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On Directional Metric Regularity, Subregularity and Optimality Conditions for Nonsmooth Mathematical Programs

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Abstract This paper mainly deals with the study of directional versions of metric regularity and metric subregularity for general set-valued mappings between infinite-dimensional spaces. Using advanced techniques of variational analysis and generalized differentiation, we derive necessary and sufficient conditions, which extend even the known results for the conventional metric regularity. Finally, these results are applied to non-smooth optimization problems. We show that that at a locally optimal solution M-stationarity conditions are fulfilled if the constraint mapping is subregular with respect to one critical direction and that for every critical direction a M-stationarity condition, possibly with different multipliers, is fulfilled.

Keywords Metric regularity · Subregularity · M-stationarity

Mathematics Subject Classifications (2010) 49J53 · 49K27 · 90C48

1 Introduction

We study in this paper non-smooth optimization problems of the form

$$\text{minimize } f(x) \quad \text{subject to} \quad 0 \in G(x) \tag{1}$$

where the objective function $f : X \to \bar{R}$ maps a Banach space X into the extended real numbers and $G : X \rightrightarrows Z$ is a set-valued mapping between Banach spaces.

Problem (1) is a very general problem and many types of constraints can be formulated in the form $0 \in G(x)$. As an example let us mention the case, where among the constraints so-called equilibrium constraints occur, where typically the equilibrium constraint are solution maps to parametric variational inequalities and complementarity problems of different types.

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In the development of optimality conditions the existence of nondegenerate multipliers is related to the validity of some constraint qualification condition. Such constraint qualification conditions are, for instance, the properties of metric regularity and subregularity.

Recall that a multifunction $M : X \rightrightarrows Y$ between Banach spaces is called *metrically regular* with modulus $\kappa > 0$ near the point $(\bar{x}, \bar{y}) \in \operatorname{gph} M := \{(x, y) \in X \times Y \mid y \in M(x)\}$ from its graph, provided there exist neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$\mathrm{d}(x, M^{-1}(y)) \leq \kappa \mathrm{d}(y, M(x)) \quad \forall (x, y) \in U \times V. \tag{2}$$

Here $\mathrm{d}(x, \Omega)$ denotes the usual distance between a point x and a set Ω. When fixing $y = \bar{y}$ in (2) we obtain the weaker property of *metric subregularity* of M at $(\bar{x}, \bar{y})$, i.e. we require the estimate

$$\mathrm{d}(x, M^{-1}(\bar{y})) \leq \kappa \mathrm{d}(\bar{y}, M(x)) \quad \forall x \in U \tag{3}$$

with some neighborhood $U$of $\bar{x}$ and a positive real $\kappa > 0$.

It is well known that a multifunction $M : X \rightrightarrows Y$ is metrically regular near $(\bar{x}, \bar{y}) \in \operatorname{gph} M$ if the inverse multifunction $F = M^{-1}$ has the *Aubin property (local Lipschitz-like property, pseudo-Lipschitzian property)* near $(\bar{y}, \bar{x})$, i.e.

$$F(y') \cap U \subset F(y) + L \left\| y - y' \right\| \mathscr{B}_X \quad \forall y', y \in V,$$

with $L \geq 0$ und neighborhoods U of $\bar{x}$ and V of $\bar{y}$. Further, the property of metric subregularity is equivalent to *calmness* of the inverse multifunction, see [5]. For a survey on the theory of metric regularity and the Aubin property we refer the reader to [16] and to the monographs [18, 24, 29] and the references therein. Various results on metric subregularity and calmness and their applications can be found, e.g., in [4, 6, 9–13, 15, 17, 19, 20, 27, 28, 30–32, 34],

Under the constraint qualification of metric (sub)regularity, so-called *Mordukhovich* or *M-stationarity* conditions are fulfilled at a locally optimal solution of (1). These optimality conditions are associated with the generalized differential calculus of Mordukhovich [24].

Let us mention that metric (sub)regularity is not the weakest possible constraint qualification under which M-stationarity can be shown, see e.g. [7]. However, metric regularity is a constraint qualification which can be actually verified by means of the generalized differential calculus. There exist equivalent characterizations of metric regularity by means of the coderivative and a partial sequential compactness property of the multifunction. We refer to [22], where this characterization was stated for the Aubin property in Asplund spaces. Characterizations of metric subregularity by generalized differentiation can be found for instance in [9–13, 17]. An important subclass of multifunctions which are known to be metrically subregular at every point of their graph, is given by polyhedral multifunctions, i.e. multifunctions whose graph is the union of finitely many polyhedral sets. This result is due to Robinson [28]. An important special case of polyhedral multifunctions is given by linear systems, where subregularity is a consequence of Hoffman's error bound [14]. Some extensions to the infinite dimensional case are given in [3, Section 2.5.7].

This paper is motivated by the observation, that for the validity of the M-stationarity conditions one only needs a regular behavior of the constraints with respect to *one* single critical direction and not the metric (sub)regularity property of the constraint mapping on the whole space. Moreover, we strengthen the concept of M-stationarity conditions by showing, that for *every* critical direction a M-stationarity condition, with possibly different multipliers, must be fulfilled. To demonstrate this let us consider the following examples:

Example 1 Given $\varphi : \mathbb{R} \to \mathbb{R}$ continuously differentiable, consider the problem

$$\min_{x_1,x_2} \varphi(x_1) \quad \text{s.t.} \quad (0,0) \in G(x_1,x_2) := \{\min\{x_1,x_2\}\} \times (x_1^2 - x_2 - \mathbb{R}_-),$$

i.e. the nonsmooth formulation of the mathematical problem with complementarity constraints

$$\min_{x_1,x_2} \varphi(x_1) \quad \text{subject to} \quad x_1 \geq 0,\ x_2 \geq 0,\ x_1 x_2 = 0,\ x_1^2 - x_2 \leq 0.$$

The feasible region is $\{0\} \times \mathbb{R}_+$ and hence $(0,0)$ is a locally optimal solution. The M-stationarity conditions read as

$$\exists \eta, \lambda, \mu \in \mathbb{R} : \varphi'(0) - \eta = 0, -\lambda + \mu = 0, \mu \geq 0, (\eta > 0, \lambda > 0 \text{ or } \eta\lambda = 0),$$

see e.g. [26, 33], and are fulfilled with $\eta = \varphi'(0)$, $\lambda = \mu = 0$, although the constraint mapping G is not metrically subregular at $((0,0),(0,0))$, since

$$t = \mathrm{d}((t,0), G^{-1}(0,0)) = \frac{1}{t}\mathrm{d}((0,0), G(t,0)) \quad \forall t > 0.$$

However, the constraints fulfill the subregularity condition (3) with respect to the direction $(0,t)$, $t > 0$. More precisely, we have

$$\mathrm{d}((\delta t, t), G^{-1}(0,0)) = |\delta| t = \mathrm{d}((0,0), G(\delta t, t))\ \forall 0 \leq t \leq 1, \forall -1 < \delta < 1$$

and we will prove in the sequel that this condition guarantees the M-stationarity of the local solution.

Example 2 Now consider the problem

$$\min_{x_1,x_2} \varphi(x_1) \quad \text{s.t.} \quad 0 \in G(x_1,x_2) := \{\min\{x_1,x_2\}\},$$

with $\varphi : \mathbb{R} \to \mathbb{R}$ continuously differentiable. Then it is easy to verify that the constraint mapping G is metrically regular near $(0,0)$ and the M-stationarity conditions at $(0,0)$ read as

$$\exists \eta, \lambda \in \mathbb{R} : \varphi'(0) - \eta = 0, -\lambda = 0, (\eta > 0, \lambda > 0 \text{ or } \eta\lambda = 0).$$

Taking $\eta = \varphi'(0)$, $\lambda = 0$ we see that $(0,0)$ is M-stationary, regardless what φ is. The simple reason is, that every point of the form $(0,t)$, $t > 0$ is a local minimizer. However, in order to reject $(0,0)$ as a local minimizer when $\varphi'(0) < 0$, we have to analyze the behavior of the problem also in direction $(t,0)$ for $t > 0$.

These observations lead us to the concepts of *directional metric regularity* and *directional metric subregularity*, respectively, where the estimates (2) and (3) are not required to hold for all points belonging to some neighborhood of $(\bar{x}, \bar{y})$ but only for points belonging to some subset of a neighborhood of a certain direction $(u, v) \in X \times Y$. We think that these concepts are not only important for the development of optimality conditions but, like the conventional metric (sub)regularity, will also play an important role in many other aspects of nonlinear analysis and its applications such as error bounds and stability analysis.

The notion of directional metric regularity was already used in a different context in the papers [1, 2], where (2) is required to hold for points belonging to some set related to a direction $v \in Y$. The concept of directional metric regularity used in [1, 2] is related to the notion of *directional regularity* (see also [3, Chapter 4.2], where regularity properties for parameter dependent problems are considered when the parameter varies in a given direction. Note that directional metric regularity in [1, 2] was defined with respect to directions $v \in Y$. Contrarily, our following Definition 1 of directional metric regularity uses directions $(u, v) \in X \times Y$.

Another type of nontraditional metric regularity has been introduced and studied in [25] under the name of *restrictive metric regularity*.

The rest of the paper is organized as follows. Section 2 contains some preliminaries from generalized differentiation used as well for characterizing metric regularity and subregularity as formulating optimality conditions. In Section 3 we introduce the concepts of directional metric regularity and subregularity and present characterizations of these properties by using directional versions of the limiting differential objects considered in Section 2. As a byproduct we show that the characterization [22] of the Aubin property is not only valid in Asplund spaces, but also in case that the domain space is Fréchet smooth whereas the image space can be an arbitrary Banach space. In Section 4 we consider the notion of mixed directional regularity/subregularity, where one part of the multifunction behaves metrically regular, whereas the other part behaves only subregular. Just by observing that at a locally optimal solution to problem (1) a certain multifunction associated with the problem cannot be mixed regular/subregular, this concept yields the optimality conditions as presented in Section 5.

Our notation is fairly standard. Throughout this paper let X, Y and Z be Banach spaces equipped with norm $\|\cdot\|$. By X^* we denote the topological dual of X with the canonical pairing $\langle\cdot,\cdot\rangle$ between X and X^*. $\mathscr{B}_X := \{x \in X \mid \|x\| \leq 1\}$ denotes the closed unit ball and $\mathscr{S}_X := \{x \in X \mid \|x\| = 1\}$ denotes the unit sphere. Unless otherwise stated, we assume that the product space $X \times Y$ of two spaces X and Y is equipped with a norm satisfying $\max\{\|x\|, \|y\|\} \leq \|(x, y)\| \leq \|x\| + \|y\|$.

Recall that a Banach space X is *Asplund* if each of its separable subspaces has a separable dual. There are many equivalent descriptions of these spaces, which can be found, e.g., in [24] and its bibliography. We use in this paper sometimes the fact, that any bounded sequence in the dual of an Asplund space has a weak* convergent subsequence. X is called *Fréchet smooth*, if it admits an equivalent norm Fréchet differentiable at any nonzero point. This class of Banach spaces is sufficiently large including, in particular, every reflexive space while the class of Asplund spaces is broader.

2 Preliminaries from Generalized Differentiation

In this section we recall some generalized differential constructions from variational analysis and their relations to metric regularity, subregularity and optimality conditions.

Let Ω be a nonempty subset of a Banach space X and let $x \in \Omega$. The *contingent cone* to Ω at x, denoted by $T(x;\Omega)$, is given by

$$T(x;\Omega) := \left\{ d \in X \mid \exists (x_k) \in \Omega,\, (t_k) \downarrow 0 : \frac{x_k - x}{t_k} \to d \right\}.$$

Given $\varepsilon \geq 0$ we denote by

$$\hat{N}_\varepsilon(x;\Omega) = \left\{ x^* \in X^* \mid \limsup_{x' \xrightarrow{\Omega} x} \frac{\langle x^*, x' - x\rangle}{\|x' - x\|} \leq \varepsilon \right\} \tag{4}$$

the *set of ε-normals* to Ω. When $\varepsilon = 0$, elements of (4) are called *Fréchet normals* or *normals in a regular sense* and their collection is denoted by $\hat{N}(x;\Omega)$. Finally, the *limiting normal cone* to Ω at x is defined by

$$N(x;\Omega) := \left\{ x^* \mid \exists (\varepsilon_k) \downarrow 0,\ (x_k) \xrightarrow{\Omega} x,\ (x_k^*) \xrightarrow{w^*} x^* : x_k^* \in \hat{N}_{\varepsilon_k}(x_k;\Omega) \forall k \right\}.$$

If $x \notin \Omega$ we put $T(x;\Omega) = \emptyset$, $N(x;\Omega) = \emptyset$ and $\hat{N}_\varepsilon(x;\Omega) = \emptyset$ for all $\varepsilon \geq 0$.

The limiting normal cone sometimes is also called basic normal cone or Mordukhovich normal cone. It is generally nonconvex whereas the Fréchet normal cone is always convex. In the case of a convex set Ω, both the Fréchet normal cone and the limiting normal cone coincide with the standard normal cone from convex analysis and moreover, the contingent cone is equal to the tangent cone in the sense of convex analysis.

Given a multifunction $M : X \rightrightarrows Y$ and a point $(\bar{x}, \bar{y}) \in \operatorname{gph} M$, the *contingent derivative* of M at $(\bar{x}, \bar{y})$ is defined as the set-valued mapping $CM(\bar{x}, \bar{y}) : X \rightrightarrows Y$ with the values $CM(\bar{x}, \bar{y})(u) := \{v \in Y \mid (u, v) \in T((\bar{x}, \bar{y}); \operatorname{gph} M)\}$, i.e. $CM(\bar{x}, \bar{y})(u)$ is the collection of all $v \in Y$ such that there are sequences $(t_k) \downarrow 0$, $(u_k, v_k) \to (u, v)$ with $(\bar{x} + t_k u_k, \bar{y} + t_k v_k) \in \operatorname{gph} M$.

