Set-Valued Var. Anal (2013) 21:297–309
DOI 10.1007/s11228-012-0225-0

Finite Convergence of the Proximal Point Algorithm for Variational Inequality Problems

Shin-ya Matsushita · Li Xu

Received: 25 July 2012 / Accepted: 14 November 2012 /
Published online: 7 December 2012

Abstract In this paper, we establish sufficient conditions for guaranteeing finite termination of an arbitrary algorithm for solving a variational inequality problem in a Banach space. Applying these conditions, it shows that sequences generated by the proximal point algorithm terminate at solutions in a finite number of iterations.

Keywords Variational inequality problem · Weak sharp minima · Finite termination · Proximal point algorithm · Banach space · Metric projection · Generalized projection

Mathematics Subject Classifications (2010) 49J40 · 47J25 · 49J53 · 47J20 · 47H05 · 47H04 · 90C25

1 Introduction

Let E be a real Banach space and E^* be its dual space. This paper deals with the variational inequality problem: find a point $\bar{x} \in C$ such that

$$\langle x - \bar{x}, F(\bar{x}) \rangle \geq 0 \text{ for all } x \in C, \tag{1}$$

where C is a closed convex subset of E, F is a mapping from C into E^* and $\langle x, x^* \rangle$ denotes the value of the continuous linear functional $x^* \in E^*$ at $x \in E$. Let S be the

The first author was partially supported by Grant-in-Aid for Young Scientists (B) No. 20740084, the Ministry of Education, Culture, Sports, Science and Technology, Japan.

S. Matsushita (✉) · L. Xu
Department of Electronics and Information Systems, Akita Prefectural University, 84-4 Yuri-Honjo, Akita, Japan
e-mail: matsushita@akita-pu.ac.jp

L. Xu
e-mail: xuli@akita-pu.ac.jp

set of solutions to (1), which we assume to be nonempty. The variational inequality problem in the form of (1) has very interesting interpretations in various fields. For instance, if C is a closed convex cone of E, then any solution $\bar{x}$ of (1) is actually a solution of the complementarity problem in mathematical programming, game theory and economic theory. For details of the variational inequality problem, we refer to [2, 3, 7, 14, 15, 22, 34, 35]. In this paper, we are interested in conditions for guaranteeing finite termination of algorithms for solving (1).

The notion of weak sharp minima plays an important role in the sensitive analysis of convex programs and variational inequality problems (see, e.g., [8, 9, 16, 18, 25, 26, 30, 36, 37, 39]). In a finite-dimensional setting, Burke and Ferris [9] have shown that solution sets of certain convex quadratic programs and linear complementarity problems are weak sharp minima that many optimization algorithms exhibit finite termination at weak sharp minima. Patriksson [30] extended this notion to variational inequality problems and Marcotte and Zhu [26] applied it to establish sufficient conditions for the finite termination of descent algorithms. For solving infinite-dimensional problems, this notion was investigated by Wu and Wu [36] and Hu and Song [18]. One of the principle advantages of this notion can be used to obtain the results of finite termination of the proximal point algorithm. The proximal point algorithm proposed by Martinet [27] and further developed by Rockafellar [33] converges weakly to a solution under the very mild assumption. Therefore, this algorithm has been investigated by a number of authors (see, e.g. [5–7, 10, 15–17, 19, 21, 23–25, 37]).

This paper focuses on analyzing the finite termination of the proximal point algorithm. Recently, Hu and Song [18] investigated the notion of weak sharp minima in the Banach space setting and applied it to prove finite termination of the proximal point algorithm. Their results based on two assumptions: (i) the solution set of (1) is weakly subsharp [18, Definition 3.2], which is a modified version of the weak sharp minima, and (ii) the sequence which is generated by the algorithm converges strongly. Under assumptions (i) and (ii), they proved that the proximal point algorithm terminates at solutions in a finite number of iterations. However, it remains an open question whether the notion of weak sharp minima is a sufficient condition that guarantees the finite termination of the proximal point algorithm in Banach spaces. Moreover, it shows in [4, 17] that the proximal point algorithm may fail to converge strongly in the infinite-dimensional case.

The main objective of this paper is to prove that the notion of weak sharp minima is a sufficient condition for guaranteeing finite termination of the proximal point algorithm in Banach spaces. In order to obtain the results, we first prove lemmas which are important for the proof of the main results. Then we establish three sufficient conditions for guaranteeing finite termination of an arbitrary algorithm under the weak sharpness assumption. It should be noted that the first and the second conditions do not require the sequences that converge strongly and the third condition unifies and extends existing results in [9, 26, 39]. As applications, we use our results to establish finite termination of the proximal point algorithm. These results are new even in the case when E is a Hilbert space and include the corresponding results in [25, 37] as special cases.

The paper is organized as follows. Section 2 introduces the main definitions. In Section 3, we first prove a lemma which is a generalization of the result by Calamai and Moré [11]. Moreover, we characterize the solution set of (1) under the paramonotonicity assumption. In Section 4, we establish three sufficient conditions for guaranteeing finite termination of algorithms for solving (1). The results from Section 4 are then applied to the proximal point algorithm in Section 5. We will prove that the proximal point algorithm has the finite termination property under the assumption of weak sharpness of S.

