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Structural Properties of Extended Normed Spaces

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Abstract We study some structural properties of real linear spaces equipped with norms that may take the value infinity but that otherwise satisfy the properties of conventional norms. A description is given of the finest locally convex topology weaker than the extended norm topology for which addition and scalar multiplication are jointly continuous. We also study bornologies and provide a characterization of relatively weakly compact sets in these spaces. It is shown that complemented and projection complemented closed subspaces can be different in extended Banach spaces. Particular attention is given to extended normed spaces whose subspace of vectors of finite norm has finite codimension.

Keywords Extended norm · Almost conventional extended normed space · Bornology · Quotient space · Weakly compact · Complemented closed subspace · Distance function · Lower semicontinuity · Locally convex topology

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1 Introduction

We will consider extended normed spaces as introduced in [2]. That is, given a vector space X over a field of scalars $\mathbb{F}$ (either $\mathbb{R}$ or $\mathbb{C}$) we let 0_X denote the origin of X and adopt the standard convention that $0 \cdot \infty = 0$. We say a function $\|\cdot\| : X \to [0, \infty]$ is an *extended norm* provided it satisfies the properties:

(i) $\|x\| = 0$ if and only if $x = 0_X$;
(ii) $\|\alpha x\| = |\alpha|\|x\|$ for each $x \in X, \alpha \in \mathbb{F}$;
(iii) $\|x + y\| \leq \|x\| + \|y\|$ for each $x, y \in X$.

When X is a vector space and $\|\cdot\|$ is an extended norm on X, we refer to $\langle X, \|\cdot\|\rangle$ as an *extended normed space*. When the extended norm is understood in the context, we may simply refer to X as an extended normed space. In this paper, we will restrict our attention to *real* extended normed spaces, that is, where the scalar field is $\mathbb{R}$.

The papers [2, 4, 5] present various arguments for using extended metrics and extended norms. With this in mind, [2] demonstrates that taking appropriate care, considerable information concerning an extended normed space $\langle X, \|\cdot\|\rangle$ can be obtained from the structure of $\langle X_{\text{fin}}, \|\cdot\|\rangle$ where $X_{\text{fin}} := \{x \in X : \|x\| < \infty\}$ is a linear subspace of X and, is itself, a conventional normed space. In the case $\langle X_{\text{fin}}, \|\cdot\|\rangle$ is a Banach space, we will refer to $\langle X, \|\cdot\|\rangle$ as an *extended Banach space* as this condition is equivalent to the requirement that each Cauchy sequence in X converges.

This note builds upon the work of [2] and complements [7] whose primary focus is separation of convex sets. The topics of the present note will explore a broader range of structural properties of extended normed space. In particular, we will:

– exhibit various characteristic properties of extended normed spaces where X/X_{fin} is finite dimensional that justify the appellation *almost conventional* that we bestow on them;
– give an attractive description of the finest locally convex topology with respect to which addition and scalar multiplication are jointly continuous weaker than the extended norm topology;
– characterize some bornologies including the subsets of weakly compacts, and the subsets of X on which each element of X^* is bounded.
– answer a question from [2] by showing that complemented and projection complemented closed subspaces of an extended Banach space can be different;
– examine the weak lower semicontinuity of distance functions and examine when the gap between a weakly compact convex set and a weakly closed convex set is attained.

Let X be an extended normed space, and let ϕ be a nonconstant linear functional on X. As in conventional normed linear spaces, ϕ is continuous if and only if $\text{Ker}(\phi)$ is closed [2, Corollary 4.6], and since translation is a homeomorphism, this occurs if and only if some/all level sets of ϕ are closed hyperplanes. On the other hand, in the extended norm setting, such hyperplanes can be open as well; this occurs if and only if $X_{\text{fin}} \subset \text{Ker}(\phi)$ [2, Corollary 3.9]. Also as in the conventional setting, a nonconstant linear functional ϕ is continuous as soon as it is either bounded above or below on some nonempty open set. From either property, it is clear that ϕ is bounded on the unit ball of the space so that ϕ is continuous at 0_X from which global continuity follows [2, Theorem 4.3].

When $\langle X, \|\cdot\|\rangle$ is an extended normed space, we let $B_X = \{x : \|x\| \le 1\}$, X' denote the algebraic dual of X, and X^* denote the continuous linear functionals on X. However, [2] shows the natural "operator norm" on X^* need not be a norm, but rather it is a seminorm on X^*. Nevertheless, we follow [2] and denote

$$\|\phi\|_{\text{op}} := \sup\{|\phi(x)| : x \in B_X\} \text{ for } \phi \in X^*.$$

Also, it is not hard to show, but important to keep in mind, that as soon as X is not a conventional normed space, X endowed with the extended norm topology is not a topological vector space in that scalar multiplication fails to be jointly continuous [2, Proposition 3.2]. However, X endowed with the weak topology remains a conventional locally convex topological vector space.

2 Finitely Compatible Conventional Norms

Given an extended normed space $\langle X, \|\cdot\|\rangle$, we will say $|\!|\!|\cdot|\!|\!|$ is a *finitely compatible norm* on X provided $|\!|\!|\cdot|\!|\!|$ is a conventional norm on X and when restricted to the subspace X_{fin}, $|\!|\!|\cdot|\!|\!|$ and $\|\cdot\|$ are equivalent conventional norms. While our primary interest in finitely compatible norms will be as a tool to exploit conventional theory in our explorations of extended normed spaces, the set of all finitely compatible norms will be used to characterize the finest locally convex topology with respect to which addition and scalar multiplication are jointly continuous that is weaker than the extended norm topology.

One basic feature of extended normed spaces is that an interior point of a (convex) set A need not be a core point; for example, this occurs for X_{fin} whenever $X \neq X_{\text{fin}}$. Recall that for a nonempty subset A of X, a_0 is said to be in the *core* of A if for each $h \in X$, there exists $\delta_h > 0$ so that $a_0 + th \in A$ whenever $0 \le t \le \delta_h$; in this case, we write $a_0 \in \text{core}(A)$.

Lemma 1 *Let $\langle X, \|\cdot\|\rangle$ be an extended normed space.*

(a) *There exists a finitely compatible norm $|\!|\!|\cdot|\!|\!|$ on X.*
(b) *Suppose K is a convex subset of X such that $0_X \in \text{core}(K) \cap \text{int}(K)$. Let U be the closed unit ball of a finitely compatible norm on X and set $B = K \cap (-K) \cap U$. Then μ_B is a finitely compatible norm on X where μ_B is the Minkowski functional of B (see* [8]*).*
(c) *Suppose $\phi \in X^*$ and ν is a finitely compatible norm on X, then $|\!|\!|x|\!|\!| = \nu(x) + |\phi(x)|$ defines $|\!|\!|\cdot|\!|\!|$ as a finitely compatible norm and $\phi \in (X, |\!|\!|\cdot|\!|\!|)^*$.*

Proof (a) Write $X = X_{\text{fin}} \oplus M$; then, for example, define $|\!|\!|x|\!|\!| = \|u\| + \nu(m)$ when $x = u + m$ with $u \in X_{\text{fin}}$ and $m \in M$, where ν is a conventional norm on M. For example, let $\{b_i\}_{i \in I}$ be an algebraic basis of M, and given $m \in M$, write $m = \sum_{i \in \Delta} \alpha_i b_i$ where Δ is a finite subset of I; put $\nu(m) = \sum_{i \in \Delta} |\alpha_i|$.

(b) The elementary details follow as in standard conventional theory (see [8]).

(c) The norm $|\!|\!|\cdot|\!|\!|$ is finitely compatible because ν is and ϕ is continuous with respect to $\|\cdot\|$, and is linear and homogeneous; moreover, ϕ is continuous with respect to $|\!|\!|\cdot|\!|\!|$ because $|\phi(x)| \le |\!|\!|x|\!|\!|$ for all $x \in X$. □

Of course, there are several other ways to construct finitely compatible norms, but these will be sufficient for our purposes.

Fact 1 *Suppose $\langle X, \|\cdot\|\rangle$ is an extended normed space. Then*

$$\|x\| = \sup\{|\!|\!|x|\!|\!| : |\!|\!|\cdot|\!|\!| \leq \|\cdot\| \text{ and } |\!|\!|\cdot|\!|\!| \text{ is a finitely compatible norm on } X\}$$

Proof As in the proof of Lemma 1, write $X = X_{\text{fin}} \oplus Z$ and define $|\!|\!|\cdot|\!|\!|_n$ by $|\!|\!|x|\!|\!|_n = \|u\| + n\nu(z)$ where $x = u + z$ with $u \in X_{\text{fin}}$, $z \in Z$ and ν is a conventional norm on Z. Then $|\!|\!|x|\!|\!|_n = \|x\|$ when $x \in X_{\text{fin}}$, and $|\!|\!|x|\!|\!|_n \to \infty$ when $x \notin X_{\text{fin}}$. □

We next make an elementary observation about continuous linear functionals.