The *normal coderivative* of M at $(\bar{x}, \bar{y})$ is a multifunction $D_N^* M(\bar{x}, \bar{y}) : Y^* \rightrightarrows X^*$, where $D_N^* M(\bar{x}, \bar{y})(y^*)$ is the collection of all $x^* \in X^*$ for which there are sequences $(\varepsilon_k) \downarrow 0$, $(x_k, y_k) \to (\bar{x}, \bar{y})$ and $(x_k^*, y_k^*) \xrightarrow{w^*} (x^*, y^*)$ with $(x_k^*, -y_k^*) \in \hat{N}_{\varepsilon_k}((x_k, y_k); \operatorname{gph} M)$.

The *reversed mixed coderivative* of M at $(\bar{x}, \bar{y})$ is a multifunction $\tilde{D}_M^* M(\bar{x}, \bar{y}) : Y^* \rightrightarrows X^*$, where $D_{\tilde{M}}^* M(\bar{x}, \bar{y})(y^*)$ is the collection of all linear functionals $x^* \in X^*$ for which there are sequences $(\varepsilon_k) \downarrow 0$, $(x_k, y_k, x_k^*) \to (\bar{x}, \bar{y}, x^*)$ and $(y_k^*) \xrightarrow{w^*} (y^*)$ with $(x_k^*, -y_k^*) \in \hat{N}_{\varepsilon_k}((x_k, y_k); \operatorname{gph} M)$.

The *mixed coderivative* of M at $(\bar{x}, \bar{y})$ can be defined as the multifunction $D_M^* M(\bar{x}, \bar{y}) : Y^* \rightrightarrows X^*$, where

$$D_M^*(\bar{x}, \bar{y})(y^*) := \{x^* \in X^* \mid y^* \in -\tilde{D}_M^* M^{-1}(\bar{y}, \bar{x})(-x^*)\},$$

i.e. the version of the reversed mixed coderivative with strong convergence in Y^* and weak* convergence in Y^*.

Note that

$$D_N^* M(\bar{x}, \bar{y})(y^*) = \{x^* \in X^* \mid (x^*, -y^*) \in N((\bar{x}, \bar{y}); \operatorname{gph} M)\}$$

and

$$D_N^* M^{-1}(\bar{y}, \bar{x})(x^*) = \{y^* \in Y^* \mid -x^* \in D_N^* M(\bar{x}, \bar{y})(-y^*)\}.$$

The mapping M is called *partially sequentially normally compact* (PSNC) at $(\bar{x}, \bar{y}) \in \operatorname{gph} M$ with respect to Y if for all sequences $(\varepsilon_k) \downarrow 0$, $(x_k, y_k) \to (\bar{x}, \bar{y})$, $(y_k^*) \stackrel{w^*}{\to} 0$ and $(x_k^*) \to 0$ with $(x_k^*, -y_k^*) \in \hat{N}_{\varepsilon_k}((x_k, y_k), \operatorname{gph} M)$ one has $\|y_k^*\| \to 0$ as $k \to \infty$.

For a detailed discussion and references concerning these generalized differential constructions we refer the reader to the monograph [24].

Mordukhovich [22] gave a complete characterization of the Aubin property of a multifunction between Asplund spaces in terms of the mixed coderivative together with the PSNC property. We state here the following result, cf. [24, Theorems 4.10, 4.18].

Theorem 1 *Let $M : X \rightrightarrows Y$ be a closed-graph multifunction between Asplund spaces and let $(\bar{x}, \bar{y}) \in \operatorname{gph} M$. Then the following statements are equivalent:*

(a) *M is metrically regular near $(\bar{x}, \bar{y})$.*
(b) *M^{-1} has the Aubin property near $(\bar{y}, \bar{x})$.*
(c) *M is PSNC at $(\bar{x}, \bar{y})$ with respect to Y and* $\ker \tilde{D}_M^* M(\bar{x}, \bar{y}) = \{0\}$.
(d) *M is PSNC at $(\bar{x}, \bar{y})$ with respect to Y and $D_M^* M^{-1}(\bar{y}, \bar{x})(0) = \{0\}$.*

In the recent paper [9] the property of metric subregularity was characterized by means of so-called limit sets critical for metric subregularity. Given a set-valued mapping $M : X \rightrightarrows Y$, a subspace $\tilde{Y} \subset Y$ and a point $(\bar{x}, \bar{y}) \in \operatorname{gph} M$, the *limit set critical for metric subregularity of M at $(\bar{x}, \bar{y})$ with respect to $\tilde{Y}$* is the set $\operatorname{Cr}_{\tilde{Y}} M(\bar{x}, \bar{y})$ of all elements $(v, x^*) \in Y \times X^*$ such that there are sequences $(t_k) \downarrow 0$, $(\varepsilon_k) \downarrow 0$, $(v_k, x_k^*) \to (v, x^*)$, $(u_k, y_k^*) \in \mathscr{S}_X \times \mathscr{S}_{Y^*}$ and a positive constant β with $(-x_k^*, y_k^*) \in \hat{N}_{\varepsilon_k}((\bar{x} + t_k u_k, \bar{y} + t_k v_k), \operatorname{gph} M)$ and $\left\| y_{k|_{\tilde{Y}}}^* \right\| := \sup\{\langle y_k^*, \tilde{y}\rangle \mid \tilde{y} \in \mathscr{B}_Y \cap \tilde{Y}\} \geq \beta$. In case that $\tilde{Y} = Y$ we simply write $\operatorname{Cr} M(\bar{x}, \bar{y})$.

Theorem 2 [9] *Let $M : X \rightrightarrows Y$ be a multifunction and let $(\bar{x}, \bar{y}) \in \operatorname{gph} M$.*

1. *Assume that M has closed graph, that either Y is Fréchet smooth or both X and Y are Asplund spaces and assume that $(0, 0) \notin \operatorname{Cr} M(\bar{x}, \bar{y})$. Then M is metrically subregular at $(\bar{x}, \bar{y})$.*
2. *If $(0, 0) \in \operatorname{Cr} M(\bar{x}, \bar{y})$, then there exists a continuously differentiable mapping $h : X \rightrightarrows Y$ with $h(\bar{x}) = 0$, $\nabla h(\bar{x}) = 0$ such that $M + h$ is not metrically subregular at $(\bar{x}, \bar{y})$.*

For any continuously differentiable mapping $h : X \to Y$ with $h(\bar{x}) = 0$, $\nabla h(\bar{x}) = 0$ we have $\operatorname{Cr}(M + h)(\bar{x}, \bar{y}) = \operatorname{Cr} M(\bar{x}, \bar{y})$. Hence the condition $(0, 0) \notin \operatorname{Cr} M(\bar{x}, \bar{y})$ is an equivalent characterization of metric subregularity in case that the property of metric subregularity is stable under smooth perturbations h with $h(\bar{x}) = 0$, $\nabla h(\bar{x}) = 0$. Let us mention that the condition $(0, 0) \notin \operatorname{Cr} M(\bar{x}, \bar{y})$ is not the weakest sufficient condition for subregularity known from the literature. Let us define the *outer limit set critical for metric subregularity* of M, denoted by $\operatorname{Cr}^{>} M(\bar{x}, \bar{y})$, as those elements $(v, x^*) \in \operatorname{Cr} M(\bar{x}, \bar{y})$ such that the corresponding sequences fulfill the additional condition $\bar{x} + t_k u_k \notin M^{-1}(\bar{y})$. Then, as already mentioned in [9, p. 1450], the condition $(0, 0) \notin \operatorname{Cr}^{>} M(\bar{x}, \bar{y})$ is also sufficient for metric subregularity of M, which is clearly not stronger than the condition $(0, 0) \notin \mathrm{Cr}\, M(\bar{x}, \bar{y})$. Other conditions for (Hölder) metric subregularity, based on so-called *outer coderivatives*, can be found e.g. in [17, 21, 35].

However, the definition of the outer limit set or outer coderivatives requires information about the set $M^{-1}(\bar{y})$, which is usually unknown. Further the outer limit set is in general not invariant under smooth perturbations having zero function value and derivative at the reference point and hence it is likely that only some very limited calculus is available for computing this set. For that reason we do not consider in the rest of the paper such sufficient conditions based on outer limit sets. On the other hand, if we have some partial information about the aforementioned set, e.g. by identifying some linear structures of M, these "outer" constructions can be very useful.

In [9] also the so-called *combined contingent coderivative* of a multifunction $M : X \rightrightarrows Y$ was introduced. At a point $(\bar{x}, \bar{y}) \in \mathrm{gph}\, M$ it is defined as the multifunction $\widehat{CD^*} M(\bar{x}, \bar{y}) : X \times Y^* \rightrightarrows Y \times X^*$, where for each $(u, y^*) \in X \times Y^*$ the set $\widehat{CD^*} M(\bar{x}, \bar{y})(u, y^*)$ is given by the collection of all $(v, x^*) \in Y \times X^*$ for which there are sequences $(t_k) \downarrow 0$, $(\varepsilon_k) \downarrow 0$, $(u_k, v_k, x_k^*) \to (u, v, x^*)$, $(y_k^*) \stackrel{w^*}{\to} y^*$ with $(-x_k^*, y_k^*) \in \hat{N}_{\varepsilon_k}((\bar{x} + t_k u_k, \bar{y} + t_k v_k), \mathrm{gph}\, M)$.

By the definition the elements $(v, x^*) \in \widehat{CD^*} M(\bar{x}, \bar{y})(u, y^*)$ satisfy $v \in CM(\bar{x}, \bar{y})(u)$ and $y^* \in D^*_M M^{-1}(\bar{y}, \bar{x})(x^*)$, or equivalently $-x^* \in \tilde{D}^*_M M(-y^*)$. In fact, the combined contingent coderivative of M is defined by elements of the contingent derivative of M and the mixed coderivative of M^{-1} which share in their definition a common sequence of points $(\bar{x} + t_k u_k, \bar{y} + t_k v_k)$.

In case that X and Y are Asplund spaces, it can be shown, e.g. by using [24, Formula (2.51)], that in the definitions above of the limiting objects $N(x, \Omega)$, $D^*_N M(\bar{x}, \bar{y})$, $\tilde{D}^*_M M(\bar{x}, \bar{y})$, $D^*_M M(\bar{x}, \bar{y})$, $\mathrm{Cr}_{|_{\hat{Y}}} M(\bar{x}, \bar{y})$, $\widehat{CD^*} M(\bar{x}, \bar{y})$ and the PSNC property we can take equivalently $\varepsilon_k = 0$.

We now recall some optimality conditions for the problem (1).

Given a function $\varphi : X \to \bar{\mathbb{R}}$ and a point $x \in X$ with $|\varphi(x)| < \infty$, we define for $\varepsilon \geq 0$ the ε-*subdifferential* of φ at x by

$$\hat{\partial}_\varepsilon \varphi(x) := \left\{ x^* \in X^* \,\middle|\, \liminf_{x' \to x} \frac{\varphi(x') - \varphi(x) - \langle x^*, x' - x \rangle}{\|x' - x\|} \geq -\varepsilon \right\}.$$

When $\varepsilon = 0$ the ε-subdifferential reduces to the *Fréchet (lower) subdifferential* $\hat{\partial}\varphi(x) := \hat{\partial}_0 \varphi(x)$, elements of $\hat{\partial}\varphi(x)$ are also called *regular subgradients*. Further the *Fréchet upper subdifferential* is described by

$$\hat{\partial}^+ \varphi(x) := \left\{ x^* \in X^* \,\middle|\, \limsup_{x' \to x} \frac{\varphi(x') - \varphi(x) - \langle x^*, x' - x \rangle}{\|x' - x\|} \leq 0 \right\}$$

and the *limiting subdifferential* of φ at x is defined as the set

$$\partial \varphi(x) := \{x^* \in X^* \mid \exists (\varepsilon_k) \downarrow 0,\ (x_k) \to x,\ (x_k^*) \stackrel{w^*}{\to} x^* : x_k^* \in \hat{\partial}_{\varepsilon_k} \varphi(x_k) \forall k\}.$$

The following theorem is a consequence of [24, Theorems 5.7(iii), 5.48] and states the M-stationarity conditions for the problem (1).

Theorem 3 *Let $\bar{x}$ be a local optimal solution of the problem* (1) *where X and Z are Asplund spaces and* gph G *is closed.*

1. *If* $D^*_M G^{-1}(\bar{y},\bar{x})(0) = 0$ *and* G *is PSNC at* $(\bar{x},\bar{y})$ *with respect to* Y*, i.e. if* G *is metrically regular near* $(\bar{x}, 0)$*, then*
$$-\hat{\partial}^+ f(\bar{x}) \subset \bigcup_{z^*\in Z^*} D^*_N G(\bar{x}, 0)(z^*) \tag{5}$$
2. *If* G *is metrically subregular at* $(\bar{x}, 0)$ *and* f *is Lipschitzian near* $\bar{x}$*, then there is some* $z^* \in Z^*$ *such that*
$$0 \in \partial f(\bar{x}) + D^*_N G(\bar{x}, 0)(z^*) \tag{6}$$

If f is Fréchet differentiable at $\bar{x}$ then $\hat{\partial}^+ f(\bar{x}) = \{\nabla f(\bar{x})\}$ and the M-stationarity condition (5) reduces to
$$\exists z^* \in Z^* : \ 0 \in \nabla f(\bar{x}) + D^*_N G(\bar{x}, 0)(z^*).$$
The use of the upper subdifferential in optimality conditions was initiated in [23]. Observe that, despite the broader applicability of (6), the upper subdifferential condition (5) is stronger for concave continuous functions f because of $\partial f(\bar{x}) \subset \hat{\partial}^+ f(\bar{x}) \neq \emptyset$.

