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An Inexact Penalty Method for Non Stationary Generalized Variational Inequalities

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Abstract We consider a set-valued (generalized) variational inequality problem in a finite-dimensional setting, where only approximation sequences are known instead of exact values of the cost mapping and feasible set. We suggest to apply a sequence of inexact solutions of auxiliary problems involving general penalty functions. Its convergence is attained without concordance of penalty, accuracy, and approximation parameters under certain coercivity type conditions.

Keywords Variational inequality · Non-stationarity · Non-monotone mappings · Set-valued mappings · Approximate solutions · Penalty method · Coercivity conditions

Mathematics Subject Classifications (2010) 90C33 · 47J20 · 65K15 · 65J20

1 Introduction

Let D be a nonempty set in the real n-dimensional space $\mathbb{R}^n$, and let $G : D \to \Pi(\mathbb{R}^n)$ be a set-valued mapping. Here and below $\Pi(A)$ denotes the family of all nonempty subsets of a set A. Then one can define the set-valued or *generalized variational inequality problem* (GVI, for short), which is to find an element $x^* \in D$ such that

$$\exists g^* \in G(x^*), \langle g^*, y - x^* \rangle \geq 0 \quad \forall y \in D. \tag{1}$$

If the cost mapping G is single-valued, GVI (1) reduces to the following usual *variational inequality problem* (VI): Find an element $x^* \in D$ such that

$$\langle G(x^*), y - x^* \rangle \geq 0 \quad \forall y \in D,$$

where $G : D \to \mathbb{R}^n$ is a given mapping.

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GVIs give a suitable common format for various problems arising in Economics, Mathematical Physics, and Operations Research and are closely related with other general problems in Nonlinear Analysis, such as fixed point, optimization, complementarity, and equilibrium problems. Now (G)VI can be treated as a differential form of equilibrium conditions in complex systems; see, e.g., [1–4] and the references therein.

Usually, most solution methods for (G)VIs relay upon exact calculation of the cost mapping G and feasible set D, but it seems better to deal with some their approximations for many real problems. On the one hand, the exact values may be unknown due to usual calculation errors and incompleteness of information about the problem under solution. On the other hand, it might be useful to replace the initial problem by a sequence of auxiliary ones with better properties, as in regularization and penalty methods. In other words, we intend to solve *non-stationary* GVIs. We note that most existing solution methods for non-stationary optimization and VI problems require (strengthened) monotonicity assumptions and specific concordance rules for approximation and iteration parameters; see, e.g., [5–8] and the references therein. These requirements seem however very restrictive for applications.

A penalty based method for non-stationary VIs without concordance of parameters was proposed in [9], its convergence requires only mild coercivity conditions. In this paper, we extend those results in several directions. Namely, we suggest a penalty based method for non-stationary GVIs, which admits inexact solutions of auxiliary problems. That is, the method becomes implementable and we do not impose any rules on concordance of parameters. Next, we introduce modified coercivity conditions, which provide existence of solutions of auxiliary problems and convergence of the iteration sequence.

2 Preliminary Properties

We now recall some auxiliary properties. Let X be a nonempty subset of a finite-dimensional space E. Recall that a function $f : X \to \mathbb{R}$ is said to be

(a) *convex* on a set $K \subseteq X$, if for each pair of points $x, y \in K$ and for all $\alpha \in [0, 1]$, it holds that

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y);$$

(b) *strongly convex* with constant $\varkappa > 0$ on $K \subseteq X$, if for each pair of points $x, y \in K$ and for all $\alpha \in [0, 1]$, it holds that

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y) - 0.5\varkappa\alpha(1-\alpha)\|x-y\|^2;$$

(c) *upper (lower) semicontinuous* on $K \subseteq X$, if for each sequence $\{x^k\} \to \bar{x}$, $x^k \in K$ we have $\limsup_{k\to\infty} f(x^k) \leq f(\bar{x})$ ($\liminf_{k\to\infty} f(x^k) \geq f(\bar{x})$);

(d) *coercive* if

$$f(x) \to +\infty \quad \text{as} \quad \|x\| \to \infty;$$

(e) *weakly coercive* with respect to a set $V \subseteq X$ (see [10]) if there exists a number ρ such that the set

$$V(f, \rho) = \{x \in V \mid f(x) \leq \rho\}$$

is nonempty and bounded.