Proposition 1 *Suppose $\langle X, \|\cdot\|_1\rangle$ and $\langle Y, \|\cdot\|_2\rangle$ are extended normed spaces, $T : X \to Y$ is a linear transformation, and $|\!|\!|\cdot|\!|\!|$ is a finitely compatible norm on X. If $T : \langle X, |\!|\!|\cdot|\!|\!|\rangle \to \langle Y, \|\cdot\|_2\rangle$ is continuous, then so is $T : \langle X, \|\cdot\|_1\rangle \to \langle Y, \|\cdot\|_2\rangle$. In particular, the $|\!|\!|\cdot|\!|\!|$-topology is coarser than the $\|\cdot\|_1$-topology on X and any linear functional ϕ that is continuous on $\langle X, |\!|\!|\cdot|\!|\!|\rangle$ is continuous on $\langle X, \|\cdot\|\rangle$.*

Proof This follows because a linear transformation between (extended) normed spaces X and Y is continuous if and only if it is continuous at 0_X; see [2, Theorem 4.3]. The statement on the relative coarseness of the topologies follows from the continuity of the identity map. □

When $\langle X, \|\cdot\|\rangle$ is an extended normed space and N is a norm closed subspace of X, a standard argument, as in the conventional setting, shows that $\|x + N\| := \inf\{\|x + n\| : n \in \mathbb{N}\}$ defines an extended norm on X/N. In the special case that $N = X_{\text{fin}}$ where $X \neq X_{\text{fin}}$, we may write

$$X = X_{\text{fin}} \oplus \text{span}(\{b_i : i \in I\}) \text{ where } \{b_i : i \in I\} \text{ a linearly independent subset of } X. \quad (1)$$

Following [2], we call a complementary basis as in (1) a *distance basis* for X; the motivation for this terminology comes from [2, Theorem 3.20] which shows that for each $x \in X$, there is a unique linear combination of the distance basis vectors that is a finite distance from x.

Proposition 2 *Suppose $\langle X, \|\cdot\|\rangle$ is an extended normed space such that X/X_{fin} is finite dimensional, and let $|\!|\!|\cdot|\!|\!|$ be a finitely compatible norm on X. Then the following are equivalent.*

(a) *The norm $|\!|\!|\cdot|\!|\!|$ is equivalent to $|\!|\!|\cdot|\!|\!|_1$ on X where $|\!|\!|\cdot|\!|\!|_1$ is defined by expressing X as an algebraic direct sum $X = X_{\text{fin}} \oplus M$ where M is finite dimensional and fixing a norm $\|\cdot\|_M$ on M, and then setting $|\!|\!|x + m|\!|\!|_1 = \|x\| + \|m\|_M$ for $x \in X_{\text{fin}}$ and $m \in M$.*
(b) *X_{fin} is $|\!|\!|\cdot|\!|\!|$-closed in X.*
(c) *The continuous linear functionals on $\langle X, |\!|\!|\cdot|\!|\!|\rangle$ and $\langle X, \|\cdot\|\rangle$ coincide.*

Proof (a) ⇒ (c): Proposition 1 shows that if ϕ is continuous on $\langle X, |\!|\!|\cdot|\!|\!|\rangle$, then it is continuous on $\langle X, \|\cdot\|\rangle$. Conversely, suppose $\phi \in X'$ is continuous on $\langle X, \|\cdot\|\rangle$. Then ϕ is bounded on $\{x : \|x\| \leq 1\}$ and because ϕ is linear on the finite dimensional space M, it is bounded on $\{m \in M : \|m\|_M \leq 1\}$. Therefore, ϕ is bounded on $\{x + m : \|x\| + \|m\|_M \leq 1\}$. Consequently ϕ is continuous on $\langle X, |\!|\!|\cdot|\!|\!|_1\rangle$, and thus also on $\langle X, |\!|\!|\cdot|\!|\!|\rangle$.

(c) $\Rightarrow$ (b): Suppose $x_0 \in X \setminus X_{\text{fin}}$. Then there exists $\phi \in X'$ such that $\phi(X_{\text{fin}}) = \{0\}$ and $\phi(x_0) = 1$. Then $\phi \in X^*$ since $\phi|_{X_{\text{fin}}}$ is continuous (see [2, Theorem 4.3]). Condition (c) then implies ϕ is continuous with respect to $\|\!| \cdot |\!\|$, and so X_{fin} is $\|\!| \cdot |\!\|$-closed.

(b) $\Rightarrow$ (a): Suppose condition (b) holds, and represent $X = X_{\text{fin}} \oplus M$ as in (a). Because $\|\!| \cdot |\!\|$ is finitely compatible and because all norms on M are equivalent, it follows from the triangle law that $\|\!| \cdot |\!\| \leq \beta \|\!| \cdot |\!\|_1$ for some $\beta > 0$. Now suppose there is no constant $\alpha > 0$ so that $\alpha \|\!| \cdot |\!\|_1 \leq \|\!| \cdot |\!\|$. Then there is a sequence $\langle u_n \rangle \subset X$ such that $\|\!| u_n |\!\| \to 0$ but $\|\!| u_n |\!\|_1 = 1$ for all n. Write $u_n = x_n + y_n$ where $x_n \in X_{\text{fin}}$ and $y_n \in M$ for all n. Since $\|y_n\|_M \leq 1$ for all n, by passing to a subsequence we may assume $\langle y_n \rangle \to y$ for some $y \in M$. We know $y \neq 0_X$, because if it were, then $\|x_n\| \to 0$, too. Because X_{fin} is $\|\!| \cdot |\!\|$-closed, by the conventional separation theorem we can choose a linear functional ϕ with $\|\!| \phi |\!\| \leq 1$ so that $\phi(X_{\text{fin}}) = \{0\}$ and $\phi(y) > 0$. Then $\liminf \|\!| u_n |\!\| \geq \liminf \phi(u_n) \geq \phi(y) > 0$; this contradiction completes the proof. □

We observe that the conditions in Proposition 2 do not necessarily hold for all finitely compatible norms, even when X/X_{fin} is one dimensional. Indeed, let $X = c_{00} \oplus \mathbb{R}z$ where c_{00} denotes the set of finitely supported elements in c_0 and $z = (1/2, 1/2^2, 1/2^3, \ldots)$. Then X is a linear subspace of c_0, but we endow X with the extended norm $\|\cdot\|$ defined by $\|x\| = \|x\|_\infty$ when $x \in c_{00}$ where $\|\cdot\|_\infty$ denotes the usual c_0 norm, and we put $\|x\| = \infty$ otherwise. Then $\|\cdot\|_\infty$ is a finitely compatible norm on X, but X_{fin} is not norm closed with respect to this finitely compatible norm. Moreover, the linear functional ϕ defined by $\phi(x) = t$ if $x = tz$ and $\phi(x) = 0$ for $x \in c_{00}$ is not continuous with respect to the finitely compatible norm $\|\cdot\|_\infty$, but $\phi \in X^*$ by [2, Theorem 4.3]. However, this type of example cannot occur when X_{fin} is a Banach space and X/X_{fin} is finite dimensional as will be seen in Corollary 1 below.

Recall that a linear functional ϕ is said to *properly separate* the nonempty sets A and B if $\sup_A \phi \leq \inf_B \phi$ and ϕ is not constant on $A \cup B$. A detailed exposition of proper, strict and strong separation theorems in extended normed spaces is given in [7]; the development therein uses classical convex algebraic methods as found in [10, 14]. However, separation theorems can also be derived using conventional theory in conjunction with finitely compatible norms.

Proposition 3 *Let $\langle X, \|\cdot\| \rangle$ be an extended normed space. Suppose A and B are nonempty convex subsets of X such that* $\text{core}(A) \cap \text{int}(A) \neq \emptyset$ *and* $\text{int}(A) \cap B = \emptyset$. *Then there exists $\phi \in X^*$ that properly separates A and B.*

Proof By translating A and B, we may assume $0_X \in \text{core}(A) \cap \text{int}(A)$. It follows from Lemma 1(b) that there is a finitely compatible norm $\|\!| \cdot |\!\|$ on X such that $\{x : \|\!| x |\!\| < 1\} \subset \text{int}(A)$ and the interior of A with respect to the $\|\!| \cdot |\!\|$-topology is contained in $\text{int}(A)$. By conventional theory [8, Corollary 2.3], there exists $\phi \in \langle X, \|\!| \cdot |\!\| \rangle^*$ that properly separates A and B; by Proposition 1, $\phi \in X^*$. □

Proposition 4 *Let $\langle X, \|\cdot\| \rangle$ be an extended normed space, and suppose A is a nonempty convex subset of X and $b \notin A$. Then the following are equivalent.*

(a) *There exists $\phi \in X^*$ such that $\phi(b) < \inf_A \phi$.*
(b) *There exists a convex set B such that $b \in$ $\text{core}(B) \cap \text{int}(B)$ *and* $B \cap A = \emptyset$.*

(c) *There is a finitely compatible norm* ν *on* X *such that* $d_\nu(b, A) > 0$.