2 Basic Definitions and Preliminaries

Let $\mathbf{N}$ and $\mathbf{R}$ denote the sets of positive integers and real numbers, respectively. Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the dual space of E. By $\langle x, x^*\rangle$ we denote the value of the continuous linear functional $x^* \in E^*$ at $x \in E$. $B(0,\epsilon) := \{y \in E : \|y\| \le \epsilon\}$ and $B^*(0,\epsilon) := \{y^* \in E^* : \|y^*\| \le \epsilon\}$ are the closed balls of E and E^* with radius $\epsilon > 0$, respectively. The duality mapping J from E into E^* is defined by

$$J(x) := \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\} \text{ for all } x \in E. \tag{2}$$

A Banach space E is said to be

(i) strictly convex if $\|\frac{x+y}{2}\| < 1$ whenever $x, y \in S(E) := \{z \in E : \|z\| \le 1\}$ with $x \neq y$;
(ii) uniformly convex if for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that $\|\frac{x+y}{2}\| \le 1 - \delta$ whenever $x, y \in S(E)$ with $\|x - y\| \ge \epsilon$.

The norm of E is said to be

(i) Gâteaux differentiable if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{3}$$

exists for all $(x, y) \in S(E) \times S(E)$. In this case, E is said to be smooth;
(ii) Fréchet differentiable if for each $x \in S(E)$, the limit (3) exists uniformly for $y \in S(E)$;
(iii) uniformly Fréchet differentiable if the limit (3) exists uniformly in $(x, y) \in S(E) \times S(E)$. In this case, E is said to be uniformly smooth.

We list the following useful properties of the duality mapping.

1. If E is strictly convex, then J is one to one, i.e.,

$$x, y \in E \text{ with } x \neq y \Rightarrow J(x) \cap J(y) = \emptyset;$$

2. if E is reflexive, then J is a mapping of E onto E^*;
3. if E is smooth, then J is single valued;
4. if E is smooth, strictly convex and reflexive, then $J^{-1} = J^*$, where J^{-1} is the inverse of J and J^* is the duality mapping on E^* defined by

$$J^*(x^*) := \{x \in E : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\} \text{ for all } x^* \in E^*;$$

5. if E^* has Fréchet differentiable norm, then E is strictly convex and reflexive.

The proofs of these results can be found in [3, 13, 34, 35].

Given a nonempty subset C of E, by int$C$ and cl$C$ we denote its interior and closure, respectively. The polar C° of C is defined by

$$C^\circ := \{x^* \in E^* : \langle x, x^*\rangle \le 0 \text{ for all } x \in C\}. \tag{4}$$

The tangent cone to the set C at $x \in C$ is defined by

$$T_C(x) := \mathrm{cl}\left(\cup_{\lambda>0}\frac{C-x}{\lambda}\right). \tag{5}$$

The normal cone to C at x is defined by $N_C(x) := T_C(x)^{\circ}$, that is,

$$N_C(x) = \{x^* \in E^* : \langle y-x, x^*\rangle \le 0 \ \text{ for all } \ y \in C\}. \tag{6}$$

Let $F : C \to E^*$ be a mapping. F is said to be

(i) monotone if

$$\langle x-y, F(x)-F(y)\rangle \ge 0 \ \text{ for all } \ x, y \in C;$$

(ii) paramonotone [12] if it is monotone and

$$\langle x-y, F(x)-F(y)\rangle = 0 \ \text{ with } \ x, y \in C \ \Rightarrow \ F(x) = F(y).$$

Example 1 Let $f : E \to \mathbf{R} \cup \{\infty\}$ be a proper, convex and lower semicontinuous function and let $\bar{x} \in (\mathrm{dom}\ f)^i$, where C^i is the algebraic interior of a set C. When f is Gâteaux differentiable at $\bar{x}$, $\partial f(\bar{x}) = \{\nabla f(\bar{x})\}$, where ∂f is the subdifferential of f and ∇f is the Gâteaux derivative of f (see, for instance [38, Theorem 2.4.4]). It has been shown in [12] that the subdifferential of a convex function is paramonotone. Therefore, ∇f is a single valued paramonotone mapping.

When F is monotone and continuous, the solution set S of (1) is closed and convex (see, for instance [14, Proposition 1] and [34, Lemma 7.1.7]).

Let C be a nonempty, closed and convex subset of a smooth, strictly convex and reflexive Banach space E. The metric projection of a point $x \in E$ onto C, denoted by $P_C(x)$, is defined as the unique solution of the problem

$$\text{minimize } \ \|x-y\| \ \text{ subject to } \ y \in C.$$

For each $x \in E$, $P_C(x)$ satisfies

$$\langle y - P_C(x), J(x-P_C(x))\rangle \le 0 \ \text{ for all } \ y \in C \tag{7}$$

(see [1, page 27] or [38, Theorem 3.8.4]). When E^* has a Fréchet differentiable norm, the metric projection P_C is continuous (see [38, Proposition 3.8.6]). Let $\phi : E \times E \to \mathbf{R}$ be a function defined by

$$\phi(x, y) := \|x\|^2 - 2\langle x, J(y)\rangle + \|y\|^2 \ \text{ for all } \ x, y \in E.$$

The generalized projection of a point $x \in E$ onto C, denoted by $\Pi_C(x)$, is defined as the unique solution of the problem

$$\text{minimize } \ \phi(y, x) \ \text{ subject to } \ y \in C. \tag{8}$$

For each $x \in E$, $\Pi_C(x)$ satisfies

$$\langle y - \Pi_C(x), J(x) - J(\Pi_C(x))\rangle \le 0 \ \text{ for all } \ y \in C \tag{9}$$

(see [1, page 35] or [20, Proposition 4]). It should be noted that if E is a Hilbert space, then $\phi(x, y) = \|x-y\|^2$ for all $x, y \in E$, and consequently $P_C = \Pi_C$. The proofs of these results can be found in [1, 20].

