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Infinite Alternative Theorems and Nonsmooth Constraint Qualification Conditions

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Abstract In this paper, some infinite alternative theorems in locally convex vector spaces are proved and some applications of these theorems in optimization theory are addressed. Extensions of constraint qualification conditions, in the absence of differentiability using Clarke's generalized gradient and Mordukhovich's subdifferential, in Banach and Asplund spaces are introduced; and relationships between them are established. Some of these relations, are proved using the provided alternative theorems.

Keywords Mordukhovich's subdifferential · Clarke's generalized gradient · Locally convex space · Alternative theorems · Optimization · Nonsmooth analysis · Banach (Asplund) space · Weak* topology

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1 Introduction

Alternative theorems play a vital role in pure and applied mathematical analysis, especially in mathematical optimization. The classical *alternative theorems* establish the equivalence between the existence of solutions for a certain ordinary linear system and the negation of another proposition that is a statement about the

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position of certain points with respect to one or two polyhedral convex sets associated with the system. Such results play a crucial role in optimization as well as game theory (see, e.g., [2, 5, 7, 13]). Thanks to their applications, studying the alternative theorems is crucial. In this paper, some infinite alternative theorems in locally convex vector spaces are proved, which generalize some results studied in [2, 5, 7]. Afterwards, an application of the established results in optimization theory is addressed.

On the other hand, the notion of generalized differentiation plays a fundamental role in modern variational analysis and optimization [1, 3, 8–10, 13]. In this paper, we utilize two well-known classes of generalized differentials: Clarke's generalized gradient and Mordukhovich's subdifferential. These consist of two main classes of generalized differentials, and play a vital role in pure and applied mathematical analysis; see [1, 3, 8–10, 14, 15] for some discussions and applications. Using these generalized differentials, nonsmooth versions of some constraint qualification conditions in optimization theory, in the absence of differentiability, in Banach and Asplund spaces are introduced and relationships between them are established. To establish some of these relationships, we utilize the provided alternative theorems.

The rest of the paper unfolds as follows: Section 2 contains some preliminaries from nonsmooth analysis and optimization; Section 3 is devoted to novel alternative theorems; and Section 4 addresses nonsmooth constraint qualification conditions.

2 Preliminaries

Throughout this paper, we consider three spaces X, Y, and Z, which are: a real locally convex space, a real Banach space, and a real Asplund space, respectively. X^*, Y^*, and Z^* denote the topological duals of X, Y, and Z, respectively, equipped with the weak* topology, and $\langle ., . \rangle$ is the duality pairing.

In this section, we review some preliminaries from nonsmooth analysis about some well-known generalized differentials: Clarke's generalized gradient and Mordukhovich's subdifferential.

Mordukhovich's subdifferential consists of a main class of generalized differentials, and plays a vital role in pure and applied mathematical analysis [1, 8–10, 17]. Definition of these subdifferentials and one of their properties are outlined in this section.

Let $\emptyset \neq \Omega \subseteq Z$. Given $x \in \Omega$ and $\epsilon \geq 0$, the set of ϵ-normals to Ω at x is defined by

$$\widehat{N}_\epsilon(x; \Omega) = \left\{ \zeta^* \in Z^* : \limsup_{v \xrightarrow{\Omega} x} \frac{\langle \zeta^*, v - x \rangle}{\|v - x\|} \leq \epsilon \right\},$$

in which the symbol $v \xrightarrow{\Omega} x$ means $v \longrightarrow x$ with $v \in \Omega$. If $x \notin \Omega$, we set $\widehat{N}_\epsilon(x; \Omega) = \emptyset$ for all $\epsilon \geq 0$. The limiting normal cone to Ω at $\bar{x} \in \Omega$ is defined as follows:

$$N(\bar{x}; \Omega) = \limsup_{\substack{x \longrightarrow \bar{x} \\ \epsilon \downarrow 0}} \widehat{N}_\epsilon(x; \Omega).$$

