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Uniqueness for Quasi-variational Inequalities

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Abstract This paper presents a uniqueness result for a quasi-variational inequality QVI(1) that, in contrast to existing results, does not require the projection mapping on a variable closed and convex set to be a contraction. Our basic idea is to find a simple QVI(0), for example a variational inequality, for which we can show the existence of a unique solution. Further, exploiting some nonsingularity condition, we will guarantee the existence of a continuous solution path from the unique solution of QVI(0) to a solution of QVI(1). Finally, we can show that the existence of a second different solution of QVI(1) contradicts the nonsingularity condition. Moreover, we present some matrix-based sufficient conditions for our nonsingularity assumption, and we discuss these assumptions in the context of generalized Nash equilibrium problems with quadratic cost and affine linear constraint functions.

Keywords Quasi-variational inequalities · Uniqueness · Continuation approach · Implicit function

Mathematics Subject Classification (2010) 47J20 · 49J40 · 90C33

1 Introduction

We consider quasi-variational inequalities (QVIs) in finite-dimensional spaces, i.e., for a map $F : \mathbb{R}^n \to \mathbb{R}^n$ and a set-valued mapping $K : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, with $K(x)$ being closed and convex and also nonempty for $x \in X$ on some subset $X \subseteq \mathbb{R}^n$, we consider the problem of finding a vector $x \in K(x)$ such that

$$\langle F(x), y - x\rangle \geq 0 \quad \forall y \in K(x). \tag{1}$$

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QVIs go back to the papers of Bensoussan and Lions [2–4], where impulse control problems are studied. They have become a powerful modeling tool in many areas, such as mechanics, statistics, biology or economics. For analytical results on QVIs we refer to [1, 11].

Using the projection operator P on closed and convex sets, it is known that we can rewrite (1) as the fixed point problem

$$x = P_{K(x)}[x - \beta F(x)] \tag{2}$$

for any $\beta > 0$. Now the application of a fixed-point theorem leads to sufficient conditions for existence results for QVIs. Exemplary we refer to [5, Theorem 5.2] for some existence result in the finite dimensional setting. However, when it comes to uniqueness the sufficient conditions are getting rare. One reason is the simple fact that many QVIs do not have unique solutions, for example QVIs coming from reformulations of jointly convex generalized Nash equilibrium problems typically have non-unique solutions, see e.g., [8]. But there are also classes of problems for which uniqueness can be shown. Unfortunately, it is not clear how uniqueness results for variational inequalities (VIs) can be generalized for QVIs. For an overview on VIs, we refer to the book [10]. To the best of our knowledge, all existing uniqueness proofs for QVIs exploit the fixed point characterization (2) and need some contraction property of the projection operator. From [13, Theorem 9] we have the following theorem.

Theorem 1 *Let F be Lipschitz continuous with constant $L > 0$ and strongly monotone with modulus $\mu > 0$, and set $\gamma := \frac{L}{\mu} \geq 1$. Assume for some constant $0 < \alpha < \frac{1}{\gamma(\gamma+\sqrt{\gamma^2-1})}$ that*

$$\|P_{K(x)}[z] - P_{K(y)}[z]\| \leq \alpha \|x - y\| \quad \forall x, y \in \mathbb{R}^n. \tag{3}$$

Then the QVI (1) has a unique solution.

Nesterov and Scrimali [12, Corollary 2] further improves the above result and requires only $\alpha < \frac{1}{\gamma}$ for the uniqueness. Nevertheless both results need $\alpha < 1$, and hence the key property is always the contraction property (3) for the projection on the set $K(\cdot)$. For example in [12, Lemma 2] it is shown that (3) holds for the moving-set case, where $K(x) := c(x) + K$ with a fixed closed and convex set K and a Lipschitz continuous map $c : \mathbb{R}^n \to \mathbb{R}^n$ with the same constant α. In more general settings (3) is hard to prove and often not satisfied as in the following simple example.

Example 1 Define $F : \mathbb{R}^2 \to \mathbb{R}^2$, $F(x_1, x_2) := \begin{pmatrix} x_1 - 1 \\ x_2 - 1 \end{pmatrix}$ and

$$K(x) := \{y \in \mathbb{R}^2 \mid g(y, x) \leq 0\}$$

with the function $g : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^4$ defined via

$$g(y, x) := (-y_1 - 1, y_1 + x_2 - 1, -y_2, y_2 - 1)^\top.$$

Then we have

$$K(x) = [-1, 1 - x_2] \times [0, 1]$$

and one can show that $\bar{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is the unique solution of the QVI, find $x \in K(x)$ such that

$$\left\langle \begin{pmatrix} x_1 - 1 \\ x_2 - 1 \end{pmatrix}, \begin{pmatrix} y_1 - x_1 \\ y_2 - x_2 \end{pmatrix} \right\rangle \geq 0 \quad \forall y \in K(x).$$

Now considering the projection map we get for example

$$\begin{aligned}
&\left\| P_{K((0,1)^\top)}\left[\begin{pmatrix}1\\1\end{pmatrix}\right] - P_{K((0,0)^\top)}\left[\begin{pmatrix}1\\1\end{pmatrix}\right]\right\| \\
&= \left\| P_{[-1,0]\times[0,1]}\left[\begin{pmatrix}1\\1\end{pmatrix}\right] - P_{[-1,1]\times[0,1]}\left[\begin{pmatrix}1\\1\end{pmatrix}\right]\right\| \\
&= \left\| \begin{pmatrix}0\\1\end{pmatrix} - \begin{pmatrix}1\\1\end{pmatrix}\right\| = 1 \\
&= \left\| \begin{pmatrix}0\\1\end{pmatrix} - \begin{pmatrix}0\\0\end{pmatrix}\right\|,
\end{aligned}$$

and hence we do not have a contraction here, i.e., the above uniqueness results can not be applied.