Clearly, we have $(b) \Longrightarrow (a)$ and $(b) \Longrightarrow (d) \Longrightarrow (e)$, but the reverse implications are not true in general.

We say that a sequence of sets $\{X_k\}$ is *Mosco convergent* to a set X (see [11]) if and only if

(i) for each sequence $\{x^k\} \to \bar{x}$, $x^k \in X_k$ we have $\bar{x} \in X$;
(ii) for each point $\bar{x} \in X$ there exists a sequence $\{x^k\} \to \bar{x}$ with $x^k \in X_k$.

Let us consider a general equilibrium problem (EP, for short) that is to find a point $u^* \in V$ such that

$$\Phi(u^*, v) \geq 0 \quad \forall v \in V, \tag{2}$$

where V is a set in $\mathbb{R}^n$, $\Phi : V \times V \to \mathbb{R}$ is an equilibrium bi-function, i.e. $\Phi(u, u) = 0$ for every $u \in V$. We give the existence result for EP (2) on compact sets as a simple adjustment of the classical Ky Fan inequality assertion from [12].

Proposition 1 *If V is a nonempty, convex and compact subset of a finite-dimensional space E, $\Phi : V \times V \to \mathbb{R}$ is an equilibrium bi-function, $\Phi(\cdot, v)$ is upper semicontinuous for each $v \in V$, and $\Phi(u, \cdot)$ is convex for each $u \in V$, then problem* (2) *has a solution.*

Recall also that a set-valued mapping $G : X \to \Pi(\mathbb{R}^n)$ is said to be

(a) *upper semicontinuous* on X, if for each point $z \in X$ and for each open set U such that $G(z) \subset U$, there is a neighborhood W of z such that $G(x) \subseteq U$ whenever $x \in X \cap W$;
(b) a *K (Kakutani)-mapping* on X if it is upper semicontinuous on X and has nonempty, convex, and compact values.

3 Existence of Solutions

Let V be a nonempty set in the real n-dimensional space $\mathbb{R}^n$, $f : V \to \mathbb{R}$ a function, and let $G : V \to \Pi(\mathbb{R}^n)$ be a set-valued mapping. Then we define the *generalized mixed variational inequality problem* (GMVI, for short), which is to find an element $x^* \in V$ such that

$$\exists g^* \in G(x^*), \langle g^*, y - x^* \rangle + f(y) - f(x^*) \geq 0 \quad \forall y \in V. \tag{3}$$

We consider the above problem under the following basic assumptions.

(H) V is a nonempty, convex and closed set, $f : V \to \mathbb{R}$ is a lower semicontinuous and convex function, $G : V \to \Pi(\mathbb{R}^n)$ is a K-mapping.

In this section, we intend to obtain some existence results for GMVI (3), which will be used for substantiation of the penalty method for non-stationary GVIs of form (1). First we give an existence result for the bounded case.

Proposition 2 *If **(H)** holds and V is bounded, then GMVI* (3) *has a solution.*

Proof Set

$$\Phi(x, y) = \sup_{g \in G(x)} \langle g, y - x \rangle + f(y) - f(x). \tag{4}$$

Under the assumptions in **(H)**,

$$\Phi_1(x, y) = \sup_{g \in G(x)} \langle g, y - x \rangle$$

is an equilibrium bi-function and $\Phi(x, \cdot)$ is convex for each $x \in V$. Besides, $\Phi_1(\cdot, y)$ is upper semicontinuous for each $y \in V$, see, e.g., [13, Section 9.2]. Then so is Φ, all the conditions of Proposition 1 hold, and EP (2) has a solution. Applying now the known minimax theorem (see, e.g., [14]), we conclude that GMVI (3) has a solution. □

Let us now consider the unbounded case. We follow the approach from [10] with proper modifications. Set

$$\Delta(g, x, y) = \langle g, y - x\rangle + f(y) - f(x).$$

Next, given a function $\mu : E \to \mathbb{R}$, we set

$$\tilde{V}(\mu, \rho) = \{x \in V \mid \mu(x) < \rho\}$$

and obtain an auxiliary existence result on reduced sets.