Set $N(\bar{x};\Omega)=\emptyset$ for $\bar{x}\notin\Omega$. Considering the extended-real-valued function $\varphi : Z \longrightarrow \overline{\mathbb{R}}=[-\infty,+\infty]$, the epigraph of φ is defined as, $epi\varphi=\{(x,a)\in Z\times\mathbb{R} : \varphi(x)\le a\}$. Considering a point $\bar{x}\in Z$ with $|\varphi(\bar{x})|<\infty$, the set

$$\partial_M\varphi(\bar{x})=\{\zeta\in Z : (\zeta,-1)\in N((\bar{x},\varphi(\bar{x})); epi\varphi)\}$$

is Mordukhovich's subdifferential of φ at $\bar{x}$, and its elements are Mordukhovich subdifferentials of φ at this point. Set $\partial_M\varphi(\bar{x})=\emptyset$ if $|\varphi(\bar{x})|=\infty$. See [9, 10] for more details and applications. One of the classes of functions whose set of Mordukhovich subdifferentials is nonempty is the class of locally Lipschitz functions. Considering this class in Z, the following result is obtained (see [9]). This theorem helps us in Section 4 of the paper.

Theorem 1 [9] *Let φ be locally Lipschitz on an open set containing* $[x,y]$. *Then*

$$\varphi(y)-\varphi(x)\le\langle\zeta, y-x\rangle$$

for some $c\in[x,y),\ \zeta\in\partial_M\varphi(c)$.

We follow this preliminary section by introducing the notion of Clarke's generalized gradient under Banach space Y. Considering $h : Y\longrightarrow\mathbb{R}$ as a locally Lipschitz function, the Clarke's generalized directional derivative of h at $\bar{x}\in Y$ in the direction $d\in Y$, denoted by $h^\circ(\bar{x};d)$, is defined as

$$h^\circ(\bar{x};d)=\limsup_{\substack{x\longrightarrow\bar{x}\\ t\downarrow 0}}\frac{h(x+td)-h(x)}{t}.$$

The Clarke's generalized gradient of h at $\bar{x}$ is given by

$$\partial_C h(\bar{x})=\{\xi^*\in Y^* : h^\circ(\bar{x};d)\ge\langle\xi^*,d\rangle,\ \ \forall d\in Y\}.$$

The following theorems, which provide some properties of the multifunction $\partial_C(.)$, help us in Section 4.

Theorem 2 [3] *Let h be Lipschitz near x with Lipschitz constant K, then*

(i) *$\partial_C h(x)$ is a nonempty, convex, and weak***-compact set.*
(ii) *$\|\xi^*\|_*\le K$ for every $\xi^*\in\partial_C h(x)$, where*

$$\|\xi^*\|_*=\sup\{\langle\xi^*,v\rangle : v\in Y, \|v\|\le 1\}.$$

(iii) *Let x_i and ξ_i^* be sequences in Y and Y^* such that $\xi_i^*\in\partial_C h(x_i)$. Suppose that x_i converges to x, and that ξ^* is a cluster point of ξ_i^* in the weak*− topology. Then $\xi^*\in\partial_C h(x)$.*

Theorem 3 [3] *Let $x, y\in Y$, and suppose that h is Lipschitz on an open set containing the line segment* $[x,y]$. *Then there exists a point $u\in(x,y)$ such that*

$$h(y)-h(x)\in\langle\partial_C h(u), y-x\rangle.$$

In addition to the above theorems, the following known results are used in the next sections of the paper as well.

Lemma 4 [6] *Let $C_1, C_2, \ldots, C_n$ be convex subsets of Y. Then the convex hull of $\cup_{i=1}^n C_i$ consists of all vectors of the form $\sum_{i=1}^n \lambda_i x_i$, where $\lambda_i \geq 0,\ x_i \in C_i\ (1 \leq i \leq n)$, and $\sum_{i=1}^n \lambda_i = 1$.*

Theorem 5 [6] *Let C_1, C_2 be disjoint nonempty closed convex subsets of a real locally convex topological vector space X, and suppose that at least one of which is compact. Then C_1 and C_2 are strictly separated by a hyperplane, i.e., there is a continuous linear functional ψ on X such that $\sup_{C_1} \psi < \inf_{C_2} \psi$.*

We close this section by providing two definitions of nonsmooth pseudo-convexity as generalizations of convexity, see [2, 12].

