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Fixed Points and Variational Principles with Applications to Capability Theory of Wellbeing via Variational Rationality

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Abstract In this paper we first develop two new results of variational analysis. One is a fixed point theorem for parametric dynamic systems in quasimetric spaces, which can also be interpreted as an existence theorem of minimal points with respect to reflexive and transitive preferences for sets in products spaces. The other one is a variational principle for set-valued mappings acting from quasimetric spaces to vector spaces with variable ordering structures, which can be treated as a far-going extension of Ekeland's variational principle to this setting. Both of these results are motivated by applications to adaptive dynamical aspects of Sen's capability theory of wellbeing. As consequence of our mathematical results, we develop new applications to such human behavioral models by using a recent variational rationality approach to behavioral sciences.

Keywords Variational analysis and optimization · Variational principles · Fixed point theory · Capacity approach to wellbeing · Variational rationality

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Dedicated to Lionel Thibault in honor of his 65th birthday.

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1 Introduction

Variational analysis has been well recognized as a fruitful area in mathematics with numerous applications to optimization, equilibria, control theory, economics, mechanics, etc.; see, e.g., the books [9, 22, 26] and the references therein. The seminal contributions of Lionel Thibault to variational and nonsmooth analysis and optimization (particularly to their infinite-dimensional aspects) as well as their various applications are difficult to overstate.

Modern *variational principles* lie at the heart of variational analysis and its applications. Among the most powerful ones is *Ekeland's variational principle* (EVP), which asserts in essence that, given a lower semicontinuous extended-real-valued function on a metric space and its ε-minimizer (that always exists whenever $\varepsilon > 0$), it is possible to *perturb* a bit both the function and the given point so that the new point gives an *exact global minimum* of the perturbed function. This result has been of great use in many areas of mathematics and its applications; see, e.g., Ekeland's survey [17], the books [9, 18, 22, 26], and their bibliographies for the history and important developments in this direction. Starting with the discovery of the EVP in the 1970s, numerous extensions of it appeared in the literature (see the references in Section 3 to those related to our study). It should be mentioned that many extensions and modifications of the EVP were given as pure mathematical results, without significant motivations and applications.

The intention of this paper is different. Our new and far-going extension of the EVP to set-valued mappings has been fully motivated by applications to *capability theory of wellbeing* initiated by Sen in welfare economics [28] (Nobel Prize in Economics, 1998) and then largely extended to other areas of behavioral sciences. We will describe major features of the capability approach to behavioral science below; the total bibliography in this field is enormous.

Our main motivation in this paper is to incorporate some *dynamical* issues into the static capability theory by using the ideas of the *variational rationality* approach to behavioral sciences suggested recently by Soubeyran [33, 34] and discussed in what follows. To proceed effectively in this way, the development of new mathematical tools of variational analysis is ultimately required. Namely, the nature of the variational rationality approach in connection with capability theory calls for such an extension of the EVP that can be applied to set-valued mappings defined on *quasimetric* spaces with values in topological vector spaces ordered by *variable preferences*. We derive the needed variational principle from a new *fixed point* theorem for parametric dynamical systems in quasimetric spaces, which is a far-reaching extension of the nonparametric Dancs-Hegedüs-Medvegyev (DHM) fixed point theorem [13] in metric spaces. Furthermore, we show that the new fixed point result can be equivalently interpreted as a *minimal point* theorem for nonempty subsets in product spaces with a preference structure satisfying general requirements.

These mathematical results are of their own interest regardless of applications to capability theory, and so we present them first. Section 2 is devoted to the aforementioned fixed point theorem and its minimal point interpretation, while in Section 3 we employ this theorem and other techniques of variational analysis to derive a new counterpart of the EVP for set-valued mappings.

Section 4 contains a brief summary of the main issues of the (static) capability theory and our justification of dynamical aspects needed for further improvements. In Sections 5 and 6 we describe the variational rationality model of human behavior, which allows us to apply the mathematical results obtained in Sections 2 and 3 to developing dynamical aspects of capability theory. Section 7 discusses the major findings in this direction. The concluding

Section 8 summarizes the main contributions of the paper and formulates some topics of the future research.

2 Fixed Point Theorem for Parametric Dynamical Systems

This section is devoted to establishing a new *parametric* version of the celebrated Dancs-Hegedüs-Medvegyev fixed point theorem for parametric dynamical systems in *quasimetric* spaces and its equivalent to *minimal point theorem* for sets in the corresponding product spaces. The results obtained seem to be new even in the classical nonparametric setting of metric spaces.

For the reader's convenience we first recall the definition of quasimetric spaces and some of their standard properties; cf., in particular, the book [11] and the references therein.

Definition 2.1 (quasimetric spaces) A quasimetric space is a set X equipped with a function $q : X \times X \longmapsto I\!R_+ := [0, \infty)$ having the following properties:

- **(i)** **(positivity)** $q(x, x') \geq 0$ for all $x, x' \in X$ with $q(x, x) = 0$ for all $x \in X$;
- **(ii)** **(triangle inequality)** $q(x, x'') \leq q(x, x') + q(x', x'')$ for all $x, x', x'' \in X$.

The function q in Definition 2.1 is known as *quasimetric*. If in addition it satisfies the symmetry property $q(x, x') = q(x', x)$ for all $x, x' \in X$, then q is a metric. We denote by (X, q) the space X with the quasimetric q. It is worth mentioning that our definition of quasimetrics is a bit different from the conventional one, which imposes the requirement: $q(x, x') = 0$ if and only if $x' = x$ for all $x, x' \in X$. Since q is not symmetric, we may have two distinct points x, x' with $q(x, x') = 0$. Observe that quasimetrics make perfect sense even in finite-dimensional (non-Euclidean) spaces.

A simple and interesting example of a quasimetric on $I\!R$ is given by

$$q(x, y) := x - y \ \text{ if } \ x \geq y \ \text{ and } \ q(x, y) := 1 \ \text{ otherwise.} \tag{2.1}$$

The topological space endowed with this quasimetric is known as the *Sorgenfrey line*.

Similarly to metric spaces, every quasimetric space (X, q) can be viewed as a topological space on which the topology is introduced by taking the collection of balls $\{I\!B_r(x) \mid r > 0\}$ as a base of the neighborhood filter for every $x \in X$, where the (left) ball $I\!B_r(x)$ is defined by

$$I\!B_r(x) := \left\{ y \in X \middle| \ q(x, y) < r \right\}.$$

According to this topology, the *convergence* $x_k \to x_* \in X$ means that $\lim_{k\to\infty} q(x_k, x_*) = 0$. Observe that the quasimetric q may not be continuous, as in the case of (2.1), but it always *lower semicontinuous* (l.s.c.) due to the triangle inequality.

Definition 2.2 (left-sequential closedness and completeness) Let (X, q) be a quasimetric space, and let Ω be a nonempty subset of X. Then:

- **(i)** Ω is LEFT-SEQUENTIALLY CLOSED if $x_* \in \Omega$ for any sequence $x_k \to x_*$ with $\{x_k\} \subset \Omega$.
- **(ii)** A sequence $\{x_k\} \subset X$ is LEFT-SEQUENTIAL CAUCHY if for each $k \in I\!N$ there is N_k with
$$q(x_n, x_m) < 1/k \ \text{ for all } \ m \geq n \geq N_k.$$
- **(iii)** (X, q) is LEFT-SEQUENTIALLY COMPLETE if each Cauchy sequence from (ii) converges.

In what follows we deal only with quasimetric spaces satisfying the *Hausdorff property*:

$$\Big[\lim_{k\to\infty} q(x_k, x_*) = 0 \ \text{ and } \ \lim_{k\to\infty} q(x_k, \widetilde{x}_*) = 0\Big] \Longrightarrow x_* = \widetilde{x}_*. \tag{2.2}$$

It is not hard to check that quasimetric (2.1) is Hausdorff in contrast, e.g., to

$$q(x, y) := \begin{cases} x - y & \text{if } x \geq y, \\ e^{x-y} & \text{otherwise.} \end{cases} \tag{2.3}$$

Now we are ready to obtain the main result of this section. Given a set-valued mapping $\Phi : X \times Z \rightrightarrows X \times Z$, we say that $\{(x_k, z_k)\}$ is a *generalized Picard sequence/iterative process* if

$$(x_2, z_2) \in \Phi(x_1, z_1),\ (x_3, z_3) \in \Phi(x_2, z_2),\ \ldots,\ (x_k, z_k) \in \Phi(x_{k-1}, z_{k-1}),\ \ldots.$$

Theorem 2.3 (**parametric fixed point theorem**) *Let (X, q) be a complete Hausdorff quasimetric space, let Z be a set of parameters, and let $\emptyset \neq \Xi \subset X \times Z$. Assume that the parametric dynamical system $\Phi : X \times Z \rightrightarrows X \times Z$ satisfies the following requirements:*

(A1) $(x, z) \in \Phi(x, z)$ *for all* $(x, z) \in \Xi$.

(A2) *For all* $(x_1, z_1), (x_2, z_2) \in \Xi$ *such that* $(x_2, z_2) \in \Phi(x_1, z_1)$ *we have* $\Phi(x_2, z_2) \subset \Phi(x_1, z_1)$.

(A3) *The* LIMITING MONOTONICITY CONDITION*: for each generalized Picard sequence* $\{(x_k, z_k)\}$ *from* Ξ *with* $x_k \to x_*$ *as* $k \to \infty$ *there is* $z_* \in Z$ *with* $(x_*, z_*) \in \Xi$ *such that*

$$(x_*, z_*) \in \Phi(x_k, z_k) \ \text{ for all } \ k \in I\!N \text{ and } (x_*, z) \in \Xi \cap \Phi(x_*, z_*) \Longrightarrow z = z_*. \tag{2.4}$$

(A4) *The* CONVERGENCE CONDITION*: for each generalized Picard sequence* $\{(x_k, z_k)\} \subset \Xi$ *the quasidistances* $q(x_k, x_{k+1})$ *tend to zero as* $k \to \infty$.

Then for every $(x_0, z_0) \in \Xi$ *there is a generalized Picard sequence* $\{(x_k, z_k)\} \subset \Xi$ *starting at* (x_0, z_0) *and ending at a fixed point* (x_*, z_*) *of* Φ *in the sense of* $\Phi(x_*, z_*) = \{(x_*, z_*)\}$.