Proposition 3 *Suppose* ***(H)*** *holds, a function* $\mu : \mathbb{R}^n \to \mathbb{R}$ *is convex, and, for some* ρ*, there exist*

$$x^\rho \in V(\mu, \rho) \text{ and } \bar{g} \in G(x^\rho) \text{ such that } \Delta(\bar{g}, x^\rho, y) \geq 0 \quad \forall y \in V(\mu, \rho), \tag{5}$$

and $z \in \tilde{V}(\mu, \rho)$ *such that* $\Delta(\bar{g}, x^\rho, z) \leq 0$*. Then* x^ρ *is a solution of GMVI* (3).

Proof Set

$$\varphi(x) = \Delta(\bar{g}, x^\rho, x),$$

then $0 \leq \varphi(z) \leq 0$. Therefore, z is a minimizer for the function φ over $V(\mu, \rho)$. Suppose that there exists a point $x' \in V \setminus V(\mu, \rho)$ such that $\varphi(x') < \varphi(z)$, and set $x(\alpha) = \alpha x' + (1 - \alpha)z$. Clearly, $x(\alpha) \in V$ for each $\alpha \in (0, 1)$. By convexity of μ, we have

$$\begin{aligned} \mu[x(\alpha)] &\leq \alpha\mu(x') + (1 - \alpha)\mu(z) \\ &= \mu(z) + \alpha[\mu(x') - \mu(z)] \leq \rho \end{aligned}$$

for $\alpha > 0$ small enough. Then $x(\alpha) \in V(\mu, \rho)$ for $\alpha > 0$ small enough, but

$$\varphi[x(\alpha)] \leq \alpha\varphi(x') + (1 - \alpha)\varphi(z) < \alpha\varphi(z) + (1 - \alpha)\varphi(z) = \varphi(z),$$

which is a contradiction. Therefore

$$0 = \varphi(z) \leq \varphi(y) \quad \forall y \in V,$$

or equivalently,

$$\Delta(\bar{g}, x^\rho, y) \geq 0 \quad \forall y \in V,$$

i.e. x^ρ solves GMVI (3). □

We take the following coercivity condition.

(C) There exist a convex function $\mu : \mathbb{R}^n \to \mathbb{R}$, which is weakly coercive with respect to the set V, and a number r such that for any point $\bar{x} \in V \setminus V(\mu, r)$ and any $\bar{g} \in G(\bar{x})$ with

$$\inf_{x \in V(\mu, r)} \Delta(\bar{g}, \bar{u}, x) \geq 0 \tag{6}$$

there is a point $z \in V$ such that

$$\begin{gathered} \min\{\Delta(\bar{g}, \bar{u}, z), \mu(z) - \mu(\bar{x})\} < 0 \\ \text{and} \\ \max\{\Delta(\bar{g}, \bar{u}, z), \mu(z) - \mu(\bar{x})\} \leq 0. \end{gathered} \tag{7}$$

Remark 1 The above coercivity condition is weaker than the streamlined specialization from [10], which was applied for EPs. That is, we take Δ instead of the bi-function Φ as in (4). This allows us to verify the conditions in (7) only for one fixed element $\bar{g} \in G(\bar{x})$, rather than for all $g \in G(\bar{x})$.

We observe that, by convexity and weak coercivity of μ, the set $V(\mu, \rho)$ is bounded if nonempty for each ρ. Moreover, the set $V(\mu, r)$ is always nonempty in **(C)**, i.e., the condition is well defined.

Proposition 4 *If* ***(H)*** *and* ***(C)*** *hold, then $V(\mu, r)$ is nonempty.*

Proof First we note that, by convexity and weak coercivity of μ, there exists $\bar{z} \in V$ such that

$$\mu(\bar{z}) = r(m) \triangleq \inf_{x \in V} \mu(x).$$

Hence the set $V(\mu, r(m))$ is nonempty, convex, and compact. Applying Proposition 2 with $V = V(\mu, r(m))$, we see that there exist $\bar{x} \in V(\mu, r(m))$ and $\bar{g} \in G(\bar{x})$ such that

$$\Delta(\bar{g}, \bar{x}, y) \geq 0 \quad \forall y \in V(\mu, r(m)).$$

If $V(\mu, r) = \varnothing$, then $r < r(m)$ and $\bar{x} \notin V(\mu, r)$. Using now (7) and noticing that $\mu(\bar{x}) \leq \mu(z)$ by definition, we obtain $\mu(\bar{x}) = \mu(z)$ and $\Delta(\bar{g}, \bar{x}, z) < 0$, which is a contradiction. □

We now give the main existence result for GMVI (3).