3 Lemmas

In this section, we present some lemmas which are important to prove our main results. Throughout this paper, we assume that E is a smooth, strictly convex and reflexive Banach space, C is a closed convex subset of E, and the solution set S of (1) is nonempty.

By (5), for any $x \in C$, $T_C(x)$ is a nonempty, closed and convex set. From the definition of the generalized projection (8), we know that

$$\Pi_{T_C(x)}(-J^*F(x)) = \operatorname*{argmin}_{v \in T_C(x)} \phi(v, -J^*F(x)). \tag{10}$$

We first prove the following lemma which generalizes the result by Calamai and Moré [11, Lemma 3.1] from the Euclidean space to a smooth, strictly convex and reflexive Banach space.

Lemma 1 *Let $F : C \to E^*$ be a mapping and let $x \in C$. Then the following properties hold:*

(a) $\left\langle \Pi_{T_C(x)}(-J^*F(x)), F(x) \right\rangle = -\|\Pi_{T_C(x)}(-J^*F(x))\|^2$;
(b) $\min\{\langle v, F(x)\rangle : v \in T_C(x), \|v\| \le 1\} = -\|\Pi_{T_C(x)}(-J^*F(x))\|$.

Proof We first show (a). Let $\bar{x} = \Pi_{T_C(x)}(-J^*F(x))$. From (9) and (10), we have

$$\langle y - \bar{x}, -F(x) - J(\bar{x})\rangle \le 0 \text{ for all } y \in T_C(x). \tag{11}$$

Since $T_C(x)$ is a cone, we have $\bar{x} \in T_C(x)$, and if $\lambda \ge 0$, then

$$(\lambda - 1)\langle \bar{x}, -F(x) - J(\bar{x})\rangle \le 0.$$

Setting $\lambda = 0$ and $\lambda = 2$ in this inequality we have (a).

We next show (b). When $\bar{x} = 0$, we have that $0 \le \langle v, F(x)\rangle$ for all $v \in T_C(x)$ with $\|v\| \le 1$ because (11) holds. It follows from $\inf\{\langle v, F(x)\rangle : v \in T_C(x), \|v\| \le 1\} \le \langle \bar{x}, F(x)\rangle$ that (b) holds. Suppose that $\bar{x} \ne 0$. From (10), if $v \in T_C(x)$ and $\|v\| \le \|\bar{x}\|$, then

$$\begin{aligned}\phi(\bar{x}, -J^*F(x)) &\le \phi(v, -J^*F(x)) \\ &\le \|\bar{x}\|^2 - 2\langle v, -F(x)\rangle + \|F(x)\|^2,\end{aligned}$$

and hence

$$\langle \bar{x}, F(x)\rangle \le \langle v, F(x)\rangle. \tag{12}$$

Combining (a) with (12), we have

$$-\|\bar{x}\|^2 \le \langle v, F(x)\rangle. \tag{13}$$

If $v \in T_C(x)$ and $\|v\| \le 1$, then (13) implies that $-\|\bar{x}\|^2 \le \langle \|\bar{x}\| v, F(x)\rangle$, and thus

$$-\|\bar{x}\| \le \langle v, F(x)\rangle. \tag{14}$$

It follows from (a) that $\left\langle \frac{\bar{x}}{\|\bar{x}\|}, F(x) \right\rangle = -\|\bar{x}\|$, and thus (14) implies that (b) holds. □

We next characterize the solution set of (1) under the paramonotonicity assumption.

Lemma 2 *Let $F : C \to E^*$ be a paramotone mapping and let $\bar{x} \in S$. Then*

$$S = \{z \in C : F(z) = F(\bar{x}) \ \textit{and} \ \langle z - \bar{x}, F(\bar{x}) \rangle = 0\}. \tag{15}$$

Proof Let $z \in S$. Then $\langle \bar{x} - z, F(z) \rangle \geq 0$ holds. It is easy to verify from $\bar{x} \in S$ that

$$\langle z - \bar{x}, F(z) - F(\bar{x}) \rangle \leq 0.$$

Using the monotonicity of F, one has $\langle z - \bar{x}, F(z) - F(\bar{x}) \rangle \geq 0$, that is,

$$\langle z - \bar{x}, F(z) - F(\bar{x}) \rangle = 0.$$

Since F is paramonotone, we have $F(\bar{x}) = F(z)$. This implies that

$$0 \leq \langle z - \bar{x}, F(\bar{x}) \rangle = \langle z - \bar{x}, F(z) \rangle \leq 0,$$

that is,

$$\langle z - \bar{x}, F(\bar{x}) \rangle = 0.$$

Conversely, let $z \in C$ such that $F(\bar{x}) = F(z)$ and $\langle z - \bar{x}, F(\bar{x}) \rangle = 0$. It follows that for any $y \in C$,

$$\begin{aligned} 0 &\leq \langle y - \bar{x}, F(\bar{x}) \rangle \\ &= \langle y - z, F(\bar{x}) \rangle + \langle z - \bar{x}, F(\bar{x}) \rangle \\ &= \langle y - z, F(z) \rangle, \end{aligned}$$

and so $z \in S$. □

4 Main Results

In this section, we establish sufficient conditions for guaranteeing finite termination of algorithms for solving (1).