Proof Without loss of generality, assume that the quasimetric $q : X \times X \to I\!R_+$ is bounded on X; otherwise, we use the equivalent quasimetric $\widetilde{q}(x, u) := \frac{q(x,u)}{1+q(x,u)}$. Given $\emptyset \neq \Omega \subset X$ and $x \in \Omega$, the *radius of the smallest ball* containing Ω and centered at x is $r(x; \Omega) := \sup_{u\in\Omega} q(x, u)$. Fix an arbitrary pair $(x_k, z_k) \in \Xi$ and denote $\Phi_k := \{x \in X \mid \exists\, z \in Z,\ (x, z) \in \Phi(x_k, z_k)\}$ for all $k \in I\!N \cup \{0\}$. We construct a *generalized Picard sequence* defined as follows:

$$(x_k, z_k) \in \Phi(x_{k-1}, z_{k-1}) \ \text{ with } \ q(x_{k-1}, x_k) \geq r\big(x_{k-1}; \Phi_{k-1}\big) - 1/2^{k-1}. \tag{2.5}$$

It is clear from (A1) that iterations (2.5) are well defined. Furthermore, by (A2) we have $\Phi_n \subset \Phi_m$ if $m, n \in I\!N$ with $n \geq m \in I\!N$. The convergence condition (A4) tells us that the quasidistances $q(x_k, x_{k+1})$ tend to zero as $k \to \infty$. Taking into account the inequality in (2.5) ensures that $r(x_k; \Phi_k) \downarrow 0$ as $k \to \infty$, which implies that for every $\varepsilon > 0$ there exists $N_\varepsilon \in I\!N$ such that

$$r(x_k; \Phi_k) < \varepsilon \ \text{ whenever } \ k \geq N_\varepsilon.$$

Picking now any $m \geq n \geq N_\varepsilon$, we have $x_m \in \Phi_m \subset \Phi_n$ and $q(x_n, x_m) \leq r(x_n; \Phi_n) \leq \varepsilon$, which verifies that the sequence $\{x_k\} \subset X$ is left-sequential Cauchy. The completeness of X ensures the existence of $x_* \in X$ such that $x_k \to x_*$ as $k \to \infty$. Applying then the limiting monotonicity condition to the chosen sequence $\{(x_k, z_k)\}$ gives us $z_* \in Z$ satisfying $(x_*, z_*) \in \Xi$ and (2.4).

Let us finally show that (x_*, z_*) is a *fixed point* of the dynamical system Φ. Arguing by contradiction, suppose that there is some $(x, z) \in \Xi \setminus \{(x_*, z_*)\}$ such that $(x, z) \in \Phi(x_*, z_*)$. It follows from (2.4) and (A2) that $(x, z) \in \Phi(x_k, z_k)$ for all $k \in I\!N$ and thus $q(x_k, x) \to 0$ due to $r(x_k; \Phi_k) \downarrow 0$ as $k \to \infty$. Since X is Hausdorff and $q(x_k, x_*) \to 0$, we have $x = x_*$. The assumption made reduces now to $(x_*, z) \in \Phi(x_*, z_*)$, which yields $z = z_*$ by (2.4). The obtained contradiction verifies that $\Phi(x_*, z_*) = \{(x_*, z_*)\}$ and thus completes the proof of the theorem. □

Next we present a direct consequence of Theorem 2.3 for the case of *nonparametric* dynamical systems, which extends even in this setting the DHM fixed point [13, Theorem 3.1] in the following two directions: **(1)** we do *not* impose the *symmetry property* of metrics (vs. quasimetrics), and **(2)** we do *not* require that all the sets $\Phi(x)$ for $x \in X$ are *closed* in X.

Corollary 2.4 (**extension of the DHM fixed point theorem**) *Let (X, q) be a complete Hausdorff quasimetric space, and let $\Phi : X \rightrightarrows X$ be a dynamical system satisfying the conditions:*

- **(A1′)** $x \in \Phi(x)$ *for all* $x \in X$.
- **(A2′)** $x_2 \in \Phi(x_1) \Longrightarrow \Phi(x_2) \subset \Phi(x_1)$ *for all* $x_1, x_2 \in X$.
- **(A3′)** *For each generalized Picard sequence* $\{x_k\} \subset X$ *convergent to* x_* *it follows that* $x_* \in \Phi(x_k)$ *for all* $k \in I\!N$, *which is automatic if* $\Phi(x)$ *is closed for all* $x \in X$ *and* (A2′) *holds.*
- **(A4′)** *For each generalized Picard sequences* $\{x_k\} \subset X$ *we have that* $q(x_k, x_{k+1}) \to 0$ *as* $k \to \infty$.

Then for every starting point $x_0 \in X$ there is a convergent generalized Picard sequence $\{x_k\} \subset X$ with the limit x_* which is a fixed point of Φ, i.e., $\Phi(x_*) = \{x_*\}$.

Proof Consider an arbitrary singleton $Z = \{z^*\}$ and define the mapping $\widetilde{\Phi}(x, z_*) := \Phi(x) \times \{z_*\}$ on $X \times Z$. Applying Theorem 2.3 to $\widetilde{\Phi}$ verifies the statement of this corollary. □

The next result gives us an *equivalent* form of Theorem 2.3 as a *minimal point* theorem for a given subset of a product space ordered by some preference satisfying the imposed requirements.

Theorem 2.5 (**parametric minimal point theorem in product spaces**) *Let (X, q) be a complete Hausdorff quasimetric space, let Z be a nonempty set of parameters, and let Ξ be a nonempty subset of the product space $X \times Z$. Endow the set Ξ with a reflexive and transitive preference $\preceq$ satisfying the following two requirements:*

- **(B1)** *The* LIMITING MONOTONICITY CONDITION: *for every* $\{(x_k, z_k)\} \subset \Xi$ *with* $(x_k, z_k) \preceq (x_{k-1}, z_{k-1})$, $k \in I\!N$, *the convergence* $x_k \to x_*$ *yields the existence of* $z_* \in Z$ *such that* $(x_*, z_*) \in \Xi$ *and*

$$(x_*, z_*) \preceq (x_k, z_k) \text{ for all } k \in I\!N \text{ and } (x_*, z) \preceq (x_*, z_*) \Longrightarrow z = z_*.$$

- **(B2)** *The* CONVERGENCE CONDITION: $q(x_k, x_{k+1}) \to 0$ *as* $k \to \infty$ *for every sequence* $\{(x_k, z_k)\}$ *entirely belonging to the set* Ξ *and decreasing with respect to* $\preceq$.

Then Ξ has a minimal point (x_*, z_*) with respect to the preference $\preceq$ in the sense that the inclusion $(x, z) \in \Xi$ with $(x, z) \preceq (x_*, z_*)$ yields $(x, z) = (x_*, z_*)$. Moreover, for every $(x_0, z_0) \in \Xi$ there is a decreasing sequence $\{(x_k, z_k)\} \subset \Xi$ starting at (x_0, z_0) and ending at a minimal point (x_*, z_*) of Ξ with respect to $\preceq$. Conversely, the above statement implies the validity of Theorem 2.3.

Proof To derive the formulated minimal point result from Theorem 2.3, consider the *level-set* mapping $\Phi : X \times Z \rightrightarrows X \times Z$ defined by

$$\Phi(x, z) := \text{Lev}\,((x, z); \preceq) = \big\{(u, v) \in \Xi \;\big|\; (u, v) \preceq (x, z)\big\} \tag{2.6}$$

and show that it satisfies all the four condition (A1)–(A4) in Theorem 2.3. Since we obviously have the equivalences (A3)$\Longleftrightarrow$(B1) and (A4)$\Longleftrightarrow$(B2), it remains to check the validity of conditions (A1) and (A2) for the level-set mapping Φ from (2.6).

It can be easily seen that the inclusion $(x, z) \in \Phi(x, z)$ for $(x, z) \in \Xi$ in (A1) follows from the reflexivity property of the preference $\preceq$. To verify (A2), we need to show that $\Phi(u, v) \subset \Phi(x, z)$ when $(u, v) \in \Phi(x, z)$. Pick $(t, w) \in \Phi(u, v)$ arbitrarily and get by (2.6) the relationships

$$\begin{aligned}\big[(t, w) \in \Phi(u, v) \text{ and } (u, v) \in \Phi(x, z)\big] &\overset{(2.6)}{\Longleftrightarrow} \big[(t, w) \preceq (u, v) \text{ and } (u, v) \preceq (x, z)\big]\\ &\overset{\text{transitivity}}{\Longrightarrow} (t, w) \preceq (x, z) \overset{(2.6)}{\Longleftrightarrow} (t, w) \in \Phi(x, z),\end{aligned}$$

where the implication holds due to the transitivity property of $\preceq$. This verifies (A2).

Applying now Theorem 2.3 to the mapping Φ from (2.6), for every $(x_0, z_0) \in \Xi$ we find a generalized Picard/decreasing sequence $\{(x_k, z_k)\} \subset \Xi$ starting at (x_0, z_0) and converging to some pair (x_*, z_*) such that

$$\Phi(x_*, z_*) = \big\{(x_*, z_*)\big\} := \big\{(u, z) \in \Xi \big|\; (u, v) \preceq (x_*, z_*)\big\}.$$

This clearly justifies the minimality of (x_*, z_*) for the set Ξ with respect to the preference $\preceq$.

To complete the proof of the theorem, we need to verify the converse implication therein, i.e., that the minimal point result ensures the validity of the fixed point assertion of Theorem 2.3. Indeed, take Φ in the setting of Theorem 2.3 and define the reflexive and transitive preference $\preceq$ on the set $\Xi \subset X \times Z$ by

$$(u, v) \preceq (x, z) :\Longleftrightarrow (u, v) \in \Phi(x, z).$$

Then proceeding as in the proof above allows us to verify assumption (B1) for this preference and to deduce the fixed point result from the minimal point formulation. □

3 Variational Principle for Mappings with Variable Orderings

In this section we derive a new *variational principle* for set-valued mappings defined on *quasimetric* spaces and taking values in linear topological spaces endowed by *variable* ordering structures. This result and its consequences presented below can be treated as far-going extensions of the classical EVP to the general setting under consideration needed for subsequent applications to capability theory of wellbeing, where both quasimetric and variable preference issues are essential; see Sections 5–7 for more discussions. In fact, the results obtained in this section reduce to the set-valued versions of the EVP established by Bao and Mordukhovich [4, 5] in the case of constant/invariable ordering structures for mappings between metric spaces and vector spaces.