Theorem 1 *If* ***(H)*** *and* ***(C)*** *are fulfilled, then GMVI* (3) *has a solution.*

Proof Since **(C)** holds, choose any $\rho > r$, then the set $V(\mu, \rho)$ is nonempty, convex, and compact due to the properties of μ. From Proposition 2 with $V = V(\mu, \rho)$, we see that there exist $\bar{x} \in V(\mu, \rho)$ and $\bar{g} \in G(\bar{x})$ such that

$$\Delta(\bar{g}, \bar{x}, y) \geq 0 \quad \forall y \in V(\mu, \rho),$$

hence the relations (5) with $x^{\rho} = \bar{x}$ and (6) hold. If $\bar{x} \in \tilde{V}(\mu, \rho)$, we set $z = \bar{x}$ in Proposition 3. Otherwise, we have $\mu(\bar{x}) = \rho$ and $\bar{x} \notin V(\mu, r)$. From **(C)** it now follows that there exists a point $z \in V$ such that $\mu(z) < \mu(\bar{x}) = \rho$ and $\Delta(\bar{g}, \bar{x}, z) = 0$ due to (7). The result follows from Proposition 3.

□

4 Penalty Method

We now intend to describe a general penalty method for GVI (1) where D is a set of the form

$$D = V \bigcap W, \tag{8}$$

V and W are convex and closed sets in the space $\mathbb{R}^n$, and $G : V \to \Pi(\mathbb{R}^n)$ is a set-valued mapping. The above partition of the feasible set means that V represents "simple" constraints whereas W corresponds to complex or "functional" ones and a suitable penalty function will be used for this set.

First we introduce the following approximation assumptions.

(A1) There exists a sequence of nonempty convex closed sets $\{V_k\}$ which is Mosco convergent to the set V;

(A2) There exists a sequence of K-mappings $G_k : V_k \to \Pi(\mathbb{R}^n)$, $k = 1, 2, \dots$, such that the relations $\{y^k\} \to \bar{y}$, $y^k \in V_k$, and $g^k \in G_k(y^k)$ imply $\{g^k\}$ is bounded and $\{g^k\} \to \bar{g}$ yields $\bar{g} \in G(\bar{y})$.

Unlike V, the set W will be approximated via general perturbed penalty functions. Let $P : \mathbb{R}^n \to \mathbb{R}$ be a general penalty function for W, i.e.

$$P(w) \begin{cases} = 0, & \text{if } w \in W, \\ > 0, & \text{if } w \notin W. \end{cases}$$

We utilize only its approximation sequence.

(B1) There exists a sequence of lower semicontinuous, convex, and non-negative functions $P_k : V_k \to \mathbb{R}$;

(B2) if $v^k \in V_k$, $\{v^k\} \to \bar{w}$, and $\liminf_{k\to\infty} P_k(v^k) = 0$, then $P(\bar{w}) = 0$;

(B3) for each point $\bar{w} \in D$ there exists a sequence $\{v^k\} \to \bar{w}$ with $v^k \in V_k$ and $P_k(v^k) = 0$.

Clearly, conditions **(B2)** and **(B3)** give a kind of the Mosco convergence of the functions $\{P_k\}$ to P.

For each $k = 1, 2, \dots$, we consider the problem of finding a point $\tilde{x}^k \in V_k$ such that

$$\exists \tilde{g}^k \in G_k(\tilde{x}^k), \langle g^k, v - \tilde{x}^k \rangle + \tau_k [P_k(v) - P_k(\tilde{x}^k)] \geq 0 \quad \forall v \in V_k, \tag{9}$$

where $\tau_k > 0$ is a penalty parameter; cf. (3). For brevity, set

$$\Delta_k(g, x, y) = \langle g, y - x \rangle + \tau_k [P_k(y) - P_k(x)].$$

However, $\tilde{x}^k$ is an exact solution of the penalized GMVI (9). In order to make the method implementable, we consider the approximate problem: find a point $x^k \in V_k$ such that

$$\begin{aligned} &\exists g^k \in G_k(x^k),\ \exists d^k \in B(\mathbf{0}, \delta_k), \\ &\langle g^k + d^k, v - x^k \rangle + \tau_k [P_k(v) - P_k(x^k)] + \varepsilon_k \geq 0 \quad \forall v \in V_k, \end{aligned} \tag{10}$$

where $\delta_k \geq 0$ and $\varepsilon_k \geq 0$ are approximation parameters, $B(\mathbf{0}, \delta_k)$ denotes the closed ball with center $\mathbf{0}$ and radius δ_k. Our approximation condition is rather general; see, e.g., [8]. We intend to prove that the sequence $\{x^k\}$ approximates a solution of GVI (1), (8) under a suitable choice of the parameters τ_k, δ_k, and ε_k. The key feature is that we insist here on the full independence of their control rules and, also, of ways of approximations of exact values of G, V, and P.