3 Directional Metric Regularity and Subregularity

To study the directional behavior of multifunctions, it is convenient to introduce the following neighborhoods of directions: Given a Banach space W, a direction $d \in W$ and positive numbers $\varepsilon, \delta > 0$, the set $V_{\varepsilon,\delta}(d)$, is given by
$$V_{\varepsilon,\delta}(d) := \{w \in \varepsilon\mathscr{B}_W \mid \big\| \|d\|w - \|w\|d\big\| \le \delta\|w\|\,\|d\|\}. \tag{7}$$
This can be written also in the form
$$V_{\varepsilon,\delta}(d) = \begin{cases} \{0\} \cup \left\{w \in \varepsilon\mathscr{B}_W \setminus \{0\} \left| \left\| \frac{w}{\|w\|} - \frac{d}{\|d\|} \right\| \le \delta \right.\right\} & \text{if } d \neq 0, \\ \varepsilon\mathscr{B}_W & \text{if } d = 0. \end{cases}$$

Note that $V_{\varepsilon,\delta}(d) = V_{\varepsilon,\delta}(\alpha d)$, $\forall \alpha > 0$ and that, given $\bar{w} \in W$ and a sequence $(w_k) \to \bar{w}$, there exist sequences $(t_k) \downarrow 0$, $(d_k) \to d$ with $w_k = \bar{w} + t_k d_k$ if and only if for every $\varepsilon > 0, \delta > 0$ there is some index $k_{\varepsilon,\delta}$ such that $w_k \in \bar{w} + V_{\varepsilon,\delta}(d)$, $\forall k \ge k_{\varepsilon,\delta}$.

Definition 1 Let $M : X \rightrightarrows Y$ be a multifunction and let $(\bar{x}, \bar{y}) \in \operatorname{gph} M$.

1. Given $w := (u, v) \in X \times Y$, M is called *metrically regular in direction* (u, v) at $(\bar{x}, \bar{y})$, provided there exist positive reals $\rho > 0$, $\delta > 0$ and $\kappa > 0$ such that
$$\mathrm{d}(x, M^{-1}(y)) \le \kappa \mathrm{d}(y, M(x)) \tag{8}$$
holds for all $(x, y) \in (\bar{x}, \bar{y}) + V_{\rho,\delta}(w)$ with $\|w\|\,\mathrm{d}((x, y), \operatorname{gph} M) \le \delta\,\|w\|\,\|(x, y)-(\bar{x}, \bar{y})\|$.
2. For given $u \in X$, M is said to be *metrically subregular in direction* u at $(\bar{x}, \bar{y})$, if there are positive reals $\rho > 0$, $\delta > 0$ and $\kappa' > 0$ such that
$$\mathrm{d}(x, M^{-1}(\bar{y})) \le \kappa' \mathrm{d}(\bar{y}, M(x)) \tag{9}$$
holds for all $x \in \bar{x} + V_{\rho,\delta}(u)$.

By the definition, M is metrically regular in direction $(0, 0)$ if and only if M is metrically regular. Similarly, metric subregularity in direction 0 is equivalent to the property of metric subregularity.

Note that we always have $\mathrm{d}((x, y), \mathrm{gph}\, M) \le \mathrm{d}(y, M(x))$. Hence, if M is metrically regular in direction $(u, v) \ne (0, 0)$, the condition (8) is fulfilled for all $(x, y) \in (\bar{x}, \bar{y}) + V_{\rho,\delta}(u, v)$ with $\mathrm{d}(y, M(x)) \le \delta \,\|(x, y) - (\bar{x}, \bar{y})\|$.

If $v \notin CM(\bar{x}, \bar{y})(u)$ then by the definition of the contingent derivative there are some $\rho > 0$, $\delta > 0$ such that $\{(x, y) \in (\bar{x}, \bar{y}) + V_{\rho,\delta}(u, v) \mid \mathrm{d}((x, y), \mathrm{gph}\, M) \le \delta \,\|(x, y) - (\bar{x}, \bar{y})\|\} = \emptyset$. Hence M is metrically regular in direction (u, v) if $v \notin CM(\bar{x}, \bar{y})(u)$.

Lemma 1 *Let the multifunction $M : X \rightrightarrows Y$ be metrically regular in direction $(u, 0)$ at $(\bar{x}, \bar{y}) \in \mathrm{gph}\, M$. Then M is also metrically subregular in direction u.*

Proof If $u = 0$ the assertion follows immediately and hence let $u \ne 0$. Let ρ, δ and κ be given according to the definition of directional metric regularity and consider an arbitrary point $x \in \bar{x} + V_{\rho,\delta}(u)$. Then $(x, \bar{y}) \in (\bar{x}, \bar{y}) + V_{\rho,\delta}(u, 0)$ and taking into account the inequalities $\mathrm{d}((x, \bar{y}), \mathrm{gph}\, M) \le \mathrm{d}(\bar{y}, M(x))$ and $\|(x, \bar{y}) - (\bar{x}, \bar{y})\| = \|x - \bar{x}\| \ge \mathrm{d}(x, M^{-1}(\bar{y}))$ we obtain

$$\mathrm{d}(x, M^{-1}(\bar{y})) \le \begin{cases} \kappa \mathrm{d}(\bar{y}, M(x)) & \text{if } \mathrm{d}(\bar{y}, M(x)) \le \delta \,\|(x, \bar{y}) - (\bar{x}, \bar{y})\| \\ \frac{1}{\delta}\mathrm{d}(\bar{y}, M(x)) & \text{if } \mathrm{d}(\bar{y}, M(x)) > \delta \,\|(x, \bar{y}) - (\bar{x}, \bar{y})\|. \end{cases}$$

Therefore (9) follows with $\kappa' = \max\left\{\kappa, \frac{1}{\delta}\right\}$. □

We now introduce directional versions of the limiting constructions presented in the preceding section by restricting the limiting process with respect to the direction under consideration.

Definition 2

1. Let $\Omega \subset X$, $x \in \Omega$ and $u \in X$ be given. The *limiting normal cone to Ω in direction u at x* is defined by

$$N(x; \Omega; u) := \left\{x^* \mid \exists (\varepsilon_k) \downarrow 0,\ (t_k) \downarrow 0,\ (u_k) \to u,\ (x_k^*) \xrightarrow{w^*} x^* : x_k^* \in \hat{N}_{\varepsilon_k}(x + t_k u_k; \Omega) \forall k\right\}.$$

2. Let $M : X \rightrightarrows Y$ and let $(\bar{x}, \bar{y}) \in \mathrm{gph}\, M$, $(u, v) \in X \times Y$.
The *normal coderivative of M in direction* (u, v) at $(\bar{x}, \bar{y})$ is defined as the set-valued mapping $D_N^* M((\bar{x}, \bar{y}); (u, v)) : Y^* \rightrightarrows X^*$, where $D_N^* M((\bar{x}, \bar{y}); (u, v))(y^*)$ is the collection of all $x^* \in X^*$ for which there exist sequences $(\varepsilon_k) \downarrow 0$, $(t_k) \downarrow 0$, $(u_k, v_k) \to (u, v)$ and $(x_k^*, y_k^*) \xrightarrow{w^*} (x^*, y^*)$ with $(x_k^*, -y_k^*) \in \hat{N}_{\varepsilon_k}((\bar{x} + t_k u_k, \bar{y} + t_k v_k); \mathrm{gph}\, M)$.
The *reversed mixed coderivative of M in direction* (u, v) at $(\bar{x}, \bar{y})$ is defined as the multifunction $\tilde{D}_M^* M((\bar{x}, \bar{y}); (u, v)) : Y^* \rightrightarrows X^*$, where $\tilde{D}_M^* M((\bar{x}, \bar{y}); (u, v))(y^*)$ is the collection of all $x^* \in X^*$ for which there are sequences $(\varepsilon_k) \downarrow 0$, $(t_k) \downarrow 0$, $(u_k, v_k, x_k^*) \to (u, v, x^*)$ and $(y_k^*) \xrightarrow{w^*} (y^*)$ with $(x_k^*, -y_k^*) \in \hat{N}_{\varepsilon_k}((\bar{x} + t_k u_k, \bar{y} + t_k v_k); \mathrm{gph}\, M)$.

The *mixed coderivative of M in direction* (u, v) at $(\bar{x}, \bar{y})$ is defined by

$$D^*_M((\bar{x}, \bar{y}); (u, v))(y^*) := \{x^* \in X^* \mid y^* \in -\tilde{D}^*_M M^{-1}((\bar{y}, \bar{x}); (v, u))(-x^*)\}, \quad (10)$$

The mapping M is called *partially sequentially normally compact (PSNC) in direction* (u, v) at $(\bar{x}, \bar{y})$ with respect to Y, if for all sequences $(\varepsilon_k) \downarrow 0$, $(t_k) \downarrow 0$, $(u_k, v_k) \to (u, v)$, $(y^*_k) \overset{w^*}{\to} 0$ and $(x^*_k) \to 0$ with $(x^*_k, -y^*_k) \in \hat{N}_{\varepsilon_k}((\bar{x} + t_k u_k, \bar{y} + t_k v_k); \text{gph}\, M)$ one has $y^*_k \to 0$ as $k \to \infty$.

3. Let $M : X \rightrightarrows Y$, let $(\bar{x}, \bar{y}) \in \text{gph}\, M$, let $\tilde{Y} \subset Y$ be a subspace and let $u \in X$. The *limit set critical for directional metric regularity of M with respect to u and* $\tilde{Y}$ at $(\bar{x}, \bar{y})$ is the set $\text{Cr}_{\tilde{Y}}\, M((\bar{x}, \bar{y}); u)$ of all elements $(v, x^*) \in Y \times X^*$ such that there are sequences $(t_k) \downarrow 0$, $(\varepsilon_k) \downarrow 0$, $(u_k, v_k, x^*_k) \to (u, v, x^*)$, $(y^*_k) \subset \mathscr{S}_{Y^*}$ and a positive constant β with $(-x^*_k, y^*_k) \in \hat{N}_{\varepsilon_k}((\bar{x} + t_k u_k, \bar{y} + t_k v_k), \text{gph}\, M)$ and $\left\| y^*_{k|_{\tilde{Y}}} \right\| \geq \beta$. In case that $\tilde{Y} = Y$ we simply write $\text{Cr}\, M((\bar{x}, \bar{y}); u)$.

For $u \neq 0$ our definition of the limiting normal cone in direction u coincides with the definition of the *basic normal cone in direction u* as presented in [8].

Note that $N(x; \Omega; 0) = N(x, \Omega)$, $D^*_N M((\bar{x}, \bar{y}); (0, 0)) = D^*_N M(\bar{x}, \bar{y})$, $D^*_M M((\bar{x}, \bar{y}); (0, 0)) = D^*_M M(\bar{x}, \bar{y})$, $\tilde{D}^*_M M((\bar{x}, \bar{y}); (0, 0)) = \tilde{D}^*_M M(\bar{x}, \bar{y})$ and that the properties PSNC in direction $(0, 0)$ and PSNC are the same, but that $\text{Cr}_{\tilde{Y}} M(\bar{x}, \bar{y}) \neq \text{Cr}_{\tilde{Y}} M((\bar{x}, \bar{y}); 0)$ in general. Further we have the relations

$$D^*_N M((\bar{x}, \bar{y}); (u, v))(y^*) = \{x^* \in X^* \mid (x^*, -y^*) \in N((\bar{x}, \bar{y}); \text{gph}\, M; (u, v))\}$$

and

$$D^*_M M^{-1}((\bar{y}, \bar{x}); (v, u))(y^*) = \left\{x^* \in X^* \mid (v, x^*) \in \widehat{CD^*} M(\bar{x}, \bar{y})(u, y^*)\right\}. \quad (11)$$

Again, if X and Y are Asplund spaces, we can equivalently take $\varepsilon_k = 0$ in the definitions above.

Theorem 4 *Let* $M : X \rightrightarrows Y$ *be a multifunction and let* $(u, v) \in X \times Y$.

1. *Assume that M has closed graph, that* $(v, 0) \notin \text{Cr}\, M((\bar{x}, \bar{y}); u)$ *and that either Y admits a Fréchet smooth renorm or both X and Y are Asplund spaces. Then M is metrically regular in direction* (u, v) *at* $(\bar{x}, \bar{y})$.
2. *If* $(v, 0) \in \text{Cr}\, M((\bar{x}, \bar{y}); u)$ *then M is not metrically regular in direction* (u, v) *at* $(\bar{x}, \bar{y})$.
3. *If* $u \neq 0$ *and* $(0, 0) \in \text{Cr}\, M((\bar{x}, \bar{y}); u)$ *then there exists a continuously differentiable function* $h : X \to Y$ *with* $h(\bar{x}) = 0$, $\nabla h(\bar{x}) = 0$ *such that* $M + h$ *is not metrically subregular in direction u at* $(\bar{x}, \bar{y})$.

Proof We proof the first part by contradiction. Let us assume on the contrary that M is not metrically regular at $\bar{z} := (\bar{x}, \bar{y})$ in direction $w := (u, v)$. Then we can find for each k some element $z_k := (x_k, y_k) \in \bar{z} + V_{\frac{1}{k}, \frac{1}{k}}(w)$ satisfying $\|w\| \text{d}(z_k, \text{gph}\, M) \leq \frac{\|w\|}{k} \|z_k - \bar{z}\|$ such that $\text{d}(x_k, M^{-1}(y_k)) > 4k\text{d}(y_k, M(x_k))$. Thus $z_k \notin \text{gph}\, M$ and consequently $z_k \neq \bar{z}$. Consider $\hat{z}_k := (\hat{x}_k, \hat{y}_k) \in \text{gph}\, M$ with $\|\hat{z}_k - z_k\| \leq 2\text{d}(z_k, \text{gph}\, M)$ and set $\varepsilon := \|\hat{y}_k - y_k\|$. Then we have $\|\hat{y}_k - y_k\| \leq \inf_{(x,y) \in \text{gph}\, M} \|y - y_k\| + \varepsilon$ and we can invoke Ekeland's variational principle to find some $\tilde{z}_k := (\tilde{x}_k, \tilde{y}_k) \in \text{gph}\, M$ such that $\|\tilde{y}_k - y_k\| \le \|\hat{y}_k - y_k\|$, $\|\tilde{z}_k - \hat{z}_k\| \le \sqrt{k}\|\hat{y}_k - y_k\|$ and $\|\tilde{y}_k - y_k\| \le \|y - y_k\| + \frac{1}{\sqrt{k}}\|(x, y) - (\tilde{x}_k, \tilde{y}_k)\|$ for all $(x, y) \in \operatorname{gph} M$. Hence we obtain

$$\|\tilde{z}_k - z_k\| \le \|\tilde{z}_k - \hat{z}_k\| + \|\hat{z}_k - z_k\| \le \sqrt{k}\|\hat{y}_k - y_k\| + \|\hat{z}_k - z_k\| \le (\sqrt{k}+1)\|\hat{z}_k - z_k\| \tag{12}$$

and consequently $\tilde{y}_k \ne y_k$, since otherwise we would obtain

$$\begin{aligned}\|\tilde{z}_k - z_k\| = \|\tilde{x}_k - x_k\| &\ge \mathrm{d}(x_k, M^{-1}(y_k)) > 4k\mathrm{d}(y_k, M(x_k))\\ &\ge 4k\mathrm{d}(z_k, \operatorname{gph} M) \ge 2k\|\hat{z}_k - z_k\|\end{aligned}$$

contradicting (12).