In the present paper we will develop a different approach to obtain a unique solution of a QVI without exploiting the contraction property (3). Instead we consider the KKT conditions for QVIs and apply some implicit function theorem. The idea to use the KKT condition of a QVI was recently exploited to design an algorithm for its solution in [9], but it was not used to get a uniqueness result before. In [14] we can find a somehow contrary approach. A uniqueness result for the solution of an optimization problem is shown via the unique solvability of a VI, where as a mathematical tool degree theory is exploited instead of an implicit function theorem. The conditions obtained there can also be used in our approach and we will compare them with our conditions after the presentation of the results. Note, however, that we will not need the strong assumption of the strict complementarity slackness condition. Let us mention that there is a further uniqueness result for complementarity problems in [15, Theorem 2], if one has only box constraints. This, however, is only applicable to the VI setting and not to our QVI setting, since the feasible set can no longer depend on the current point.

Our notation is standard, we use $\langle\cdot,\cdot\rangle$ for the Euclidean scalar product and $\|A\|$ stands for the spectral norm of A, i.e., the square root of the maximum eigenvalue of the matrix $A^\top A$. For a function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$, $(x, y) \mapsto f(x, y)$ we denote by $\nabla_y f(x, x) \in \mathbb{R}^{m\times p}$ the partial gradient of f with respect to the second variable, evaluated at $y = x$, and the gradients of the component functions are written column wise. In contrast $J_y f(x, x) \in \mathbb{R}^{p\times m}$ stands for the Jacobian of f with respect to the second variable, evaluated at $y = x$, and the gradients of the components are written row wise.

The rest of the paper is organized as follows. In the next section we present our uniqueness result for QVIs. After that Section 3 provides matrix-dependent sufficient conditions for our main nonsingularity assumption. In Section 4 we consider our sufficient condition in the context of generalized Nash equilibrium problems (GNEPs) with quadratic cost and affine linear constraint functions, before we summarize the results in Section 5.

2 Uniqueness for QVIs by a Continuation Approach

In our approach we will consider a family of parametrized QVIs. Therefore we define the mapping $F : \mathbb{R}^n \times [0, 1] \to \mathbb{R}^n$ and the set-valued mapping $K : \mathbb{R}^n \times [0, 1] \rightrightarrows \mathbb{R}^n$. Then we have for every $t \in [0, 1]$ the QVI(t): Find $x = x(t) \in K(x, t)$ such that

$$\langle F(x, t), y - x\rangle \geq 0 \quad \forall y \in K(x, t).$$

We write $F(x) := F(x, 1)$ and $K(x) := K(x, 1)$, i.e., we embed problem (1) as QVI(1). Our basic idea is to find a simple QVI(0) for which we can show the existence of a unique solution that satisfies some regularity property. Then we seek for a continuous solution path from the unique solution of QVI(0) to a solution of QVI(1) via solutions of QVI(t). Assuming the existence of a second different solution of QVI(1) we seek for a solution path back to the unique solution of QVI(0) whose existence then contradicts the regularity condition and proves the uniqueness of the solution.

For this task we use the KKT conditions of a QVI and therefore we assume that K is explicitly represented by inequalities (to be understood for each component), i.e.,

$$K(x, t) := \{y \in \mathbb{R}^n \mid g(y, x, t) \leq 0\},$$

with the function $g : \mathbb{R}^n \times \mathbb{R}^n \times [0, 1] \to \mathbb{R}^m$ and the component functions $g_i(\cdot, x, t)$ being convex and continuously differentiable in $\mathbb{R}^n$ for each $x \in \mathbb{R}^n, t \in [0, 1]$ and all $i = 1, \dots, m$. Obviously we have $x \in K(x, t)$, if and only if $g(x, x, t) \leq 0$. The convexity assumption ensures that the set $K(x, t)$ is convex and continuity of g implies that it is closed. The continuous differentiability is required in order to write down the KKT system for the QVI(t), which means that we can find a multiplier $\lambda = \lambda(x, t) \in \mathbb{R}^m$ such that

$$\begin{aligned} F(x, t) + \nabla_y g(x, x, t)\lambda &= 0, \\ 0 \leq \lambda \perp g(x, x, t) &\leq 0. \end{aligned}$$

From [9, Theorem 1] we get that the x-part of a KKT point is a solution of the QVI (t), and for each solution of the QVI(t) we can find a multiplier λ in order to satisfy the KKT conditions, if any standard constraint qualification (like the Slater condition or the linear independence constraint qualification LICQ) is satisfied for the constraints $g(\cdot, x, t)$. Therefore, and since the upcoming analysis requires further smoothness properties we make the following assumption.

Assumption 1 (a) *$F : \mathbb{R}^n \times [0, 1] \to \mathbb{R}^n$ is continuously differentiable with respect to x.*

(b) *$g : \mathbb{R}^n \times \mathbb{R}^n \times [0, 1] \to \mathbb{R}^m$ is continuously differentiable with respect to y, and $\nabla_y g$ is continuously differentiable with respect to x and y. Further $g_i(\cdot, x, t)$ is convex.*

(c) *LICQ is satisfied, i.e., $\nabla_y g_{I(x,t)}(x, x, t)$ has full column rank for the set of active constraints*

$$I(x, t) := \{i \in \{1, \dots, m\} \mid g_i(x, x, t) = 0\}.$$

Using the Fischer-Burmeister function

$$\phi(a, b) := \sqrt{a^2 + b^2} - a - b$$

and defining

$$\varphi(\lambda, -g(x, x, t)) := \begin{pmatrix} \phi(\lambda_1, -g_1(x, x, t)) \\ \vdots \\ \phi(\lambda_m, -g_m(x, x, t)) \end{pmatrix},$$

we can reformulate the KKT-system as a nonsmooth equation system

$$0 = \Psi(x, \lambda, t) := \begin{pmatrix} F(x, t) + \nabla_y g(x, x, t)\lambda \\ \varphi(\lambda, -g(x, x, t)) \end{pmatrix}. \tag{4}$$

The function Ψ is known to be locally Lipschitz and hence Clarke's (partial) generalized Jacobian $\partial_{(x,\lambda)}\Psi(x, \lambda, t)$ of the function Ψ is well defined. Assuming that all the matrices in Clarke's (partial) generalized Jacobian are nonsingular, we can use Clarke's implicit function theorem [6, Section 7.2]. Now we can state our uniqueness result.