Definition 1 The locally Lipschitz function $h : Y \longrightarrow \mathbb{R}$ is called a pseudo-convex function with respect to ∂_C (in short, CPC), if for each $x, y \in Y$,

$$\exists \zeta^* \in \partial_C h(y);\quad \langle \zeta^*, x - y\rangle \geq 0 \Longrightarrow h(x) \geq h(y).$$

Definition 2 The locally Lipschitz function $h : Z \longrightarrow \mathbb{R}$ is called a pseudo-convex function with respect to ∂_M (in short, MPC), if for each $x, y \in Z$,

$$\exists \zeta^* \in \partial_M h(y);\quad \langle \zeta^*, x - y\rangle \geq 0 \Longrightarrow h(x) \geq h(y).$$

3 Infinite Alternative Theorems

In this section, we use the locally convex space X. Thanks to their applications, the alternative theorems are worth studying in pure and applied mathematical analysis. One of the most important alternative theorems is Motzkin's theorem, which is generalized in the following result. If $X = \mathbb{R}^n$, then the following theorem collapses in Theorem 3.5 in [5]. Furthermore, if $X = \mathbb{R}^n$ and I_i sets are finite, then the following theorem collapses in Theorem 4.2 in [7].

Theorem 6 *Let $\{x_i^*\}_{i \in I_1 \cup I_2 \cup I_3}$ be a family of members in X^* such that $I_1 \neq \emptyset$ and*

$$conv\ \{x_i^* : i \in I_1\} + cone\ \{x_j^* : j \in I_2\} + span\ \{x_k^* : k \in I_3\}$$

be closed. Then exactly one of the following statements holds:

(i) $0 \in conv\ \{x_i^* : i \in I_1\} + cone\ \{x_j^* : j \in I_2\} + span\ \{x_k^* : k \in I_3\}$,

(ii) *there exists $x \in X$ such that* $\begin{cases} \langle x_i^*, x\rangle > 0 & \text{for all}\ \ i \in I_1, \\ \langle x_j^*, x\rangle \geq 0 & \text{for all}\ \ j \in I_2, \\ \langle x_k^*, x\rangle = 0 & \text{for all}\ \ k \in I_3. \end{cases}$

Proof If (i) holds, then it is clear that system (ii) has no solution. So, assume that (i) does not hold. Thus

$$\{0\} \cap (conv\ \{x_i^* : i \in I_1\} + cone\ \{x_j^* : j \in I_2\} + span\ \{x_k^* : k \in I_3\}) = \emptyset.$$

Since one of the above disjoint convex sets is compact, the other is closed and X^* is a locally convex space, by a separation theorem, [4] (See Theorem 5), there exists a continuous linear functional ψ on X^* such that

$$\langle \psi, 0 \rangle < \langle \psi, x^* \rangle,$$

for each $x^* \in conv\ \{x_i^* : i \in I_1\} + cone\ \{x_j^* : j \in I_2\} + span\ \{x_k^* : k \in I_3\}$. Since X^* has been equipped with weak* topology, $X^{**} = X$, and hence $\psi \in X$. Also we have

$$\left\langle \psi, x_i^* \right\rangle > 0 \text{ for each } i \in I_1,$$

and

$$\left\langle \psi, x_i^* + \lambda x_j^* \right\rangle > 0 \text{ for each } i \in I_1,\ \ j \in I_2,\ \ \lambda > 0.$$