If $w \ne 0$ we obtain from (12)

$$\|\tilde{z}_k - z_k\| \le \frac{2(\sqrt{k}+1)}{k}\|z_k - \bar{z}\| \le \frac{4}{\sqrt{k}}\|z_k - \bar{z}\|,$$

which implies

$$\begin{aligned}0 < \frac{1}{2}\|z_k - \bar{z}\| &\le \left(1 - \frac{4}{\sqrt{k}}\right)\|z_k - \bar{z}\| \le \|\tilde{z}_k - \bar{z}\|\\ &\le \left(1 + \frac{4}{\sqrt{k}}\right)\|z_k - \bar{z}\| \le \frac{3}{2}\|z_k - \bar{z}\| \le \frac{3}{2k}, \ \forall k \ge 64\end{aligned} \tag{13}$$

and

$$\begin{aligned}\big\|\,\|w\|\,(\tilde{z}_k - \bar{z}) - \|\tilde{z}_k - \bar{z}\|\,w\big\| &\le \big\|\,\|w\|\,(z_k - \bar{z}) - \|z_k - \bar{z}\|\,w\big\| + 2\,\|\tilde{z}_k - z_k\|\,\|w\|\\ &\le \left(\frac{1}{k} + \frac{8}{\sqrt{k}}\right)\|z_k - \bar{z}\|\,\|w\|\\ &\le \frac{18}{\sqrt{k}}\|\tilde{z}_k - \bar{z}\|\,\|w\|, \ \forall k \ge 64\end{aligned} \tag{14}$$

showing $\tilde{z}_k - \bar{z} \in V_{\frac{3}{2k}, \frac{18}{\sqrt{k}}}(w)$.

On the other hand, if $w = 0$, by using the estimate $\|\hat{z}_k - z_k\| \le 2\mathrm{d}(z_k, \operatorname{gph} M) \le 2\|z_k - \bar{z}\| \le \frac{2}{k}$, we obtain from (12) the bounds $\|\tilde{z}_k - z_k\| \le \frac{2(\sqrt{k}+1)}{k} \le \frac{4}{\sqrt{k}}$ and $\|\tilde{z}_k - \bar{z}\| \le \frac{1}{k} + \frac{4}{\sqrt{k}} \le \frac{5}{\sqrt{k}}$ showing $\tilde{z}_k - \bar{z} \in V_{\frac{5}{\sqrt{k}}, \frac{1}{k}}(w)$.

Defining the sequences

$$t_k := \begin{cases} \max\left\{\frac{1}{2}\|z_k - \bar{z}\|, \|\tilde{z}_k - \bar{z}\|\right\} \Big/ \|w\| & \text{if } w \ne 0,\\ k^{-\frac{1}{4}} & \text{if } w = 0,\end{cases}$$

$(\tilde{u}_k, \tilde{v}_k) := (\tilde{z}_k - \bar{z})/t_k$ and taking into account (13) we conclude $(\tilde{u}_k, \tilde{v}_k) \to w$ as $k \to \infty$.

In case that Y admits a Fréchet smooth renorm we can assume without loss of generality that the original norm $\|\cdot\|$ on Y is Fréchet smooth since the properties directional metric regularity and $(v, 0) \in \operatorname{Cr} M((\bar{x}, \bar{y}); u)$ are invariant with respect to equivalent norms. Let $\tilde{y}_k^* \in Y^*$ denote the Fréchet derivative of $\|\cdot\|$ at $\tilde{y}_k - y_k$, then $\tilde{y}_k^*$ belongs to the subdifferential of convex analysis and therefore $\tilde{y}_k^* \in \mathscr{S}_{Y^*}$ and $\langle \tilde{y}_k^*, \tilde{y}_k - y_k\rangle = \|\tilde{y}_k - y_k\|$. Due to the defintion of Fréchet differentiability we can find some positive δ_k such that $\|y - y_k\| \le \|\tilde y_k - y_k\| + \langle \tilde y_k^*, y - \tilde y_k\rangle + \frac{1}{\sqrt{k}}\|y - \tilde y_k\|$ for all $y \in \tilde y_k + \delta_k \mathscr{B}_Y$. Hence for all $(x, y) \in \operatorname{gph} M \cap ((\tilde x_k, \tilde y_k) + \delta_k \mathscr{B}_{X\times Y})$ we have

$$\begin{aligned}\|\tilde y_k - y_k\| &\le \|y - y_k\| + \frac{1}{\sqrt{k}}\|(x, y) - (\tilde x_k, \tilde y_k)\| \\ &\le \|\tilde y_k - \bar y\| + \langle \tilde y_k^*, y - \tilde y_k\rangle + \frac{2}{\sqrt{k}}\|(x, y) - (\tilde x_k, \tilde y_k)\|\end{aligned}$$

showing $(0, -\tilde y_k^*) \in \hat N_{\frac{2}{\sqrt{k}}}((\tilde x_k, \tilde y_k), \operatorname{gph} M)$. Thus, by setting $\varepsilon_k = \frac{2}{\sqrt{k}}$, $(u_k, v_k) = (\tilde u_k, \tilde v_k)$, $x_k^* = 0$ and $y_k^* = -\tilde y_k^*$ we can conclude $(v, 0) \in \operatorname{Cr} M((\bar x, \bar y); u)$, a contradiction.

In case that both X and Y are Asplund so also is $X \times Y$. Since $(\tilde x_k, \tilde y_k)$ minimizes the function $(x, y) \to \chi_{\operatorname{gph} M}(x, y) + \psi_k(x, y)$, where $\psi_k(x, y) := \|y - y_k\| + \frac{1}{\sqrt{k}}\|(x, y) - (\tilde x_k, \tilde y_k))\|$ and $\chi_{\operatorname{gph} M}$ denotes the characteristic function of gph M, by the fuzzy (semi-Lipschitzian) sum rule (see for example [24, Theorem 2.33])) for arbitrary $\delta_k > 0$ we can find points $z_k' = (x_k', y_k')$, $z_k'' = (x_k'', y_k'')$ in $\tilde z_k + \delta_k \mathscr{B}_{X\times Y}$ with $\chi_{\operatorname{gph} M}(z_k') \le \chi_{\operatorname{gph} M}(\tilde z_k) + \delta_k = \delta_k$, $\psi_k(z_k'') \le \psi_k(\tilde z_k) + \delta_k$ and linear functionals $(\tilde x_k^*, \tilde y_k^*) \in \hat\partial \chi_{\operatorname{gph} M}(x_k', y_k')$ and $(\hat x_k^*, \hat y_k^*) \in \hat\partial \psi_k(x_k'', y_k'')$ such that $\|(\tilde x_k^*, \tilde y_k^*) + (\hat x_k^*, \hat y_k^*)\| \le \delta_k$. Taking $\delta_k := \min\left\{\frac{t_k}{k}, \frac{1}{2}\|\tilde y_k - y_k\|\right\}$ we can deduce that $\|y_k'' - y_k\| \ge \|\tilde y_k - y_k\|/2 > 0$ and therefore, as a consequence of the sum rule of convex analysis, $|\,\|\hat y_k^*\| - 1| \le \frac{1}{\sqrt{k}}$ and $\|\hat x_k^*\| \le \frac{1}{\sqrt{k}}$ follows. Therefore $|\,\|\tilde y_k^*\| - 1| \le \|\tilde y_k^* + \hat y_k^*\| + |\,\|\hat y_k^*\| - 1| \le \delta_k + \frac{1}{\sqrt{k}} \to 0$ and $\|\tilde x_k^*\| \le \|\hat x_k^*\| + \delta_k \to 0$ follows. Then we can set $\varepsilon_k = 0$, $(u_k, v_k) := (z_k' - \bar z)/t_k$, $x_k^* = -\tilde x_k^*/\|\tilde y_k^*\|$, $y_k^* = \tilde y_k^*/\|\tilde y_k^*\|$ to obtain the contradiction $(v, 0) \in \operatorname{Cr} M((\bar x, \bar y); u)$. Indeed, finiteness of $\chi_{\operatorname{gph} M}(x_k', y_k')$ implies $\hat\partial \chi_{\operatorname{gph} M}(x_k', y_k') = \hat N((x_k', y_k'), \operatorname{gph} M)$ and consequently $(-x_k^*, y_k^*) \in \hat N((x_k', y_k'), \operatorname{gph} M)$. Since $\|\tilde y_k^*\| \to 1$, $\|\tilde x_k^*\| \to 0$ we have $x_k^* \to 0$. Finally, $\|(u_k, v_k) - (\tilde u_k, \tilde v_k)\| \le \|z_k' - \tilde z_k\|/t_k \le \frac{1}{k}$ showing $(u_k, v_k) \to w$ and the first part of the theorem is proved.

To show the second part of the theorem, let $(v, 0) \in \operatorname{Cr} M((\bar x, \bar y); u)$. Then we can find sequences $(t_k) \downarrow 0$, $(\varepsilon_k) \downarrow 0$, $(u_k, v_k, x_k^*) \to (u, v, 0)$, $(y_k^*) \subset \mathscr{S}_{Y^*}$ with $(-x_k^*, y_k^*) \in \hat N_{\varepsilon_k}((\bar x + t_k u_k, \bar y + t_k v_k), \operatorname{gph} M)$. By passing to a subsequence we can assume that $\varepsilon_k + \|x_k^*\| \le \frac{1}{4k}$ $\forall k$. For each k we can find some positive radius $\rho_k < t_k$ such that $\langle -x_k^*, x - x_k\rangle + \langle y_k^*, y - y_k'\rangle \le (\frac{1}{4k} + \varepsilon_k)\|(x, y) - (x_k, y_k')\|$ for every $(x, y) \in ((x_k, y_k') + \rho_k \mathscr{B}_{X\times Y}) \cap \operatorname{gph} M$, where we set $(x_k, y_k') := (\bar x, \bar y) + t_k(u_k, v_k)$. Next we choose elements $z_k \in \mathscr{S}_Y$ such that $\langle y_k^*, z_k\rangle \ge \frac{1}{2}$ and set $y_k = y_k' + \frac{\rho_k}{k} z_k$. Then we have $y_k \notin M(x)$ for all $x \in x_k + (1 - \frac{1}{k})\rho_k \mathscr{B}_X$. Indeed, assuming on the contrary that $y_k \in M(x)$ for some $x \in x_k + (1 - \frac{1}{k})\rho_k \mathscr{B}_X$, then $(x, y_k) \in ((x_k, y_k') + \rho_k \mathscr{B}_{X\times Y}) \cap \operatorname{gph} M$ and

$$\begin{aligned}\frac{\rho_k}{2k} &\le \langle y_k^*, y_k - y_k'\rangle \le \langle -x_k^*, x - x_k\rangle + \langle y_k^*, y_k - y_k'\rangle + \|x_k^*\|(x - x_k) \\ &\le \left(\varepsilon_k + \frac{1}{4k} + \|x_k^*\|\right)\rho_k < \frac{\rho_k}{2k},\end{aligned}$$

a contradiction. Hence $y_k \notin M(x)$ for all $x \in x_k + (1 - \frac{1}{k})\rho_k \mathscr{B}_X$ and consequently

$$\mathrm{d}(x_k, M^{-1}(y_k)) \ge \left(1 - \frac{1}{k}\right)\rho_k = \left(1 - \frac{1}{k}\right)k\,\|y_k - y_k'\| \ge (k-1)\mathrm{d}(y_k, M(x_k)).$$

Further we have $((x_k, y_k) - (\bar{x}, \bar{y}))/t_k = (u_k, v_k + \frac{\rho_k}{kt_k} z_k) \to (u, v)$ and, if $(u, v) \neq (0, 0)$,

$$\mathrm{d}((x_k, y_k), \mathrm{gph}\, M) \leq \frac{\rho_k}{k} \leq \frac{2}{k\, \|(u, v)\|} \|(x_k, y_k) - (\bar{x}, \bar{y})\|$$

for all k sufficiently large such that $\|(x_k, y_k) - (\bar{x}, \bar{y})\| / t_k \geq \|(u, v)\| /2$. This shows that M is not metrically regular at $(\bar{x}, \bar{y})$ in direction (u, v).