Theorem 2 *Suppose Assumption 1 holds. Let the functions* $F(x,\cdot)$ *and* $g(y,x,\cdot)$ *be continuous on* $[0,1]$ *with* $F(x,1)=F(x)$ *and* $g(y,x,1)=g(y,x)$, *and suppose that*

$$X := \bigcup_{t\in[0,1]} \{x \in \mathbb{R}^n \mid g(x,x,t) \le 0\}$$

is bounded. Assume that for every $t \in [0,1]$ *and every solution* $(\bar{x}(t),\bar{\lambda}(t))$ *of* $\Psi(x,\lambda,t)=0$ *all matrices in* $\partial_{(x,\lambda)}\Psi(\bar{x}(t),\bar{\lambda}(t),t)$ *are nonsingular. Further assume that QVI(0): find* $x \in K(x,0)$ *such that*

$$\langle F(x,0), y-x\rangle \ge 0 \quad \forall y \in K(x,0),$$

has a unique solution. Then QVI(t) has for all $t \in [0,1]$ *a unique solution.*

Proof Let $x(0)$ be the unique solution of QVI(0). Using LICQ from Assumption 1, we obtain a unique multiplier $\lambda(0)$ such that $\Psi(x(0),\lambda(0),0)=0$. The nonsingularity of all elements in $\partial_{(x,\lambda)}\Psi(x(0),\lambda(0),0)$ allows the application of Clarke's implicit function theorem, and we get a $\tilde{t}>0$ and a uniquely defined continuous function $t \mapsto (\bar{x}(t),\bar{\lambda}(t))$ such that $(\bar{x}(0),\bar{\lambda}(0))=(x(0),\lambda(0))$ and $\Psi(\bar{x}(t),\bar{\lambda}(t),t)=0$ for all $t\in[0,\tilde{t})$. Let $\tilde{t}>0$ be maximal.

Assume for contradiction $\tilde{t} \le 1$: $\bar{x}(t)$ is bounded, because $\Psi(\bar{x}(t),\bar{\lambda}(t),t)=0$ implies the feasibility $g(\bar{x}(t),\bar{x}(t),t)\le 0$ and hence $\bar{x}(t)\in X$, which is bounded by assumption. Using the LICQ condition from Assumption 1 we thus also get boundedness of the unique multiplier $\lambda(t)$. But then we can find a sequence $\{t_k\}\in[0,\tilde{t})$ with $t_k \to \tilde{t}$ such that $(\bar{x}(t_k),\bar{\lambda}(t_k))$ converges to some point $(\tilde{x},\tilde{\lambda})$. Now continuity of Ψ implies that $\Psi(\tilde{x},\tilde{\lambda},\tilde{t})=0$. By assumption we have for $\tilde{t}\le 1$ that all elements in $\partial_{(x,\lambda)}\Psi(\tilde{x},\tilde{\lambda},\tilde{t})$ are nonsingular and we can once again use Clarke's implicit function theorem, to get a neighbourhood $(\tilde{t}-\varepsilon,\tilde{t}+\varepsilon)$, $\varepsilon>0$ and a unique continuous function $t\mapsto(\tilde{x}(t),\tilde{\lambda}(t))$ with $(\tilde{x}(\tilde{t}),\tilde{\lambda}(\tilde{t}))=(\tilde{x},\tilde{\lambda})$ and $\Psi(\tilde{x}(t),\tilde{\lambda}(t),t)=0$ for all $t\in(\tilde{t}-\varepsilon,\tilde{t}+\varepsilon)$. Now the functions $(\bar{x}(t),\bar{\lambda}(t))$ and $(\tilde{x}(t),\tilde{\lambda}(t))$ must be equal on the intersection $(\tilde{t}-\varepsilon,\tilde{t})$ of the intervals, and hence there exists a continuation of $(\bar{x}(t),\bar{\lambda}(t))$ on the interval $[\tilde{t},\tilde{t}+\varepsilon)$. This contradicts that $\tilde{t}$ is maximal and hence we must have $\tilde{t}>1$.

So far we have shown the existence of a continuous function $t\mapsto(\bar{x}(t),\bar{\lambda}(t))$ such that $\bar{x}(t)$ is a solution of QVI(t) for all $t\in[0,1]$, and in particular the existence of a solution of QVI(1). It remains to prove the uniqueness.