Therefore

$$\frac{\langle \psi, x_i^* \rangle}{-\lambda} < \left\langle \psi, x_j^* \right\rangle \text{ for each } i \in I_1,\ \ j \in I_2,\ \ \lambda > 0,$$

which implies that

$$\left\langle \psi, x_j^* \right\rangle \geq 0 \text{ for each } j \in I_2.$$

Furthermore, we have

$$\left\langle \psi, x_i^* + \lambda x_k^* \right\rangle > 0 \text{ for each } i \in I_1,\ \ k \in I_3,\ \ \lambda \in \mathbb{R},$$

which implies that

$$\left\langle \psi, x_k^* \right\rangle = 0 \text{ for each } k \in I_3.$$

Thus ψ is a solution to system (ii) and the proof is complete. □

Setting $I_2 = I_3 = \emptyset$, the above theorem leads to an infinite generalization of Gordan's theorem [5], which is an important alternative theorem. If $X = \mathbb{R}^n$ and $I_2 = I_3 = \emptyset$, then the above theorem collapses in Theorem 3.2 in [5]. And if $X = \mathbb{R}^n$, $I_2 = I_3 = \emptyset$, and I_1 is finite, then the above theorem leads to Corollary 1 of Theorem 2.4.5 in [2].

The following two theorems provide Farkas-type and Gale-type infinite alternative theorems for locally convex vector spaces.

Theorem 7 *Let $\{x_i^*\}_{i\in I}$ and $\{\alpha_i\}_{i\in I}$ be two families of members in X^* and $\mathbb{R}$, respectively. Also suppose that $x_0^* \in X^*$ and $\alpha \in \mathbb{R}$. Then exactly one of the following statements holds:*

(i) $\begin{pmatrix} x_0^* \\ \alpha \end{pmatrix} \in cl\ cone \left\{ \begin{pmatrix} x_i^* \\ \alpha_i \end{pmatrix} : i \in I,\ \begin{pmatrix} 0_{X^*} \\ -1 \end{pmatrix} \right\}$
or $\begin{pmatrix} 0_{X^*} \\ 1 \end{pmatrix} \in cl\ cone \left\{ \begin{pmatrix} x_i^* \\ \alpha_i \end{pmatrix} : i \in I \right\}$,

(ii) *there exists $x \in X$ such that* $\begin{cases} \langle x_i^*, x \rangle \geq \alpha_i & \text{for } i \in I, \\ \langle x_0^*, x \rangle < \alpha. \end{cases}$

Proof First assume that (i) holds, and by contradiction suppose that there exists an $\bar{x} \in X$ which satisfies system (ii). There are two possible cases:

Case a. $\begin{pmatrix} x_0^* \\ \alpha \end{pmatrix} \in cl\ cone \left\{ \begin{pmatrix} x_i^* \\ \alpha_i \end{pmatrix} : i \in I,\ \begin{pmatrix} 0_{X^*} \\ -1 \end{pmatrix} \right\}$

In this case, there exists a sequence $\left\{ \begin{pmatrix} y_j^* \\ \beta_j \end{pmatrix} \right\}$ such that

$$\lim_{j \longrightarrow \infty} \begin{pmatrix} y_j^* \\ \beta_j \end{pmatrix} = \begin{pmatrix} x_0^* \\ \alpha \end{pmatrix},$$

and

$$\begin{pmatrix} y_j^* \\ \beta_j \end{pmatrix} = \sum_{i \in I} \lambda_i^j \begin{pmatrix} x_i^* \\ \alpha_i \end{pmatrix} + \lambda_0^j \begin{pmatrix} 0_{X^*} \\ -1 \end{pmatrix}, \quad j = 1, 2, \ldots$$

for some $\lambda^j \in \mathbb{R}_+^{(I)}$ and $\lambda_0^j \geq 0$. Now we have

$$\begin{aligned} \langle x_0^*, \bar{x} \rangle &= \lim_{j \longrightarrow \infty} \langle y_j^*, \bar{x} \rangle = \lim_{j \longrightarrow \infty} \sum_{i \in I} \lambda_i^j \langle x_i^*, \bar{x} \rangle \\ &\geq \lim_{j \longrightarrow \infty} \sum_{i \in I} \lambda_i^j \alpha_i = \lim_{j \longrightarrow \infty} (\beta_j + \lambda_0^j) \geq \lim_{j \longrightarrow \infty} \beta_j = \alpha. \end{aligned}$$

Hence, $\langle x_0^*, \bar{x} \rangle \geq \alpha$. This contradicts the feasibility of $\bar{x}$ for system (ii) and completes the proof in this case.