To show the third part, let us assume that $(0, 0) \in \mathrm{Cr}\, M((\bar{x}, \bar{y}); u)$ for some $u \neq 0$ and let us choose according to the definition the sequences $(\tilde{t}_k) \downarrow 0$, $(\varepsilon_k) \downarrow 0$, $(\tilde{u}_k, \tilde{v}_k, x_k^*) \to (u, 0, 0)$ and $(y_k^*) \subset \mathscr{S}_{Y^*}$ with $(-x_k^*, y_k^*) \in \hat{N}_{\varepsilon_k}((\bar{x} + \tilde{t}_k \tilde{u}_k, \bar{y} + \tilde{t}_k \tilde{v}_k), \mathrm{gph}\, M)$. We may assume that $\tilde{u}_k \neq 0$, $\forall k$ and therefore the sequences $t_k = \tilde{t}_k \left\|\tilde{u}_k\right\| \downarrow 0$, $u_k = \tilde{u}_k / \left\|\tilde{u}_k\right\| \to u/\|u\|$, $v_k = \tilde{v}_k / \left\|\tilde{u}_k\right\| \to 0$ are well defined. Exactly as in the proof of the second part of [9, Theorem 3.2] we can find a continuously differentiable mapping $h : X \to Y$ satisfying $h(\bar{x}) = 0$, $\nabla h(\bar{x}) = 0$ and

$$\mathrm{d}(x_k, (M + h)^{-1}(\bar{y})) \geq \frac{\sqrt{k}}{2} \mathrm{d}(\bar{y}, (M + h)(x_k)) > 0 \quad \forall k \geq 17,$$

where $x_k := \bar{x} + t_k u_k = \bar{x} + \tilde{t}_k \tilde{u}_k$ and we have eventually passed to a subsequence. Since $(x_k - \bar{x})/ \|x_k - \bar{x}\| \to u/ \|u\|$ we conclude that $M + h$ is not metrically subregular at $(\bar{x}, \bar{y})$ in direction u. □

Theorem 5 *Let $M : X \rightrightarrows Y$ be a closed-graph multifunction, $(\bar{x}, \bar{y}) \in \mathrm{gph}\, M$, $(u, v) \in X \times Y$ and assume that either Y is Fréchet smooth or both X and Y are Asplund spaces. Then the following statements are equivalent:*

(a) *M is metrically regular in direction (u, v) at $(\bar{x}, \bar{y})$.*
(b) $(v, 0) \notin \mathrm{Cr}\, M((\bar{x}, \bar{y}); u)$.
(c) *M is PSNC in direction (u, v) at $(\bar{x}, \bar{y})$ with respect to Y and $(v, 0) \notin \widehat{CD^*} M(\bar{x}, \bar{y})(u, y^*)$, $\forall y^* \neq 0$.*
(d) *M is PSNC in direction (u, v) at $(\bar{x}, \bar{y})$ with respect to Y and* $\ker \tilde{D}^*_M M((\bar{x}, \bar{y}); (u, v)) = \{0\}$.
(e) *M is PSNC in direction (u, v) at $(\bar{x}, \bar{y})$ with respect to Y and $D^*_M M^{-1}((\bar{y}, \bar{x}); (v, u))(0) = \{0\}$.*

Proof The equivalence (a) $\Longleftrightarrow$ (b) follows immediately from Theorem 4. Next we prove (b) $\Rightarrow$ (c) by contradiction. If M is not PSNC in direction (u, v) at $(\bar{x}, \bar{y})$ with respect to Y or $(v, 0) \notin \widehat{CD^*} M(\bar{x}, \bar{y})(u, y^*)$ for some $y^* \neq 0$ then there are sequences $(\varepsilon_k, t_k, v_k, u_k, x_k, \tilde{y}_k^*, \tilde{x}_k^*)$ with $\varepsilon_k \to 0$, $t_k \downarrow 0$, $(u_k, v_k) \to (v, u)$, $(-\tilde{x}_k^*, \tilde{y}_k^*) \in \hat{N}_{\varepsilon_k}((\bar{x} + t_k u_k, \bar{y} + t_k v_k), \mathrm{gph}\, M)$, $\|\tilde{x}_k^*\| \to 0$ and $\liminf_{k\to\infty} \|\tilde{y}_k^*\| > 0$. Hence, by taking $y_k^* := \tilde{y}_k^*/\|\tilde{y}_k^*\|$ and $x_k^* := \tilde{x}_k^*/\|\tilde{y}_k^*\|$ we have $(-x_k^*, y_k^*) \in \hat{N}_{\varepsilon_k}((\bar{x} + t_k u_k, \bar{y} + t_k v_k), \mathrm{gph}\, M)$ and $x_k^* \to 0$ verifying $(v, 0) \in \mathrm{Cr}\, M(\bar{x}, \bar{y})(u)$, a contradiction. The reverse implication (c) $\Rightarrow$ (b) is also shown by contradiction. Let us assume $(v, 0) \in \mathrm{Cr}\, M(\bar{x}, \bar{y})(u)$ and consider the sequences $(\varepsilon_k, t_k, v_k, u_k, x_k, y_k^*, x_k^*)$ according to the definition. Since Fréchet smooth spaces are also Asplund spaces, Y is an Asplund space and hence the sequence (y_k^*) has a weak* convergent subsequence and without loss of generality we may assume that the sequence (y_k^*) itself weakly* converges to some y^*. We conclude $(v, 0) \in \widehat{CD^*}M(\bar{x}, \bar{y})(u, y^*)$ and , if M^{-1} is $PSNC(v, u)$, we obtain the contradiction $y^* \neq 0$. Hence (c) $\Rightarrow$ (b) also holds. Finally, the equivalences (c) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e) are consequences of (11) and (10). □

It is easy to see that for any smooth perturbation $h : X \to Y$ with $h(\bar{x}) = 0$, $\nabla h(\bar{x}) = 0$ we have $\mathrm{Cr}\,(M + h)((\bar{x}, \bar{y}); u) = \mathrm{Cr}\,M((\bar{x}, \bar{y}); u)$. Hence the property of directional metric regularity is invariant under such smooth perturbations. Further, by combining Lemma 1 and Theorem 4 we see that the condition $(0, 0) \notin \mathrm{Cr}\,M((\bar{x}, \bar{y}); u)$ is both necessary and sufficient for metric subregularity in direction $u \neq 0$, when the property of directional subregularity is stable under smooth perturbations with zero function value and derivative at the reference point. Note that in general the property of directional metric subregularity, like metric subregularity, is not stable under such smooth perturbations. We conjecture that in this case some second-order characterizations of directional metric subregularity can be formulated, similar to the second-order conditions [9] for smooth constraint systems.

To illustrate Theorems 4 and 5 we consider the following example.

Example 3 Consider the system

$$\begin{aligned} x_1 &\geq 0 \\ x_2 &\geq 0 \\ \min\{x_1 - x_2, x_1 + x_2\} &= 0 \end{aligned}$$

which can be equivalently written in the form

$$0 \in M(x_1, x_2) := F(x_1, x_2) - P,$$

where $P := \mathbb{R}^2_+ \times \{0\}$, $F(x_1, x_2) := (x_1, x_2, \min\{x_1 - x_2, x_1 + x_2\}$ and let $\bar{x} = (0, 0)$, $\bar{y} := (0, 0, 0)$. Obviously $M^{-1}(\bar{y}) = \{(t, t) \mid t \geq 0\}$ and therefore $\bar{x}$ is a non-isolated solution of the inclusion $\bar{y} \in M(x)$. Given $x = (x_1, x_2) \in \mathbb{R}^2$, $p = (p_1, p_2, p_3) \in P$, straightforward calculations yield

$$\hat{N}((x, F(x) - p); \mathrm{gph}\, M) = \left\{ (x_1^*, x_2^*, y_1^*, y_2^*, y_3^*) \;\middle|\; \begin{aligned} y_1^* \geq 0,\ y_1^* p_1 &= 0, \\ y_2^* \geq 0,\ y_2^* p_2 &= 0, \\ x_1^* + y_1^* + y_3^* &= 0, \\ x_2^* + y_2^* - y_3^* &= 0 \ \text{ if } x_2 > 0, \\ |x_2^* + y_2^*| - y_3^* &\leq 0 \ \text{ if } x_2 = 0, \\ x_2^* + y_2^* + y_3^* &= 0 \ \text{ if } x_2 < 0. \end{aligned} \right\}$$

It follows that for given $u = (u_1, u_2)$ the limit set $\mathrm{Cr}\, M((\bar{x}, \bar{y}); u)$ critical for metric regularity with respect to u is the collection of all $(v_1, v_2, v_3, x_1^*, x_2^*)$ such that there exists some $y^* = (y_1^*, y_2^*, y_3^*)$, $\|y^*\| = 1$ fulfilling the conditions

$$\begin{aligned}
&v_1 \le u_1,\ \ y_1^* \ge 0,\ \ y_1^*(u_1 - v_1) = 0,\\
&v_2 \le u_2,\ \ y_2^* \ge 0,\ \ y_2^*(u_2 - v_2) = 0,\\
&v_3 = \min\{u_1 - u_2, u_1 + u_2\},\\
&x_1^* = y_1^* + y_3^*,\\
&x_2^* = y_2^* - y_3^* \text{ if } u_2 > 0,\\
&x_2^* = y_2^* + y_3^* \text{ if } u_2 < 0,\\
&x_2^* \in \{y_2^* \pm y_3^*\} \cup [y_2^* - y_3^*, y_2^* + y_3^*] \text{ if } u_2 = 0.
\end{aligned}$$

Hence the set $\{(u, v) \mid (v, 0) \in \mathrm{Cr}\, M((\bar{x}, \bar{y}); u)\}$, i.e. the set of all directions (u, v) such that M is not directionally metrically regular, is given by

$$\begin{aligned}
&\{(u, v) \mid (v, 0) \in \mathrm{Cr}\, M((\bar{x}, \bar{y}); u)\}\\
&\quad = \{((u_1, u_2), (v_1, v_2, v_3)) \mid v_1 = u_1,\ \ v_2 = u_2 \le 0,\ \ v_3 = u_1 + u_2\}\,.
\end{aligned}$$

Indeed, let $u = (u_1, u_2)$ with $u_2 \le 0$ be given, put $v := (u_1, u_2, u_1 + u_2)$ and for each $t \in (0, \frac{1}{2})$ consider the points $x^t = (tu_1, tu_2 - t^2)$, $y^t = (tu_1, tu_2 - t^2 + t^3, t(u_1 + u_2) - t^2)$. Then

$$M(x^t) = \left\{(y_1, y_2, y_3) \mid y_1 \le tu_1,\ \ y_2 \le tu_2 - t^2,\ \ y_3 = t(u_1 + u_2) - t^2\right\}$$

showing $\mathrm{d}(y^t, M(x^t)) = t^3$. On the other hand,

$$\begin{aligned}
&M^{-1}(y^t)\\
&\ = \left\{(x_1, x_2) \mid tu_1 \le x_1,\ \ tu_2 - t^2 + t^3 \le x_2,\ \ t(u_1 + u_2) - t^2 = \min\{x_1 - x_2, x_1 + x_2\}\right\}\\
&\ = \left\{(x_1, x_2) \mid x_1 \ge tu_1,\ \ x_2 \ge 0,\ \ x_1 - x_2 = t(u_1 + u_2) - t^2\right\}
\end{aligned}$$

implying $\mathrm{d}(x^t, M^{-1}(y^t)) \ge t^2 - tu_2 = \frac{1}{t}\mathrm{d}(y^t, M(x^t))$ for $t \in (0, \frac{1}{2})$. Since $(x_t - \bar{x})/t \to u$, $(y_t - \bar{y})/t \to v$ we conclude that M is not metrically regular in direction (u, v).

From these considerations it also follows that M is not metrically regular near $(\bar{x}, \bar{y})$. However, since the underlying space $X = \mathbb{R}^2$ is finite dimensional, we have

$$\mathrm{Cr}\, M(\bar{x}, \bar{y}) = \bigcup_{u \in \mathcal{S}_{\mathbb{R}^2}} \mathrm{Cr}\, M((\bar{x}, \bar{y}); u)$$

and therefore for every $(v, x^*) \in \mathrm{Cr}\, M(\bar{x}, \bar{y})$ we have $v = (u_1, u_2, \min\{u_1 - u_2, u_1 + u_2\})$ for some $u = (u_1, u_2) \in \mathcal{S}_{\mathbb{R}^2}$ and consequently $v \ne 0$. Hence $(0, 0) \notin \mathrm{Cr}\, M(\bar{x}, \bar{y})$ and from Theorem 2 we conclude that M is metrically subregular at $(\bar{x}, \bar{y})$.

Applying Theorem 5 with $(u, v) = (0, 0)$ and taking into account the well-known equivalence between the property of metric regularity of a multifunction and the Aubin property of the inverse, we obtain the following corollary:

Corollary 1 *Let $M : X \rightrightarrows Y$ be a closed-graph multifunction, $(\bar{x}, \bar{y}) \in \operatorname{gph} M$ and assume that either Y is Fréchet smooth or both X and Y are Asplund spaces. Then the following statements are equivalent:*

(a) *M is metrically regular near $(\bar{x}, \bar{y})$.*
(b) *M^{-1} has the Aubin property near $(\bar{y}, \bar{x})$.*
(c) *M is PSNC at $(\bar{x}, \bar{y})$ with respect to Y and* $\ker \tilde{D}^*_M M(\bar{x}, \bar{y}) = \{0\}$.
(d) *M is PSNC at $(\bar{x}, \bar{y})$ with respect to Y and $D^*_M M^{-1}(\bar{y}, \bar{x})(0) = \{0\}$.*

This corollary extends the characterizations of metric regularity and the Aubin property by means of the PSNC condition and the mixed coderivative in Asplund spaces, as presented in Theorem 1, to the case when the space Y is Fréchet smooth, whereas the other space X can be an arbitrary Banach space.

4 Mixed Regularity and Subregularity of Multifunctions

The results of the preceding section show that the criterion $(0, 0) \notin \operatorname{Cr} M((\bar{x}, \bar{y}); u)$ is optimal for directional subregularity under the viewpoint of invariance under smooth perturbations. But many problems have a specific structure which is not invariant under smooth perturbations, e.g. as a consequence of Hoffman's error bound [14], multifunctions associated with linear systems are always metrically subregular , but the property of linearity is obviously not invariant under smooth perturbations. Now we try to develop a theory which takes into account such a specific structure. In fact, we consider now an extension of the sufficient condition of Theorem 4 under the additional information that some part of M is known to be metrically subregular in advance, e.g. when this part is a polyhedric multifunction.

In this section we consider the case that our multifunction M is composed by two multifunctions $M_i : X \rightrightarrows Y_i$, $i = 1, 2$, i.e. M has the form

$$M = (M_1, M_2) : X \rightrightarrows Y := Y_1 \times Y_2. \tag{15}$$

In what follows we denote the components of $y \in Y = Y_1 \times Y_2$ by y_i, i.e. $y = (y_1, y_2)$ and we set $\tilde{Y}_1 := Y_1 \times \{0_{Y_2}\}$.