Proof (a) $\Rightarrow$ (b): Let ϕ be as given in (a) and choose α such that $\phi(b) < \alpha < \inf_A \phi$. Then the set $B = \{x : \phi(x) \leq \alpha\}$ does the job.

(b) $\Rightarrow$ (c): Suppose such a set B exists. Then let $C = B - b$, and let

$$K = C \cap (-C) \cap U$$

where U is the unit ball of some finitely compatible norm on X (see Lemma 1(b)). Then letting $\nu = \mu_K$, the Minkowski functional of K, does the job.

(c) $\Rightarrow$ (a): According to the conventional strict separation theorem there is a linear functional $\phi \in \langle X, \nu\rangle^*$ such that $\phi(b) < \inf_A \phi$; Proposition 1 ensures $\phi \in \langle X, \|\cdot\|\rangle^*$. □

We conclude this section by examining the finest locally convex topology coarser than the extended norm topology. For this, suppose $\langle X, \|\cdot\|\rangle$ is an extended normed linear space where $X \neq X_{\text{fin}}$. While there is a local base of convex neighborhoods at each point, as we know, scalar multiplication is not jointly continuous with respect to the extended norm topology, and so such a space cannot be a locally convex topological vector space. Nevertheless, there is a finest locally convex topology with respect to which addition and scalar multiplication are jointly continuous on X coarser than the norm topology that we now describe. We shall henceforth refer to such topologies simply as locally convex topologies as understood in the conventional lexicon.

Let P_X be the set of finitely compatible norms for X. Since P_X is stable under multiplication by positive scalars and $\{\nu_1, \nu_2\} \subset P_X \Rightarrow \max\{\nu_1, \nu_2\} \in P_X$, all sets of the form

$$\{x \in X : \nu(x) < 1\}, \quad \nu \in P_X,$$

form a local base at the origin for the coarsest locally convex topology with respect to which each member of P_X is continuous [13, p. 15]. By Proposition 1, this topology, which we now denote by τ_{P_X}, is coarser than the extended norm topology.

Theorem 1 *Let* $\langle X, \|\cdot\|\rangle$ *be an extended normed linear space where* $X \neq X_{fin}$. *Then* τ_{P_X} *is the finest locally convex topology on* X *coarser than the extended norm topology. Further, each element of* X^* *is continuous with respect to* τ_{P_X}.

Proof Let τ be a locally convex topology on X coarser than the extended norm topology, and let V be an absolutely convex (absorbing) τ-neighborhood of 0_X. Fix $\nu \in P_X$; by statement (b) of Lemma 1, the Minkowski function μ for $V \cap \{x : \nu(x) \leq 1\}$ belongs to P_X and clearly

$$\{x : \mu(x) < 1\} \subset V \cap \{x : \nu(x) \leq 1\} \subset V.$$

Since such τ-neighborhoods form a local base for τ at 0_X, this proves that $\tau \subset \tau_{P_X}$.

For the second claim, let $\phi \in X^*$, let $\{b_i\}_{i\in I}$ be an algebraic basis for a subspace M complementary to X_{fin}, and put $\lambda_i = \phi(b_i)$ for $i \in I$. We next define a conventional norm $\nu(\cdot)$ on M: for each $m \in M$ other than 0_X, there is a finite subset Δ of I such that $m = \sum_{i\in\Delta} \alpha_i b_i$; put $\nu(m) = \sum_{i\in\Delta} (|\lambda_i| + 1)|\alpha_i|$. Now define $\mu \in P_X$ by $\mu(x) = \|u\| + \nu(m)$ where $x = u + m$ and where $u \in X_{\text{fin}}$ and $m \in M$. We will be done if we can show that ϕ is $\mu(\cdot)$-continuous.

To this end, suppose $\mu(x) \leq 1$ so that both $\|u\| \leq 1$ and $\nu(x) \leq 1$. We compute

$$\begin{aligned}|\phi(x)| &\leq |\phi(u)| + |\phi(m)| \leq \|\phi\|_{\rm op} + |\sum_{i\in\Delta} \lambda_i \alpha_i| \\ &\leq \|\phi\|_{\rm op} + \sum_{i\in\Delta} |\lambda_i\|\alpha_i| \leq \|\phi\|_{\rm op} + \nu(m) \leq \|\phi\|_{\rm op} + 1.\end{aligned}$$

We have shown that ϕ is $\mu(\cdot)$-continuous as required. □

We conclude this section with a result that shows Proposition 2 can be sharpened considerably when $X_{\rm fin}$ is a Banach space.

Corollary 1 *Suppose $\langle X, \|\cdot\|\rangle$ is an extended Banach space. The following are equivalent.*

(a) *$X/X_{\rm fin}$ is finite dimensional.*
(b) *All finitely compatible norms for X are equivalent.*
(c) *The topology τ_{P_X} is normable.*

Proof (a) ⇒ (b): $X_{\rm fin}$ is a Banach space with respect to each finitely compatible norm $\|\!|\cdot|\!\|$ and is thus closed with respect to each compatible norm topology, and so $\|\!|\cdot|\!\|$ is equivalent to $\|\!|\cdot|\!\|_1$ on X as described in condition (a) of Proposition 2.

(b) ⇒ (c): This is immediate from the definition of τ_{P_X}.

(c) ⇒ (a): Suppose $X_{\rm fin}$ has infinite codimension and $\|\!|\cdot|\!\|$ is the finitely compatible norm whose topology is τ_{P_X}. Write $X = X_{\rm fin} \oplus M$ and let ϕ_0 be a $\|\!|\cdot|\!\|$-discontinuous linear functional on M. Extending ϕ_0 to ϕ on X by letting it be zero on $X_{\rm fin}$, we get $\phi \in \langle X, \|\cdot\|\rangle^*$ while $\phi \notin \langle X, \|\!|\cdot|\!\|\rangle^* = \langle X, \tau_{P_X}\rangle^*$. This contradicts Theorem 1. □

3 Almost Conventional Extended Normed Spaces

It was shown in [7, Corollary 2.8] that two nonempty convex sets A and B in a finite dimensional extended normed space X can be separated properly by some $\phi \in X^*$ provided $\text{int}(A) \neq \emptyset$ and $\text{int}(A) \cap B \neq \emptyset$. On the other hand [7, Proposition 3.1] showed such a result can fail in infinite dimensional extended normed spaces. The next result shows that the natural proper separation theorem is valid precisely when $X/X_{\rm fin}$ is finite dimensional. Because of this and other nice properties shared by this class of spaces (e.g. Proposition 2 and Corollary 1), we will say an extended normed space X is *almost conventional* if $X/X_{\rm fin}$ is finite dimensional. In the following, we denote the *gap* between nonempty sets A and B by $d(A, B)$ where

$$d(A, B) := \inf\{\|a - b\| \ : \ a \in A, \ b \in B\}.$$

Theorem 2 *Let X be an extended normed space with $X_{\rm fin} \neq X$. Then the following are equivalent.*

(a) *$X/X_{\rm fin}$ is finite dimensional, that is, X is almost conventional.*
(b) *Nonempty convex subsets A and B of X can be separated properly by some $\phi \in X^*$ whenever $\text{int}(A) \neq \emptyset$ and $\text{int}(A) \cap B = \emptyset$.*
(c) *Disjoint nonempty convex sets A and B in X can be separated properly by some $\phi \in X^*$ whenever $\text{int}(A) \neq \emptyset$.*
(d) *Nonempty convex subsets A and B of X can be separated properly by some $\phi \in X^*$ whenever $d(A, B) > 0$.*

(e) *A nonempty convex subset A and point b in X can be separated properly by some $\phi \in X'$ whenever $d(b, A) = \infty$.*
(f) *Whenever A is a convex subset of X with $0_X \in \text{int}(A)$ and $\text{span}(A) = Y$, one has $\text{core}(A) \neq \emptyset$ when the core is with respect to the overspace Y.*

Proof (a) $\Rightarrow$ (f): Because $0_X \in \text{int}(A)$, $X_{\text{fin}} \subset Y$. In the event $Y = X_{\text{fin}}$, the result is clear. Thus we write $Y = X_{\text{fin}} \oplus \text{span}(\{e_1, e_2, \ldots, e_n\})$ where $\{e_1, e_2, \ldots, e_n\} \subset A$ is a linearly independent subset of X, and $n \geq 1$. Let $x_0 = \sum_{i=1}^n \frac{1}{3n} e_i$. We will show that $x_0 \in \text{core}(A)$ when the core is considered with respect to Y. Because $0_X \in \text{int}(A)$ we choose $\delta > 0$ so that $x \in A$ whenever $\|x\| < \delta$. Now for any h with $\|h\| < \delta/3$ and for any $(\alpha_i)_{i=1}^n$ with $|\alpha_i| \leq 1/3n$, one has

$$x_0 + h + \sum_{i=1}^n \alpha_i e_i = \frac{1}{3}(3h) + \left(\frac{2}{3} - \sum_{i=1}^n \left(\frac{1}{3n} + \alpha_i\right)\right) 0_X + \sum_{i=1}^n \left(\frac{1}{3n} + \alpha_i\right) e_i \qquad (2)$$

is in A since the right hand side is a convex combination of elements in A.