Note that vector optimization problems with variable ordering structures have already been studied in the literature, especially during recent years, due to their theoretical interest and many important applications to operations research, economics, engineering design, behavioral sciences, etc.; see, e.g., [6–8, 15, 16, 32, 36, 37] and the references therein. It seems that our previous papers [7, 8], motivated by applications to various models in behavioral sciences and dealing with single-valued objectives, were the first attempts to extend the EVP to problems with ordering structures defined on image spaces. A different approach to vectorial variational principles for problems with variable structures acting on both domain and image spaces was developed in the concurrent preprint by Soleimani and Tammer [32] based on a nonlinear scalarization technique. On the other hand, there is a number of publications invoking quasimetric structures, either on domain spaces or as perturbations, into variational principles of the Ekeland type, with no variable ordering structures involved; see, e.g., [11, 24, 35] and the references therein. These results and approaches are essentially different from ours obtained below. Observe also that, in contrast to [4, 5, 7, 8], we derive here the new variational principle in the general framework by using the *parametric fixed point theorem* from Section 3.

We begin with describing the *set optimization* setting considered in what follows. Let Z be a (real) linear topological space with $\emptyset \neq \Xi \subset Z$, and let $\Theta \subset Z$ be a proper convex *ordering cone*. We say that $z_* \in \Xi$ is a *minimal point* of Ξ with respect to Θ, written as $z_* \in \operatorname{Min}(\Xi; \Theta)$, if

$$\Xi \cap (z_* - \Theta) = \{z_*\}. \tag{3.7}$$

To describe the class of variable preferences invoking in our main result, take vectors $z_1, z_2 \in Z$, denote $d := z_1 - z_2$, and say that z_2 is *preferred* by the decision maker to z_1 with the *domination factor d* for z_1. The set of all the domination factors for z_1 together with the zero vector $\mathbf{0} \in Z$ is denoted by $K[z_1]$. Then the set-valued mapping $K : Z \rightrightarrows Z$ is called a *variable ordering structure*. We define a *binary/ordering relation* induced by the variable ordering structure K as

$$z_2 \ \leq_{K[z_1]} z_1 \ \text{ if and only if } \ z_2 \in z_1 - K[z_1] \tag{3.8}$$

and say that $z_* \in \Xi$ is *Pareto efficient/minimal point* to the set Ξ in Z *with respect to the variable ordering structure* K if there is no other vector $z \in \Xi \setminus \{z_*\}$ such that $z \leq_{K[z_*]} z_*$, i.e.,

$$\big(z_* - K[z_*]\big) \cap \Xi = \{z_*\}.$$

It is worth observing that generally the binary/ordering relation $\leq_{K[\cdot]}$ is *nontransitive* (see (C3) in Theorem 3.1) and not even compatible with positive scalar multiplication.

Consider next a set-valued mapping $F\colon X \rightrightarrows Z$ between a quasimetric space (X, q) and a linear topological vector space Z equipped with an ordering structure $K : Z \rightrightarrows Z$. Denote the *domain* and *graph* of F by, respectively,

$$\operatorname{dom} F := \big\{x \in X \big|\ F(x) \neq \emptyset\big\} \ \text{ and } \ \operatorname{gph} F := \big\{(x, z) \in X \times Z \big|\ z \in F(x)\big\}.$$

We say that a pair $(x_*, z_*) \in \operatorname{gph} F$ is a (Pareto) *minimizer* of F with respect to the ordering structure K if $z_* \in \operatorname{Min}\big(F(X); K[z_*]\big)$ is a minimal point of the image set $F(X) := \bigcup_{x \in X} F(x)$ with respect to K, i.e., $F(X) \cap (z_* - K[z_*]) = \{z_*\}$. Fix further a direction $\xi \in Z \setminus \{\mathbf{0}\}$ and a *threshold/accuracy* $\varepsilon > 0$. We say that a pair $(x_*, z_*) \in \operatorname{gph} F$ is an $\varepsilon\xi$*-approximate minimizer* of F with respect to the ordering structure K if

$$F(X) \cap \big(z_* - \varepsilon\xi - K[z_*]\big) = \emptyset.$$

Now we are ready to derive the aforementioned main result of this section.

Theorem 3.1 (**variational principle for set-valued mappings with variable ordering structures**) *Let $F : (X, q) \rightrightarrows Z$ be a set-valued mapping between a quasimetric space (X, q) to a linear topological space Z, let $K : Z \rightrightarrows Z$ be an ordering structure on Z, and let $\Theta_K := \bigcap_{z \in F(X)} K[z]$ with $F(X) := \bigcup_{x \in X} F(x)$. Impose the following assumptions on the initial data:*

(C1) *The quasimetric space (X, q) is (left-sequentially) complete and Hausdorff.*
(C2) *For every $z \in F(X)$ the domination set $K[z]$ is a proper, closed, and convex subcone of Z.*
(C3) *The ordering structure K satisfies the monotonicity property: for any $z, v \in F(X)$, if $v \leq_{K[z]} z$, then $K[v] \subset K[z]$; this condition implies the transitivity of the binary relation $\leq_K [\cdot]$.*
(C4) *The mapping F is quasibounded from below with respect to a cone Θ, i.e. there is a bounded subset $M \subset Z$ such that $F(X) \subset M + \Theta$.*
(C5) *The mapping F satisfies the limiting decreasing monotonicity condition on dom F with respect to K in the sense that for any sequence $\{(x_k, z_k)\} \subset \operatorname{gph} F$ such that $x_k \to x_* \in X$ as $k \to \infty$ and that $\{z_k\}$ is decreasing with respect to $\leq_{K[\cdot]}$ (i.e. $z_{k+1} \leq_{K[z_k]} z_k$ for all $k \in \mathbb{N}$), it follows that there is a minimal point $z_* \in \operatorname{Min}\big(F(x_*); K[z_*]\big)$ for which $z_* \leq_{K[z_k]} z_k$ as $k \in \mathbb{N}$.*

Then given any $\gamma > 0$, $(x_0, z_0) \in \operatorname{gph} F$, and $\xi \in \Theta_K \setminus \operatorname{cl}(-\Theta - K[z_0])$, there exists a pair $(x_*, z_*) \in \operatorname{gph} F$ with $z_* \in \operatorname{Min}\big(F(x_*); K[z_*]\big)$ satisfying the relationships

$$z_* + \gamma q(x_0, x_*)\xi \leq_{K[z_0]} z_0, \tag{3.9}$$

$$z + \gamma q(x_*, x)\xi \not\leq_{K[z_*]} z_* \ \textit{for all} \ \ (x, z) \in \operatorname{gph} F \setminus \{(x_*, z_*)\}. \tag{3.10}$$

If furthermore (x_0, z_0) is an $\varepsilon\xi$-approximate minimizer of F with respect to K, then x_* can be chosen so that in addition to (3.9) and (3.10) with $\gamma = (\varepsilon/\lambda)$ we have

$$q(x_0, x_*) \leq \lambda. \tag{3.11}$$

Proof Without loss of generality, assume that $\gamma = 1$. The general case can be easily reduced to it by rescaling the quasimetric on X as $\widetilde{q}(x, u) := \gamma q(x, u)$. Define $\Phi : X \times Z \rightrightarrows X \times Z$ by

$$\Phi(x, z) := \big\{(u, v) \in X \times Z \big|\ v + q(x, u)\xi \leq_{K[z]} z\big\} \tag{3.12}$$

and get due to the choice of ξ and the convexity of the cone $K[z]$ that $v \leq_{K[z]} z$, which yields

$$K[v] \subset K[z] \tag{3.13}$$

by the imposed condition (C3).

Let us verify that the assumptions made in this theorem ensure that the mapping Φ from (3.12) satisfies all the assumptions (A1)–(A4) of Theorem 2.3.

(A1) By (3.12) we always have $(x,z) \in \Phi(x,z)$ for all $(x,z) \in \operatorname{gph} F$ since $z + q(x,x)\xi = z \leq_{K[z]} z$.

(A2) For any $(x,z),(u,v) \in \operatorname{gph} F$ with $(u,v) \in \Phi(x,z)$ and for any $(t,w) \in \Phi(u,v)$ we have

$$\begin{aligned}
&(u,v) \in \Phi(x,z) \ \text{ and } \ (t,w) \in \Phi(u,v)\\
\overset{(3.12)}{\Longleftrightarrow}\ & v + q(x,u)\xi \leq_{K[z]} z \ \text{ and } \ w + q(u,t)\xi \leq_{K[v]} v\\
\overset{(3.8)}{\Longleftrightarrow}\ & \exists\, \theta_z \in K[z],\ \theta_v \in K[v]:\ v + q(x,u)\xi = z - \theta_z \ \text{ and } \ w + q(u,t)\xi = v - \theta_v\\
\Longrightarrow\ & \exists\, \theta := \theta_z + \theta_v + \big(q(x,u) + q(u,t) - q(x,t)\big)\xi \in K[z]:\ w + q(x,t)\xi = z - \theta\\
\overset{(3.8)}{\Longleftrightarrow}\ & w + q(x,t)\xi \leq_{K[z]} z \overset{(3.12)}{\Longleftrightarrow} (t,w) \in \Phi(x,z)
\end{aligned}$$

provided that $\theta := \theta_z + \theta_v + (q(x,u) + q(u,t) - q(x,t))\xi \in K[z]$. The latter condition holds due to the choice of $\xi \in \Theta_K \subset K[z]$, the triangle inequality for the quasimetric, the convexity of the cone $K[z]$ ($K[z]+K[z] = K[z]$), and the inclusion $K[v] \subset K[z]$ in (3.13). Indeed, we have

$$\theta := \theta_z + \theta_v + (q(x,u) + q(u,t) - q(x,t))\xi \in K[z] + K[z] + K[z] = K[z].$$

Due to the arbitrary choice of $(t,w) \in \Phi(u,v)$, it yields $\Phi(u,v) \subset \Phi(x,z)$, which justifies (A2).