Definition 3 Let M be given by (15), $(\bar{x}, \bar{y}) \in \operatorname{gph} M$ and let $(u, v_1) \in X \times Y_1$. We say that M is *mixed regular/subregular in direction* (u, v_1) at $(\bar{x}, \bar{y})$ if there are numbers $\rho > 0$, $\delta > 0$ and $\kappa > 0$ such that

$$\mathrm{d}(x, M^{-1}(y_1, \bar{y}_2)) \le \kappa \mathrm{d}((y_1, \bar{y}_2), M(x))$$

holds for all $(x, y_1) \in (\bar{x}, \bar{y}_1) + V_{\rho,\delta}(u, v_1)$ satisfying

$$\|(u, v_1)\| \, \mathrm{d}((x, (y_1, \bar{y}_2)), \operatorname{gph} M) \le \delta \, \|(u, v_1)\| \, \|(x, y_1) - (\bar{x}, \bar{y}_1)\| \, .$$

We call M *mixed regular/subregular* at $(\bar{x}, \bar{y})$ if it is mixed regular/subregular in direction $(0, 0)$ at $(\bar{x}, \bar{y})$.

Using the same arguments as in the proof of Lemma 1 one can show the following lemma.

Lemma 2 *Let the multifunction* $M : X \rightrightarrows Y$ *be given by* (15) *and let* M *be mixed regular/subregular in direction* $(u, 0)$ *at* $(\bar{x}, \bar{y}) \in \operatorname{gph} M$. *Then* M *is also metrically subregular in direction* u.

Definition 4 Let M be given by (15), $(\bar{x}, \bar{y}) \in \operatorname{gph} M$ and let $(u, v_1) \in X \times Y_1$. We say that M_2 is *proper subregular in direction* u *relative to* M_1 *and* v_1 at $(\bar{x}, \bar{y})$, if there are positive constants $\kappa', \rho', \delta', L > 0$ such that for all $(x, y_1) \in ((\bar{x}, \bar{y}_1) + V_{\rho',\delta'}(u, v_1)) \cap \operatorname{gph} M_1$ there is some $\check{x} \in M_2^{-1}(\bar{y}_2)$ satisfying $\|x - \check{x}\| \le \kappa' \mathrm{d}(\bar{y}_2, M_2(x))$ and $\mathrm{d}(y_1, M_1(\check{x})) \le L\|x - \check{x}\|$.

We present the following simple sufficient condition for M_2 being proper subregular relative to M_1.

Lemma 3 *Let* M *be given by* (15), $(\bar{x}, \bar{y}) \in \operatorname{gph} M$ *and let* $u \in X$. *Assume that* M_2 *is metrically subregular in direction* u *at* $(\bar{x}, \bar{y}_2)$ *and that* M_1 *has the Aubin property near* $(\bar{x}, \bar{y}_1)$. *Then,* M_2 *is proper subregular in direction* u *relative to* M_1 *and every* $v_1 \in Y_1$ *at* $(\bar{x}, \bar{y})$.

Proof Let $\rho' > 0, \delta > 0, \kappa' > 0$, $L > 0$ be chosen such that $\mathrm{d}(x, M_2^{-1}(\bar{y}_2)) \le \frac{\kappa'}{2}\mathrm{d}(\bar{y}_2, M_2(x))$ and $M_1(x') \cap (\bar{y}_1 + \rho'\mathscr{B}_Y) \subset M_1(x'') + L\left\|x'' - x'\right\|\mathscr{B}_{Y_1}$ holds for all $x \in \bar{x} + V_{\rho',\delta}(u)$ and all $x', x'' \in \bar{x} + 2\rho' B_X$. For arbitrarily fixed $v_1 \in Y_1$ let $\delta' := \delta\left\|u\right\|/(2\left\|(u, v_1)\right\|)$ if $u \ne 0$ and $\delta' = 1$ if $u = 0$ and consider $(x, y_1) \in ((\bar{x}, \bar{y}_1) + V_{\rho',\delta'}(u, v_1)) \cap \operatorname{gph} M_1$. If $x = \bar{x}$ or $u = 0$ then $x \in \bar{x} + V_{\rho',\delta}(u)$ obviously holds. On the other hand, if $x \ne \bar{x}$ and $u \ne 0$, we have

$$\left\|\frac{x - \bar{x}}{\|(x - \bar{x}, y_1 - \bar{y}_1)\|} - \frac{u}{\|(u, v_1)\|}\right\| \le \left\|\frac{(x - \bar{x}, y_1 - \bar{y}_1)}{\|(x - \bar{x}, y_1 - \bar{y}_1)\|} - \frac{(u, v_1)}{\|(u, v_1)\|}\right\| \le \delta'$$

and therefore $\|\lambda x - u/\left\|u\right\|\| \le \delta'\left\|(u, v_1)\right\|/\left\|u\right\| = \delta/2$, where $\lambda := \frac{\|(u,v_1)\|}{\|u\|\|(x-\bar{x},y_1-\bar{y}_1)\|}$, showing $|\lambda\left\|x\right\| - 1| \le \delta/2$,

$$\left\|\frac{x}{\|x\|} - \frac{u}{\|u\|}\right\| \le \left\|\lambda x - \frac{u}{\|u\|}\right\| + \left\|\frac{x}{\|x\|} - \lambda x\right\| \le \delta/2 + |1 - \lambda\left\|x\right\||\le \delta.$$

and consequently $x \in \bar{x} + V_{\rho',\delta}(u)$. Hence there is some $\check{x} \in M_2^{-1}(\bar{y}_2)$ such that $\left\|\check{x} - x\right\| \le \kappa'\mathrm{d}(\bar{y}_2, M_2(x))$ and, because of $\bar{x} \in M_2^{-1}(\bar{y}_2)$, $\left\|\check{x} - x\right\| \le \|x - \bar{x}\|$. Thus, $\check{x} \in \bar{x} + 2\rho'\mathscr{B}_X$ and therefore $y_1 \in M_1(x) \cap (\bar{y}_1 + \rho'\mathscr{B}_Y) \subset M_1(\check{x}) + L\left\|x - \hat{x}\right\|\mathscr{B}_{Y_1}$ yielding the assertion. □

Theorem 6 *Let* M *be given by* (15), $(\bar{x}, \bar{y}) \in \operatorname{gph} M$ *and let* $(u, v_1) \in X \times Y_1$. *Assume that* M *has closed graph, that* M_2 *is proper subregular in direction* u *relative to* M_1 *and* v_1 *at* $(\bar{x}, \bar{y})$, *that* $((v_1, 0), 0) \notin \mathrm{Cr}_{\tilde{Y}_1}((\bar{x}, \bar{y}); u)$ *and assume that either* Y_1 *and* Y_2 *are Fréchet smooth or* X, Y_1 *and* Y_2 *are Asplund spaces. Then* M *is mixed regular/subregular in direction* (u, v_1) *at* $(\bar{x}, \bar{y})$.

Proof Let the constants $\kappa', \rho', \delta', L$ be given in accordance with Definition 4 and assume that Y is equipped with the norm

$$|||(y_1, y_2)||| := \sqrt{\|y_1\|^2 + (L\kappa' + 2)^2 \|y_2\|^2}.$$

Assuming that M is not mixed regular/subregular in direction (u, v_1) we can find a sequence $z_k := (x_k, y_k) \in \bar{z} + V_{\frac{1}{k},\frac{1}{k}}(w)$, $y_{k2} = \bar{y}_2$ satisfying $\|w\|\, \mathrm{d}(z_k, \mathrm{gph}\, M) \le \frac{\|w\|}{k} \|z_k - \bar{z}\|$ such that $\mathrm{d}(x_k, M^{-1}(y_k)) > 4k\mathrm{d}(y_k, M(x_k))$, where $w := (u, (v_1, 0))$. Then we can proceed as in the proof of the first part of Theorem 4 with $\|y\|$ replaced by $|||y|||$ to find the sequences $(\tilde{z}_k) \to \bar{z}$, $(t_k) \downarrow 0$, $(\tilde{u}_k) \to u$, $\tilde{v}_k \to (v_1, 0)$. Hence, by eventually passing to a subsequence we can assume that $(\tilde{x}_k, \tilde{y}_{k1}) \in ((\bar{x}, \bar{y}_1) + V_{\rho',\delta'}(u, v_1)) \cap \mathrm{gph}\, M_1$, $\forall k$.

Next, let $\bar{k}$ be chosen such that

$$\left(1 + \frac{1}{\sqrt{k}}\right)\left(L\kappa' + \frac{1}{k} + 1\right) + \frac{\kappa'}{\sqrt{k}} \le \left(1 - \frac{1}{\sqrt{k}}\right)(L\kappa' + 2), \ \forall k > \bar{k}. \tag{16}$$

and we will show that $\|\tilde{y}_{k2} - \bar{y}_2\| \le \|\tilde{y}_{k1} - y_{k1}\|$, $\forall k > \bar{k}$ by contradiction. Assume that $\|\tilde{y}_{k2} - \bar{y}_2\| > \|\tilde{y}_{k1} - y_{k1}\|$ for some $k > \bar{k}$. Then we can find some $\check{x}_k \in M_2^{-1}(\bar{y}_2)$ and some $\check{y}_{k1} \in M_1(\check{x}_k)$ satisfying $\|\check{x}_k - \tilde{x}_k\| \le \kappa' \|\tilde{y}_{k2} - \bar{y}_2\|$ and $\|\check{y}_{k1} - \tilde{y}_{k1}\| \le (L\kappa' + \frac{1}{k}) \|\tilde{y}_{k2} - \bar{y}_2\|$. Since $\check{y}_k := (\check{y}_{k1}, \bar{y}_2) \in M(\check{x}_k)$ we obtain by the minimizing property of $\tilde{z}_k$

$$\begin{aligned}
|||\tilde{y}_k - y_k||| &\le |||\check{y}_k - y_k||| + \frac{1}{\sqrt{k}} \|(\check{x}_k, \check{y}_k) - (\tilde{x}_k, \tilde{y}_k)\| \\
&\le \left(1 + \frac{1}{\sqrt{k}}\right) |||\check{y}_k - y_k||| + \frac{1}{\sqrt{k}} \|\check{x}_k - \tilde{x}_k\| + \frac{1}{\sqrt{k}} |||\tilde{y}_k - y_k||| \\
&\le \left(1 + \frac{1}{\sqrt{k}}\right) (\|\check{y}_{k1} - \tilde{y}_{k1}\| + \|\tilde{y}_{k1} - y_{k1}\|) \\
&\quad + \frac{1}{\sqrt{k}} \|\check{x}_k - \tilde{x}_k\| + \frac{1}{\sqrt{k}} |||\tilde{y}_k - y_k||| \\
&< \left(1 + \frac{1}{\sqrt{k}}\right)\left(L\kappa' + \frac{1}{k} + 1\right) \|\tilde{y}_{k2} - \bar{y}_2\| \\
&\quad + \frac{\kappa'}{\sqrt{k}} \|\tilde{y}_{k2} - \bar{y}_2\| + \frac{1}{\sqrt{k}} |||\tilde{y}_k - y_k|||
\end{aligned}$$

and, after rearranging,

$$\begin{aligned}
\left(\left(1 + \frac{1}{\sqrt{k}}\right)\left(L\kappa' + \frac{1}{k} + 1\right) + \frac{\kappa'}{\sqrt{k}}\right) \|\tilde{y}_{k2} - \bar{y}_2\| &> \left(1 - \frac{1}{\sqrt{k}}\right) |||\tilde{y}_k - y_k||| \\
&\ge \left(1 - \frac{1}{\sqrt{k}}\right)(L\kappa' + 2) \|\tilde{y}_{k2} - \bar{y}_2\|
\end{aligned}$$

follows, a contradiction to (16). Hence, $\|\tilde{y}_{k2} - \bar{y}_2\| \le \|\tilde{y}_{k1} - y_{k1}\|$, $\forall k > \bar{k}$ and we can conclude $\|\tilde{y}_{k1} - y_{k1}\| \ne 0$, since $\tilde{y}_k \ne y_k$. If the norms on Y_1 and Y_2 are Fréchet smooth then the norm $|||\cdot|||$ on Y is also and the functional $\tilde{y}_k^*$ representing the Fréchet derivative at $\tilde{y}_k - y_k$ has the form

$$\langle \tilde{y}_k^*, h\rangle = \frac{\left\|\tilde{y}_{k1} - y_{k1}\right\| \langle \nabla \left\|\tilde{y}_{k1} - y_{k1}\right\|, h_1\rangle + (L\kappa' + 2)^2 \left\|\tilde{y}_{k2} - \bar{y}_2\right\| \langle \xi_k^*, h_2\rangle}{|||\tilde{y}_k - y_k|||},$$
$$\forall h = (h_1, h_2) \in Y$$

where $\xi_k^* \in \mathscr{B}_{Y_2^*}$. Hence

$$\langle \tilde{y}_k^*, (\tilde{y}_{k1} - y_{k1}, 0)\rangle = \frac{\left\|\tilde{y}_{k1} - y_{k1}\right\| \langle \nabla \left\|\tilde{y}_{k1} - y_{k1}\right\|, \tilde{y}_{k1} - y_{k_1}\rangle}{|||\tilde{y}_k - y_k|||} = \frac{\left\|\tilde{y}_{k1} - y_{k1}\right\|^2}{|||\tilde{y}_k - y_k|||}$$
$$\geq \frac{\left\|\tilde{y}_{k1} - y_{k1}\right\|}{\sqrt{1 + (L\kappa' + 2)^2}}$$

showing $|||\tilde{y}^*_{k|_{\tilde{Y}_1}}||| \geq (1 + (L\kappa' + 2)^2)^{-1/2}$. Using the same arguments as in the first part of the proof of Theorem 4, the contradiction $((v_1, 0), 0) \in \mathrm{Cr}_{\tilde{Y}_1}((\bar{x}, \bar{y}); u)$ follows.