Since the feasible set need not be bounded, we introduce certain coercivity conditions.

(C1) For each $k = 1, 2, \dots$ there exist a convex function $\mu_k : \mathbb{R}^n \to \mathbb{R}$, which is weakly coercive with respect to the set V_k, and a number σ_k such that for any point $u \in V_k \setminus V_k(\mu_k, \sigma_k)$ and any $g \in G_k(u)$ with

$$\inf_{x \in V_k(\mu_k, \sigma_k)} \Delta_k(g, u, x) \geq 0,$$

there is a point $z \in V_k$ such that

$$\min\{\Delta_k(\bar{g}, u, z), \mu_k(z) - \mu_k(u)\} < 0$$
$$\text{and}$$
$$\max\{\Delta_k(\bar{g}, u, z), \mu_k(z) - \mu_k(u)\} \leq 0.$$

(C2) There exist a number $\alpha > 0$ and a point $\bar{v} \in D$ such that for any sequences $\{u^k\}$, $\{v^k\}$, $\{g^k\}$, and $\{d^k\}$, satisfying the conditions:

$$u^k \in V_k,\, v^k \in V_k,\, g^k \in G_k(u^k),\, \{v^k\} \to \bar{v},\, \{\|u^k\|\} \to +\infty,\, \{d^k\} \to \mathbf{0};$$

it holds that

$$\liminf_{k\to\infty}\langle g^k + d^k, v^k - u^k\rangle \le -\alpha. \tag{11}$$

Clearly, **(C1)** is a specialization of **(C)** for the auxiliary problems of form (9). Observe also that **(C2)** does not require for $\alpha = \infty$ to be fulfilled in (11), unlike the usual coercivity conditions, which are applied for providing convergence of penalty methods applied to stationary VI's; see e.g. [15, 16].

We now establish the main convergence result.

Theorem 2 *Suppose that assumptions **(A1)–(A2)**, **(B1)–(B3)**, and **(C1)–(C2)** are fulfilled, the sequences $\{\tau_k\}$, $\{\delta_k\}$, and $\{\varepsilon_k\}$ satisfy*

$$\{\tau_k\} \nearrow +\infty,\, \{\delta_k\} \searrow 0,\, \{\varepsilon_k\} \searrow 0. \tag{12}$$

Then:

- (i) *problem* (10) *has a solution for any $\tau_k > 0$, $\delta_k \ge 0$, and $\varepsilon_k \ge 0$;*
- (ii) *each sequence $\{x^k\}$ of solutions of* (10) *has limit points and all these limit points are solutions of GVI* (1)*, where D is defined by* (8).

Proof We first observe that **(C1)** implies that each auxiliary GMVI (9) has a solution due to Theorem 1. It follows that problem (10) is also solvable (at least with $\delta_k = 0$ and $\varepsilon_k = 0$). Hence, assertion (i) is true.

By (i), the sequence $\{x^k\}$ is well-defined. We have to show that it is bounded. Conversely, suppose that $\{\|x^k\|\} \to +\infty$. By definition, $x^k \in V_k$, besides, by **(B3)** and **(C2)** there exists a sequence $\{v^k\} \to \bar{v}$ such that $v^k \in V_k$ and $P_k(v^k) = 0$. Applying (10), we have for some $g^k \in G_k(x^k)$ and $d^k \in B(\mathbf{0}, \delta_k)$:

$$\begin{aligned} 0 &\le \langle g^k + d^k, v^k - x^k\rangle + \tau_k[P_k(v^k) - P_k(x^k)] + \varepsilon_k \\ &= \langle g^k + d^k, v^k - x^k\rangle - \tau_k P_k(x^k) + \varepsilon_k \\ &\le \langle g^k + d^k, v^k - x^k\rangle + \varepsilon_k. \end{aligned}$$