Therefore, we assume that for some $\tilde{t}\in(0,1]$, QVI(t) has two solution $x_1(\tilde{t})\neq x_2(\tilde{t})$. With the LICQ condition we get unique multipliers $\lambda_1(\tilde{t})$ and $\lambda_2(\tilde{t})$ such that $\Psi(x_i(\tilde{t}),\lambda_i(\tilde{t}),\tilde{t})=0$ for i=1,2. By the assumed nonsingularity of the elements in $\partial_{(x,\lambda)}\Psi(x_i(\tilde{t}),\lambda_i(\tilde{t}),\tilde{t})$ we can apply the implicit function theorem to get neighborhoods $(\tilde{t}-\varepsilon_i,\tilde{t}+\varepsilon_i)$, $\varepsilon_i>0$ and continuous functions $t\mapsto(\bar{x}_i(t),\bar{\lambda}_i(t))$ such that $(\bar{x}_i(\tilde{t}),\bar{\lambda}_i(\tilde{t}))=(x_i(\tilde{t}),\lambda_i(\tilde{t}))$ and $\Psi(\bar{x}_i(t),\bar{\lambda}_i(t),t)=0$ for all $t\in(\tilde{t}-\varepsilon_i,\tilde{t}]$ and $i=1,2$. Repeating the contradiction argument from above, we can show that both solution paths can be continued onto the interval $[0,\tilde{t}]$. By the assumed nonsingularity, all KKT points of QVI(t) must be isolated, and hence the solution paths cannot intersect in $[0,\tilde{t}]$, i.e., we cannot have $(\bar{x}_1(t),\bar{\lambda}_1(t))=(\bar{x}_2(t),\bar{\lambda}_2(t))$ for any $t\in[0,\tilde{t}]$. This in particular implies $(\bar{x}_1(0),\bar{\lambda}_1(0))\neq(\bar{x}_2(0),\bar{\lambda}_2(0))$. Since the solution of QVI(0) must be unique, we have $\bar{x}_1(0)=\bar{x}_2(0)$ and by LICQ also $\bar{\lambda}_1(0)=\bar{\lambda}_2(0)$, a contradiction. Therefore we must have $x_1(\tilde{t})=x_2(\tilde{t})$ for all $t\in[0,1]$, and hence a unique solution of QVI(t) for all $t\in[0,1]$, which completes the proof. □

In contrast to QVIs the conditions for uniqueness for variational inequalities (VIs) are much simpler. Therefore in order to ensure the existence of a unique solution of QVI(0) one can define for some constant vector $c \in \mathbb{R}^n$ the function

$$g(y, x, t) := \bar{g}(y, t \cdot x + (1 - t)c).$$

Then one has $g(y, x, 0) = \bar{g}(y, c)$ and hence one has a VI for $t = 0$, since the function $\bar{g}(y, c)$ is independent of x. Thus, if one can guarantee $F(x, 0)$ to be strictly monotone, for example by $J_x F(x, 0)$ being positive definite, the uniqueness of the solution follows from the compactness assumption on the set $K(x, 0) = \{y \in \mathbb{R}^n \mid \bar{g}(y, c) \leq 0\}$. Alternatively one can also use some coerciveness conditions for $F(x, 0)$.

If F is the gradient of a scalar valued function $f : \mathbb{R}^n \to \mathbb{R}$ one could obtain uniqueness of the solution of QVI(1) via the KKT-system also by [14, Corollary 5.1], which deals with complementarity problems and requires the following conditions:

(a) $M := \{x \in \mathbb{R}^n \mid g(x, x, 1) \leq 0\}$ is compact.
(b) There is an open neighborhood U of M such that $f : U \to \mathbb{R}$ is twice continuously differentiable.
(c) We have for the Euler characteristic $\chi(M) = 1$.
(d) LICQ is satisfied at any KKT point, i.e., at any solution of (4).
(e) Any KKT point satisfies the strict complementarity slackness condition, i.e., $g_i(x, x, 1) = 0$ implies $\lambda_i > 0$.
(f) At any KKT point the matrix

$$\Lambda(x) := V(x)^\top J_x L(x, \lambda, 1) V(x)$$

is nonsingular, where

$$L(x, \lambda, 1) := Jf(x) + \nabla_y g(x, x, 1)\lambda,$$

is the Lagrange function of the optimization problem and $V(x)$ is the change-of-coordinates matrix from tangent coordinates to standard coordinates, whose columns are an orthonormal basis of the tangent space to the active constraints in x.
(g) At any KKT point we have $sign(\det(\Lambda(x))) = 1$.

First of all, let us mention that our Theorem 2 is also applicable to functions F that are not gradients of a scalar valued function. Further, we also have the advantage that we do not require the strict complementarity condition (e), which is a strong assumption. Considering Example 1 it is easy to see that at the solution $\bar{x} = (0, 1)^\top$ we have $g_3(\bar{x}, \bar{x}) = 0$ and $\lambda_3 = 0$ and hence the strict complementarity slackness condition is violated. The advantage of [14, Corollary 5.1] is that one has to consider only a single problem and not a family of parametrized problems as in our approach.

To check the nonsingularity assumption in Theorem 2 is not easy and therefore we provide some sufficient conditions for it in the next section, which will allow to prove uniqueness of the solution of Example 1.

3 Nonsingularity Conditions

Let us first show a general result to obtain nonsingularity of all elements in $\partial_{(x,\lambda)} \Psi(\bar{x}, \bar{\lambda}, t)$ at a KKT point $(\bar{x}, \bar{\lambda})$ for some fixed $t \in [0, 1]$. Therefore we have to compute the structure of its elements. Defining

$$L(x, \lambda, t) := F(x, t) + \nabla_y g(x, x, t)\lambda,$$

we have

$$\Psi(x, \lambda, t) = \begin{pmatrix} L(x, \lambda, t) \\ \varphi(\lambda, -g(x, x, t)) \end{pmatrix}.$$

Using standard calculation we can show that all elements of Clarke's (partial) generalized Jacobian $\partial_{(x,\lambda)}\Psi(x, \lambda, t)$ of the function Ψ have the form

$$\begin{pmatrix} J_x L(x, \lambda, t) & \nabla_y g(x, x, t) \\ -D^a(x, \lambda, t)(\nabla_y g(x, x, t) + \nabla_x g(x, x, t))^\top & D^b(x, \lambda, t) \end{pmatrix}$$

where the matrices $D^a(x, \lambda, t), D^b(x, \lambda, t) \in \mathbb{R}^{m\times m}$ are diagonal matrices

$$\begin{aligned} D^a(x, \lambda, t) &:= \operatorname{diag}(a_1(x, \lambda_1, t), \ldots, a_m(x, \lambda_m, t)), \\ D^b(x, \lambda, t) &:= \operatorname{diag}(b_1(x, \lambda_1, t), \ldots, b_m(x, \lambda_m, t)), \end{aligned}$$