Case b. $\begin{pmatrix} 0_{X^*} \\ 1 \end{pmatrix} \in cl\ cone \left\{ \begin{pmatrix} x_i^* \\ \alpha_i \end{pmatrix} : i \in I \right\}$

In this case, there exists a sequence $\left\{ \begin{pmatrix} y_j^* \\ \beta_j \end{pmatrix} \right\}$ such that

$$\lim_{j \longrightarrow \infty} \begin{pmatrix} y_j^* \\ \beta_j \end{pmatrix} = \begin{pmatrix} 0_{X^*} \\ 1 \end{pmatrix},$$

and

$$\begin{pmatrix} y_j^* \\ \beta_j \end{pmatrix} = \sum_{i \in I} \lambda_i^j \begin{pmatrix} x_i^* \\ \alpha_i \end{pmatrix}, \quad j = 1, 2, \ldots$$

for some $\lambda^j \in \mathbb{R}_+^{(I)}$. Now we have

$$0 = \lim_{j \longrightarrow \infty} \langle y_j^*, \bar{x} \rangle = \lim_{j \longrightarrow \infty} \sum_{i \in I} \lambda_i^j \langle x_i^*, \bar{x} \rangle \geq \lim_{j \longrightarrow \infty} \sum_{i \in I} \lambda_i^j \alpha_i = \lim_{j \longrightarrow \infty} \beta_j = 1.$$

This is an obvious contradiction and completes the proof in this case.

Now let us to prove the converse statement. Assume that (ii) does not hold, then there are two possible cases:

Case 1 The system $\{\langle x_i^*, x \rangle \geq \alpha_i,\ \ i \in I\}$ has solution, but all of its solutions satisfy the relation $\langle x_0^*, x \rangle \geq \alpha$. Considering

$$\mathcal{A} = cone \left\{ \begin{pmatrix} x_i^* \\ \alpha_i \end{pmatrix} : i \in I,\ \begin{pmatrix} 0_{X^*} \\ -1 \end{pmatrix} \right\},$$

in this case we show that $\begin{pmatrix} x_0^* \\ \alpha \end{pmatrix} \in cl\mathcal{A}$. By contradiction suppose that

$$\left\{\begin{pmatrix} x_0^* \\ \alpha \end{pmatrix}\right\} \cap cl\mathcal{A} = \emptyset.$$

Since one of the above disjoint convex sets is compact, the other is closed and X^* is a locally convex space, by a separation theorem, [4] (See Theorem 5), there exists $(\psi, \theta) \in X^{**} \times \mathbb{R}$ such that

$$\langle \psi, x_0^* \rangle + \theta\alpha := \gamma < \langle \psi, x^* \rangle + \theta\beta, \qquad \forall \begin{pmatrix} x^* \\ \beta \end{pmatrix} \in cl\mathcal{A}.$$

Since X^* has been equipped with weak* topology, $X^{**} = X$, and hence $\psi \in X$. Also since $cl\mathcal{A}$ is a cone we have

$$\gamma < \lambda \left(\langle \psi, x^* \rangle + \theta\beta\right),$$

for each $\lambda > 0$, which implies that

$$\langle \psi, x^* \rangle + \theta\beta \geq 0, \qquad \forall \begin{pmatrix} x^* \\ \beta \end{pmatrix} \in cl\mathcal{A}.$$

