)|$$

and does not depend on u . This implies that condition vii) of theorem 31 is satisfied for every $u \in \{-1, 1\}$. So we can choose an arbitrary control $\bar{\sigma}(t) = \delta_{u(t)}$ with, for instance, $u(t) \equiv 1$. Plugging such a control into the hyperrelaxed dynamics and looking at the function $y(\bar{\sigma} \otimes \pi)(1)$, we observe that the maximum of theorem 31, condition v) is achieved when $\pi(t, u)(dv) = \delta_{u(t)}(dv)$ and that an optimal solution is given by solving the ordinary differential equation $\dot{y}(\bar{\sigma} \otimes \pi)(t) = |y(\bar{\sigma} \otimes \pi)(t)|$, $y(\bar{\sigma} \otimes \pi)(0) = 1$. It follows that $\bar{\alpha} = y(\bar{\sigma} \otimes \pi)(1)$.

On the other hand, if we consider the relaxed extension of problem (E) , we obtain a problem with the same data, but where the dynamics is now expressed by

$$y(\sigma \times \sigma_P)(t) = 1 + \int_0^t ds \int_{\mathbf{U} \times \mathbf{V}} |y(s)| u v \sigma(s) (du) \times \sigma_P(s) (dv).$$

Arguing as before, it follows from theorem 32 that there exists a sequence $\hat{k}^\varepsilon(\sigma_P)(t)$ converging uniformly w.r.t. $t \in [0, 1]$, pointwise w.r.t. $\sigma_P \in \mathcal{S}_P$, to $\hat{k}(\sigma_P)(t)$. Since $y(0) = 1$, there exists a positive measure set $T' := [0, c) \subset [0, 1]$, such that, for every $\sigma_P \in \mathcal{S}_P$ and $t \in T'$, $y(\bar{\sigma} \times \sigma_P)(t)$ is not vanishing. Now we use the ‘averaged function’ $\mathfrak{h}(\cdot, \dots)$ which satisfies the equality

$$\begin{aligned} \mathfrak{h}(\sigma_P, t, s) &= \max_{v \in \{-1, 1\}} \hat{k}(\sigma_P)(t) |y(\bar{\sigma} \times \sigma_P)(t)| \int_{\mathbf{U}} u v s (du) = \\ &= |\hat{k}(\sigma_P)(t)| \int_{\mathbf{U}} u s (du) |y(\bar{\sigma} \times \sigma_P)(t)| \end{aligned}$$

for $s \in \mathcal{S}$ and $t \in T'$. Furthermore condition $v)$ of theorem 32 implies that ω is not the zero measure (otherwise, this would imply that $\max_{\sigma_P} y(\bar{\sigma} \times \sigma_P)(1) < \bar{\alpha}$, a contradiction with the definition of minimizer for problem (E)). Condition $iv)$ of theorem 32 can be applied on the set T' : if $|\hat{k}(\sigma_P)(t)| \neq 0$, then the optimal relaxed control has the form $\bar{\sigma}(t) = \frac{1}{2}(\delta_{\{-1\}}(t) + \delta_{\{+1\}}(t))$ on the set T' ; if $|\hat{k}(\sigma_P)(t)| \equiv 0$, from the definition of relaxed derivative we find that also in this case the adjoint equation has to be generated by a chattering control like $\bar{\sigma}(t)$. Plugging such a control into the dynamic equation, we observe that the optimal trajectory satisfies the equation $\dot{y}(\bar{\sigma} \times \sigma_P)(t) = 0$, $y(\bar{\sigma} \times \sigma_P)(0) = 1$ on T' . Then also $\hat{k}(\sigma_P)$ is constant on T' . In particular, this implies that $c = 1$, the optimal trajectory is determined by the equations $\dot{y}(\bar{\sigma} \times \sigma_P)(t) = 0$, $\bar{y}(0) = 1$ and $\bar{\alpha} = y(\bar{\sigma} \times \sigma_P)(1) = 1$.

This example shows some of the typical difficulties of relaxation of adverse control problems. First of all, notice that for the same problem, the cost of the relaxed extension is lower than the cost of the hyperrelaxed extension. Furthermore, the ‘real’ pointwise maximum principle $vii)$ of theorem 31 cannot be extended in the general case to theorem 32. With reference to the relaxed extension of problem (E) , it is easy check that the use at the end of the analysis of condition $vii)$ of theorem 31 would imply a contradiction. This phenomenon confirms the results showed in [11], theorem 2.4, where special assumptions on the dynamic equations or on the properties of the optimal trajectories were made (cfr. with [11], Sections 2 and 3).

Acknowledgments This work was co-funded by the European Union under the 7th Framework Program FP7-PEOPLE-2010-ITN, grant agreement number 264735-SADCO. The author is indebted to the anonymous referees for their helpful and constructive comments.

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