(A3) To verify that Φ from (3.12) satisfies the limiting monotonicity condition on $\Xi := \Phi(x_0, z_0)$, take any generalized Picard sequence $\{(x_k, z_k)\} \subset \Xi$ with $x_k \to x_* \in X$ and

$$\begin{aligned}
(x_k, z_k) \in \Phi(x_{k-1}, z_{k-1}) &\overset{(3.12)}{\Longleftrightarrow} z_k + q(x_{k-1}, x_k)\xi \leq_{K[z_{k-1}]} z_{k-1}\\
&\overset{(3.13)}{\Longrightarrow} z_k \leq_{K[z_{k-1}]} z_{k-1}, \quad k \in I\!N.
\end{aligned}$$

Then by (C5) we get $x_* \in \operatorname{dom} F$ and $z_* \in \operatorname{Min}\big(F(x_*); K[z_*]\big)$ with $z_* \leq_{K[z_k]} z_k$ for all $k \in I\!N$. Invoking now (A2) gives us $\Phi(x_{k+n}, z_{k+n}) \subset \Phi(x_k, z_k)$ for all $k, n \in I\!N$. This together with the choice of $\xi \in \Theta_K \subset K[z_k]$ and the triangle inequality for q yields the following estimate:

$$\begin{aligned}
z_* + q(x_k, z_*)\xi &\in z_{k+n} - K[z_{k+n}] + q(x_k, x_*)\xi\\
&= z_{k+n} + q(x_k, x_{k+n})\xi - K[z_{k+n}] + (q(x_k, x_*) - q(x_k, x_{k+n}))\xi\\
&\subset z_k - K[z_k] - K[z_{k+n}] + q(x_{k+n}, x_*)\xi - \Theta_K\\
&\subset z_k - K[z_k] + q(x_{k+n}, x_*)\xi.
\end{aligned}$$

Passing there to the limit as $n \to \infty$ with taking into account the closedness of $K[z_k]$ and the convergence $q(x_{k+n}, x_*) \to 0$ as $n \to \infty$, we arrive at the equivalence

$$z_* + q(x_k, z_*)\xi \in z_k - K[z_k] \overset{(3.8)}{\Longleftrightarrow} z_* + q(x_k, z_*)\xi \leq_{K[z_k]} z_k \overset{(3.12)}{\Longleftrightarrow} (x_*, z_*) \in \Phi(x_k, z_k).$$

Since $k \in \mathbb{N}$ was chosen arbitrarily and since $z_* \in \text{Min}\big(F(x_*); K[z_*]\big)$, it follows that

$$(x_*, z) \in \Phi(x_*, z_*) \overset{(3.12)}{\Longleftrightarrow} z = z + q(x_*, x_*)\xi \leq_{K[z_*]} z_* \overset{(C5)}{\Longrightarrow} z = z_*,$$

which justifies the validity of assumption (A3) in Theorem 2.3.

(A4) To verify the convergence condition for (3.12), let $\Xi := \Phi(x_0, z_0)$ and take a generalized Picard sequence $\{(x_k, z_k)\} \subset \Xi$ meaning in this setting that

$$(x_k, z_k) \in \Phi(x_{k-1}, z_{k-1}) \Longleftrightarrow z_k + q(x_{k-1}, x_k)\xi \leq_{K[z_{k-1}]} z_{k-1}, \quad k \in \mathbb{N}. \tag{3.14}$$

We need to show that $q(x_k, x_{k+1}) \to 0$ as $k \to \infty$. Observe by (3.13) that the sequence $\{K[z_k]\}$ is *nonexpansive*, i.e., $K[z_k] \subset K[z_{k-1}]$ for all $k \in \mathbb{N}$. It follows from the convexity of $K[z_0]$ that

$$\sum_{k=0}^{m} K[z_k] = K[z_0] \text{ for all } m \in \mathbb{N} \cup \{0\}.$$

Summing up the inequalities in (3.14) from $k = 0$ to m and letting $t_m := \sum_{k=0}^{m} q(x_k, x_{k+1})$ yield

$$t_m\xi \in z_0 - z_{m+1} - K[z_0] \subset z_0 - M - \Theta - K[z_0] \text{ for all } m \in \mathbb{N} \cup \{0\}, \tag{3.15}$$

where the set $M \subset Z$ and the cone Θ are taken from the definition of quasiboundedness assumed in (C4). Now we claim that

$$\sum_{k=0}^{\infty} q(x_k, x_{k+1}) < \infty. \tag{3.16}$$

Arguing by contradiction, suppose that (3.16) does not hold, i.e., the increasing sequence $\{t_m\}$ tends to ∞ as $m \to \infty$. By using (3.15) and the boundedness of the set M, find a bounded sequence $\{w_m\} \subset z_0 - M$ satisfying

$$t_m\xi - w_m \in -\Theta - K[z_0] \Leftrightarrow \xi - w_m/t_m \in -\Theta - K[z_0], \quad m \in \mathbb{N} \cup \{0\}.$$

This gives us by letting $m \to \infty$ and taking into account the boundedness of $\{w_m\}$ and the divergence of $\{t_m\}$ that $\xi \in \text{cl}(-\Theta - K[z_0])$, which contradicts the choice of $\xi \in \Theta_K \setminus \text{cl}(-\Theta - K[z_0])$. Thus (3.16) holds and so $q(x_k, x_{k+1}) \to 0$ as $k \to \infty$.

Since all the assumptions of Theorem 2.3 are satisfied for the mapping Φ in (3.12) with $\Xi = \Phi(x_0, z_0)$, we now apply its conclusion and find $(x_*, z_*) \in \text{gph} F$ such that $(x_*, z_*) \in \Phi(x_0, z_0)$ and $\Phi(x_*, z_*) = \{(x_*, z_*)\}$. This clearly gives us (3.9) and (3.10). To complete the proof of the theorem, it remains to obtain estimate (3.11) when (x_0, z_0) is chosen as an $\varepsilon\xi$-approximate minimizer of F with $\gamma = (\varepsilon/\lambda)$. Arguing by contradiction, suppose that $q(x_0, x_*) > \lambda$. Since $(x_*, z_*) \in \Phi(x_0, z_0)$, where the quasimetric in question is now $(\varepsilon/\lambda)q(\cdot, \cdot)$ instead of $q(\cdot, \cdot)$ and since $\xi \in \Theta_K \subset K[z_0]$, we get the following estimate:

$$z_* \in z_0 - (\epsilon/\lambda)q(x_0, x_*)\xi - K[z_0] \subset z_0 - \epsilon\xi - K[z_0],$$

which clearly contradicts the choice of (x_0, z_0) as an $\varepsilon\xi$-approximate minimizer of F with respect to the ordering structure K. Thus we arrive at (3.11) and complete the proof of the theorem. □

Remark 3.2 (**on closedness assumptions**) If K is a constant ordering structure $K[z] \equiv \Theta$ with some convex ordering cone $\Theta \subset Z$ and if (X, q) is a metric space, then Theorem 3.1 reduces to the set-valued version of the Ekeland variational principle established in [5, Theorem 4.3], but with the following improvement: we do *not* assume now that F is *level-closed*, i.e., all the level sets

$$\mathrm{Lev}\,(z; F) := \left\{x \in X \middle|\, \exists\, v \in F(x) \text{ with } v \le_\Theta z\right\}, \quad z \in Z,$$

are closed in X. This closedness means that for any sequence $\{(x_k, z_k)\} \subset \mathrm{gph}\, F$ the existence of $z \in Z$ with $z_k \le_\Theta z$ for all $k \in I\!N$, and $x_k \to x_*$ implies the existence of $z_* \in F(x_*)$ with $z_* \le_\Theta z$. In our new version, even for constant ordering structure and metric space settings, we need this property only for sequences $\{(x_k, z_k)\}$ with the *monotonicity* of $\{z_k\}$ in the sense that $z_{k+1} \le_\Theta z_k$ for all $k \in I\!N$. It is called the *level-decreasing-closedness property*. To illustrate the difference, consider the following simple example of the extended-real-valued and lower semicontinuous (as in the original Ekeland principle) function $\varphi : I\!R \to \overline{I\!R} := I\!R \cup \{\infty\}$ defined by

$$\varphi(x) := \begin{cases} \infty & \text{if } x \ge 1, \\ x & \text{if } 0 < x < 1, \\ 0 & \text{if } x \le 0. \end{cases}$$

It is easy to check that φ is level-decreasingly-closed while not level-closed. Indeed, the 1-level set of φ is $(-\infty, 1)$ being an open subset of $I\!R$.

The next consequence of Theorem 3.1 provides a new quasimetric version of the classical Ekeland principle for extended-real-valued functions, which incorporates also the closedness improvement discussed in Remark 3.2. The later means that we do *not* require anymore the *lower semicontinuity* of the function in question, which has always been imposed in the literature.

Corollary 3.3 (**quasimetric version of the classical Ekeland variational principle**) *Let* (X, q) *be a* (*left-sequentially*) *complete Hausdorff quasimetric space, and let* $\varphi : X \to \overline{I\!R}$ *be a proper level-decreasingly-closed and bounded below function on* X. *Take any* $\varepsilon, \lambda > 0$ *and* $x_0 \in \mathrm{dom}\,\varphi$ *satisfying* $\varphi(x_0) \le \inf_{x \in X} \varphi(x) + \varepsilon$. *Then there is* $x_* \in \mathrm{dom}\,\varphi$ *such that*

$$\varphi(x_*) + (\varepsilon/\lambda) q(x_0, x_*) \le \varphi(x_0) \quad \left(\Longrightarrow \; \varphi(x_* \le \varphi(x_0) \text{ and } q(x_0, x_*) \le \lambda \right),$$

$$\varphi(x) + (\varepsilon/\lambda) q(x_*, x) > \varphi(x_*) \text{ for all } x \in \mathrm{dom}\,\varphi \setminus \{x_*\}.$$

Proof It follows directly from Theorem 3.1. □

Finally in this section, we demonstrate that the *Hausdorff property* (2.2) of the quasimetric space in question *cannot be dropped* in Theorem 3.1 and Corollary 3.3. To illustrate it, consider the quasimetric space $(I\!R, q)$ with q is defined in (2.3), which is not Hausdorff as discussed above. Take $\varphi(x) := e^{-x}$, $x_0 = 0$, $\xi = 1$ and $\varepsilon = \lambda = 1$ in the framework of Corollary 3.3. Since φ is decreasing, a candidate for x_* must be positive, say $x_* = a > 0$. Then

$$\varphi(x) + q(x_*, x) = e^{-x} + e^{a-x} < e^{-a} = \varphi(x_*)$$

for any x sufficiently large ($x > 2a + \ln 2$), which indicates that the minimization conclusion in Corollary 3.3 does not hold for $x_* = a$. Since a was arbitrarily chosen as long as $\varphi(a) < \varphi(x_0)$, this shows the failure of the EVP for l.s.c. functions on non-Hausdorff spaces.

It is worth mentioning that the Hausdorff requirements *was missed* in the formulations and proofs of quasimetric extensions of the EVP in [35] and some other publications.

4 Capability Approach and Dynamical Challenges

Here we discuss some principal issues of the famous *capability approach* to wellbeing. This approach is usually contrasted with two other main theories: the traditional income-based measures of welfare, and the subjective measures of wellbeing. The capability theory was first suggested by Amartya Sen in the 1980s as an approach to welfare economics [28] and then was extended by Sen and his followers to a broader range of behavior sciences developments; see, e.g., the books [20, 23, 25, 27, 29] and the references therein. The core focus of the capability approach is on what individuals/agents are able (capable of) to do or to be along their life, rather than on what bundle of commodities they can command with their income and how much utilities they derive from this bundle. In this section we first briefly recall basic elements of the static capability theory and then discuss challenging dynamical issues addressed and partly resolved in this paper.