In case that X, Y_1 and Y_2 are Asplund, consider as in the proof of Theorem 4 the linear functionals $(\tilde{x}_k^*, \tilde{y}_k^*) \in \hat{\partial}\chi_{\mathrm{gph}\, M}(x_k', y_k')$ and $(\hat{x}_k^*, \hat{y}_k^*) \in \hat{\partial}\psi_k(x_k'', y_k'')$ such that $\|(\tilde{x}_k^*, \tilde{y}_k^*) + (\hat{x}_k^*, \hat{y}_k^*)\| \leq \delta_k$, where we now choose $0 < \delta_k \leq \min\{\frac{t_k}{k}, |||\tilde{y}_k - \bar{y}|||\}$ so small, such that $\max\left\{\|y_{k1}'' - \tilde{y}_{k1}\|, \|y_{k2}'' - \tilde{y}_{k2}\|\right\} \leq \frac{1}{4}\|\tilde{y}_{k1} - \bar{y}_1\|$ holds. Then $\|y_{k1}'' - \bar{y}_1\| \geq \frac{3}{4}\|\tilde{y}_{k1} - \bar{y}_1\| \geq \frac{3}{5}\|y_{k2}'' - \bar{y}_2\|$ and by convex analysis there is a linear functional $\check{y}_k^* \in \hat{\partial}|||\cdot - \bar{y}|||(y_k'')$ satisfying $|||\check{y}_k^* - \hat{y}_k^*||| \leq \frac{1}{\sqrt{k}}$. Using similar arguments as before in the case of a Frechét smooth renorm we obtain the bound

$$\langle \check{y}_k^*, y_{k1}'' - \bar{y}_1)\rangle = \frac{\|y_{k1}'' - \bar{y}_1)\|^2}{|||y_k'' - \bar{y}|||} \geq \frac{\|y_{k1}'' - \bar{y}_1)\|}{\sqrt{1 + \frac{25}{9}(L\kappa + 2)^2}}$$

showing $|||\check{y}^*_{k|_{\tilde{Y}_1}}||| \geq (1 + \frac{25}{9}(L\kappa + 2)^2)^{-1/2}$ and now we can proceed as in the proof of the first part of Theorem 4 to obtain the contradiction $((v_1, 0), 0) \in \mathrm{Cr}_{\tilde{Y}_1}((\bar{x}, \bar{y}); u)$. □

5 Necessary Optimality Conditions in Mathematical Programming

We associate with the mathematical program (1) the multifunction

$$M_P : X \rightrightarrows \mathbb{R} \times Z, \quad M_P(x) := (f(x) - \mathbb{R}_-) \times G(x). \tag{17}$$

Given a linear functional $x^* \in X^*$, we also consider the multifunction

$$M_{x^*} : X \rightrightarrows \mathbb{R} \times Z, \quad M_{x^*}(x) := (\langle x^*, x\rangle - \mathbb{R}_-) \times G(x). \tag{18}$$

The multifunctions M_P and M_{x^*}, respectively, are of the form (15) with $Y_1 = \mathbb{R}$, $M_1(x) = f(x) - \mathbb{R}_-$ respectively $M_1(x) = \langle x^*, x\rangle - \mathbb{R}_-$ and $Y_2 = Z$, $M_2 = G$.

Definition 5 Let $\bar{x}$ be feasible for the problem (1). We call $u \in X$ *a critical direction* for the problem (1) at $\bar{x}$ if $(0, 0) \in CM_P(\bar{x}, (f(\bar{x}), 0))(u)$.

Obviously $u = 0$ is always a critical direction. By the definition, u is a critical direction if and only if there exist sequences $(t_k) \downarrow 0$, $(u_k) \to u$ satisfying

$$\limsup_{k\to\infty} \frac{f(\bar{x} + t_k u_k) - f(\bar{x})}{t_k} \le 0, \quad \lim_{k\to\infty} \frac{\mathrm{d}(0, G(\bar{x} + t_k u_k))}{t_k} = 0. \tag{19}$$

Now we introduce the following directional subdifferentials:

Definition 6 Let $\varphi : X \to \bar{R}$, $x \in X$ with $|\varphi(x)| < \infty$ and $u \in X$. The *limiting subdifferential of φ in direction u* at x is defined as the set

$$\partial\varphi(x; u)$$
$$:= \left\{x^* \in X^* \mid \exists(\varepsilon_k) \downarrow 0,\ (t_k) \downarrow 0,\ (u_k) \to u,\ (x_k^*) \overset{w^*}{\to} x^* : x_k^* \in \hat{\partial}_{\varepsilon_k}\varphi(x + t_k u_k) \forall k\right\}$$

and the *Fréchet upper subdifferential in direction u* is defined as

$$\hat{\partial}^+\varphi(x; u) := \left\{ x^* \in X^* \,\middle|\, \limsup_{\delta\downarrow 0} \frac{1}{\delta} \limsup_{u' \overset{V_{1,\delta}(u)}{\to} 0} \frac{\varphi(x + u') - \varphi(x) - \langle x^*, u'\rangle}{\|u'\|} \le 0 \right\}.$$

One can easily show that in case $u \ne 0$ our limiting directional subdifferential is the same as the *basic directional subdifferential* used in [8].

Obviously we have $\partial\varphi(x; 0) = \partial\varphi(x)$ and $\hat{\partial}^+\varphi(x; 0) = \hat{\partial}^+\varphi(x)$. If φ is Fréchet differentiable at x then $\hat{\partial}^+\varphi(x; u) = \{\nabla\varphi(x)\}, \forall u$.

Proposition 1 *Let $\bar{x}$ be a local optimal solution to problem (1). Then, for every critical direction u for (1) at $\bar{x}$ the multifunction M_P is not mixed regular/subregular in direction $(u, 0)$ at $(\bar{x}, (f(\bar{x}), 0))$. Further, for every direction $u \in X$ with $0 \in CG(\bar{x}, 0)(u)$ and every $x^* \in \hat{\partial}^+ f(\bar{x}; u)$ with $\langle x^*, u\rangle \le 0$ the multifunction M_{x^*} is not mixed regular/subregular in direction $(u, 0)$ at $(\bar{x}, (\langle x^*, \bar{x}\rangle, 0))$.*

Proof To show the first part, assume on the contrary that there is some critical direction u such that M_P is mixed regular/subregular in direction $(u, 0)$ at $(\bar{x}, (f(\bar{x}), 0))$. Let the positive constants ρ, δ, κ be chosen according to Definition 3 and consider sequences $(t_k) \downarrow 0$ and $(u_k) \to u$ fulfilling (19). Next choose $\tau \in (0, \frac{\delta}{4}]$ and set $x_k := \bar{x} + t_k u_k$ and

$$f_k := \begin{cases} f(\bar{x}) - \tau t_k \|u_k\| & \text{if } u \ne 0, \\ f(\bar{x}) - \tau t_k & \text{if } u = 0. \end{cases}$$

If $u \ne 0$, by passing to subsequences if necessary, we can assume that $f(x_k) \le f(\bar{x}) + \tau t_k \|u_k\|$, $\mathrm{d}(0, G(x_k)) \le \tau t_k \|u_k\|$ and $\|\tilde{u}_k - \tilde{u}\| \le \tau$, where $\tilde{u}_k := u_k/\|u_k\|$, $\tilde{u} := u/\|u\|$. Then we obtain, since $1 = \|\tilde{u}_k\| \le \|(\tilde{u}_k, -\tau)\| \le 1 + \tau$

$$\begin{aligned} \left\| \frac{(x_k - \bar{x},\ f_k - f(\bar{x}))}{\|(x_k - \bar{x},\ f_k - f(\bar{x}))\|} - \frac{(u, 0)}{\|(u, 0)\|} \right\| &= \left\| \frac{(\tilde{u}_k, -\tau)}{\|(\tilde{u}_k, -\tau)\|} - (\tilde{u}, 0) \right\| \\ &\le \left\| \frac{\tilde{u}_k}{\|(\tilde{u}_k, -\tau)\|} - \tilde{u}_k \right\| + \|\tilde{u}_k - \tilde{u}\| + \frac{\tau}{\|(\tilde{u}_k, -\tau)\|} \\ &\le 3\tau < \delta \end{aligned}$$

showing $(x_k,\ f_k) \in (\bar{x},\ f(\bar{x})) + V_{\rho,\delta}(u, 0)$, and

$$\begin{aligned} \mathrm{d}((x_k, (f_k, 0)), \mathrm{gph}\, M_P) &\le \mathrm{d}((f_k, 0), M_P(x_k)) \\ &\le (\max\{f(x_k) - f_k, 0\} + \mathrm{d}(0, G(x_k))) \\ &\le 2\tau t_k \|u_k\| \le \delta \|(x_k,\ f_k) - (\bar{x},\ f(\bar{x}))\|. \end{aligned}$$

If $u = 0$ we can assume that $f(x_k) \le f(\bar{x}) + \tau t_k$, $\mathrm{d}(0, G(x_k)) \le \tau t_k$ and $(x_k,\ f_k) \in (\bar{x},\ f(\bar{x})) + V_{\rho,\delta}(u, 0)$. Hence in both cases there exists $\tilde{x}_k \in M_P^{-1}(f_k, 0)$ with

$$\|\tilde{x}_k - x_k\| \le \kappa \mathrm{d}((f_k, 0), M_P(x_k)) \le \begin{cases} 2\kappa\tau t_k \|u_k\| & \text{if } u \ne 0, \\ 2\kappa\tau t_k & \text{if } u = 0, \end{cases}$$

showing $\lim \tilde{x}_k = \bar{x}, 0 \in G(\tilde{x}_k)$ and $f(\tilde{x}_k) \le f_k < f(\bar{x})$, a contradiction to the optimality of $\bar{x}$ and therefore the first assertion is proved.

To show the second part, consider some fixed $u \in X$ with $0 \in CG(\bar{x}, 0)(u)$ and some $x^* \in \hat{\partial}^+ f(\bar{x}; u)$ with $\langle x^*, u\rangle \le 0$ and assume on the contrary that M_{x^*} is mixed regular/subregular in direction $(u, 0)$ at $(\bar{x}, (\langle x^*, \bar{x}\rangle, 0))$ with constants ρ, δ, κ. Then we choose $\bar{\eta} > 0$ such that

$$\limsup_{u' \overset{V_{1,\eta}(u)}{\to} 0} \frac{f(\bar{x} + u') - f(\bar{x}) - \langle x^*, u'\rangle}{\|u'\|} \le \frac{\eta}{4(1 + 8\kappa)}, \forall 0 < \eta \le \bar{\eta}$$

and set $\tau := \min\left\{\frac{\delta}{4}, \frac{1}{4\kappa}, \frac{\bar{\eta}}{1+8\kappa}\right\}$. Since $0 \in CG(\bar{x}, 0)(u)$ there are sequences $(t_k) \downarrow 0$, $(u_k) \to u$ satisfying $\lim_{k\to\infty} \mathrm{d}(0, G(\bar{x} + t_k u_k))/t_k = 0$ and

$$\limsup_{k\to\infty} \frac{\langle x^*, \bar{x} + t_k u_k\rangle - \langle x^*, \bar{x}\rangle}{t_k} = \limsup_{k\to\infty} \langle x^*, u_k\rangle = \langle x^*, u\rangle \le 0$$

and we can proceed as before, with $f(x)$ replaced by $\langle x^*, x\rangle$, to find the sequence $(\tilde{x}_k) \subset G^{-1}(0)$ satisfying

$$\langle x^*, \tilde{x}_k - \bar{x}\rangle \le \begin{cases} -\tau t_k \|u_k\| & \text{if } u \ne 0, \\ -\tau t_k & \text{if } u = 0. \end{cases}$$

If $u \ne 0$, by setting $\xi_k := (\tilde{x}_k - x_k)/(2\tau\kappa t_k \|u_k\|)$ we obtain, since $\|\xi_k\| \le 1$,

$$\begin{aligned} \left\| \frac{\tilde{x}_k - \bar{x}}{\|\tilde{x}_k - \bar{x}\|} - \tilde{u} \right\| &= \left\| \frac{\tilde{u}_k + 2\tau\kappa\xi_k}{\|\tilde{u}_k + 2\tau\kappa\xi_k\|} - \tilde{u} \right\| \le \|\tilde{u}_k - \tilde{u}\| + \frac{\left|\|\tilde{u}_k + 2\tau\kappa\xi_k\| - 1\right| + 2\tau\kappa}{\|\tilde{u}_k + 2\tau\kappa\xi_k\|} \\ &\le \tau + \frac{4\tau\kappa}{1 - 2\tau\kappa} \le \tau(1 + 8\kappa) := \eta \le \bar{\eta}, \end{aligned}$$

implying

$$\limsup_{k\to\infty} \frac{f(\tilde{x}_k) - f(\bar{x}) - \langle x^*, \tilde{x}_k - \bar{x}\rangle}{\|\tilde{x}_k - \bar{x}\|} \le \frac{\eta}{4(1 + 8\kappa)} = \frac{\tau}{4}.$$

Since $\|\tilde{x}_k - \bar{x}\| \le 2\kappa\tau t_k \|u_k\| + t_k \|u_k\| \le \frac{3}{2} t_k \|u_k\|$ we obtain for all k sufficiently large

$$f(\tilde{x}_k) - f(\bar{x}) < \langle x^*, \tilde{x}_k - \bar{x}\rangle + \frac{4}{3}\frac{\tau}{4}\|\tilde{x}_k - \bar{x}\| \le -\frac{\tau}{2} t_k \|u_k\| < 0,$$

contradicting the optimality of $\bar{x}$.