Hence regarding the strong separation,

$$\langle \psi, x^* \rangle + \theta\beta \geq 0 > \gamma, \qquad \forall \begin{pmatrix} x^* \\ \beta \end{pmatrix} \in cl\mathcal{A}.$$

By setting $\begin{pmatrix} 0_{X^*} \\ -1 \end{pmatrix}$ in the above inequality we have $\theta \leq 0$. If $\theta = 0$, then we get

$$\langle \psi, x^* \rangle \geq 0 > \gamma = \langle \psi, x_0^* \rangle, \qquad \forall \begin{pmatrix} x^* \\ \beta \end{pmatrix} \in cl\mathcal{A}.$$

Thus considering $\bar{x}$ as a solution of the system $\{\langle x_i^*, x \rangle \geq \alpha_i, \;\; i \in I\}$, for a large $r > 0$, $x = \bar{x} + r\psi$ is a solution of the system $\{\langle x_i^*, x \rangle \geq \alpha_i, \;\; i \in I\}$ which satisfies the relation $\langle x_0^*, x \rangle < \alpha$. This contradicts the assumption of case 1.

If $\theta < 0$. Defining $\hat{x} = \frac{-1}{\theta}\psi$, $\hat{x}$ is a solution to the system $\{\langle x_i^*, x \rangle \geq \alpha_i, \;\; i \in I\}$ which does not satisfy the relation $\langle x_0^*, x \rangle \geq \alpha$. This contradicts the assumption of case 1, and the proof is complete in this case.

Case 2 The system $\{\langle x_i^*, x \rangle \geq \alpha_i, \;\; i \in I\}$ has no solution $x \in X$. Therefore the system

$$\left\{\langle x_i^*, x \rangle + \alpha_i t \geq 0, \; i \in I, \; t < 0\right\}$$

has no also solution $(x, t) \in X \times \mathbb{R}$. Thus for every solution (x, t) of the feasible system

$$\left\{\langle x_i^*, x \rangle + \alpha_i t \geq 0, \; i \in I\right\}$$

we have $t \geq 0$. Therefore, regarding Case 1, we have

$$\begin{pmatrix} 0_{X^*} \\ 1 \\ 0 \end{pmatrix} \in cl\; cone \left\{\begin{pmatrix} x_i^* \\ \alpha_i \\ 0 \end{pmatrix} : i \in I, \; \begin{pmatrix} 0_{X^*} \\ 0 \\ -1 \end{pmatrix}\right\},$$

which leads to

$$\begin{pmatrix} 0_{X^*} \\ 1 \end{pmatrix} \in cl\ cone \left\{ \begin{pmatrix} x_i^* \\ \alpha_i \end{pmatrix} : i \in I \right\},$$

and the proof is complete. □

Theorem 8 *Let $\{x_i^*\}_{i\in I}$ and $\{\alpha_i\}_{i\in I}$ be two families of members in X^* and $\mathbb{R}$, respectively. Then exactly one of the following statements holds:*

(i) $\begin{pmatrix} 0_{X^*} \\ 1 \end{pmatrix} \in cl\ cone \left\{ \begin{pmatrix} x_i^* \\ \alpha_i \end{pmatrix} : i \in I \right\},$

(ii) *there exists $x \in X$ such that $\langle x_i^*, x\rangle \geq \alpha_i$,* for each $i \in I$.

Proof To prove this result, we only have to consider Cases 2 and b in the proof of Theorem 7. □

The alternative theorems provided in this section can be useful tools for establishing some new results in optimization in infinite-dimensional spaces. Also, these can help us to provide shortest proofs for some results existing in the optimization theory. For instance, by studying the proofs of Theorem 3.1 in [13] and Theorem 4.1 in [16], it is not difficult to see that some parts of the proofs of these theorems can be shorted using Theorem 6 (Remark 1) of the present paper.

Furthermore, in the next section some nonsmooth Constraint Qualification (CQ) conditions in Banach (Asplund) spaces are studied, and relationships between them are proved. To prove some of these relations, we utilize the alternative theorems given in this paper.