4.1 Basic Notions of Static Capability Theory of Wellbeing

Let us start with some fundamental concepts of Sen's static capability approach.

Bundles of Commodities Agents have access to *bundle of commodities/resources* $x \in X$, where $\overline{X} \subset X$ signifies the agent's subset of resources.

Conversion Factors Every bundle of resources x is mapped into a *vector of characteristics* $c(x) \in C$ via a *conversion function* $c(\cdot)$. Characteristics are *independent* of the agent. However, the capability approach left *unspecified* whether a certain good is a resource or a conversion factor.

Utilization Functions Each agent has a *personal utilization function* $f : c \in C \longmapsto b = f(c/\varphi) \in \mathbf{B}$, which converts characteristics c into beings $b = f(c/\varphi) \in \mathbf{B}$, or more generally into functionings (beings and doings); see below. The vector b represents the beings (or functionings) that the agent has to manage in order to accomplish by using the commodities he/she possesses and choosing an utilization function f from $\mathbf{F}$. The vector of parameters $\varphi = (\varphi_i, \varphi_s, \varphi_e)$ represents individual (φ_i), social (φ_s) and environmental (φ_e) influences; see [20].

Functionings In the general sense, *functionings* consist of *beings and doings* and are crucial to adequately understand the capacity approach; capacity is conceptualized as a reflection of the freedom to achieve valuable functionings; see [29]. Each functioning can be described as a function $x \in X \longmapsto b = f(c(x)/\varphi)$ that tells us what the individual has achieved (a being b) given his/her choice of a utilization function $f \in \mathbf{F}$.

Capabilities The individual's overall quality of life, or wellbeing, is determined not only by what he/she has done (actions, i.e., doings) and has achieved (outcomes of his/her actions, i.e., beings), but also by the individual's *capability set* including all the other actions he/she could have done and all the other beings that could have reached. Sen defined [28] the set $P(x)$ of functioning vectors feasible for the agent in the form

$$P(x) := \{b \in \mathbf{B} \mid b = f\big(c(x)/\varphi\big) \text{ for some } f \in \mathbf{F}\}.$$

If the agent has only access to a subset of commodity bundles $\overline{X} \subset X$, his/her capability set is

$$Q(\varphi) := \{b \in \mathbf{B} \mid b = f\big(c(x)/\varphi\big) \text{ for some } f \in \mathbf{F} \text{ and some } x \in \overline{X}\}.$$

This subset defines the effective "freedom" that the individual has, given his/her commands over commodities and possibilities of converting the characteristics of goods into functionings. To summarize this brief discussion, we conclude that Sen's formalization goes *from commodities via functionings to capability*. Other *static* formalizations are given, e.g., in [19, 27].

4.2 Dynamical Aspects of Capability Theory

Sen's original model and subsequent developments offer a *static view* of capabilities and consider characteristics and utilization functions as given exogenously, which is not the case as recognized by Sen himself [28]. Several authors have considered that this static aspect is the main limitation of Sen capability approach. In particular, Brandolini and D'Alessio [10] wrote: "...the lack of a dynamic orientation makes the (Sen) approach ill-suited to deal with scenarios of the following sort: a person's capability set at time k might reject voluntary choices of the person at time $k-1$. A student willingly commits to years of study in relative poverty in order to later secure a better job. That student might initially have foregone a bigger opportunity set (through work without study) to later have an even larger opportunity set." In this paper we develop a *dynamical approach* to capability theory, which modelizes the co-evolution of functionings and preferences.

Course Pursuit between Functionings and Preferences Sen's capability approach is driven by the two main concepts: **(i)** *functionings and capabilities*, and **(ii)** *adaptive preferences*. These two basic ingredients have strong *dynamical aspects*. First of all, functionings (beings and doings, i.e., human behaviors) can change or stay. They are determined endogenously by agents. Given the current period, what is really important for the agent is not only his/her current functionings, but mainly what the agent can be able to do or to be in the future, i.e., his/her capabilities. Furthermore, it has been one of the starting points of the capacity approach that preferences should adapt to favorable or adverse circumstances.

It what follows we propose to take into account the two interrelated dynamical issues: *dynamics of functionings and capabilities*, and *dynamics of preferences*. Let us proceed in this way by offering a *course pursue* between functionings and *variable* preferences with possible intermediate stays.

How Capabilities Evolve As in Sen's model, $x \in X$ represents a bundle of commodities and $c(x)$ signifies characteristics of this bundle. Let $\omega \in \Omega(x) \subset \Omega$ be a way of using these commodities (a *way of functioning*) let $f(c(x), \cdot/\varphi) : \omega \in \Omega(x) \longmapsto z = f(c(x), \omega/\varphi) \in Z$ be an utilization function (a *functioning itself*), and let $z = b = f(c(x), \omega/\phi) \in Z = \mathbf{B}$ be what the agent achieves (a vector of *beings*). As before, the vector $\varphi = (\varphi_i, \varphi_s, \varphi_e)$ represents the given influences: individual φ_i, social φ_s, and environmental φ_e. By $F(x) := \{z = f(c(x), \omega/\varphi) | \ \omega \in \Omega(x)\} \subset Z$ we denote the subset of beings relative to the given bundle of commodities $x \in X$, i.e., the subset of beings that the commodity bundle x can command. Given the agent's resources, let $\overline{X} \subset X$ be the subset of commodity bundles that the agent can access. Then the *capability set* is

$$Q := \left\{z \in Z \middle| \ z = f(c(x), \omega/\varphi) \ \text{ for some } \ \omega \in \Omega(x) \ \text{ and } \ x \in \overline{X}\right\} = \bigcup_{x \in \overline{X}} F(x).$$

Note that the set-valued mapping $F(\cdot) : x \in \overline{X} \subset X \longmapsto F(x) \subset Z = \mathbf{B}$ defined above signifies the subset of *beings reachable* from each x. This means that the inclusion $z \in F(x)$ is understood as the existence of a *way of using* $\omega \in \Omega(x)$ such that $z = f(c(x), \omega/\varphi)$.

In this discrete dynamical setting, at each period $k \in I\!N$ the agent can use a commodity bundle $x_k \in \overline{X}$ to reach any being from $z_{k+1} \in F(x_k)$. This defines a *dynamic of the reachable beings* as soon as we determine a *dynamic of the commodity bundles* $x_k \in \overline{X}$.

How Preferences Adapt The problem of *adaptive preferences* is one of the tenets and major starting points of the capability approach, and it is known therein as the "hopeless beggar" or "adaptation problem." Sen emphasized that agents adapt their preferences to what they are able to get. For example, individuals who are denied decent conditions of living can consider themselves to be happy or satisfied if they accept their situations. To the best of our knowledge, the current developments in capability theory do not present adequate models of preference adaptation. A major point of this paper is to offer a *multidimensional modelization* of how preferences adapt based on *dynamical issues* presented in the developed *variational technique*.

Wellbeing and Illbeing/Poverty Traps Considering the capability approach as a *dynamical process*, where functionings and adaptive preferences interact along the whole transition period, requires the description of *endpoints* of this dynamics. In real-life problems such endpoints correspond to *wellbeing* or *illbeing traps*. The concept of traps can be found almost everywhere in behavior sciences (cognitive, decision, affective, motivational, technological, social traps, etc.). In particular, in psychology a behavioral trap is "easy to enter and difficult to exit" [1]. Another striking example comes from economics, where a poverty trap is "any self-reinforcing mechanism, which causes poverty to persist" [3, p. 33]. The end of the dynamical process is treated there as an equilibrium at a low-level of wellbeing. Let us also mention the recent source [14] related to the *hope* model, with well documented and strong empirical findings; see more details on this model in Section 7. A major issue is to find a rigorous formalization of such traps and derive practical conclusions based on mathematical results, which is done below.

Lack of Resources and Capabilities As Comim [12] wrote, "one of the most important contributions of the capability approach is its emphasis on the processes that allow individuals to expand their capability set (exercise their freedoms)." This process aspect could also be seen in Sen's distinction between *culmination outcomes* (i.e., outcomes that ignore the processes of getting there) and *comprehensive outcomes*, i.e., outcomes that consider such processes; see [29].

Poverty Traps as Lack of Aspirations In an influential contribution, Appadurai [2] argued that the poor may lack the capacity to aspire, and policies that strengthen this capacity could help them to " contest and alter the conditions of their poverty" (p. 59). Aspiration is a desire or ambition to achieve something. As written in [2], the "capacity to aspire" involves not only setting goals but also knowing how to reach them: the poor may lack the capacity to aspire to contest and alter the conditions of their own poverty.

5 Variational Rationality: Prototype Model

In this section we summarize and discuss basic elements of the *variational rationality* (VR) approach for the prototype model from [33, 34]. In Sections 6 and 7 we extend the VR approach to a new *adaptive dynamical model* in capability theory, which is of our main interest in this paper. To be able to do that, we split into two steps the presentation of the VR model. This allows us to better correlate with the mathematical developments given in Section 2 and 3. The first step is to consider dynamics of acceptable stays and changes. The second step is to make more precise what is the concrete content of acceptable stays or changes. The answer is: *worthwhile stays and changes*.

5.1 Variational Rationality: Succession of Acceptable Stays and Changes

Adaptive Processes of Acceptable Temporary Stays and Changes Agent's behavior is defined as a succession $\{x_0, \ldots, x_k, \ldots\} \subset X$ of actions entwining possible *stays* $x_k \in X \curvearrowright x_{k+1} \in X$, $x_{k+1} = x_k$ and possible *changes* $x_k \in X \curvearrowright x_{k+1} \in X$, $x_{k+1} \neq x_k$. This behavior is said to be *variationally rational* if at each period $k+1$ the agent chooses to *change or to stay* depending on what he/she accepts to consider as an acceptable change in contrast to a stay. Such acceptable changes balance satisfactions and sacrifices to change rather than to stay. Agents accept to change if the satisfactions to change are high enough with respect to the sacrifices. This VR approach generalizes and extends to dynamic settings, and in several ways, the well-recognized Simon static satisficing theory of *rational choice* [30] concerning behavior of bounded and procedural rational agents. It focuses the attention not only on *satisfactions* (as in Simon's case) but also on *sacrifices*. Then the agent is supposed to be rational enough trying to implement at each period $k+1$ a succession of acceptable (satisficing with not too much sacrifices) stays and changes $x_{k+1} \in W_{e_k,\gamma_{k+1}}(x_k)$, $\gamma_{k+1} \in \Upsilon$ as $k \in I\!N$, where $e_k \in E$ stands for the past experience. Details follow.