In [18], Hu and Song introduced the notion of weak sharpness for the solution set of (1) as follows: The solution set S is said to be weak sharp if

$$-F(x) \in \operatorname{int} \bigcap_{z \in S} \left(T_C(z) \cap J^* N_S(z)\right)^\circ \quad \text{for all} \ \ x \in S. \tag{16}$$

This condition is regarded as a natural extension of notions by Burke and Ferris [9] and Patriksson [30]. Some characterizations of the weak sharpness were given in the literature [8, 9, 18, 26, 30, 36, 37]. However, it remains an open question whether or not the notion (16) can be applied to prove finite termination of the proximal point algorithm in the Banach space case.

Now, we first apply (16) to establish sufficient conditions for guaranteeing finite termination of an arbitrary algorithm for solving (1).

Theorem 1 *Let $F : C \to E^*$ be a continuous mapping and let $\{x_n\}$ be a sequence in C such as (16) and*

$$\lim_{n\to\infty} \Pi_{T_C(x_n)}\left(-J^* F(x_n)\right) = 0 \tag{17}$$

hold. Then $x_n \in S$ for all sufficiently large n, if one of the following conditions holds:

(i) *F is paramonotone;*
(ii) *F is monotone and $\{P_S(x_n)\}$ converges strongly to some $\bar{x} \in S$;*
(iii) *$\{x_n\}$ converges strongly to some $\bar{y} \in S$ and S is convex.*

Proof Assume that the conclusion does not hold. Then there exists $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \notin S$ for each $i \in \mathbf{N}$. Let $x_i = x_{n_i}$ and $z_i = P_S(x_i)$ for each $i \in \mathbf{N}$. From (5) and (6), we have

$$x_i - z_i \in T_C(z_i) \cap J^* N_S(z_i) \ \text{ and } \ z_i - x_i \in T_C(x_i). \tag{18}$$

Let $\bar{z} \in S$. By (16), there exists $\alpha > 0$ such that

$$-F(\bar{z}) + B^*(0, \alpha) \subset \left(T_C(z) \cap J^* N_S(z)\right)^\circ \ \text{ for all } \ z \in S. \tag{19}$$

From (18) and (19), we have that

$$\left\langle x_i - z_i, -F(\bar{z}) + \alpha \frac{J(x_i - z_i)}{\|x_i - z_i\|} \right\rangle \le 0 \ \text{ for all } \ i \in \mathbf{N}. \tag{20}$$

It is easy to verify from (20) that for each $i \in \mathbf{N}$,

$$\alpha \le \left\langle \frac{z_i - x_i}{\|x_i - z_i\|}, -F(\bar{z}) \right\rangle. \tag{21}$$

The proof is divided into three cases.

Suppose that (i) holds. It follows from (21), the monotonicity of F, and Lemmas 1 and 2 that

$$\begin{aligned}
\alpha &\le \left\langle \frac{z_i - x_i}{\|x_i - z_i\|}, -F(\bar{z}) \right\rangle \\
&= \left\langle \frac{z_i - x_i}{\|x_i - z_i\|}, -F(z_i) \right\rangle \\
&= \left\langle \frac{z_i - x_i}{\|x_i - z_i\|}, F(x_i) - F(z_i) \right\rangle + \left\langle \frac{z_i - x_i}{\|x_i - z_i\|}, -F(x_i) \right\rangle \\
&\le \left\langle \frac{z_i - x_i}{\|x_i - z_i\|}, -F(x_i) \right\rangle \\
&\le \max\{\langle v, -F(x_i)\rangle : v \in T_C(x_i), \|v\| \le 1\} \\
&= \|\Pi_{T_C(x_i)}\left(-J^* F(x_i)\right)\|.
\end{aligned}$$

It follows from (17) that $\alpha \le 0$ and this is a contradiction.