4 Constraint Qualification Conditions

Let us to consider the following optimization problem:

$$\min\{f(x):\ x \in C,\ \ g_i(x) \leq 0,\ i = 1, 2, \ldots, m\}, \tag{P}$$

in which C is a nonempty convex subset of space T, that T can be Banach space Y or Asplund space Z. Also, $f, g_i : T \longrightarrow \mathbb{R}$ for $i = 1, 2, \ldots, m$ are locally Lipschitz functions. Suppose that

$$S = \{x :\ x \in C,\ \ g_i(x) \leq 0,\ i = 1, 2, \ldots, m\}$$

is the feasible set of Problem (P). Considering $x^0 \in S$, we assume that

$$I_{x^0} = \{i \in \{1, 2, \ldots, m\} :\ g_i\left(x^0\right) = 0\}.$$

The Constraint Qualification (CQ) conditions are some assumptions (conditions) which help to derive optimality conditions in optimization theory. These conditions have been defined and studied in many publications, see e.g. Chapter 5 in [2] for a review. In addition to deriving the optimality conditions, CQ conditions were utilized for other purposes too, see e.g. [2, 13, 16].

In this section, we provide some novel nonsmooth CQ conditions, for problem (P) at $x^0 \in S$, with respect to Clarke's generalized gradient and Mordukhovich's subdifferential, in Banach and Asplund spaces, respectively. To this end, we need to define some sets as follows. The following four sets are defined using $\partial_C(.)$ in Banach space Y.

$$H_{x^0}^{C} = \left\{d \in Y : \forall i \in I_{x^0}\ \exists \zeta_i^* \in \partial_C g_i\left(x^0\right) \text{ such that } \langle \zeta_i^*, d\rangle < 0\right\},$$

$$H_{x^0}^{C'} = \left\{d \in Y : \forall i \in I_{x^0}\ \exists \zeta_i^* \in \partial_C g_i\left(x^0\right) \text{ such that } \langle \zeta_i^*, d\rangle \le 0\right\},$$

$$G_{x^0}^{C} = \left\{d \in Y : \langle \zeta^*, d\rangle < 0,\ \forall \zeta^* \in \bigcup_{i \in I_{x^0}} \partial_C g_i\left(x^0\right)\right\},$$

$$G_{x^0}^{C'} = \left\{d \in Y : \langle \zeta^*, d\rangle \le 0,\ \forall \zeta^* \in \bigcup_{i \in I_{x^0}} \partial_C g_i\left(x^0\right)\right\}.$$

The following four sets are defined using $\partial_M(.)$ in Asplund space Z:

$$H_{x^0}^{M} = \left\{d \in Z : \forall i \in I_{x^0}\ \exists \zeta_i^* \in \partial_M g_i\left(x^0\right) \text{ such that } \langle \zeta_i^*, d\rangle < 0\right\},$$

$$H_{x^0}^{M'} = \left\{d \in Z : \forall i \in I_{x^0}\ \exists \zeta_i^* \in \partial_M g_i\left(x^0\right) \text{ such that } \langle \zeta_i^*, d\rangle \le 0\right\},$$

$$G_{x^0}^{M} = \left\{d \in Z : \langle \zeta^*, d\rangle < 0,\ \forall \zeta^* \in \bigcup_{i \in I_{x^0}} \partial_M g_i\left(x^0\right)\right\},$$

$$G_{x^0}^{M'} = \left\{d \in Z : \langle \zeta^*, d\rangle \le 0,\ \forall \zeta^* \in \bigcup_{i \in I_{x^0}} \partial_M g_i\left(x^0\right)\right\}.$$

The following three sets are called the cones of feasible directions, attainable directions, and tangents of S at x^0 (see [2]). In these sets, we used the universal space T, which can be Y or Z.