Acceptable Transitions Given the *degree of acceptability* $\gamma_k \in \Upsilon$ at step $k \in I\!N$ (discussed later), the agent performs his/her action x_k. Using this at step $k+1$, the agent *adapts* his/her behavior in the following way: chooses a new degree of acceptability $\gamma_{k+1} \in \Upsilon$ of the next acceptable change $x_{k+1} \in W_{e_k,\gamma_{k+1}}(x_k)$, which may be the same as before. This degree of acceptability satisfies some tolerable sacrifices and depends on how much acceptable the agent considers that the action must be to accept to change rather than to stay. There are two possible cases:

(i) **A temporary acceptable stay** is defined by $x_k \curvearrowright x_{k+1} = x_k$, which is the case when $W_{e_k,\gamma_{k+1}}(x_k) = \{x_k\}$. In this case the agents chooses, in a rational variational way, to stay at $x_k = x_{k+1}$. If furthermore at the subsequent steps $k+2, k+3, \ldots$ the agent does not change the degree of acceptability, he/she chooses to stay there forever at least if his experience is of Markov type. This defines an "acceptable to stay" *trap*, which is a *permanent acceptable stay*.

(ii) **A temporary acceptable change** is defined by $x_k \curvearrowright x_{k+1} \neq x_k$. In this the case $W_{e_k,\gamma_{k+1}}(x_k) \neq \{x_k\}$ and the agent can find $x_{k+1} \in W_{e_k,\gamma_{k+1}}(x_k)$ with $x_{k+1} \neq x_k$. Then the agent chooses to move from x_k to $x_{k+1} \in W_{e_k,\gamma_{k+1}}(x_k)$, and so on.

Endpoints as Variational Traps Given the final "acceptable to change" rate $\gamma_* > 0$, and a final experience $e_* \in E$, we say that the endpoint $x_* \in X$ ends in an *acceptable stay* and that the worthwhile change process $x_{k+1} \in W_{e_k,\gamma_{k+1}}(x_k)$ as $k \in I\!N$ is a *stationary trap* if $W_{e_*,\gamma_*}(x_*) = \{x_*\}$. Furthermore, we say that $x_* \in X$ is an *aspiration point* of the acceptable stay and change process $x_{k+1} \in W_{e_k,\gamma_{k+1}}(x_k)$, $k \in I\!N$, if it can be reached by a direct acceptable change $x_* \in W_{e_k,\gamma_{k+1}}(x_k)$. An aspiration point x_* is *feasible* if there are enough resources to be able to reach the endpoint starting from the initial point x_0. Finally, a *variational trap* is both a stationary trap and an aspiration point of a worthwhile stay and change dynamic.

Variational Rationality Objectives include the following major components. Starting with any given initial point $x_0 \in X$ and depending on the satisfaction and sacrifice functions, we want to find a *path of acceptable changes* so that:

(i) the steps go to zero and have *finite length*;
(ii) the corresponding iterations converge to a *variational trap*;
(iii) the *convergence rate* and *stopping criteria* are determined;
(iv) the *efficiency* or *inefficiency* of such acceptable to change processes are studied to clarify whether the acceptable to change process ends at a *critical point*, a *local or global optimum*, a *local or global equilibrium*, an *epsilon-equilibrium*, a *Pareto solution*, etc.

5.2 Acceptable Changes as Worthwhile Changes

Dynamics of acceptable stays and change being defined as before, the VR approach considers *worthwhile changes*, $x_{k+1} \in W_{e_k,\gamma_{k+1}}(x_k)$ as specific instances of acceptable changes, where satisfactions refer to motivations to change and sacrifices refer to resistances to change rather than to stay. In this paper we limit the impact of experience $e = e_k \in E$ to the last action $e_k = x_k$ (the Markov case). Then a dynamic of worthwhile temporary stays and changes simplifies in $x_{k+1} \in W_{\gamma_{k+1}}(x_k)$.

Motivation to Change $M(x, x') = U[A(x, x')]$ is defined as the *pleasure* or *utility* $U[A]$ of the advantage to change $A(x, x') \in I\!R$ from x to x'. In the simplest case we define the *advantages to change* as the difference $A(x, x') = g(x') - g(x)$ between a payoff to be improved (e.g., performance, revenue, profit) $g(x') \in I\!R$ when the agent performs a new action x' and the payoff $g(x) \in I\!R$ when he/she repeats a past action x supposing that repetition gives the same payoff as before. On the other hand, the advantages to change $A(x, x') = f(x) - f(x')$ can also be the difference between a payoff $f(x)$ to be decreased (e.g., cost, unsatisfied need) when the agent repeats the same old action x and the payoff $f(x')$ the agent gets when he/she performs a new action x'. The *pleasure function* $U[\cdot] : A \in I\!R \longmapsto U[A] \in I\!R_+$ is strictly increasing with the initial condition $U[0] = 0$.

Resistance to Change $R(x, x') = D[I(x, x')]$ is defined as the *pain* or *disutility* $D[I]$ of the inconveniences to change $I(x, x') = C(x, x') - C(x, x) \in I\!R_+$, which is the difference between the costs to be able to change $C(x, x') \in I\!R_+$ from x to x' and the costs $C(x, x) \in I\!R_+$ to be able to stay at x. In the simplest case we suppose that $C(x, x) = 0$ for all $x \in X$ while the costs to be able to change are defined as the *quasidistances* $C(x, x') = q(x, x') \in I\!R_+$ satisfying the conditions: **(a)** $q(x, x') \geq 0$, **(b)** $q(x, x') = 0 \iff x' = x$, and **(c)** $q(x, x'') \leq q(x, x') + q(x', x'')$ for all $x, x', x'' \in X$. The *pain function* $D[\cdot] : I \in I\!R_+ \longmapsto D[I] \in I\!R_+$ is strictly increasing with $D[0] = 0$.

Worthwhile Change and Stay Processes The VR approach defines as acceptable any change or stay that is worthwhile or not worthwhile by the following algorithm. Consider step $k+1$, and let $x = x_k$ be the preceding action. At this step the agent first take the acceptability ratio $\gamma' = \gamma_{k+1} \in I\!R_+$. To choose further a new action $x' = x_{k+1}$, the agent calculates the motivation to change $M(x, x') \in I\!R$ and the resistance to change $R(x, x') \in I\!R_+$ from x to x'. Then the agent decides that it is worthwhile to move from x to x' if his/her motivation to change exceeds the resistance to change up to the acceptability ratio γ_{k+1}, i.e., if $M(x, x') \geq \gamma' R(x, x')$. Thus the worthwhile to change and stay process satisfies the alternative conditions

$$g(x_{k+1}) - g(x_k) \geq \gamma_{k+1} q(x_k, x_{k+1}) \quad \text{or} \quad f(x_k) - f(x_{k+1}) \geq \gamma_{k+1} q(x_k, x_{k+1})$$

for each $k \in I\!N$. This yields the alternative representations for worthwhile to change or stay sets

$$\begin{aligned} W_{\gamma'}(x) &= \left\{ x' \in X \middle| \, g(x') - g(x) \geq \gamma' q(x, x') \right\} \text{ or } W_{\gamma'}(x) \\ &= \left\{ x' \in X \middle| \, f(x) - f(x') \geq \gamma' q(x, x') \right\}. \end{aligned}$$

It is worth mentioning that the defined worthwhile transitions are *path dependent* in the sense that when worthwhile to change sets are nested, the set of decisions the agent can take today is limited by the decisions he/she has made in the past, even though past circumstances may no longer be relevant. We refer the reader to [33, 34] for more details and discussions in this direction.

Entering Ekeland's Variational Principle To illustrate the connections with Ekeland's variational principle, consider the simplest model with $\gamma_{k+1} = \gamma_* = \varepsilon/\lambda > 0$ for all $k \in I\!N$ and $g(x_0) \geq \overline{g} - \varepsilon$, where $\overline{g} = \sup\{g(y), y \in X\} < \infty$. In this model we have that $x_* \in X$ is:

(i) a *stationary trap* $W_{\gamma_*}(x_*) = \{x_*\}$ if $g(x') - g(x_*) < \gamma_* q(x_*, x')$ or $f(x_*) - f(x') < \gamma_* q(x_*, x')$ whenever $x' \in X$ with $x' \neq x_*$;

(ii) an *aspiration point* if $x_* \in W_{\gamma_*}(x_k)$, i.e., $g(x_*) - g(x_{k+1}) \geq \gamma_* q(x_{k+1}, x_*)$ or $f(x_{k+1}) - f(x_*) \geq \gamma_* q(x_{k+1}, x_*)$ for all $x_{k+1} \neq x_*$, $k \in I\!N$;

(iii) a *feasible aspiration point* if $q(x_0, x_*) \leq \lambda$, where λ represents the initial level of resources.

Applying to this model of variational rationality, the classical Ekeland's principle supposes a *constant rate* of acceptability, and hence the corresponding stay and change dynamic is *not adaptive*. It also supposes the boundedness from above and upper semicontinuity of the maximizing payoff $g(\cdot)$ and a *symmetric* cost to be able to change function, which is the distance function $C(x, x') = d(x, x') = d(x', x)$. As follows from the discussions above, this is not sufficient to cover adaptive dynamical issues coming from realistic requirements of capability theory.

6 Variational Models in Dynamical Capability Theory

Among major requirements to mathematical modeling of adaptive dynamical processes in capability theory from the viewpoint of extended variational rationality approach we list the following:

- **(i)** *Multidimensional set-valued framework* dealing with *Pareto efficiency* in the space of beings.
- **(ii)** *Dynamics* of the improving process, where motivation and resistance to change matter.
- **(iii)** *Adaptive/variable preferences*, which help balancing at each step between variable weights to solve the tradeoff between different reachable beings.

Our presentation follows the logic of the VR approach, and it moves from the vague concept of acceptable stay or change to the much more precise concept of worthwhile stay or change. First we consider modeling the functionings/preferences dynamics in term of acceptable stays and changes, which mainly relates to the fixed point theorem of Section 2. Then our attention is paid to the functionings/preferences dynamics in term of worthwhile stays and changes, which relates to the obtained variational principle for mappings with variable preferences.