Suppose that (ii) holds. We may assume without the loss of generality that $\bar{z} = \bar{x}$. Using monotonicity of F and Lemma 1, we have

$$\begin{aligned}
\alpha &\le \left\langle \tfrac{z_i - x_i}{\|x_i - z_i\|}, -F(\bar{x}) \right\rangle \\
&= \left\langle \tfrac{z_i - x_i}{\|x_i - z_i\|}, -F(x_i) \right\rangle + \left\langle \tfrac{z_i - x_i}{\|x_i - z_i\|}, F(x_i) - F(\bar{x}) \right\rangle \\
&= \left\langle \tfrac{z_i - x_i}{\|x_i - z_i\|}, -F(x_i) \right\rangle + \left\langle \tfrac{z_i - x_i}{\|x_i - z_i\|}, F(x_i) - F(z_i) + F(z_i) - F(\bar{x}) \right\rangle \\
&\le \left\langle \tfrac{z_i - x_i}{\|x_i - z_i\|}, -F(x_i) \right\rangle + \left\langle \tfrac{z_i - x_i}{\|x_i - z_i\|}, F(z_i) - F(\bar{x}) \right\rangle \\
&\le \max\{\langle v, -F(x_i)\rangle : v \in T_C(x_i), \|v\| \le 1\} + \|F(z_i) - F(\bar{x})\| \\
&= \|\Pi_{T_C(x_i)} \left(-J^* F(x_i)\right)\| + \|F(z_i) - F(\bar{x})\|.
\end{aligned}$$

Since F is continuous and $\{z_i\}$ converges strongly to $\bar{x}$, we have $\|F(z_i) - F(\bar{x})\| \to 0$. Combining this with (17), letting $i \to \infty$, we obtain $\alpha \le 0$. This is a contradiction.

Suppose that (iii) holds. We may assume without the loss of generality that $\bar{z} = \bar{y}$. It follows from (21) that

$$\begin{aligned}
\alpha &\le \left\langle \tfrac{z_i - x_i}{\|x_i - z_i\|}, -F(\bar{y}) \right\rangle \\
&= \left\langle \tfrac{z_i - x_i}{\|x_i - z_i\|}, -F(x_i) \right\rangle + \left\langle \tfrac{z_i - x_i}{\|x_i - z_i\|}, F(x_i) - F(\bar{y}) \right\rangle \\
&\le \|\Pi_{T_C(x_i)} \left(-J^* F(x_i)\right)\| + \|F(x_i) - F(\bar{y})\|.
\end{aligned}$$

Since F is continuous and $\{x_i\}$ converges strongly to $\bar{x}$, we have $\|F(x_i) - F(\bar{y})\| \to 0$. Combining this with (17), letting $i \to \infty$, we obtain $\alpha \le 0$. This is a contradiction.

Therefore, we obtain that $x_n \in S$ for all sufficiently large n. □

Remark 1 (i) and (ii) of Theorem 1 do not require $\{x_n\}$ converges strongly. (iii) of Theorem 1 includes the results of Burke and Ferris [9, Theorem 4.7], Marcotte and Zhu [26, Theorem 5.2] and Zhou and Wang [39, Theorem 2] as special cases.

5 Applications

In this section, we apply Theorem 1 to obtain results of finite termination of the proximal point algorithm for solving the variational inequality problem. Here, we obtain two finite convergence results.

Throughout this section, we assume that $F : C \to E^*$ is a single valued monotone continuous mapping. A monotone mapping T is maximal if its graph is not contained in the graph of any other monotone mapping. Here, we consider the mapping $T : E \to 2^{E^*}$ defined by

$$T(x) := \begin{cases} F(x) + N_C(x) & \text{if } x \in C; \\ \emptyset & \text{otherwise.} \end{cases} \tag{22}$$

By Rockafellar's result [32, Theorem 3], T is maximal monotone and $T^{-1}(0) = S$. It has been known in [3, 32, 34] that a monotone mapping T is maximal if and only if the range of $J + rT$ is the whole space E^* for all $r > 0$. Then for each $r > 0$ and $x \in E$, there corresponds a unique element $x_r \in E$ satisfying

$$J(x) \in J(x_r) + rT(x_r) \tag{23}$$

(see [19, 20, 23]). From (23), we define the resolvent of T by $J_r(x) := x_r$. In other words, $J_r = (J + rT)^{-1}J$ for all $r > 0$. We can also define the Yosida approximation of T by $A_r := r^{-1}(J - JJ_r)$. It shows that $A_r(x) \in T(J_r(x))$ for all $r > 0$ and $x \in E$. For the theory of monotone mappings, we refer to [2, 3, 32, 34].

The proximal point algorithm which was first proposed by Martinet [27] is known for its theoretically nice convergence properties. Several authors [5–7, 10, 16, 17, 19, 21, 24, 25, 28, 29, 33, 37] studied this algorithm under appropriate assumptions. The proximal point algorithm generates, for any initial point $x_1 \in E$, a sequence $\{x_n\}$ converging to an element of $T^{-1}(0)$ by the iterative scheme:

$$x_{n+1} = J_{r_n}(x_n) \ \ (n = 1, 2, \dots), \tag{24}$$

where $\{r_n\}$ is a positive sequence. The sequence $\{x_n\}$ generated by (24) has the following property.