Now let $y \in Y$ be arbitrary. We write y as $x + \sum_{i=1}^n k_i e_i$ where $x \in X_{\text{fin}}$. Choose $\lambda > 0$ sufficiently small so that $\|\lambda x\| < \delta/3$ and $|\lambda k_i| \leq \frac{1}{3n}$ for $1 \leq i \leq n$. Then, using (2) we have

$$x_0 + ty = x_0 + tx + \sum_{i=1}^n t k_i e_i \in A \ \text{ whenever } 0 \leq t \leq \lambda.$$

Thus $x_0 \in \text{core}(A)$ when the core is considered with respect to the overspace Y.

(f) $\Rightarrow$ (c): Suppose A and B are disjoint nonempty convex subsets of X with $\text{int}(A) \neq \emptyset$. Then $A - B$ is a nonempty convex subset of X with $\text{int}(A - B) \neq \emptyset$. Let $x_0 \in \text{int}(A - B)$ and let $C = A - B - x_0$. Then $0_X \in \text{int}(C)$ and $-x_0 \notin C$ because $0_X \notin A - B$. Now let $Z = \text{span}(C)$. Then X_{fin} is a subspace of Z and the condition in (f) ensures that $\text{core}(C) \neq \emptyset$ when C is considered as a subset of Z and the core is taken with respect to Z. Now apply Proposition 3 to C, $\{-x_0\}$ as subsets of Z to find a linear functional, say ϕ, that is continuous on Z and properly separates C and $\{-x_0\}$. Take any linear extension $\tilde{\phi}$ of ϕ to all of X. Then $\tilde{\phi}$ is continuous because $\tilde{\phi}|_{X_{\text{fin}}} = \phi|_{X_{\text{fin}}}$ is continuous (see [2, Theorem 4.3]). It is now easy to check that $\tilde{\phi}$ properly separates A and B.

(c) $\Rightarrow$ (b): Suppose A and B are convex sets in X with $\text{int}(A) \neq \emptyset$ and $\text{int}(A) \cap B = \emptyset$. Without loss of generality, we may assume $0_X \in \text{int}(A)$. We apply (c), to properly separate $\text{int}(A)$ and B with some continuous linear functional ϕ. Say $\sup_{\text{int}(A)} \phi \leq k \leq \inf_B \phi$ where necessarily $k \geq 0$. Choose $\delta > 0$ so that $\delta B_X \subset \text{int}(A)$. Now for any $x \in A \setminus \text{int}(A)$, the convexity of A ensures that

$$\lambda x + (1 - \lambda)\delta B_X \subset A \ \text{ for } \ 0 < \lambda < 1.$$

Therefore, $\lambda x \in \text{int}(A)$ for $0 < \lambda < 1$. Now the linearity of ϕ implies

$$\phi(x) = \lim_{\lambda \to 1^-} \lambda \phi(x) = \lim_{\lambda \to 1^-} \phi(\lambda x) \leq k.$$

Therefore, $\sup_A \phi \leq k$ and so ϕ properly separates A and B.

(b) $\Rightarrow$ (d): Suppose $d(A, B) > 0$. Then fix $\delta > 0$ with $\delta < d(A, B)$ and let $C = A + \delta B_X$. Then $\text{int}(C) \neq \emptyset$ while $\text{int}(C) \cap B = \emptyset$, so by (b) some $\phi \in X^*$ properly separates C and B, say $\sup_C \phi \leq \inf_B \phi$. If $\|\phi\|_{\text{op}} = 0$, then $\phi(\delta B_X) = \{0\}$, and so $\phi(A \cup B) = \phi(C \cup B)$, and we conclude ϕ is not constant on $A \cup B$. If $\|\phi\|_{\text{op}} > 0$ then $\sup_A \phi + \delta \|\phi\|_{\text{op}} \leq \inf_B \phi$. In either case, ϕ properly separates A and B.

Now (d) $\Rightarrow$ (e) is trivial, so we conclude the proof by establishing (e) $\Rightarrow$ (a) by contraposition. For this, suppose the quotient X/X_{fin} is infinite dimensional. Write $X = X_{\text{fin}} \oplus Z$

where Z is infinite dimensional. It follows [7, Proposition 3.1] there is a nonempty convex set $A \subset Z$ with $0_X \notin A$ such that A and 0_X cannot be properly separated by any linear functional ϕ on Z. Thus 0_X and A cannot be properly separated by any linear functional in X'. Because the extended norm topology on Z is discrete, we deduce that $\|x - y\| = \infty$ whenever $x, y \in Z$ and $x \neq y$; see [2]. Then $d(0_X, A) = \infty$ and we are done. □

The next two observations concern the preservation of spaces that are almost conventional.

Proposition 5 *Let X be an almost conventional extended normed space.*

(a) *Each linear subspace of X is almost conventional.*
(b) *Each extended normed space Y that is the image of X under a continuous linear transformation T is almost conventional.*
(c) *Each quotient of X by a closed subspace F is almost conventional.*

Proof Statement (a) is obvious. For (b), if either X or Y is conventional, then the statement is clearly true. Otherwise, write $X = X_{\text{fin}} \oplus M$ where $\dim(M) < \infty$. By continuity, $T(X_{\text{fin}}) \subset Y_{\text{fin}}$, and by surjectivity of T,

$$Y = T(X_{\text{fin}}) + T(M) \subset Y_{\text{fin}} + T(M).$$

This means that $Y = Y_{\text{fin}} + T(M)$, and so $T(M)$ must contain a (finite dimensional) subspace N with $Y = Y_{\text{fin}} \oplus N$. Clearly, Y/Y_{fin} has the same dimension as N. Finally, statement (c) is a consequence of statement (b), as the quotient is the continuous image of X since $\|x + X_{\text{fin}}\| \leq \|x\|, x \in X$. □

Proposition 6 *Let $(X, \|\cdot\|_1)$ and $(Y, \|\cdot\|_2)$ be almost conventional extended normed spaces and suppose $W = X \oplus Y$, equipped with the norm $\|w\| := \|x\|_1 + \|y\|_2$ where $w = x + y$ with $x \in X$ and $y \in Y$. Then W is almost conventional.*

Proof Put $X = X_{\text{fin}} \oplus M$ and $Y = Y_{\text{fin}} \oplus N$. Any element in W and be expressed uniquely as $(x + m) + (y + n)$ where $x \in X_{\text{fin}}$, $y \in Y_{\text{fin}}$, $m \in M$ and $n \in N$. Since $\|x + m\|_1 < \infty$ if and only if $m = 0_X$ and $\|y + n\|_2 < \infty$ if and only if $n = 0_Y$, we have $W_{\text{fin}} = X_{\text{fin}} + Y_{\text{fin}}$. Thus, W_{fin} has finite codimension and the result follows. □

Our next characterization of almost conventional normed linear spaces involves the topology of uniform convergence on bounded subsets τ_{ucb} for continuous linear transformations. As we would like the bounded subsets of an extended normed space to (1) be stable under finite unions and (2) contain the norm compact subsets, it is not appropriate to require that a bounded set be contained in a ball, i.e, that $\text{diam}(A) < \infty$. Rather, we will say that A is *bounded* provided that A is contained in a finite union of balls, as in [2]. This definition has also been used more generally in the context of extended metric spaces [4, 5]. Of course, for conventional normed spaces, this reduces to the standard notion, but there are other possibilities that do so as well that we will touch on later. Note that our definition aligns well with total boundedness as it is normally understood.

Given extended normed linear spaces $\langle X, \|\cdot\|_1\rangle$ and $\langle Y, \|\cdot\|_2\rangle$, τ_{ucb} is a locally convex topology on the continuous linear transformations $\mathbf{B}(X, Y)$ from X to Y having a local base at the zero transformation consisting of all sets of the form $\{T : T(B) \subset \varepsilon B_Y\}$ where B runs over the bounded subsets of X and $\varepsilon > 0$. It can be shown that τ_{ucb} convergence occurs for a sequence or net in $\mathbf{B}(X, Y)$ if and only if there is operator norm convergence plus pointwise convergence [2, Theorem 4.11] .