with

$$(a_i(x, \lambda_i, t), b_i(x, \lambda_i, t)) \begin{cases} = \dfrac{(-g_i(x, x, t), \lambda_i)}{\sqrt{(\lambda_i)^2 + g_i(x, x, t)^2}} - (1, 1), & \text{if } (-g_i(x, x, t), \lambda_i) \neq (0, 0), \\ \in cl(\mathbb{B}_1(0, 0)) - (1, 1), & \text{if } (-g_i(x, x, t), \lambda_i) = (0, 0), \end{cases}$$

for all $i = 1, \ldots, m$, where $cl(\mathbb{B}_1(0, 0))$ is the closure of the ball with center $(0, 0)$ and radius 1. Note that the matrices $D^a(x, \lambda, t)$, $D^b(x, \lambda, t)$ are negative semidefinite diagonal matrices and their sum $D^a(x, \lambda, t) + D^b(x, \lambda, t)$ is negative definite.

For notational simplicity we will from now on suppress the dependence on x and t when we refer to the set of active constraints, i.e., we will write

$$I := I(\bar{x}, t) = \{i \in \{1, \ldots, m\} \mid g_i(\bar{x}, \bar{x}, t) = 0\}.$$

Next, we define the diagonal matrices $D_I^a(\bar{x}, \bar{\lambda}, t)$ and $D_I^b(\bar{x}, \bar{\lambda}, t)$ as those matrices where all rows and columns corresponding to indices of inactive constraints $i \notin I$ are dropped.

Let us recall a characterization and some property of a P-matrix, see e.g., [7]: A matrix $M \in \mathbb{R}^{n\times n}$ is a P-matrix, if the determinant of each principal submatrix has positive signum. Further we will use that for a P-matrix M and negative semidefinite diagonal matrices D^a and D^b with $D^a + D^b$ being negative definite, the matrix $D^a \cdot M + D^b$ is nonsingular.

Theorem 3 *Let $t \in [0, 1]$ be fixed and let $(\bar{x}, \bar{\lambda})$ be a KKT point of QVI(t). Suppose that in addition to Assumption 1, we have*

(a) $J_x L(\bar{x}, \bar{\lambda}, t)$ *is nonsingular,*
(b) $(\nabla_y g_I(\bar{x}, \bar{x}, t) + \nabla_x g_I(\bar{x}, \bar{x}, t))^\top J_x L(\bar{x}, \bar{\lambda}, t)^{-1} \nabla_y g_I(\bar{x}, \bar{x}, t)$ *is a P-matrix.*

Then all elements of $\partial_{(x,\lambda)}\Psi(\bar{x}, \bar{\lambda}, t)$ are nonsingular.

Proof Let M be an arbitrary element of $\partial_{(x,\lambda)}\Psi(\bar{x}, \bar{\lambda}, t)$. Since $a_i(\bar{x}, \bar{\lambda}_i, t) = 0$ and $b_i(\bar{x}, \bar{\lambda}_i, t) < 0$ for all $i \notin I$ we have that M is nonsingular if and only if the matrix

$$\tilde{M} := \begin{pmatrix} J_x L(\bar{x}, \bar{\lambda}, t) & \nabla_y g_I(\bar{x}, \bar{x}, t) \\ -D_I^a(\bar{x}, \bar{\lambda}, t)(\nabla_y g_I(\bar{x}, \bar{x}, t) + \nabla_x g_I(\bar{x}, \bar{x}, t))^\top & D_I^b(\bar{x}, \bar{\lambda}, t) \end{pmatrix}$$

is nonsingular. By the nonsingularity of $J_x L(\bar{x}, \bar{\lambda}, t)$, the matrix $\tilde{M}$ is nonsingular, if

$$D_I^a(\bar{x}, \bar{\lambda}, t)(\nabla_y g_I(\bar{x}, \bar{x}, t) + \nabla_x g_I(\bar{x}, \bar{x}, t))^\top J_x L(\bar{x}, \bar{\lambda}, t)^{-1} \nabla_y g_I(\bar{x}, \bar{x}, t) + D_I^b(\bar{x}, \bar{\lambda}, t)$$

is nonsingular. But this matrix is nonsingular due to the assumed P-property and the fact that the diagonal matrices $D_I^a(\bar{x}, \bar{\lambda}, t)$, $D_I^b(\bar{x}, \bar{\lambda}, t)$ are negative semidefinite and their sum is negative definite. This shows that M is nonsingular. □

Note that a P-matrix is in particular nonsingular and hence a necessary condition to satisfy the assumption (b) is the full row rank of the matrix

$$(\nabla_y g_I(\bar{x}, \bar{x}, t) + \nabla_x g_I(\bar{x}, \bar{x}, t))^\top,$$

which is not a consequence of the LICQ from Assumption 1, since this only deals with $\nabla_y g_I(x, x, t)$. Further note, that while the matrix condition coming from [14, Corollary 5.1], and stated as condition (f) in Section 2, is based on tangent coordinate matrices V, our condition (b) in Theorem 3 is based on normal coordinate matrices depending on gradients of g.

If the constraint function $g(\cdot, \cdot, t)$ is affine linear, the conditions in Theorem 3 are independent of the multiplier λ, since $L(x, \lambda, t) = F(x, t)$. If further also $F(\cdot, t)$ is affine linear, condition (a) becomes independent of x, whereas condition (b) is implicitly dependent on x via the set $I = I(\bar{x}, t)$ of active constraints. For an application of our Theorem 3, let us consider once again the QVI from Example 1.