6.1 Wellbeing Adaptive Dynamics of Acceptable Stays and Changes

Given a subset $\Xi \subset X \times Z$ of the commodity-being product space, we say that a wellbeing pair (x, z) is *feasible* if $(x, z) \in \Xi$. Note that Ξ defines a functioning set-valued mapping $F\colon X \rightrightarrows Z$ by $z \in F(x)$ if and only if $(x, z) \in \Xi$. Suppose now that the agent follows a feasible wellbeing dynamics $(x_{k+1}, z_{k+1}) \in \Phi(x_k, z_k)$ from a current wellbeing pair $(x_k, z_k) \in \Xi$ to the next one $(x_{k+1}, z_{k+1}) \in \Xi$, where $\Phi(x, z) \subset \Xi$. This feasible wellbeing dynamics allows the agent to *stay* if $(x_k, z_k) \in \Phi(x_k, z_k)$ for all $k \in I\!N$. It is *nested* if the inclusion $(x_{k+1}, z_{k+1}) \in \Phi(x_k, z_k)$ yields $\Phi(x_{k+1}, z_{k+1}) \subset \Phi(x_k, z_k)$ for all $k \in I\!N$. Consider the following *classes of preferences*:

- **(i)** Abstract *preorders* $\preceq$ on the wellbeing (commodity-being) product space $X \times Z$, which a *reflexive* and *transitive* preferences. Then we define a feasible and improving dynamics of feasible wellbeing pairs $(x, z) \in \Xi$ by $(x_{k+1}, z_{k+1}) \in \Phi(x_k, z_k)$, $k \in I\!N$, where

$$\Phi(x, z) := \left\{(u, v) \in X \times Z \,\Big|\, (u, v) \preceq (x, z)\right\}.$$

- **(ii)** *variable preferences* $\geq_{K(z)}$ on the space of beings Z, which show how to solve, in an adaptive way, tradeoff problems between *multidimensional* components of beings $z \in Z$. More precisely, associate with each vector of beings z from a topological space Z a variable proper cone $K(z) \subset Z$, which is supposed to be closed and convex while satisfying also the monotonicity property: $v \in K(z)$ implies that $K(v) \subset K(z)$. Under some consistency assumptions, this defines a *variable domination structure* on the space of beings.

We are interested in the following major issues:

- **(1)** The *convergence* of such an improving wellbeing dynamic $(x_{k+1}, z_{k+1}) \in \Phi(x_k, z_k)$ as $k \to \infty$ to some wellbeing limiting position $(x_*, z_*) \in \Xi$.

(2) Effective conditions ensuring that the wellbeing limiting position $(x_*, z_*) \in \Xi$ is an *endpoint* of this improving dynamic, i.e., $\Phi(x_*, z_*) = \{(x_*, z_*)\}$.

(3) The *efficiency* of this endpoint in the sense that it is a *Pareto optimal solution* and it can describe *wellbeing/illbeing traps*.

6.2 Towards Wellbeing Adaptive Dynamics of Worthwhile Stays and Changes

The usage of an *abstract* improving dynamic description to modelize how functionings and capabilities can change or stay is *too vague* and does not seem to be of practical help for offering a realistic theory of wellbeing. Indeed, what does it mean to "improve" wellbeing and how to describe wellbeing or illbeing/poverty traps that happen in real world? In Section 7 we justify, based on the obtained mathematical results, that the wellbeing dynamics follows a succession of *worthwhile* temporary wellbeing stays and changes. These variational rationality processes define worthwhile to change reference dependent preferences. To proceed in this way and having in mind the *multiobjective* and *adaptive* aspects of the wellbeing concept, we have to consider *variable preferences* $\preceq_r$ that take into account variable weights on different criteria on the product space of actions and payoffs, where r indicates the current reference point depending on the current experience of the agent and a future satisficing level. Recall that in applications to capability theory, actions represent commodity bundles $x \in X$ and payoffs refer to beings $z \in Z$. In Section 4 we have already discussed various types of wellbeing and illbeing/poverty traps, which arise in real-life models of behavioral sciences and which can be considered from dynamical viewpoints of capability theory. Now we intend to modelize these traps as *variational traps* in adaptive dynamical modeling of variational rationality. Furthermore, the challenge is to constructively describe them as endpoints of the corresponding worthwhile stay and change adaptive dynamical processes similarly to the prototype model of human behavior discussed above.

6.3 Variational Rationality Challenges to Variational Principles

It follows from the previous discussions that proper versions of variational principles of the Ekeland type can provide the key impacts to meet our goals in dynamical issues of capability theory via the adaptive approach of variational rationality. Indeed, even the classical Ekeland principle and its proof can catch the dynamics in worthwhile change and stay processes with endpoint variational traps. However, the classical Ekeland framework and its known extensions are not sufficient to meet our purposed in adaptive dynamical models of capability theory. The following three major mathematical challenges should be imposed simultaneously to fits its requirements:

(i) the possibility of being applied to *set-valued mappings*;
(ii) *quasimetric structures* on the domain spaces of such mappings;
(iii) *variable preference structures* on the ordering image spaces.

As the reader can see, all these requirements are met in the mathematical framework of Section 3; see [8] for detailled psychological aspects, and so now we are able to apply the obtained results to capability theory of wellbeing. It is done in the next section, where applications of these results, as well as the related fixed point/minimal point results of Section 2, to capability theory are given via the variational rationality approach. Furthermore, we also interpreted behavioral meanings of the assumptions made in the theorems.

7 Major Findings and Applications to Hope Model

Now we are in a position to consider applications of the developed variational approach and results to dynamical issues of capability theory. We first reveal the *behavioral sense* of the *assumptions* and *conclusions* of the main theorems of Section 2 and 3. It opens the gate of their applications to a broad spectrum of adaptive dynamical problems of capability theory, including those formulated and discussed in Section 4. We consider these results among major findings of this paper.

Then we address in more detail the *hope model* mentioned in Section 4. After considering empirical data, we present a *variational description* of the hope model and, based on the developed VR approach and obtained mathematical results, show how capability traps can be seen as the ends of a succession of worthwhile stays and changes.

7.1 Behavioral Sense and Applications of the Fixed Point Theorem

We start with Theorem 2.3, which is a new fixed point result for parametric dynamical systems. This variational result plays a crucial role in our proof of the variational principle in Theorem 3.1 and shares the major assumptions with the latter. Applying in the framework of capability theory to feasible pairs $(x, z) \in \text{gph}\, F \subset X \times Z$ of commodities and beings, it reflects an *acceptable* stay and change dynamic and justifies the existence of a *fixed point* at the end as a variational trap resulting from convergent and acceptable stay and change processes, where the agent *accepts* at each intermediate step to change while *stops accepting* to change at the end of the process.

Let us first discuss the behavioral content of the major assumptions of this theorem, which it mostly shares with the variational principle of Theorem 3.1, and then reveal the behavioral meaning of the established conclusions for dynamical issues of capability theory.

Costs to be Able to Change as Quasidistances They are given as $C(x, x') = q(x, x')\xi \in Z$ and represent for each individual his/her costs to be able to acquire the usage of the bundle of commodities $x' \in X$ provided that this agent owns and is able to use ex ante the bundle of commodities $x \in X$. It gives us $C(x, x) = 0$ for all $x \in X$, which corresponds to the VR settings discussed in Sections 5 and 6. The costs to acquire *new abilities* (commodities in this case), starting from having old ones, significantly help to modelize *inertia* and *learning aspects* within capability theory. They mainly concern durable goods that survive to their first utilization. The costs to acquire of *nondurable goods* (ingredients, efforts, etc.) are embedded in the *utilization function*. The triangular inequality for quasimetrics comes from the definition of a change as a *path of operations* (e.g., acquisitions, conservations, deletions), where each operation generates a cost. Then the costs to be able to change are the sum of these costs. Their infimum defines a *quasidistance* (not the distance due to the obvious lack of symmetry) and modelizes ability costs as costs to be able to change. It fully justifies the *quasimetric* requirement on the spaces in questions in the mathematical results of Sections 2 and 3.

Acceptable Stays or Changes In behavioral models not all changes are acceptable; some can be rejected. Changes that are acceptable (i.e., are not rejected) are described in Theorem 2.3 as follows: the transition from the feasible means-end pair $(x, z) \in \text{gph}\, F$ to the new feasible means-end pair $(x', z') \in \text{gph}\, F$ is *acceptable* if and only if we have $(x', z') \in \Phi(x, z) \subset X \times Z$, which is condition (A1) of Theorem 2.3.

Dynamics is Path Dependent The improving processes are path dependent because they are *nested*, which follows from condition (A2).

Aspiration Points Exist This corresponds to the *limiting monotonicity* condition (A3) realistic in general for models of behavioral sciences (see [7, 8] for more details); in particular, for dynamical issues of capability theorem discussed in Section 4.

Agents Make Small Steps This is the *convergent condition* (A4), which reflects one of the main realistic characteristics of bounded rational and muddling through processes; see [21, 30] and a short survey of these two famous approaches in [8].

Endpoints are Variational Traps This is the *conclusion* of Theorem 3.1 about the existence of a *fixed point* of the acceptable stay and change dynamics.

Parametric Minimal Point Theorem This result, Theorem 2.5, provides an equivalent interpretation of the acceptable stay and change dynamics in the product space $X \times Z$ and justifies the existence of a variational trap (the end of the improving process) as a *minimal point/Pareto solution* with respect to a certain reflexive and transitive preference. The latter assumptions are satisfied in the dynamical models of capability theory discussed above.

7.2 Impacts of the Main Variational Principle

The major assumptions of Theorem 3.1, which are interrelated with (A1)–(A4) in Theorem 2.3, as well as necessity of the quasimetric requirement have been discussed in the previous subsection. Let us reveal now the remaining behavioral features specific for the obtained variational principle for set-valued mappings with variable preferences.

Variational Principle as Dynamic Process Analyzing Theorem 3.1 and its proof shows that the *proof itself* provides a dynamical *algorithmic* iterative process of *making decisions* at each step on worthwhile change or stay *adaptive dynamics* and establishes verifiable sufficient conditions for the *existence* of a *variational trap*, which can be interpreted as the corresponding wellbeing or illbeing trap in dynamical problems of capability theory including those those listed in Section 4.

Variable Ordering Structures The presence of adaptive preferences is at the core of Sen's capability approach. These variable preferences $\ge_{K[z]}$ are applied on the space of beings $z \in Z$ and are defined by variable cones $K[z] \subset Z$. The cones $K[z]$ and their intersection play a major role in reflecting a *minimal coherence* between the changing preferences in behavioral science models and dynamical issues of capability theory discussed in Section 4.

Functionings are Described by Set-Valued Mappings Such a mapping is given by $F : x \in X \longmapsto F(x) \subset Z$ in Theorem 3.1 and defines, for each commodity bundle $x \in X$, the agent's feasible functionings, which he or she possesses as

$$F(x) := \left\{ z = f(x, \omega) \middle| \, \omega \in \Omega(x) \right\}.$$

Hypothesis (C4) on the quasiboundedness of F from below means actually that illbeings are bounded from below. Hypothesis (C5) supposes the existence of weak aspiration points. If F has the domination property, this tells us that the agent may hope for better.