Adjoining linearly independent vectors $\{b_i : i \in I\}$ to a basis for X_{fin} to produce a distance basis for X, it is clear that each bounded subset of X lies in a finite union of balls of the form $A + \mu B_X$ where A is a finite subset of span $(\{b_i : i \in I\})$ and $\mu > 0$ (see [2]).

Theorem 3 *Let $\langle X, \|\cdot\|_1\rangle$ be an extended normed linear space. The following conditions are equivalent.*

(a) $\langle X, \|\cdot\|_1\rangle$ *is almost conventional.*
(b) *For each target space $\langle Y, \|\cdot\|_2\rangle$, $\langle \boldsymbol{B}(X, Y), \tau_{ucb}\rangle$ is normable.*
(c) $\langle X^*, \tau_{ucb}\rangle$ *is normable.*

Proof (a) $\Rightarrow$ (b): If X is conventional, there is nothing to prove. Otherwise let $\{b_1, b_2, \ldots, b_n\}$ be a distance basis. We claim that

$$\|T\|_{\text{ucb}} := \|T(b_1)\|_2 + \|T(b_2)\|_2 + \cdots + \|T(b_n)\|_2 + \|T\|_{\text{op}}$$

is a compatible norm for $\mathbf{B}(X, Y)$ equipped with τ_{ucb}. It is clear that $\|\cdot\|_{\text{ucb}}$ is a norm, in particular, if T is not the zero transformation, then $\|T\|_{\text{ucb}} \neq 0$, as either some point of B_X or some b_i is not in the kernel.

We first show that the norm topology is finer than τ_{ucb}. From our earlier discussion, it suffices to show that if $A = \{a_1, a_2, \ldots, a_k\} \subset \text{span}(\{b_i : i \leq n\})$ and $\mu > 0$ and $\varepsilon > 0$, then $\{T : T(A + \mu B_X) \subset \varepsilon B_Y\}$ contains some norm ball about the zero transformation. Write $a_j = \sum_{i=1}^{n} \beta_{ij} b_i$ for $j = 1, 2, \ldots, k$ and put $\beta = \max\{|\beta_{ij}| : j \leq k, i \leq n\} + 1$. If $\|T\|_{\text{ucb}} < \varepsilon(\beta + \mu)^{-1}$, then for $j \leq k$ and $x \in \mu B_X$, we get

$$\left\| T\Big(\sum_{i=1}^{n} \beta_{ij} b_i + x\Big)\right\|_2 \leq \beta(\|T(b_1)\|_2 + \|T(b_2)\|_2 + \cdots + \|T(b_n)\|_2) + \mu\|T\|_{op} < \varepsilon.$$

The other inclusion is easier: given $\varepsilon > 0$,

$$\{T : \|T\|_{\text{ucb}} < \varepsilon\} \supseteq \bigcap_{i=1}^{n} \{T : T(b_i) \subset \frac{\varepsilon}{2n} B_Y\} \cap \{T : T(B_X) \subset \frac{\varepsilon}{3} B_Y\}.$$

(b) $\Rightarrow$ (c): This is trivial.

(c) $\Rightarrow$ (a): Suppose (a) fails, that is, X has an infinite distance basis $\{b_i : i \in I\}$. To show that (c) fails as well, we show each τ_{ucb}-neighborhood of the zero transformation fails to be absorbed by some other neighborhood [10, p. 56]. To this end it suffices to consider all neighborhoods of the form $\{\phi \in X^* : \phi(A + \mu B_X) \subset (-\varepsilon, \varepsilon)\}$ where A is a finite subset of span$(\{b_i : i \in I\})$, $\mu > 0$, and $\varepsilon > 0$. Denote this neighborhood by $U(A, \mu, \varepsilon)$. Since each vector in A is a finite linear combination of distance basis elements, there exist $i_0 \in I$ such that b_{i_0} arises from no linear combination of $\{b_i : i \in I\}$ yielding the points of A. Hence no scalar multiple of $\{\phi \in X^* : \phi(b_{i_0}) \in (-1, 1)\}$ contains $U(A, \mu, \varepsilon)$, because $\phi_n \in X^*$ defined by

$$\phi_n(x) = \begin{cases} n\alpha & \text{if } x = \alpha b_{i_0} \\ 0 & \text{otherwise} \end{cases}$$

satisfies $\phi_n(A + \mu B_X) \subset (-\varepsilon, \varepsilon)$ for each n. $\square$

A locally convex Hausdorff topology is of course metrizable if it has a countable local base at the origin. It is not hard to show that $\langle \mathbf{B}(X, Y), \tau_{\text{ucb}}\rangle$ is metrizable for all target spaces Y if and only X has a countable distance basis.

4 Bornologies in Extended Normed Spaces

In this section we study the large structure of extended normed spaces, as captured by bornologies (see, e.g., [3, 4, 6, 9, 11, 12]).

Definition 1 Let X be a nonempty set. By a *bornology* on on X, we mean a family $\mathscr{B}$ of nonempty subsets that contains the singles, is stable under finite unions, and that is stable under taking nonempty subsets.

A representative set of bornologies in the context of topological spaces are the following:

- the finite nonempty subsets of a topological space X, which is the smallest bornology on X;
- the nonempty subsets of a topological space X each of which is contained in some compact subset; in our discussion, X will be a Hausdorff space, and so this is the bornology of *relatively compact sets* in that their closures are compact;
- the nonempty subsets of a topological space each of which is bounded with respect to each member of a family $\{f_i : i \in I\}$ of real-valued functions on X;
- the nonempty subsets of a metrizable space that are metrically bounded subsets with respect to some admissible metric, as characterized by Hu [11];
- the nonempty subsets of a metrizable space that form the metrically totally bounded subsets with respect to some admissible metric, as characterized by Beer, Costantini and Levi [6].
- the bounded nonempty subsets of an extended normed space as we have defined them in the previous section.

In this section we intend to characterize the following bornologies in a general real extended normed space: (1) the bornology of nonempty relatively norm compact sets; (2) the bornology of nonempty relatively weakly compact sets; (3) the bornology of X^*-bounded nonempty subsets (we say a nonempty subset A of X is X^*-*bounded* provided $\forall \phi \in X^*, \phi(A)$ is a bounded set of reals).

We first look at the family of nonempty relatively norm compact sets. We use the following facts discussed in [2]: For a fixed $x_0 \in X$, the flat $x_0 + X_{\text{fin}}$ is the equivalence class of x_0 under the equivalence relation $x \sim y$ provided $\|x - y\| < \infty$ and is called the *metric component* of x_0. The family of flats $\{x + X_{\text{fin}} : x \in X\}$ partitions X into clopen subsets. Such flats also enumerate the connected components of the space.

Theorem 4 *Let A be a nonempty subset of an extended normed space X. Then A is a relatively norm compact set if and only if for some norm compact subset K of X_{fin} and some finite subset $\{x_1, x_2, \ldots, x_n\}$ of X we have $A \subset K + \{x_1, x_2, \ldots, x_n\}$.*

Proof The conditions are sufficient because compact sets are stable under finite unions and translation is a homeomorphism. For necessity, let A_0 be a norm compact superset of A. Since $\{x + X_{\text{fin}} : x \in X\}$ is an open cover of A_0, there exists a finite subfamily $\{x_j + X_{\text{fin}} :$

$1 \le j \le n\}$ that covers A_0 and such that each flat hits A_0. Put $K_j = (x_j + X_{\text{fin}}) \cap A_0$ for $1 \le j \le n$, a norm compact set as well. Then $\bigcup_{j=1}^{n}(-x_j + K_j)$ is norm compact subset of X_{fin} and

$$A \subset A_0 \subset \bigcup_{j=1}^{n}(-x_j + K_j) + \{x_1, x_2, \ldots, x_n\},$$

as required. □

To characterize the X^*-bounded subsets (respectively, the relatively weakly compact sets), we state and prove a preliminary lemma.