Take a subsequence $\{k_s\}$ such that

$$\lim_{k_s\to\infty}\langle g^{k_s} + d^{k_s}, v^{k_s} - u^{k_s}\rangle = \liminf_{k\to\infty}\langle g^k + d^k, v^k - u^k\rangle,$$

then, by **(C2)**, we have

$$0 \le \lim_{k_s\to\infty}\langle g^{k_s} + d^{k_s}, v^{k_s} - u^{k_s}\rangle \le -\alpha < 0,$$

for k_s large enough, a contradiction. Therefore, the sequence $\{x^k\}$ is bounded and has limit points. Let $\bar{x}$ be an arbitrary limit point for $\{x^k\}$, i.e.

$$\bar{x} = \lim_{s\to\infty} x^{k_s}.$$

Since $x^k \in V_k$, we have $\bar{x} \in V$ due to **(A1)**. From (10) it follows that

$$0 \le P_{k_s}(x^{k_s}) \le \tau_{k_s}^{-1}\langle g^{k_s} + d^{k_s}, v - x^{k_s}\rangle + P_{k_s}(v) + \tau_{k_s}^{-1}\varepsilon_{k_s} \quad \forall v \in V_{k_s},$$

where $g^{k_s} \in G_{k_s}(x^{k_s})$ and $d^{k_s} \in B(\mathbf{0}, \delta_{k_s})$, besides, the sequence $\{g^{k_s}\}$ is bounded due to **(A2)**.

For any $w \in D$ there exists a sequence $\{v^k\} \to w$ with $v^k \in V_k$ and $P_k(v^k) = 0$ due to **(B3)**. Taking $v = v^{k_s}$ above, we obtain

$$0 \le \liminf_{s\to\infty} P_{k_s}(x^{k_s}) \le \limsup_{s\to\infty} \left[\tau_{k_s}^{-1}\langle g^{k_s} + d^{k_s}, v^{k_s} - x^{k_s}\rangle\right] = 0$$

on account of **(A2)** and (12), i.e.

$$\lim_{s\to\infty} P_{k_s}(x^{k_s}) = 0.$$

Due to **(B2)**, this gives $\bar{x} \in W$, i.e., $\bar{x} \in D$.

Now, by **(B3)**, there exists a sequence $\{v^k\} \to \bar{x}$ with $v^k \in V_k$ and $P_k(v^k) = 0$. Again from (10), for some $g^{k_s} \in G_{k_s}(x^{k_s})$ and $d^{k_s} \in B(\mathbf{0}, \delta_{k_s})$, we have

$$0 \le \tau_{k_s} P_{k_s}(x^{k_s}) \le \langle g^{k_s} + d^{k_s}, v^{k_s} - x^{k_s}\rangle + \varepsilon_{k_s} \to 0$$

as $s \to \infty$, hence

$$\lim_{s\to\infty} \left[\tau_{k_s} P_{k_s}(x^{k_s})\right] = 0.$$

Take an arbitrary point $w \in D$, then, again by **(B3)**, there exists a sequence $\{v^k\} \to w$ with $v^k \in V_k$ and $P_k(v^k) = 0$. Using again (10), for some $g^{k_s} \in G_{k_s}(x^{k_s})$ and $d^{k_s} \in B(\mathbf{0}, \delta_{k_s})$, we have

$$\begin{aligned}
&\langle g^{k_s} + d^{k_s}, v^{k_s} - x^{k_s}\rangle - \tau_{k_s} P_{k_s}(x^{k_s}) + \varepsilon_{k_s} \\
&= \langle g^{k_s} + d^{k_s}, v^{k_s} - x^{k_s}\rangle + \tau_{k_s}[P_{k_s}(v^{k_s}) - P_{k_s}(x^{k_s})] + \varepsilon_{k_s} \ge 0.
\end{aligned}$$

By **(A2)**, we can suppose that $\{g^{k_s}\} \to \bar{g}$ without loss of generality, then $\bar{g} \in G(\bar{x})$. It now follows that

$$\begin{aligned}
\langle \bar{g}, w - \bar{x}\rangle &= \lim_{s\to\infty} \langle g^{k_s}, v^{k_s} - x^{k_s}\rangle = \lim_{s\to\infty} \langle g^{k_s} + d^{k_s}, v^{k_s} - x^{k_s}\rangle \\
&\ge \lim_{s\to\infty} \left[\tau_{k_s} P_{k_s}(x^{k_s})\right] = 0,
\end{aligned}$$

therefore $\bar{x}$ solves GVI (1), (8) and assertion (ii) holds true. □

We observe that the above proof implies that the feasible set D is nonempty and that GVI (1), (8) has a solution.