Example 2 We want to clarify uniqueness for the QVI:
find $(x_1, x_2) \in [-1, 1 - x_2] \times [0, 1]$ such that

$$\langle F(x, 1), y - x \rangle \geq 0 \quad \forall y \in [-1, 1 - x_2] \times [0, 1].$$

We have $F(x, t) := \begin{pmatrix} x_1 - 1 \\ x_2 - 1 \end{pmatrix}$ (independent of t) and we define the function

$$g(y, x, t) := (-y_1 - 1, y_1 + t x_2 - 1, -y_2, y_2 - 1)^\top.$$

Obviously the affine linear functions F and g are arbitrary often continuously differentiable, $g(\cdot, x, t)$ is convex, and for any $(x_1, x_2) \in [-1, 1 - x_2] \times [0, 1]$, LICQ is satisfied. Thus Assumption 1 holds. Further, the set

$$X := \bigcup_{t \in [0,1]} \{x \in \mathbb{R}^n \mid g(x, x, t) \leq 0\} = [-1, 1] \times [0, 1]$$

is bounded. The problem QVI(0) is to find $(x_1, x_2) \in [-1, 1] \times [0, 1]$ such that

$$\langle F(x, 0), y - x \rangle \geq 0 \quad \forall y \in [-1, 1] \times [0, 1],$$

which is a VI with a strictly monotone function $F(x, 0)$ and a compact feasible set. Hence QVI(0) has a unique solution. Moreover, we have $J_x L(x, \lambda, t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ which is nonsingular, and we have

$$\begin{aligned} &(\nabla_y g(\bar{x}, \bar{x}, t) + \nabla_x g(\bar{x}, \bar{x}, t))^\top J_x L(\bar{x}, \bar{\lambda}, t)^{-1} \nabla_y g(\bar{x}, \bar{x}, t) \\ &= \begin{pmatrix} -1 & 0 \\ 1 & t \\ 0 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & -t & t \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}. \end{aligned}$$

Since for all feasible points the lower and the upper constraint for each variable can not be active at the same time, we only have to show that each principal submatrix with rows and columns from $\{\{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}\}$ is a P-matrix, in order to show condition (b) of Theorem 3. This, however, is easy to see, since we always get upper triangular matrices with 1 on the diagonal. Therefore, Theorem 3 provides the missing nonsingularity condition in order to apply Theorem 2, which shows that the considered QVI has a unique solution.

Assuming a special structure of the constraints one can develop further conditions for nonsingularity, building on Theorem 3. The obtainable results here are similar to those guaranteeing the nonsingularity of a certain matrix in the constrained equation approach of [9]. Note, however, that their approach is based on an interior point method and the nonsingularity condition is not required at the solution but at the interior of the feasible set. Let us consider the following situation, which is termed linear constraints with variable right-hand side in [9]:

$$K(x,t) := \{y \in \mathbb{R}^n \mid g(y,x,t) := Ey - b - c(x,t) \le 0\},$$

where $E \in \mathbb{R}^{m\times n}$ is a given matrix, $b \in \mathbb{R}^m$ is a vector and $c : \mathbb{R}^n \times [0,1] \to \mathbb{R}^m$ is an arbitrary function. This setting fits into Assumption 1(b).

Theorem 4 *Let $t \in [0,1]$ be fixed and let $(\bar{x},\bar{\lambda})$ be a KKT point of QVI(t). Suppose that, in addition to Assumption 1, $J_xF(\bar{x},t)$ is positive definite and we have*

$$\|\nabla_x c(\bar{x},t)^\top\| < \frac{\mu^+(\bar{x},t)}{\|J_xF(\bar{x},t)^{-1}\|\,\|E\|},$$

where

$$\mu^+(\bar{x},t) = \min\Big\{\mu^+(A) \mid A \text{ is a principal submatrix of } \frac{1}{2}E(J_xF(\bar{x},t)^{-1} + J_xF(\bar{x},t)^{-T})E^\top\Big\},$$

and $\mu^+(A)$ denotes the minimum positive eigenvalue of the matrix A, and A^{-T} is the transpose of the inverse of A. Then all element of $\partial_{(x,\lambda)}\Psi(\bar{x},\bar{\lambda},t)$ are nonsingular.

Proof We will show that the conditions of Theorem 3 are satisfied. Since $g(\cdot,x,t)$ is linear, we have $J_xL(\bar{x},\bar{\lambda},t) = J_xF(\bar{x},t)$ and hence the nonsingularity of $J_xL(\bar{x},\bar{\lambda},t)$ follows from the positive definiteness of $J_xF(\bar{x},t)$. It remains to show that

$$\begin{aligned}&(\nabla_y g_I(\bar{x},\bar{x},t) + \nabla_x g_I(\bar{x},\bar{x},t))^\top J_xL(\bar{x},\bar{\lambda},t)^{-1}\nabla_y g_I(\bar{x},\bar{x},t)\\ &= (E_I - \nabla_x c_I(\bar{x},t)^\top)J_xF(\bar{x},t)^{-1}E_I^\top\end{aligned}$$

is a P-matrix. Using the positive definiteness of $J_xF(\bar{x},t)$ and its inverse and the full column rank of E_I coming from LICQ in Assumption 1 we have

$$\begin{aligned}v^\top E_I J_xF(\bar{x},t)^{-1}E_I^\top v &= v^\top\left(\frac{1}{2}E_I(J_xF(\bar{x},t)^{-1} + J_xF(\bar{x},t)^{-T})E_I^\top\right)v\\ &\ge \mu^+(\bar{x},t)\|v\|^2\end{aligned}$$

for all $v \in \mathbb{R}^n$. By the inequality for $\mu^+(\bar{x},t)$, and the fact that the spectral norm of a submatrix is less or equal the spectral norm of the full matrix, we get

$$\begin{aligned}\mu^+(\bar{x},t)\|v\|^2 &> \|\nabla_x c(\bar{x},t)^\top\|\,\|J_xF(\bar{x},t)^{-1}\|\,\|E\|\,\|v\|^2\\ &\ge \|\nabla_x c_I(\bar{x},t)^\top\|\,\|J_xF(\bar{x},t)^{-1}\|\,\|E_I\|\,\|v\|^2\\ &\ge v^\top\nabla_x c_I(\bar{x},t)^\top J_xF(\bar{x},t)^{-1}E_I v\end{aligned}$$

for all $v \in \mathbb{R}^n \setminus \{0\}$. Altogether we obtain

$$v^\top(E_I - \nabla_x c_I(\bar{x},t)^\top)J_xF(\bar{x},t)^{-1}E_I\,v > 0$$

for all $v \in \mathbb{R}^n \setminus \{0\}$. Hence the matrix

$$(E_I - \nabla_x c_I(\bar{x},t)^\top)J_xF(\bar{x},t)^{-1}E_I$$

is positive definite, which implies that it is a P-matrix, and the conditions of Theorem 3 are satisfied. □