Lemma 3 (Kamimura-Kohsaka-Takahashi [19]) *Let E be a uniformly smooth and uniformly convex Banach space and let $T : E \to 2^{E^*}$ be a maximal monotone mapping with $T^{-1}(0) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined by* (24) *such that* $\liminf_{n\to\infty} r_n > 0$. *Then*

$$A_{r_n}(x_n) \to 0 \ \ as \ \ n \to \infty.$$

It can been seen from (23), (24) and $T = F + N_C$ that

$$A_{r_n}(x_n) - F(J_{r_n}(x_n)) \in N_C(J_r(x_n)),$$

and hence

$$\langle y - J_{r_n}(x_n), A_{r_n}(x_n) - F(J_{r_n}(x_n))\rangle \leq 0 \ \text{ for all } \ y \in C. \tag{25}$$

From the definition of A_r, we have

$$\left\langle y - J_{r_n}(x_n), \tfrac{1}{r_n}(J - JJ_{r_n})(x_n) - F(J_{r_n}(x_n))\right\rangle \leq 0,$$

and hence

$$\langle y - J_{r_n}(x_n), JJ^*(Jx_n - r_n F(J_{r_n}(x_n))) - JJ_{r_n}(x_n)\rangle \leq 0 \ \text{ for all } \ y \in C.$$

It follows from (9) that

$$J_{r_n}(x_n) = \Pi_C(J^*(J(x_n) - r_n F(J_{r_n}(x_n)))).$$

Therefore, (24) is equivalent to

$$x_{n+1} = \Pi_C(J^*(J(x_n) - r_n F(x_{n+1}))) \ \ (n = 1, 2, \dots). \tag{26}$$

Our first main result in this section is stated as follows.

Theorem 2 *Let E be a uniformly smooth and uniformly convex Banach space, let C be a closed convex subset of E and let $F : C \to E^*$ be a monotone continuous* *mapping. Let* $\{x_n\}$ *be the sequence generated by* (26) *such that* $\liminf_{n\to\infty} r_n > 0$. *Then* $x_n \in S$ *for all sufficiently large n if condition* (16) *and one of the following conditions hold:*

(i) F *is paramonotone;*
(ii) $\{x_n\}$ *converges strongly to some* $\bar{x} \in S$.

Proof We first show that (17) holds. Let $\epsilon > 0$. From (b) of Lemma 1, there exists $v_n \in T_C(x_n)$ with $\|v_n\| \leq 1$ such that

$$\|\Pi_{T_C(x_n)}\left(-J^* F(x_n)\right)\| \leq \langle v_n, -F(x_n)\rangle + \epsilon. \tag{27}$$

From (25), we have

$$\begin{aligned}\langle x - x_n, -F(x_n)\rangle &\leq \left\langle x_n - x, A_{r_{n-1}}(x_{n-1})\right\rangle \\ &\leq \|x_n - x\|\|A_{r_{n-1}}(x_{n-1})\| \end{aligned} \tag{28}$$

for all $x \in C$. From (5) and $v_n \in T_C(x_n)$, there exist $\{w_m\} \in C$ and $\{\lambda_m\} \subset (0, \infty)$ such that $\frac{1}{\lambda_m}(w_m - x_n) \to v_n$ as $m \to \infty$. Let $z_m = \frac{1}{\lambda_m}(w_m - x_n)$. From $w_m = x_n + \lambda_m z_m \in C$ and (28), we have

$$\langle z_m, -F(x_n)\rangle \leq \|z_m\|\|A_{r_{n-1}}(x_{n-1})\|.$$

Letting $m \to \infty$, we obtain

$$\begin{aligned}\langle v_n, -F(x_n)\rangle &\leq \|v_n\|\|A_{r_{n-1}}(x_{n-1})\| \\ &\leq \|A_{r_{n-1}}(x_{n-1})\|. \end{aligned} \tag{29}$$

It can be seen from (27) and (29) that

$$\|\Pi_{T_C(x_n)}\left(-J^* F(x_n)\right)\| \leq \|A_{r_{n-1}}(x_{n-1})\| + \epsilon.$$

It follows from (b) of Lemma 3 that

$$\limsup_{n\to\infty} \|\Pi_{T_C(x_n)}\left(-J^* F(x_n)\right)\| \leq \epsilon.$$

Since ϵ is arbitrary, (17) holds.

Suppose that (i) holds. By Theorem 1, $x_n \in S$ for all sufficiently large n. On the other hand, suppose that (ii) holds. By applying (ii) or (iii) of Theorem 1, $x_n \in S$ for all sufficiently large n. □

Remark 2 (i) It should be noted that (i) of Theorem 2 does not require $\{x_n\}$ converges strongly. (ii) of Theorem 2 includes the results of Mangasarian [25, Theorem 2.13] and Xiu and Zhang [37, Theorems 4.2] as special cases.
(ii) Some conditions for guaranteeing strong convergence of the proximal point algorithm are discussed in [10, 21, 33].

We next apply Theorem 1 to prove that the proximal point algorithm has one step termination property, i.e., there exists $r > 0$ such that $J_r(x) \in S$, where x is an arbitrary initial point. For any initial point $x \in E$, we consider the following iterative scheme:

$$x_n = J_{r_n}(x) \ \ (n = 1, 2, \dots), \tag{30}$$

where $\{r_n\} \subset (0, \infty)$. The sequence $\{x_n\}$ generated by (30) was analyzed by Reich [31].

Theorem 3 (Reich [31, page 342]) *Let E be a smooth Banach space with E^* which has a Fréchet differentiable norm and let $T : E \to 2^{E^*}$ be a maximal monotone operator with $T^{-1}(0) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by (30) such that* $\lim_{n\to\infty} r_n = \infty$. *Then $\{x_n\}$ converges strongly to some $\bar{x} \in T^{-1}(0)$.*

It should be noted that Reich's result does not guarantee the finite termination of the algorithm. Here, we investigate the finite termination of the sequence generated by (30) for solving (1).