Lemma 2 *Let A be an X^*-bounded subset of an extended normed space X not contained in X_{fin}. Then there exists a maximal finite linearly independent subset $\{b_1, b_2, \ldots, b_n\}$ of A such that $\forall j \le n$, $\|b_j\| = \infty$ and such that*

$$A \subset X_{\text{fin}} + \text{span}(\{b_1, b_2, \ldots, b_n\}) = X_{\text{fin}} \oplus \text{span}(\{b_1, b_2, \ldots, b_n\}).$$

Proof Consider $\mathscr{B} = \{B : B$ is a linearly independent subset of A with $\text{span}(B) \cap X_{\text{fin}} = \{0_x\}\}$. Since A contains an element of infinite norm, $\mathscr{B}$ is nonempty and by Zorn's lemma, $\mathscr{B}$ has a maximal element B_0. If B_0 were infinite, we could find a linear functional ϕ on X mapping X_{fin} to 0 and B_0 onto $\mathbb{N}$. Since ϕ restricted to X_{fin} is continuous, ϕ is continuous on X by [2, Theorem 4.3]. Clearly, ϕ is not bounded on A, contradicting X^*-boundedness. Now writing $B_0 = \{b_1, b_2, \ldots, b_n\}$, by construction we have $X_{\text{fin}} + \text{span}(\{b_1, b_2, \ldots, b_n\}) = X_{\text{fin}} \oplus \text{span}(\{b_1, b_2, \ldots, b_n\})$. To see that $A \subset X_{\text{fin}} + \text{span}(\{b_1, b_2, \ldots, b_n\})$, let $a \in A$ be arbitrary. If either $a \in X_{\text{fin}}$ or $a \in \text{span}(B_0)$, we are done. Otherwise a has infinite norm and X_{fin} nontrivially intersects span $(B_0 \cup \{a\})$ which allows one to express a as a linear combination of some nonzero element of X_{fin} and elements of B_0. □

As a side observation, the previous lemma leads to another characterization of almost conventional spaces.

Proposition 7 *Let X be an extended normed space. The following conditions are equivalent.*

(a) *X is almost conventional.*
(b) *X contains an absorbing absolutely convex set A on which each element of X^* is bounded.*
(c) *X contains an absorbing convex set A on which each element of X^* is bounded.*

Proof (a) $\Rightarrow$ (b): Let $\|\cdot\|_1$ be a finitely compatible (sum) norm as in Proposition 2(a), then $\langle X, \|\cdot\|\rangle^* = \langle X, \|\cdot\|\rangle^*$ and so $A = \{x \in X : \|x\| \le 1\}$ does the job.

(b) $\Rightarrow$ (c): This is trivial.

(c) $\Rightarrow$ (a): Because $\text{span}(A) = X$, Lemma 2 implies X/X_{fin} is finite dimensional. □

The following provides a characterization of X^*-bounded sets.

Theorem 5 *Let $\langle X, \|\cdot\|\rangle$ be an extended normed space, and suppose A is a nonempty subset of X. Then A is X^*-bounded if and only if there exists a ball B in X_{fin} and a polytope P in X such that $A \subset B + P$.*

Proof Suppose A is an X^*-bounded subset of X. In the case $A \subset X_{\text{fin}}$ the result follows from conventional theory. Otherwise, by Lemma 2 we know $A \subset Z$ where the subspace Z can be expressed as the direct sum $Z = X_{\text{fin}} \oplus M$ and M is a finite dimensional subspace of X. Next, we endow Z with the finitely compatible norm $|\!|\!| \cdot |\!|\!|$ defined as follows. For $z \in Z$, we write $z = x + m$ with $x \in X_{\text{fin}}$ and $m \in M$ and we define $|\!|\!| z |\!|\!| = \|x\| + \|m\|_M$ where $\| \cdot \|_M$ is a norm on M. It follows from Proposition 1, any $\phi \in Z'$ that is continuous with respect to $|\!|\!| \cdot |\!|\!|$ can be written $\phi = \psi|_Z$ where $\psi \in X^*$. Thus A is $\langle Z, |\!|\!| \cdot |\!|\!| \rangle^*$-bounded. Let $\langle \widetilde{Z}, |\!|\!| \cdot |\!|\!| \rangle$ be the completion of $\langle Z, |\!|\!| \cdot |\!|\!| \rangle$. Then $\langle \widetilde{Z}, |\!|\!| \cdot |\!|\!| \rangle^* = \langle Z, |\!|\!| \cdot |\!|\!| \rangle^*$ and so A is weakly bounded in the Banach space $\langle \widetilde{Z}, |\!|\!| \cdot |\!|\!| \rangle$. The Banach-Steinhaus theorem in conventional theory (see [8, Theorem 3.15]) ensures that A is $|\!|\!| \cdot |\!|\!|$-bounded. Because M is finite dimensional, any bounded subset of M lies in a polytope, thus we conclude that $A \subset B + P$ where B is a ball in X_{fin} and P is a polytope in M.

Conversely, suppose $A \subset B + P$ where B is ball in X_{fin} and P is a polytope. Then A is a subset of some subspace $Z \supset X_{\text{fin}}$ such that Z/X_{fin} is finite dimensional. Next we write $Z = X_{\text{fin}} \oplus M$ and consider the finitely compatible norm $|\!|\!| \cdot |\!|\!|$ on $X_{\text{fin}} \oplus M$ as defined in the previous paragraph. It follows that A is $|\!|\!| \cdot |\!|\!|$-bounded in Z. Thus, given any $\phi \in X^*$, Proposition 2 ensures that $\phi|_Z$ is continuous with respect to $|\!|\!| \cdot |\!|\!|$. Consequently, $\phi(A)$ is bounded for every $\phi \in X^*$. □

Theorem 6 *Let $\langle X, \|\cdot\| \rangle$ be an extended normed space, and suppose A is a nonempty subset of X. Then A is relatively weakly compact if and only if there is a weakly compact subset K of X_{fin} and a polytope P such that $A \subset K + P$.*

Proof Suppose A is a relatively weakly compact subset of X. Then $A \subset W$ where W is weakly compact. As in the proof Theorem 5, using Lemma 2 we know $W \subset Z$ where $Z = X_{\text{fin}} \oplus M$ and M is a finite dimensional subspace of X. Let $P_1 : Z \to Z$ and $P_2 : Z \to Z$ be the canonical projections such that $P_1(Z) = X_{\text{fin}}$ and $P_2(Z) = M$. It follows from [2, Theorem 4.3] that P_1 and P_2 are continuous on Z with respect to the inherited extended norm since they are continuous at 0_Z. Consequently, for any $\phi \in Z^*$, $\phi \circ P_1$ and $\phi \circ P_2$ are continuous linear functionals on Z, and then it is straightforward to check that P_1 and P_2 are weak-to-weak continuous. Therefore, $P_1(W) \subset X_{\text{fin}}$ and $P_2(W) \subset M$ are weakly compact sets, and $W \subset P_1(W) + P_2(W)$. Finally, $P_2(W)$ is norm compact in M because M is finite dimensional, and so $P_2(W)$ is a subset of some polytope P.

For the converse, observe that the polytope P is weakly compact. Also, X is a topological vector space in its weak topology and so the sum of weakly compact sets is weakly compact, and thus A is a subset of the weakly compact set $K + P$. □

Because closed balls are weakly compact in reflexive Banach spaces, the previous two theorems immediately yield the following result.

Corollary 2 *Suppose X is an extended Banach space such that X_{fin} is reflexive. Then a subset of X is X^*-bounded if and only if it is relatively weakly compact.*

In view of the direct sum described in Lemma 2, let us denote a polytope P as given in Theorem 5 by $\text{conv}(\{\rho b_1, -\rho b_1, \ldots, \rho b_n, -\rho b_n\})$. Then:

– any two distinct points of the polytope lie an infinite distance apart, i.e, the relative norm topology on P is discrete;

– each element of our X^*-bounded set A can be written uniquely as a sum of an element of P plus a vector of finite norm (actually of norm at most λ).

In consideration of these points, each metric component of X either does not hit A or has nonempty intersection with A which is contained in a ball. Thus, A is weakly bounded as defined in [2, 4]. In fact A is *uniformly weakly bounded*, in that the trace of A on each metric component is contained in a ball of some common radius. Clearly the uniformly weakly bounded sets as we have just defined them form a bornology that lies between the bounded subsets and the weakly bounded subsets as studied in [2, 4].

This yields the following bornological characterization of subsets of weakly compact bounded sets.

Corollary 3 *Let X be an extended normed space. Then the bornology of subsets of X consisting of nonempty subsets of weakly compact bounded subsets is exactly*

$$\{A \subset X : A \neq \emptyset \text{ and } \exists K \subset X_{\text{fin}} \text{ weakly compact } \exists F \text{ finite with } A \subset K + F\}.$$

Proof Suppose A is weakly compact and bounded. By definition A is contained in a finite union of balls and thus can hit only finitely many metric components, say $x_1 + X_{\text{fin}}, x_2 + X_{\text{fin}}, \ldots, x_n + X_{\text{fin}}$ where, without loss of generality, we may assume $x_j \in A$ for $j \leq n$. Since flats are weakly closed by [7, Theorem 3.11], for each $j \leq n$, $A \cap (x_j + X_{\text{fin}})$ is weakly compact. Put $K_j = (A \cap (x_j + X_{\text{fin}})) - x_j$; then K_j is a weakly compact subset of X_{fin} and we have $A \subset (\bigcup_{j=1}^{n} K_j) + \{x_1, x_2, \ldots, x_n\}$ as required.