Although the conditions in Theorem 4 are stronger than in Theorem 3, we have the advantage of being independent of the set of active constraints here. However, one might need some rescaling to be able to apply the theorem, as the following example illustrates.

Example 3 Consider the QVI(t) with linear constraints with variable right hand side defined by

$$F(x,t) := \begin{pmatrix} x_1 - 2 \\ x_2 - 2 \end{pmatrix} \quad \text{and} \quad g(y,x,t) := \begin{pmatrix} 3 & 0 \\ -1 & 0 \\ 0 & 3 \\ 0 & -1 \end{pmatrix} y - \begin{pmatrix} 3 \\ 0 \\ 3 \\ 0 \end{pmatrix} - \begin{pmatrix} -tx_2 \\ 0 \\ -tx_1 \\ 0 \end{pmatrix}.$$

Here we have

$$J_x F(x,t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad E = \begin{pmatrix} 3 & 0 \\ -1 & 0 \\ 0 & 3 \\ 0 & -1 \end{pmatrix}, \quad \text{and} \quad \nabla_x c(\bar{x},t)^\top = \begin{pmatrix} 0 & -t \\ 0 & 0 \\ -t & 0 \\ 0 & 0 \end{pmatrix}.$$

This implies

$$\frac{1}{2} E(J_x F(\bar{x},t)^{-1} + J_x F(\bar{x},t)^{-T}) E^\top = EE^\top = \begin{pmatrix} 9 & -3 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 0 & 0 & 9 & -3 \\ 0 & 0 & -3 & 1 \end{pmatrix},$$

and the eigenvalues of all principal submatrices are in $\{0, 1, 9, 10\}$. Hence we have $\mu^+(x,t) = 1$ for the smallest positive eigenvalue. Further, since $\|E\|^2$ is the maximum eigenvalue of $E^\top E = \text{diag}(10, 10)$ we get $\|E\| = \sqrt{10}$. Finally, we have $\|\nabla_x c(\bar{x},t)^\top\| = t$. Therefore the condition in Theorem 4

$$t = \|\nabla_x c(\bar{x},t)^\top\| < \frac{\mu^+(\bar{x},t)}{\|J_x F(\bar{x},t)^{-1}\| \, \|E\|} = \frac{1}{1 \cdot \sqrt{10}}$$

is only satisfied for $t \in \left[0, \frac{1}{\sqrt{10}}\right)$. If we rescale the constraints by dividing the first and the third one by 3, we obtain an equivalent QVI with

$$g(y,x,t) := \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix} y - \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} -\frac{1}{3}tx_2 \\ 0 \\ -\frac{1}{3}tx_1 \\ 0 \end{pmatrix}.$$

Then we have

$$\frac{1}{2} E(J_x F(\bar{x},t)^{-1} + J_x F(\bar{x},t)^{-T}) E^\top = EE^\top = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix},$$

and therefore we still get $\mu^+(x,t) = 1$. But now we have $\|E\| = \sqrt{2}$, and further we have $\|\nabla_x c(\bar{x},t)^\top\| = \frac{t}{3}$. Thus the condition from Theorem 4

$$\frac{t}{3} = \|\nabla_x c(\bar{x},t)^\top\| < \frac{\mu^+(\bar{x},t)}{\|J_x F(\bar{x},t)^{-1}\|\,\|E\|} = \frac{1}{1\cdot\sqrt{2}}$$

is satisfied for all $t \in [0,1]$, and we obtain the nonsingularity condition. Since we can also verify the remaining assumptions of Theorem 2 we get a unique solution for QVI(1).

4 Generalized Nash Equilibrium Problems

Clearly, it is hard to check the conditions of Theorem 3 or 4 in general, mainly because they are dependent on the KKT points. However, if the functions F and g are affine linear, the difficult conditions get independent of a particular point (x,λ), and one can hope for approving them. In this section we consider generalized Nash equilibrium problems (GNEPs), where the cost functions are quadratic in the players' variables and the constraint functions are affine linear. It is known that these problems can be reformulated as a QVI with an affine function F and affine constraints g, see e.g., [8], which also provides a survey on GNEPs. It is further known that GNEPs where two players share at least one common constraint, which is active at a solution, typically do not have unique solutions. However, in the case where all players have different constraints (non-shared constraints) one might hope for uniqueness. We will consider the following GNEPs:

$$\begin{aligned}\min_{x^\nu\in\mathbb{R}^{n_\nu}} \quad & \frac{1}{2}(x^\nu)^\top Q_{\nu\nu}x^\nu + \sum_{\mu\neq\nu}(x^\nu)^\top Q_{\nu\mu}x^\mu + d_\nu^\top x^\nu \\ \text{subject to} \quad & A_{\nu\nu}x^\nu + \sum_{\mu\neq\nu} A_{\nu\mu}x^\mu - b_\nu \le 0,\end{aligned}$$

for all players $\nu = 1,\dots,N$. The matrix dimensions are $A_{\nu\mu} \in \mathbb{R}^{m_\nu\times n_\mu}$, $Q_{\nu\mu} \in \mathbb{R}^{n_\nu\times n_\mu}$ for all $\nu,\mu = 1,\dots,N$. Let us define the vectors