It can be seen from (23), (30) and $T = F + N_C$ that

$$\left\langle y - x_n, \tfrac{1}{r_n}(J(x) - J(x_n)) - F(x_n)\right\rangle \leq 0 \text{ for all } y \in C. \tag{31}$$

From (9), (30) is equivalent to

$$x_n = \Pi_C(J^*(J(x) - r_n F(x_n))) \ \ (n = 1, 2, \cdots). \tag{32}$$

The second theorem of this section states for any given $x \in E$, and the sequence $\{x_n\}$ generated by (32) terminates after a finite number of iterations if r_n is chosen to be sufficiently large with weak sharp S.

Theorem 4 *Let E be a smooth Banach space with E^* which has a Fréchet differentiable norm and let C be a closed convex subset of E and let $F : C \to E^*$ be a monotone continuous mapping. Let $\{x_n\}$ be a sequence generated by (32) such that* $\lim_{n\to\infty} r_n = \infty$. *If condition (16) holds, then $x_n \in S$ for all sufficiently large n.*

Proof By Theorem 3, $\{x_n\}$ converges strongly to some $\bar{x} \in S$. Thus, it is sufficient to show that (17) holds.

Let $\epsilon > 0$. From (b) of Lemma 1, there exists $v_n \in T_C(x_n)$ with $\|v_n\| \leq 1$ such that

$$\|\Pi_{T_C(x_n)}\left(-J^*F(x_n)\right)\| \leq \langle v_n, -F(x_n)\rangle + \epsilon. \tag{33}$$

From (31), we have

$$\begin{aligned}\langle y - x_n, -F(x_n)\rangle &\leq \left\langle y - x_n, \tfrac{1}{r_n}(J(x_n) - J(x))\right\rangle \\ &\leq \frac{1}{r_n}\|y - x_n\|\|J(x_n) - J(x)\| \end{aligned} \tag{34}$$

for all $y \in C$. From (5) and $v_n \in T_C(x_n)$, there exist $\{w_m\} \in C$ and $\{\lambda_m\} \subset (0, \infty)$ such that $\frac{1}{\lambda_m}(w_m - x_n) \to v_n$ as $m \to \infty$. Let $z_m = \frac{1}{\lambda_m}(w_m - x_n)$. From $w_m = x_n + \lambda_m z_m \in C$ and (34), we have

$$\langle z_m, -F(x_n)\rangle \leq \frac{1}{r_n}\|z_m\|\|J(x_n) - J(x)\|.$$

Letting $m \to \infty$, we obtain

$$\begin{aligned}\langle v_n, -F(x_n)\rangle &\leq \frac{1}{r_n}\|v_n\|\|J(x_n)-J(x)\| \\ &\leq \frac{1}{r_n}\|J(x_n)-J(x)\|. \end{aligned} \tag{35}$$

It can be seen from (33) and (35) that

$$\|\Pi_{T_C(x_n)}\left(-J^*F(x_n)\right)\| \leq \frac{1}{r_n}\|J(x_n)-J(x)\| + \epsilon.$$

Since $\{J(x_n)\}$ is bounded and $r_n \to \infty$ as $n \to \infty$, we obtain

$$\limsup_{n\to\infty} \|\Pi_{T_C(x_n)}\left(-J^*F(x_n)\right)\| \leq \epsilon.$$

Since ϵ is arbitrary, (17) holds. Applying (ii) or (iii) of Theorem 1, we conclude that $x_n \in S$ for all sufficiently large n. □

Remark 3 Theorems 4 includes the results of Mangasarian [25, Theorem 2.13] and Xiu and Zhang [37, Theorem 4.3] as special cases.

Acknowledgements The authors are grateful to Professor W. Takahashi of Tokyo Institute of Technology and Professor D. Kuroiwa of Shimane University for their helpful support.

References

1. Alber, Y.I.: Metric and generalized projection operators in Banach spaces:properties and applications. In: Kartsatos, A.G. (ed.) Theory and Applications of Nonlinear Operators of Accretive and Monotone Type Lecture Notes in Pure and Appl. Math., vol. 178, pp. 15–50. Marcel Dekker, New York (1996)
2. Aubin, J.-P., Ekeland, I.: Applied Nonlinear Analysis. Pure and Applied Mathematics. Wiley, New York (1984)
3. Barbu, V., Precupanu, T.: Convexity and Optimization in Banach Spaces, 2nd edn. Reidel, Dordrecht (1986)
4. Bauschke, H.H., Burke, J.V., Deutsch, F.R., Hundal, H.S., Vanderwerff, J.D.: A new proximal point iteration that converges weakly but not in norm. Proc. Am. Math. Soc. **133**, 1829–1835 (2005)
5. Brézis, H., Lions, P.-L.: Produits infinis de résolvantes. Israel J. Math. **29**, 329–345 (1978)
6. Burachik, R.S., Iusem, A.N.: Set-valued Mappings and Enlargements of Monotone Operators. Springer Optim. Appl., Springer, New York (2008)
7. Burachik, R.S., Scheimberg, S.: A proximal point method for the variational inequality problem in Banach spaces. SIAM J. Control Optim. **39**, 1633–1649 (2001)
8. Burke, J.V., Deng, S.: Weak sharp minima revisited, part I: basic theory. Control Cybern. **31**, 439–469 (2002)
9. Burke, J.V., Ferris, M.C.: Weak sharp minima in mathematical programming. SIAM J. Control Optim. **31**, 1340–1359 (1993)
10. Butnariu, D., Iusem, A.I., Zălinescu, C.: On uniformly convexity, total convexity and convergence of the proximal point and outer Bregman projection algorithms in Banach spaces. J. Convex Anal. **10**, 35–61 (2003)
11. Calamai, P.H., Moré, J.J.: Projected gradient methods for linearly constrained problems. Math. Program. **39**, 93–116 (1987)
12. Censor, Y., Iusem, A.N., Zenios, S.A.: An interior point method with Bregman functions for the variational inequality problem with paramonotone operators. Math. Program. **31**, 373–400 (1998)