5 Accuracy Estimates for Auxiliary Problems

In the previous section, we described a penalty method for GVI (1), (8), which consists in sequential approximate solution of the auxiliary penalized GMVI (9), that is, we have to find a point satisfying (10) within the given tolerances $\delta_k \ge 0$ and $\varepsilon_k \ge 0$. Clearly, we can apply a great number of solution methods for GMVI (9); see, e.g., [2, 4] and the references therein. Now we suggest a way of estimation of a desired accuracy indicated in (10) by using the gap function approach.

Let us consider GMVI (3), since GMVI (9) falls into its format. Our problem is then consists in finding a point $\tilde{x} \in V$ such that

$$\begin{aligned}
&\exists \tilde{g} \in G(\tilde{x}),\ \exists \tilde{d} \in B(\mathbf{0}, \delta), \\
&\langle \tilde{g} + \tilde{d}, v - \tilde{x}\rangle + f(v) - f(\tilde{x}) + \varepsilon \ge 0 \quad \forall v \in V,
\end{aligned} \tag{13}$$

where $\delta \geq 0$ and $\varepsilon \geq 0$; cf. (10). We also observe that GMVI (3) reduces to EP (2), (4). Therefore, following the approach from [17], we can consider the so-called regularized gap function for the latter problem. Fix a number $\lambda > 0$ and consider the function

$$\varphi_\lambda(x) = \max_{y \in V}\{-\Phi(x, y) - 0.5\lambda \|x - y\|^2\}. \tag{14}$$

If the assumptions in **(H)** hold, the inner problem in (14) has the unique solution $y(x)$, i.e.

$$\varphi_\lambda(x) = -\Phi(x, y(x)) - 0.5\lambda \|x - y(x)\|^2 .$$

In [17, Propositions 3.1 and 3.3], the following basic properties of φ_λ were established.

Proposition 5 *Let the assumptions in **(H)** hold. Then:*

(*i*) $\varphi_\lambda(x) \geq 0$ *for every* $x \in V$;
(*ii*) $x^* \in V$ *and* $\varphi_\lambda(x^*) = 0 \iff x^*$ *solves EP* (2), (4) $\iff x^* = y(x^*)$;
(*iii*) *it holds that*

$$\varphi_\lambda(x) \geq 0.5\lambda \|x - y(x)\|^2 \quad \forall x \in V. \tag{15}$$

It follows that a point x is close to a solution of the EP, if $\varphi_\lambda(x)$ is small enough. Next, from (14) we have

$$\Phi(x, y(x)) + 0.5\lambda\|x - y(x)\|^2 \leq \Phi(x, y) + 0.5\lambda\|x - y\|^2 \quad \forall y \in V.$$

Using the optimality condition from [18, Proposition 2.2.2], we can rewrite equivalently this relation as follows:

$$\Phi(x, y) - \Phi(x, y(x)) + \lambda\langle y(x) - x, y - y(x)\rangle \geq 0 \quad \forall y \in V.$$

For any $y \in V$, we then have

$$\begin{aligned}
&\Phi(x, y) + \varphi_\lambda(x) + \lambda\langle y(x) - x, y - x\rangle - 0.5\lambda\|x - y(x)\|^2\\
&= \Phi(x, y) - \Phi(x, y(x)) - 0.5\lambda\|x - y(x)\|^2\\
&+\lambda\langle y(x) - x, y - x\rangle - 0.5\lambda\|x - y(x)\|^2\\
&= \Phi(x, y) - \Phi(x, y(x)) - \lambda\|x - y(x)\|^2 + \lambda\langle y(x) - x, y - x\rangle \geq 0,
\end{aligned}$$

hence x satisfies (13) if $\lambda\|x - y(x)\| \leq \delta$ and

$$\varphi_\lambda(x) - 0.5\lambda\|x - y(x)\|^2 \leq \varphi_\lambda(x) \leq \varepsilon.$$

In view of (15), both these inequalities hold if $\varphi_\lambda(x)$ is small enough. Then the point x attains the desired accuracy.

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