$$b := \begin{pmatrix} b_1 \\ \vdots \\ b_N \end{pmatrix}, \quad d := \begin{pmatrix} d_1 \\ \vdots \\ d_N \end{pmatrix},$$

the matrices

$$Q := \begin{pmatrix} Q_{11} & \dots & Q_{1N} \\ \vdots & & \vdots \\ Q_{N1} & \dots & Q_{NN} \end{pmatrix}, \quad A := \begin{pmatrix} A_{11} & \dots & A_{1N} \\ \vdots & & \vdots \\ A_{N1} & \dots & A_{NN} \end{pmatrix}, \quad E := \begin{pmatrix} A_{11} & & \\ & \ddots & \\ & & A_{NN}, \end{pmatrix}$$

and further the set

$$K(x) := \{y \in \mathbb{R}^{n_1+\dots n_N} \mid Ey + (A-E)x - b \le 0\}.$$

With these definitions the above defined GNEP is equivalent to the QVI: Find $x \in K(x)$ such that

$$\langle Qx - d, y - x\rangle \ge 0 \qquad \forall y \in K(x).$$

Next we define the family of parametrized problems. Therefore we define for $t \in [0,1]$ and a suitable constant vector $c \in \mathbb{R}^{m_1+\dots+m_N}$ (that allows to preserve LICQ) the sets

$$K(x,t) := \{y \in \mathbb{R}^{n_1+\dots n_N} \mid Ey + t(A-E)x + (1-t)c - b \le 0\}.$$

Then we obtain QVI(t): Find $x \in K(x,t)$ such that

$$\langle Qx - d, y - x\rangle \geq 0 \quad \forall y \in K(x,t).$$

Now we can show the following Corollary of Theorem 3.

Corollary 1 *Assume that at any KKT point $(\bar{x}, \bar{\lambda})$ of QVI(t) the matrix E_I has full row rank, with I being the set of active constraints. Further let Q be positive definite and $AQ^{-1}E^\top$ be positive semidefinite. Then $\partial_{(x,\lambda)}\Psi(\bar{x}(t), \bar{\lambda}(t), t)$ is nonsingular for all $t \in [0, 1)$.*

Proof The positive definiteness of Q implies that $J_x L(\bar{x}, \bar{\lambda}, t) = Q$ is nonsingular. Further we have

$$\begin{aligned}
&(\nabla_y g_I(\bar{x}, \bar{x}, t) + \nabla_x g_I(\bar{x}, \bar{x}, t))^\top J_x L(\bar{x}, \bar{\lambda}, t)^{-1} \nabla_y g_I(\bar{x}, \bar{x}, t)\\
&= (E_I + t(A_I - E_I))Q^{-1}E_I^\top\\
&= (1-t)E_I Q^{-1} E_I^\top + t A_I Q^{-1} E_I^\top.
\end{aligned}$$

Now positive definiteness of Q^{-1} together with the full row rank of E_I imply that $E_I Q^{-1} E_I^\top$ is positive definite. Since $AQ^{-1}E^\top$ is positive semidefinite this also holds true for $A_I Q^{-1} E_I^\top$. Thus for all $t \in [0, 1)$

$$(1-t)E_I Q^{-1} E_I^\top + t A_I Q^{-1} E_I^\top$$

is positive definite and hence a P-matrix. Together with the LICQ condition of Assumption 1(c), which follows from the full row rank of E_I, we can apply Theorem 3 to complete the proof. □

Note that this Corollary can not be used to obtain the uniqueness of the solution of the GNEP for the parameter $t = 1$. Here we still have to check that $A_I Q^{-1} E_I$ is a P-matrix, and this condition is independent of t and λ but implicitly dependent on x via the set of active constraints I. It is clear that this condition will not hold for GNEPs with active shared constraints, since A_I has identical rows. But it might be possible for the non-shared case. We can use Corollary 1 to get a unique solution of the perturbed GNEP (which is equivalent to QVI(t)) with Theorem 2. Therefore, one has to verify the remaining boundedness condition and then the unique solvability of QVI(0), which follows for the strictly monotone VI (Q is positive definite) if we have a bounded set.

For the QVI(t), $t \in [0, 1)$ the (semi-)definiteness conditions in Corollary 1 are independent of the KKT point $(\bar{x}, \bar{\lambda})$ and of the parameter $t \in [0, 1)$, and hence verifiable.

With Corollary 1 it is possible to obtain unique solutions of the perturbed problems QVI(t) for all $t \in [0, 1)$, even if QVI(1) does not have a unique solution, since by the perturbation there are no more shared constraints. Then using a path-following method can provide some specific solution of QVI(1). In this sense the uniqueness result may also be meaningful for GNEPs with shared constraints and it paves the way for a numerical approach.

5 Conclusion

Observing that the projection operator on variable sets often violates the contraction property, we proposed a new approach to obtain a unique solution for a QVI. We consider a family of parametrized QVIs, where the parameter t varies in [0, 1]. The problem QVI(0) can actually be a VI, for which we can guarantee a unique solution by various known conditions. QVI(1) is the problem for which uniqueness of the solution has to be shown.

Considering a nonsmooth reformulation of the KKT conditions of the QVI, which is under some differentiability and convexity assumptions together with a constraint qualification equivalent to the QVI, we were able to use Clarke's implicit function theorem to get unique solutions of QVI(t) for all $t \in [0, 1]$, and in particular for $t = 1$. The main assumption to apply the implicit function theorem is a nonsingularity condition.

We provided some matrix-dependent sufficient conditions for this nonsingularity to hold in a general setting and also in a more specific setting with linear constraints with variable right-hand side. Further, we discussed sufficient conditions in the context of GNEPs with quadratic cost and linear constraint functions, that provide uniqueness for the perturbed problems, and hence the possibility to design a path-following method for the solution of GNEPs.

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