Conversely suppose $A \subset K + F$ where K is a weakly compact subset of X_{fin} and $F = \{x_1, x_2, \ldots, x_n\}$. Since translation is continuous with respect to the weak topology, $\forall j \leq n, x_j + K$ is weakly compact and so $K + F = \bigcup_{j=1}^{n}(x_j + K)$ is weakly compact. From conventional linear analysis, K is contained in a single ball, so $K + F$ is contained in a finite union of balls. □

5 Complemented Subspaces

Let M and N be complementary closed subspaces of the extended normed linear space $\langle X, \|\cdot\|\rangle$. Following [2], we say that M is a *projection complement* of N if the mapping $m + n \mapsto m$ where $m \in M$ and $n \in N$ is continuous. In a conventional Banach space, it is well known that closed subspaces M and N are complemented if and only if M is a projection complement of N; see [8, Proposition 5.3]. However, [2, Example 6.2] exhibits complemented norm closed subspaces of a two-dimensional extended Banach space where the projection map fails to be continuous. Nevertheless, each of the subspaces in that example has a projection complement because any finite dimensional subspace of an extended normed space has a projection complement [2, Proposition 6.6]. This leads to an intriguing question posed in [2]: if a closed subspace of an extended Banach space has a closed complement, does it have a projection complement as well?

To answer this question we need the following two results from [2] that we state for reference.

Proposition 8 (Proposition 3.15, [2]) *A linear subspace N of an extended normed space $\langle X, \|\cdot\|\rangle$ is closed if and only if $N \cap X_{\text{fin}}$ is closed.*

Theorem 7 (Theorem 6.9, [2]) *Let N be a closed linear subspace of an extended Banach space $\langle X, \|\cdot\|\rangle$. Then N has a projection complement if and only if there is a closed subspace W of X_{fin} with $X_{\text{fin}} = W \oplus (N \cap X_{\text{fin}})$.*

Building upon [2, Example 6.2] we provide an example in an extended Banach space of a closed subspace with a closed complement that has no projection complement.

Example 1 Let $\langle Y, \|\cdot\|_Y\rangle$ be a Banach space that is not isomorphic to a Hilbert space. Then Y contains a closed subspace V that is not complemented; see [8, Theorem 5.7]. Now write $Y = V \oplus W$ as an algebraic direct sum where W is a linear subspace of Y that is necessarily not closed. Now let X be the vector space $Y \times W$ where we will write $x \in X$ as $x = (y, w)$ with $y \in Y$ and $w \in W$. We define the extended norm, $\|\cdot\|$, on X by

$$\|(y, w)\| = \begin{cases} \|y\|_Y & \text{if } w = 0_W, \\ \infty & \text{otherwise.} \end{cases}$$

Notice that $X_{\text{fin}} = Y \times \{0_W\}$. We let $M = \{(w, w) \in X : w \in W\}$ and $N = \{(v, w) \in X : v \in V,\ w \in W\}$. Then M is a subspace of X such that $M \cap X_{\text{fin}} = \{0_X\}$ and so M is a closed subspace of X by Proposition 8; similarly, $N \cap X_{\text{fin}} = V \times \{0_W\}$ and so N is closed according to Proposition 8.

If $(y, w) \in M \cap N$, then $y = w$ and so $y \in V \cap W$ which means $(y, w) = 0_X$. Moreover, $M + N = X$. Indeed, for $(y, w) \in X$ we choose $v_1 \in V$ and $w_1 \in W$ such that $y = v_1 + w_1$. Then

$$(y, w) = (w_1, w_1) + (v_1, w - w_1) \in M + N$$

Therefore, M and N are closed complemented subspaces of X. However, $N \cap X_{\text{fin}} = V \times \{0_W\}$ does not have a closed complement in X_{fin} because V does not have one in Y. Consequently, Theorem 7 ensures N does not have a projection complement in X.

The subspace W in the above example is necessarily infinite dimensional, and hence X/X_{fin} is infinite dimensional. Thus a natural question is: Does every complemented closed subspace of an almost conventional extended Banach space have a projection complement? Our final result of this section provides an affirmative answer to this question, which again highlights a nice structural property of almost conventional spaces.

Theorem 8 *Let $\langle X, \|\cdot\|\rangle$ be a an almost conventional extended Banach space. If M is a norm closed complemented subspace of X, then M has a projection complement.*

Proof Write $X = M \oplus N$ where M and N are norm closed subspaces of X. Write $X = X_{\text{fin}} \oplus Y$ where Y is finite dimensional and endow X with the conventional norm $|\!|\!|\cdot|\!|\!|$ defined by $|\!|\!|x+y|\!|\!| = \|x\| + \|y\|_Y$ when $x \in X_{\text{fin}}$, $y \in Y$ and $\|\cdot\|_Y$ is a norm on the finite dimensional space Y. According to [7, Theorem 3.11] a norm closed subspace of an extended normed space is weakly closed, so both M and N are weakly closed. It follows from Proposition 2, M and N are weakly closed in $\langle X, |\!|\!|\cdot|\!|\!|\rangle$ and thus M and N are $|\!|\!|\cdot|\!|\!|$-closed in $\langle X, |\!|\!|\cdot|\!|\!|\rangle$. Therefore, M and N are complemented in $\langle X, |\!|\!|\cdot|\!|\!|\rangle$. Moreover, $\langle X, |\!|\!|\cdot|\!|\!|\rangle$ is a Banach space since X_{fin} is. Consequently there is a continuous projection $P : \langle X, |\!|\!|\cdot|\!|\!|\rangle \to \langle X, |\!|\!|\cdot|\!|\!|\rangle$ such that $P(X) = M$ (see [8, Proposition 5.3]).

Now choose a maximal linearly independent set $\{b_1, b_2, \ldots, b_k\} \subset M$ such that $X_{\text{fin}} + M = X_{\text{fin}} \oplus \text{span}(\{b_1, b_2, \ldots, b_k\})$ (the special case $M \subset X_{\text{fin}}$ can be handled directly, or by letting $Z = X_{\text{fin}}$ in what follows). Let $Z = X_{\text{fin}} \oplus F$ where $F = \text{span}(\{b_1, b_2, \ldots, b_k\})$. Then $\langle Z, |\!|\!|\cdot|\!|\!|\rangle$ is a Banach space where $|\!|\!|\cdot|\!|\!|$ is the inherited norm. Moreover, X_{fin} and F

are closed and complemented subspaces in $\langle Z, ||| \cdot ||| \rangle$ so we let $Q : \langle Z, ||| \cdot ||| \rangle \to \langle Z, ||| \cdot ||| \rangle$ be the continuous projection such that $Q(Z) = X_{\text{fin}}$. Observe that $P|_{X_{\text{fin}}} : X_{\text{fin}} \to Z$, and let $T = (Q \circ P|_{X_{\text{fin}}})$. Then $T : \langle X_{\text{fin}}, ||| \cdot ||| \rangle \to \langle X_{\text{fin}}, ||| \cdot ||| \rangle$ is a continuous linear mapping.

Moreover, T is a projection with $T(X_{\text{fin}}) = M \cap X_{\text{fin}}$. Indeed, suppose $x \in M \cap X_{\text{fin}}$, then $P(x) = x$ and so $Q(P(x)) = x$. On the other hand suppose $x \in X_{\text{fin}} \setminus M$. Then $P(x) = u + f$ where $u \in X_{\text{fin}} \cap M$ and $f \in F \subset M$, and then $Q(u + f) = u$, and so $T(x) \in M \cap X_{\text{fin}}$ as desired. Because $T : \langle X_{\text{fin}}, ||| \cdot ||| \rangle \to \langle X_{\text{fin}}, ||| \cdot ||| \rangle$ is a continuous projection with $T(X_{\text{fin}}) = M \cap X_{\text{fin}}$, it follows that $M \cap X_{\text{fin}}$ is complemented in $\langle X_{\text{fin}}, ||| \cdot ||| \rangle$. That is, there is a $||| \cdot |||$-closed subspace W of X_{fin} such that $X_{\text{fin}} = (M \cap X_{\text{fin}}) \oplus W$. Finally, W is necessarily $\| \cdot \|$-closed and so Theorem 7 ensures M has a projection complement in $\langle X, \| \cdot \| \rangle$ as desired. □

6 Distance Functions to Convex Sets

Distance functions to convex sets play an important role in optimization and approximation; see for example [1]. In an extended normed space, a closed convex set need not be weakly closed [7], and so distance functions to closed convex sets need not be weakly lower semicontinuous. Moreover, a distance function need not be finite valued, but rather its effective domain is the union of metric components that hit the set. This section will examine the weak lower semicontinuity of distance functions to weakly closed convex sets. This is potentially of interest because when a function is weakly lower semicontinuous on a weakly compact set, it will attain its minimum on the set. Hence, in Hilbert or reflexive Banach spaces where closed bounded convex sets are weakly compact, one can ensure the existence of minimizers in a variety of applications; see [1]. Ultimately, the results of this section will illustrate that weakly compact convex sets are also well-suited for nearest point problems in extended normed spaces.