Justification of Worthwhile Changes Theorem 3.1 and its proof tells us that

$$A(z, z') \geq_{K[z]} \gamma I(x, x')$$

in our notation, which provides the justification of worthwhile changes.

Variational Traps *Conclusions* (3.9)–(3.11) describe the achieved properties of variational traps as endpoints of the convergent iterative process of worthwhile changes. Their behavioral interpretations are similar to those given for the prototype VR model in Section 5 with the major addition that now we deal with real-life *adaptive* models of capability theory (and generally of behavioral sciences) by employing *variable multidimensional preferences*. This vectorial aspect is very important, given the huge amounts of different needs and criteria to define wellbeing.

Let us finally apply the VR approach and obtained results to the case of *hope model*, which can be considered as a dynamical outgrowth of capacity theory; cf. Section 4.

7.3 Hope Model via Variational Rationality

Let us first describe the hope problem and some conclusions based in empirical findings.

Empirical Findings: Poverty Traps as Lack of Hope It has been widely accepted that hope is characterized by expectations that desired goals will be attained. Snyder [31] went a step further. He showed that hope comprises two beliefs occurring simultaneously: **(1)** the individual is capable of executing the means to attain desired goals (agency thinking), and **(2)** the individual is capable of generating those means (pathways thinking). Then hope is defined as “the perceived capability to derive pathways to desired goals and to motivate oneself via agency thinking to use those pathways.” In other words, hopeful people believe that they are able to do something to obtain their goals. More recently, Duflot [14] has argued that a strong lack of hope, and not just capitals, credits, skills, or food, could create and sustain a poverty trap. Drawing on the results of a number of empirical studies, she listed several reasons why the poor might neglect opportunities to improve their economic conditions and discussed how hope can help breaking poverty traps.

Applications of the Variational Principle From the viewpoint of the VR approach, hope can be defined in term of the existence of appropriate *worthwhile changes*. Indeed, the VR approach leads us to the following formulation of the *hope variational problem*:

$$\text{given } (x, z) \in \text{gph}\, F \text{ and } z' \in Z, \text{ find } x' \in X \text{ with } z' \in F(x'),\ z - z' \geq_{K[z]} \gamma q(x, x')\xi, \quad (7.17)$$

where hope corresponds to the being $z' \in Z$, and where variable preferences $\geq_{K[z]}$ adapt to the current being $z \in Z$. In this interpretation the condition $z - z' \geq_{K[z]} \gamma q(x, x')\xi$ tells us that it is *worthwhile to change* from $(x, z) \in \text{gph}\, F(x)$ to $(x', z') \in \text{gph}\, F(x')$. Observe that in format (7.17) the notion of wellbeing is actually *illbeing* (poverty) and that the cone “minimization” is applied to *unsatisfied needs* (i.e., payoffs to be decreased) described by $z = f(x, \omega)$ and $z' = f(x', \omega')$. This means that *advantages to change* expressed by $A(z, z') = z - z'$ are higher than some proportion $\gamma > 0$ of *inconvenience to change* $I(x, x') = C(x, x') - C(x, x) \geq_{K[z]} 0$, where the *costs* to be able to change are given by $C(x, x') = q(x, x')\xi$ and $C(x, x) = 0$ for all $x \in X$. In other words, the hope variational problem (7.17) can be reformulates as: given $z \in F(x)$ and $z' \in Z$, find $x' \in X$ with

$$z' \in F(x') \text{ and } A(z, z') \geq_{K[z]} \gamma I(x, x').$$

In this variational model the *lack of hope* results from the lack of motivation and/or too much inertia and resistance to change. Applying our main *variational principle* of Theorem 3.1 with $\xi = \gamma$ shows that this leads us to a *poverty trap*, which is described as $z_* \in F(x_*)$ and turns out to be a particular instance of *variational traps*, where

$$z_* - z' <_{K(z_*)} \gamma q(x_*, x')\xi \quad \text{for all} \quad (x', z') \in \operatorname{gph} F \neq (x_*, z_*) \in \operatorname{gph} F$$

in the notation of Theorem 3.1 with their behavioral interpretation given in this section.

8 Concluding Remarks

This paper is one of the first attempts to apply variational principles and techniques of variational analysis to adaptive dynamical models of behavioral sciences. We mainly concentrate here on dynamical aspects of capability theory by applying and developing the VR approach to multiobjective and set-valued frameworks of Pareto efficiency arising in capability theory of wellbeing. The major challenge for our consideration is provided by *multidimensional variable preferences*, which motivate us to derive new variational principles and related mathematical results that are able to address these needs. In this way we offer new insights into dynamical problems of capability theory of wellbeing by developing it as a course pursuit between functionings and adaptive preferences.

Our future research directs to more detailed studies of various adaptive and dynamical aspects of capability theory (including those discussed in Section 4) by using the VR approach and adequate tools of variational analysis, which should be developed to meet new requirements for applications. We also plan to investigate the possibility for applications to capability theory of the recent variational principle derived in [32] for problems with variable preferences, which is different from ours. Furthermore, our future goals include applications of the developed variational principles and techniques to other adaptive models of behavioral sciences.

References

1. Alber, S.R., Heward, W.L.: Twenty-five behavior traps guaranteed to extend your students' academic and social skills. Interv. School Clin. **31**, 285–289 (1996)
2. Appadurai, A.: The capacity to aspire: culture and the terms of recognition. In: Rao, V., Walton, M. (eds.) Culture and Public Action: A cross Disciplinary Dialog in Development Policy. Stanford University, CA (2004)
3. Azariadis, C., Stachurski, J.: Poverty traps. In: Aghion, P., Durlauf, S. (eds.) Handbook of Economic Growth. Elsevier, Amsterdam (2004)
4. Bao, T.Q., Mordukhovich, B.S.: Variational principles for set-valued mappings with applications to multiobjective optimization. Control Cybern. **36**, 531–562 (2007)
5. Bao, T.Q., Mordukhovich, B.S.: Relative Pareto minimizers for multiobjective problems: existence and optimality conditions. Math. Program. **122**, 301–347 (2010)
6. Bao, T.Q., Mordukhovich, B.S.: Necessary nondomination conditions for set and vector optimization with variable structures. J. Optim. Theory Appl. **162**, 350–370 (2014)
7. Bao, T.Q., Mordukhovich, B.S., Soubeyran, A.: Variational analysis in psychological modeling. J. Optim. Theory Appl. doi:10.1007/s10957-014-0569-8 (2014)
8. Bao, T.Q., Mordukhovich, B.S., Soubeyran, A.: Variational principles in models of behavioral sciences. Arxiv:1311.6017 (2013)
9. Borwein, J.M., Zhu, Q.J.: Techniques of Variational Analysis. Springer, New York (2005)
10. Brandolini, A., D'Alessio, G., Chiappero-Martinetti, E.: Measuring well-being in the functioning space, pp. 91–156. Fondazione Giangiacomo Feltrinelli, Milan (2009)

11. Cobzaş, S.: Functional Analysis in Asymmetric Normed Spaces. Birkhäuser, Basel (2013)
12. Comim, F.: Capability dynamics: the importance of time to capability assessments, Capability and Sustainability Centre, St. Edmund's College, University of Cambridge, UK. http://cfs.unipv.it/sen/papers/Comim.pdf (2003)
13. Dancs, S., Hegedüs, M., Medvegyev, P.: A general ordering and fixed point principle in complete metric space. Acta Scient. Math. **46**, 381–388 (1983)
14. Duflo, E.: Hope, aspirations and the design of the fight against poverty, Stanford University, CA. https://events.stanford.edu/events/393/39325 (2013)
15. Eichfelder, G.: Variable Ordering Structures in Vector Optimization. Springer, Berlin (2014)
16. Eichfelder, G., Ha, T.X.D.: Optimality conditions for vector optimization problems with variable ordering structures. Optimization **62**, 597–627 (2013)
17. Ekeland, I.: Nonconvex minimization problems. Bull. Amer. Math. Soc. **1**, 432–467 (1979)
18. Göpfert, A., Riahi, H., Tammer, C., Zălinescu, C.: Variational Methods in Partially Ordered Spaces. Springer, New York (2003)
19. Herrero, C.: Capabilities and utilities. Econ. Des. **2**, 69–88 (1996)
20. Kuklys, W.: Amartya Sen's Capability Approach: Theoretical Insights and Empirical Applications. Springer, Berlin (2005)
21. Lindblom, C.E.: The science of "muddling through". Public Admin. Rev. **19**, 79–88 (1959)
22. Mordukhovich, B.S.: Variational Analysis and Generalized Differentiation, I: Basic Theory, II: Applications. Springer, Berlin (2006)
23. Nissbaum, M.C., Sen, A.: The Quality of Life. Clarendon Press, Oxford (1993)
24. Qiu, J.-H.: Set-valued quasimetrics and a general Ekeland's variational principle in vector optimization. SIAM J. Control Optim. **51**, 1350–1371 (2013)
25. Rath, T., Harter, J.: Wellbeing: The Five Essential Elements. Gallup Press, New York (2010)
26. Rockafellar, R.T., Wets, J.-B.: Variational Analysis. Springer, Berlin (1998)
27. Roemer, J.: Theories of Distributive Justice. Harvard University Press, Cambridge (1996)
28. Sen, A.: Commodities and Capabilities. Oxford University Press, Oxford (1985)
29. Sen, A.: Development as Freedom. Knopf Press, New York (1999)
30. Simon, H.: A behavioral model of rational choice. Quart. J. Econ. **69**, 99–188 (1955)
31. Snyder, C.R.: The Psychology of Hope: You Can Get There from Here. Free Press, New York (1994)
32. Soleimani, B., Tammer, C.: Scalarization of approximate solutions of vector optimization with variable order structure based on nonlinear scalarization, Report No. 3. Martin-Luther-University of Halle-Wittenberg (2013)
33. Soubeyran, A.: Variational rationality, a theory of individual stability and change, worthwhile and ambidextry behaviors, preprint, GREQAM, Aix-Marseille University (2009)
34. Soubeyran, A.: Variational rationality and the unsatis ed man: Routines and the course pursuit between aspirations, capabilities and beliefs, preprint, GREQAM, Aix-Marseille University (2010)
35. Ume, J.S.: A minimization theorem in quasimetric spaces and its applications. Inter. J. Math. Maths. Sci. **31**, 443–447 (2002)
36. Yu, P.L.: Cone convexity, cone extreme points, and nondominated solutions in decision problems with multiobjectives. J. Optim. Theory Appl. **14**, 319–377 (1974)
37. Yu, P.L.: Multiple-Criteria Decision Making: Concepts, Techniques and Extensions. Plenum Press, New York (1985)