If $u = 0$, then $x^* \in \hat{\partial}^+ f(\bar{x})$ and therefore

$$\limsup_{k\to\infty} \frac{f(\tilde{x}_k) - f(\bar{x}) - \langle x^*, \tilde{x}_k - \bar{x}\rangle}{\|\tilde{x}_k - \bar{x}\|} \le 0.$$

Together with $\|u_k\| \to 0$ we obtain for all k sufficiently large

$$f(\tilde{x}_k) - f(\bar{x}) < \langle x^*, \tilde{x}_k - \bar{x}\rangle + \tau\|\tilde{x}_k - \bar{x}\| \le -\tau t_k + \tau(2\kappa\tau t_k + t_k \|u_k\|) < -\frac{1}{4}\tau t_k < 0,$$

again contradicting the optimality of $\bar{x}$. This completes the proof of the proposition. □

Theorem 7 *Let $\bar{x}$ be a locally optimal solution and let $u \in X$ be a critical direction for (1) at $\bar{x}$. Assume that G has closed graph and that G is metrically subregular in direction u at $(\bar{x}, 0)$.*

(i) *If Z is Fréchet smooth or both X and Z are Asplund spaces, then*

$$-\{x^* \in \hat{\partial}^+ f(\bar{x}; u) \mid \langle x^*, u\rangle \le 0\} \subset \bigcup_{z^*\in Z^*} \tilde{D}^*_M G((\bar{x}, 0); (u, 0))(z^*).$$

(ii) *If f is Lipschitzian near $\bar{x}$ and both X and Z are Asplund spaces, then there is some $z^* \in Z^*$ such that*

$$0 \in \partial f(\bar{x}; u) + D^*_N G((\bar{x}, 0); (u, 0))(z^*).$$

Proof To prove (i), consider for fixed $x^* \in \hat{\partial}^+ f(\bar{x}; u)$ with $\langle x^*, u\rangle \le 0$ the multifunction M_{x^*}. Since $M_1(x) = \langle x^*, x\rangle - \mathbb{R}_-$ has the Aubin property, by Lemma 3 the multifunction M_2 is proper subregular in direction u with respect to M_1 and 0. By Proposition 1 it follows that M_{x^*} is not mixed regular/subregular in direction $(u, 0)$ and from Theorem 6 we conclude that $((0, 0), 0) \in \mathrm{Cr}_{\mathbb{R}\times\{0\}} M_{x^*}((\bar{x}, (\langle x^*, \bar{x}\rangle, 0)); u)$, i.e. there exist a constant $\beta > 0$ and sequences $(\varepsilon_k) \downarrow 0$, $(t_k) \downarrow 0$, $(u_k, x^*_k) \to (u, 0)$, $(v_k) = (\tau_k, w_k) \to (0, 0) \in \mathbb{R} \times Z$, $(y^*_k) = (\alpha_k, z^*_k) \subset \mathscr{S}_{\mathbb{R}\times Z^*}$ with $(-x^*_k, y^*_k) \in \hat{N}_{\varepsilon_k}((\bar{x} + t_k u_k, (\langle x^*, \bar{x}\rangle + t_k\tau_k, 0 + t_k w_k)); \mathrm{gph}\, M_{x^*})$ and $\|y^*_{k|_{\mathbb{R}\times\{0\}}}\| = |\alpha_k| \ge \beta$. Taking $x_k := \bar{x} + t_k u_k$, $(\gamma_k, z_k) := (\langle x^*, \bar{x}\rangle + t_k\tau_k, 0 + t_k w_k)$ we have $(x_k, (\gamma_k, z_k)) \in \mathrm{gph}\, M_{x^*}$ and for each k there is some positive radius $\eta_k > 0$ such that

$$\langle -x^*_k, x - x_k\rangle + \alpha_k(\gamma - \gamma_k) + \langle z^*_k, z - z_k\rangle \le \varepsilon'_k \|(x, (\gamma, z)) - (x_k, (\gamma_k, z_k))\|$$

for every $(x, (\gamma, z)) \in \mathrm{gph}\, M_{x^*} \cap ((x_k, (\gamma_k, z_k)) + \eta_k \mathscr{B}_{X\times(\mathbb{R}\times Z)})$, where $\varepsilon'_k := \varepsilon_k + \frac{1}{k}$. Taking $x = x_k$, $z = z_k$, $\gamma = \gamma_k + \eta_k$ and $\gamma = \gamma_k + \tau_k(\langle x^*, x_k\rangle - \gamma_k)$ with $\tau_k > 0$ sufficiently small yields $\alpha_k\eta_k \le \varepsilon'_k\eta_k$ and $\alpha_k\tau_k(\langle x^*, x_k\rangle - \gamma_k) \le \varepsilon'_k\tau_k(\langle x^*, x_k\rangle - \gamma_k)$. Since $|\alpha_k| \ge \beta > 0$ and $(\varepsilon'_k) \to 0$ we obtain $\alpha_k \le -\beta < 0$ and $\gamma_k = \langle x^*, x_k\rangle$ for all k sufficiently large. Defining $(\tilde{x}^*_k, \tilde{z}^*_k, \tilde{\varepsilon}_k) := (x^*_k, z^*_k, \varepsilon'_k)/(-\alpha_k)$ we obtain

$$\langle -\tilde{x}^*_k - x^*, x - x_k\rangle + \langle \tilde{z}^*_k, z - z_k\rangle \le \tilde{\varepsilon}_k \left\|(x - x_k, (\langle x^*, x - x_k\rangle, z - z_k))\right\|$$

for all $(x, z) \in \mathrm{gph}\, G$ with $\|(x - x_k, (\langle x^*, x - x_k\rangle, z - z_k))\| \le \eta_k$. Since

$$\|(x - x_k, (\langle x^*, x - x_k\rangle, z - z_k))\| \le 2(1 + \|x^*\|)\, \|(x - x_k, z - z_k)\|$$

we conclude $(-\tilde{x}^*_k - x^*, \tilde{z}^*_k) \in \hat{N}_{2(1+\|x^*\|)\tilde{\varepsilon}_k} G(x_k, z_k)$. In any case Z is Asplund and therefore, by eventually passing to a subsequence, the bounded sequence $\tilde{z}^*_k$

weakly* converges to some element $-z^*$. Since $-\tilde{x}_k^* - x^* \to -x^*$, we conclude $-x^* \in \tilde{D}_M^* G((\bar{x}, 0); (u, 0))(z^*)$. This justifies (i).

To prove (ii), by again invoking Lemma 3, Proposition 1 and Theorem 6 and using similar arguments as in the first part of the proof we obtain the existence of sequences $(\tilde{\varepsilon}_k) \downarrow 0$, $(t_k) \downarrow 0$, $(u_k, x_k^*) \to (u, 0)$, $(w_k) \to 0$, $(\eta_k) > 0$ and a bounded sequence $(\tilde{z}_k^*)$, such that

$$\begin{aligned}\langle -\tilde{x}_k^*, x - x_k\rangle - (f(x) - f(x_k)) + \langle \tilde{z}_k^*, z - z_k\rangle &\le \tilde{\varepsilon}_k \,\|(x - x_k, (f(x) - f(x_k), z - z_k))\| \\ &\le 2\tilde{\varepsilon}_k((1 + L)\,\|(x - x_k, z - z_k)\|\end{aligned}$$

for all $(x, z) \in \operatorname{gph} G$ satisfying $2(1 + L)\,\|(x - x_k, z - z_k)\| \le \eta_k$, where $x_k := \bar{x} + t_k u_k$, $z_k = 0 + t_k w_k$ and L denotes the Lipschitz module of f near $\bar{x}$. Hence, (x_k, z_k) is a local minimizer of the problem

$$\min_{x,z} f(x) - \langle \tilde{z}_k^*, z\rangle + \langle \tilde{x}_k^*, x\rangle + \check{\varepsilon}_k \,\|(x - x_k, z - z_k)\| + \chi_{\operatorname{gph} G}(x, z),$$

where $\check{\varepsilon}_k := 2(1 + L)\tilde{\varepsilon}_k$ and by the fuzzy (semi-Lipschitzian) sum rule (see, e.g., [24, Theorem 2.33]) we can find points $(x_k', z_k'), (x_k'', y_k'') \in (x_k, z_k) + \frac{t_k}{k}\mathscr{B}_{X\times Z}$ and linear functionals $(x_k'^*, z_k'^*) \in \hat{\partial}\varphi_1(x_k', z_k')$, $(x_k''^*, z_k''^*) \in \hat{\partial}\varphi_2(x_k'', z_k'')$ with $\varphi_1(x, z) := f(x) - \langle \tilde{z}_k^*, z\rangle + \langle \tilde{x}_k^*, x\rangle$, $\varphi_2(x, z) := \chi_{\operatorname{gph} G}(x, z) + \check{\varepsilon}_k \,\|(x - x_k, z - z_k)\|$, such that $\left\|(x_k'^*, z_k'^*) + (x_k''^*, z_k''^*)\right\| \le \frac{t_k}{k}$. Thus $x_k'^* \in \tilde{x}_k^* + \partial f(x_k')$, $z_k'^* = -\tilde{z}_k^*$, $(x_k''^*, z_k''^*) \in \hat{N}_{\check{\varepsilon}_k}((x_k'', z_k''), \operatorname{gph} G)$ and consequently

$$(-\tilde{x}_k^* - \hat{x}_k^*, \tilde{z}_k^*) \in \hat{N}_{\check{\varepsilon}_k + \frac{t_k}{k}}((x_k'', z_k''), \operatorname{gph} G)$$

with $\hat{x}_k^* \in \partial f(x_k')$. By Lipschitz continuity of f, $\|\hat{x}_k^*\| \le L$ and since both X and Z are Asplund, some subsequence of the bounded sequence $(\hat{x}_k^*, \tilde{z}_k^*)$ weakly* converges to some $(x^*, -z^*)$. Since $\lim_k (x_k' - \bar{x})/t_k = \lim_k (x_k'' - \bar{x})/t_k = u$, $\lim (z_k'' - 0)/t_k = 0$, $\lim_k \tilde{x}_k^* = 0$ we obtain $-x^* \in D_N^* G((\bar{x}, 0); (u, 0))(z^*)$ and $x^* \in \partial f(\bar{x}; u)$. This completes the proof of the theorem. □

Let us compare the results of Theorem 7 in case $u = 0$ with those of Theorem 3. Even in this case Theorem 7 improves the known results for the upper Fréchet subdifferential by weakening the assumption of metric regularity of G to metric subregularity and enlarging the range of applicability to the situation when Y is Fréchet smooth and X is an arbitrary Banach space. One can show that, by assuming that G is metrically regular, the condition (5) of Theorem 3 states that for every $x^* \in \hat{\partial}^+ f(\bar{x})$ the multifunction M_{x^*} is not metrically regular near $(\bar{x}, (\langle x^*, \bar{x}\rangle, 0))$, whereas Theorem 7 in case $u = 0$ is based on the stronger statement of Proposition 1, that for a local minimizer $\bar{x}$ the multifunction M_{x^*} is not mixed regular/subregular at $(\bar{x}, (\langle x^*, \bar{x}\rangle, 0))$.

Note that for every $u \in X$ we have the inclusions $\tilde{D}_M^* G((\bar{x}, 0); (u, 0))(z^*) \subset \tilde{D}_M^* G(\bar{x}, 0)(z^*)$, $D_N^* G((\bar{x}, 0); (u, 0))(z^*) \subset D_N^* G(\bar{x}, 0)(z^*)$ and $\partial f(\bar{x}; u) \subset \partial f(\bar{x})$. Further, if f is Fréchet differentiable at $\bar{x}$, then $\langle \nabla f(\bar{x}), u\rangle \le 0$ for every critical direction u. Hence we obtain the following corollary.

Corollary 2 *Let $\bar{x}$ be a locally optimal solution and assume that G has closed graph. Further assume that G is metrically subregular in some critical direction $u \in X$ at $(\bar{x}, 0)$.*

(i) *If f is Fréchet differentiable at $\bar{x}$ and if Z is Fréchet smooth or both X and Z are Asplund spaces, then there is some $z^* \in Z^*$ such that*

$$0 \in \nabla f(\bar{x}) + \tilde{D}^*_M G(\bar{x}, 0)(z^*).$$

(ii) *If f is Lipschitzian near $\bar{x}$ and both X and Z are Asplund spaces, then there is some $z^* \in Z^*$ such that*

$$0 \in \partial f(\bar{x}) + D^*_N G(\bar{x}, 0)(z^*).$$

For the sake of completeness we state the necessary optimality conditions if no constraint qualification condition is fulfilled. For every critical direction we obtain optimality conditions of *Fritz–John-type* with a nonnegative multiplier corresponding to the cost functional. If this multiplier is zero these necessary conditions reflect the circumstance that the constraint mapping is not metrically regular with respect to the critical direction.

Theorem 8 *Let $\bar{x}$ be a locally optimal solution and let $u \in X$ be a critical direction for* (1) *at $\bar{x}$. Assume that G has closed graph and that G is PSNC in direction $(u, 0)$ at $(\bar{x}, 0)$ with respect to Z.*

(i) *If Z is Fréchet smooth or both X and Z are Asplund spaces, then for each $x^* \in \hat{\partial}^+ f(\bar{x}; u)$ with $\langle x^*, u\rangle \leq 0$ there are some $\lambda_0 \geq 0$ and some $z^* \in Z^*$ such that $(\lambda_0, z^*) \neq (0, 0)$ and*

$$0 \in \lambda_0 x^* + \tilde{D}^*_M G((\bar{x}, 0); (u, 0))(z^*).$$

(ii) *If f is Lipschitzian near $\bar{x}$, $\partial f(\bar{x}; u) \neq \emptyset$ and both X and Z are Asplund spaces, then there are some $\lambda_0 \geq 0$ and some $z^* \in Z^*$ such that $(\lambda_0, z^*) \neq (0, 0)$ and*

$$0 \in \lambda_0 \partial f(\bar{x}; u) + D^*_N G((\bar{x}, 0); (u, 0))(z^*).$$

Proof If G is metrically subregular in direction u then the assertion follows from Theorem 7 with $\lambda_0 = 1$. Hence we assume that G is not metrically subregular in direction u and by Lemma 1 and Theorem 4 we obtain $(0, 0) \in \mathrm{Cr}\, G((\bar{x}, 0); u)$. Then there exist sequences $(t_k) \downarrow 0$, $(\varepsilon_k) \downarrow 0$, $(u_k, v_k, x^*_k) \to (u, 0, 0)$, $(z^*_k) \subset \mathcal{S}_{Z^*}$ with $(-x^*_k, z^*_k) \in \hat{N}_{\varepsilon_k}((\bar{x} + t_k u_k, 0 + t_k v_k), \mathrm{gph}\, G)$. In any case Z is Asplund and hence, by passing to a subsequence we can assume that (z^*_k) weakly* converges to some z^*. Since G is PSNC in direction $(u, 0)$ with respect to Z we conclude $z^* \neq 0$ and, taking into account $\tilde{D}^*_M G(\bar{x}, 0); (u, 0)) \subset D^*_N G(\bar{x}, 0); (u, 0))$, the assertion follows with $\lambda_0 = 0$. □

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