Proposition 9 *Let $\langle X, \| \cdot \| \rangle$ be an extended normed space. Suppose W is a weakly compact convex subset of $\langle X, \| \cdot \| \rangle$. Then $d_W(\cdot)$ is weakly lower semicontinuous.*

Proof Suppose $\langle x_\lambda \rangle$ converges weakly to x. It suffices to show $\liminf d_W(x_\lambda) \geq d_W(x)$. In the case $\liminf d_W(x_\lambda) = \infty$ there is nothing to do, so we suppose $\liminf d_W(x_\lambda) = r$ for some $r \geq 0$. Let $\varepsilon > 0$. Choose a subnet $\langle x_{\lambda_\alpha} \rangle$ so that $d_W(x_{\lambda_\alpha}) < r + \varepsilon$ for all λ_α. Then $x_{\lambda_\alpha} \in W + (r + \varepsilon)B_X$. Because X is a topological vector space under its weak topology, it follows that $W + (r + \varepsilon)B_X$ is weakly closed as it is the sum of a weakly compact set and weakly closed set. Therefore $x \in W + (r + \varepsilon)B_X$ and so $d_W(x) \leq r + \varepsilon$. □

The following has a proof very similar to that of the previous proposition.

Proposition 10 *Let $\langle X, \| \cdot \| \rangle$ be an extended Banach space where X_{fin} is reflexive. Suppose W is a weakly closed convex subset of X. Then $d_W(\cdot)$ is weakly lower semicontinuous.*

Proof Suppose $\langle x_\lambda \rangle$ converges weakly to x. It suffices to show $\liminf d_W(x_\lambda) \geq d_W(x)$. In the case $\liminf d_W(x_\lambda) = \infty$ there is nothing to do, so we suppose $\liminf d_W(x_\lambda) = r$ for some $r \geq 0$. Let $\varepsilon > 0$. Choose a subnet $\langle x_{\lambda_\alpha} \rangle$ so that $d_W(x_{\lambda_\alpha}) < r + \varepsilon$ for all λ_α. Then $x_{\lambda_\alpha} \in W + (r + \varepsilon)B_X$. Because X is a topological vector space under its weak topology, it follows that $W + (r + \varepsilon)B_X$ is weakly closed because $(r + \varepsilon)B_X$ is weakly compact. Therefore $x \in W + (r + \varepsilon)B_X$ and so $d_W(x) \leq r + \varepsilon$. □

The previous results ensure the gap between two sets is attained in certain cases.

Corollary 4 *Let $\langle X, \|\cdot\|\rangle$ be an extended normed space, and suppose C and W are nonempty weakly closed convex subsets of X.*

(a) *If C and W are weakly compact, then the gap $d(C, W)$ is attained if it is finite.*
(b) *If X_{fin} is a reflexive Banach space and C is X^*-bounded, then the gap $d(C, W)$ is attained if it is finite.*

Proof (a) The distance function $d_W(\cdot)$ is weakly lower semicontinuous by Proposition 9, thus it attains it minimum (if finite) on the weakly compact set C.

(b) It follows from Corollary 2 that C is weakly compact. Proposition 10 ensures that $d_W(\cdot)$ is weakly lower semicontinuous and thus attains its minimum (if finite) on C. □

The following simple example illustrates that it would not be sufficient to assume C is an X^*-bounded norm closed convex set in Corollary 4(b).

Example 2 Let $\langle X, \|\cdot\|\rangle$ be the extended normed space where $X = \mathbb{R}^2$ and we define the norm by $\|(s,t)\| = |s|$ if $t = 0$, and $\|(s,t)\| = \infty$ otherwise. Let $A = \{(s,t) : -1 \leq s < 0, 1 + s \leq t < 1\}$ and $B = \{(s,t) : 0 < s \leq 1, 1 - s \leq t < 1\}$. Then A and B are X^*-bounded norm closed convex sets in X. However, $d(A, B)$ is 0, but the gap is not attained.

Our final example illustrates limitations on extending Propositions 9 and 10, by showing, among other things, that a distance function to a weakly closed convex set need not be weakly lower semicontinuous.

Example 3 We consider the extended normed space $\langle X, \|\cdot\|\rangle$ where $X := c_0 \oplus \mathbb{R}z$ with $z = (1, 1, 1, \ldots)$ and $\|\cdot\|$ is defined by letting $\|x\|$ be the usual c_0-norm of x when $x \in c_0$ and $\|x\| = \infty$ otherwise. For each $n \in \mathbb{N}$, we fix $z_n \in X$ such that $z_n(i) = 1$ if $1 \leq i \leq n$, and $z_n(i) = \frac{1}{2^n}$ if $i > n$ where $z_n(i)$ denotes the i-th coordinate of z_n. Then let W be the weak closure of the convex hull of $\{z_n\}_{n=1}^{\infty}$.

We first show $W \cap c_0 = \emptyset$ and so $d(W, c_0) = \infty$. Indeed, suppose $\langle x_\lambda\rangle$ converges weakly to some $x \in c_0$ where $x_\lambda \in \operatorname{conv}(\{z_n\}_{n=1}^{\infty})$. Let $e_k \in X^*$ denote the k-th basis element of the usual basis of ℓ_1. Because $x \in c_0$, we have $e_m(x) < 1/4$ for some $m \in \mathbb{N}$, hence we fix λ_1 so that $e_m(x_\lambda) < 1/4$ for all $\lambda \geq \lambda_1$. Now choose $n > m$ such that $e_n(x) < 1/4^m$, and choose $\lambda_2 > \lambda_1$ such that $e_n(x_\lambda) < 1/4^m$ for all $\lambda \geq \lambda_2$. For each λ, write x_λ as the convex combination $x_\lambda = \sum_{k=1}^{m_\lambda} t_{\lambda,k} z_k$. Since $e_n(x_\lambda) < 1/4^m$ for each $\lambda \geq \lambda_2$, we have $\sum_{k \leq m} t_{\lambda,k} \leq 1/2^m$ (because $z_k(m) \geq 1/2^m$ for $1 \leq k \leq m$). Thus, $\sum_{k>m} t_{\lambda,k} \geq 1 - 1/2^m$ for each $\lambda \geq \lambda_2$. Since $z_k(m) = 1$ for $m < k$, we have $e_m(x_\lambda) \geq 1 - 1/2^m > 1/4$ for each $\lambda \geq \lambda_2$. This contradiction shows that $W \cap c_0 = \emptyset$. Consequently, $d(y, W) = \infty$ for each $y \in c_0$.

Next let C be the convex hull of 0_X and y where $y := z - (2, 0, 0, \ldots) = (-1, 1, 1, 1, \ldots)$. Then C is a weakly compact convex set, and we will show that the gap between C and W is not attained. Indeed, let $x \in C$, if $x = 0_X$, then $d(x, W) = \infty$. Otherwise, let $x_\lambda = \lambda y + (1 - \lambda)0_X$ for $0 < \lambda \leq 1$. In this case $d(x_\lambda, W) \geq 1 + \lambda > 1$ since

$x_\lambda(1) = -\lambda$ and $w(1) = 1$ for all $w \in W$. On the other hand, for $\lambda = 1/2^n$ we see that

$$z_n - x_\lambda = (1+\lambda)e_1 + \sum_{i=2}^{n}(1 - 1/2^n)e_i, \quad \text{where } \{e_i\}_{i=1}^{\infty} \text{ is the usual basis of } c_0.$$

Thus $\|z_n - x_\lambda\| = 1 + \lambda$ when $\lambda = 1/2^n$, and so $d(C, W) \le 1 + 1/2^n$ for each $n \in \mathbb{N}$. Consequently, $d(C, W) = 1$, but that gap is not attained.

Because C is weakly compact and the gap $d(C, W)$ is not attained, it follows that $d_W(\cdot)$ is not weakly lower semicontinuous (even though W is a weakly closed convex set). This is also easy to see directly. Indeed, define $x_n = \frac{1}{2^n}z$. Then $\langle x_n \rangle$ converges weakly to 0_X, and $d(x_n, W) = 1 - 1/2^n$ however, $d(0_X, W) = \infty$.

Contrasting Example 3 with Proposition 9 reinforces the importance of understanding the structure of weakly compact sets as discussed in Section 4.

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