13. Cioranescu, I.: Geometry of Banach spaces, Duality Mapping and Nonlinear Problems. Kluwer Academic Publishers, Amsterdam (1990)
14. Daniilidis, A., Hadjisavvas, N.: Coercivity conditions and variational inequalities. Math. Program. **86**, 433–438 (1999)
15. Facchinei, F., Pang, J.S.: Finite-Dimensional Variational Inequalities and Complementarity Problems I. Springer, New York (2003)
16. Ferris, M.C.: Finite termination of the proximal point algorithm. Math. Program. **50**, 359–366 (1991)
17. Güler, O.: On the convergence of the proximal point algorithm for convex minimization. SIAM J. Control Optim. **29**, 403–419 (1991)
18. Hu, Y.H., Song, W.: Weak sharp solutions for variational inequalities in Banach spaces. J. Math. Anal. Appl. **374**, 118–132 (2011)
19. Kamimura, S., Kohsaka, F., Takahashi, W.: Weak and strong convergence theorems for maximal monotone operators in a Banach space. Set-Valued Anal. **12**, 417–429 (2004)
20. Kamimura, S., Takahashi, W.: Strong convergence of a proximal-type algorithm in a Banach space. SIAM J. Optim. **13**, 938–945 (2002)
21. Kassay, G.: The proximal point algorithm for reflexive Banach spaces. Stud. Univ. Babeş-Bolyai Math. **30**, 9–17 (1985)
22. Kinderlehrer, D., Stampacchia, G.: An Introduction to Variational Inequalities and Their Applications. Academic, New York (1980)
23. Kohsaka, F., Takahashi, W.: Strong convergence of an iterative sequence for maximal monotone operators in a Banach space. Abstr. Appl. Anal. **2004**, 239–249 (2004)
24. Luque, F.J.: Asymptotic convergence analysis of the proximal point algorithm. SIAM J. Control Optim. **22**, 277–293 (1984)
25. Mangasarian, O.L.: Error bounds for nondegenerate monotone linear complementarity problems. Math. Program. **48**, 437–445 (1990)
26. Marcotte, P., Zhu, D.L.: Weak sharp solutions of variational inequalities. SIAM J. Optim. **9**, 179–189 (1999)
27. Martinet, B.: Régularisation d'inéquations variationnelles par approximations successives. Rev. Fr. Inform. Rech. Opér. **4**, 154–158 (1970)
28. Matsushita, S., Takahashi, W.: Weak and strong convergence theorems for relatively nonexpansive mappings in Banach spaces. Fixed Point Theory Appl. **2004**, 37–47 (2004)
29. Matsushita, S., Takahashi, W.: A strong convergence theorem for relatively nonexpansive mappings in a Banach space. J. Approx. Theory **134**, 257–266 (2005)
30. Patriksson, M.: Unified framework of descent algorithms for nonlinear programs and variational inequalities. Ph.D. Thesis, Linköping Institute of Technology (1993)
31. Reich, S.: Constructive techniques for accretive and monotone operators. In: Lakshmikantham, V. (ed.) Applied Nonlinear Analysis, pp. 335–345. Academic Press, New York (1979)
32. Rockafellar, R.T.: On the maximality of sums of nonlinear monotone operators. Trans. Am. Math. Soc. **149**, 75–88 (1970)
33. Rockafellar, R.T.: Monotone operators and the proximal point algorithm. SIAM J. Control Optim. **14**, 877–898 (1976)
34. Takahashi, W.: Nonlinear Functional Analysis. Yokohama Publishers, Yokohama (2000)
35. Takahashi, W.: Convex Analysis and Approximation of Fixed Points. Yokahama Publishers, Yokahama, (2000) (Japanese)
36. Wu, Z.L., Wu, S.Y.: Weak sharp solutions of variational inequalities in Hilbert spaces. SIAM J. Optim. **14**, 1011–1027 (2004)
37. Xiu, N., Zhang, J.: On finite convergence of proximal point algorithms for variational inequalities. J. Math. Anal. Appl. **312**, 148–158 (2005)
38. Zălinescu, C.: Convex Analysis in General Vector Spaces. World Scientific, Singapore (2002)
39. Zhou, J., Wang, C.: A note on finite termination of iterative algorithms in mathematical programming. Oper. Res. Lett. **36**, 715–717 (2008)