$$D_{x^0}^{T} = \{d \in T : d \neq 0,\ \exists \delta > 0 \text{ such that } x^0 + \lambda d \in S\ \ \forall \lambda \in (0, \delta)\},$$

$$A_{x^0}^{T} = \Bigg\{d \in T : d \neq 0,\ \exists(\delta > 0,\ \alpha : \mathbb{R} \to Y);\ \ \alpha(\lambda) \in S\ \ \forall \lambda \in (0, \delta), \\ \alpha(0) = x^0,\ d = \lim_{\lambda \downarrow 0} \frac{\alpha(\lambda) - \alpha(0)}{\lambda}\Bigg\}.$$

and

$$T_{x^0}^{T} = \left\{d \in T : \exists\left(\{\lambda_k\} \subseteq (0, +\infty),\ \{x_k\} \subseteq S\right);\ \ x_k \to x^0,\ d = \lim_{k \to +\infty} \lambda_k (x_k - x^0)\right\}.$$

Now, we extend some CQ conditions for (P) as follows. These conditions generalize some popular CQ conditions: Slater CQ, Linear independence CQ, Cottle CQ, Zangwill CQ, KT CQ, and Abadie CQ. See Chapter 5 in [2] for more details.

Slater–Clarke–CQ (SCCQ) C is open, g_i is CPC at x^0 for each $i \in I_{x^0}$, and there exists $\bar{x} \in S$ such that $g_i(\bar{x}) < 0$, for each $i \in I_{x^0}$;

Slater–Mordukhovich–CQ (SMCQ) C is open, g_i is MPC at x^0 for each $i \in I_{x^0}$, and there exists $\bar{x} \in S$ such that $g_i(\bar{x}) < 0$, for each $i \in I_{x^0}$;

Linear Independent–Clarke-CQ (LICCQ) C is open and, $0 \notin \sum_{i \in I_{x^0}} u_i \partial_C g_i(x^0)$, for each nonzero vector $(u_i;\ i \in I_{x^0})$;

Linear Independent–Mordukhovich–CQ (LIMCQ) C is open and, $0 \notin conv(\bigcup_{i \in I_{x^0}} \partial_M g_i(x^0))$;

Cottle–Clarke–CQ (CCCQ) C is open and, $cl G^C_{x^0} = G^{C'}_{x^0}$;

Cottle–Mordukhovich–CQ (CMCQ) C is open and, $cl G^M_{x^0} = G^{M'}_{x^0}$;

Zangwill–Clarke–CQ (ZCCQ) $cl D^Y_{x^0} = G^{C'}_{x^0}$;

Zangwill–Mordukhovich–CQ (ZMCQ) $cl D^Z_{x^0} = G^{M'}_{x^0}$;

Kuhn–Tucker-Clarke–CQ (KTCCQ) $cl A^Y_{x^0} = G^{C'}_{x^0}$;

Kuhn–Tucker–Mordukhovich-CQ (KTMCQ) $cl A^Z_{x^0} = G^{M'}_{x^0}$;

Abadie–Clarke–CQ (ACCQ) $T^Y_{x^0} = G^{C'}_{x^0}$;

Abadie–Mordukhovich–CQ (AMCQ) $T^Z_{x^0} = G^{M'}_{x^0}$.

Furthermore, we define some new CQ conditions using two above-defined H-sets, as follows:

CQ-1: C is open and, $cl H^C_{x^0} = H^{C'}_{x^0}$;
CQ-2: $cl D^Y_{x^0} = H^{C'}_{x^0}$;
CQ-3: $cl A^Y_{x^0} = H^{C'}_{x^0}$;
CQ-4: $T^Y_{x^0} = H^{C'}_{x^0}$;
CQ-5: C is open and, $cl H^M_{x^0} = H^{M'}_{x^0}$;
CQ-6: $cl D^Z_{x^0} = H^{M'}_{x^0}$;
CQ-7: $cl A^Z_{x^0} = H^{M'}_{x^0}$;
CQ-8: $T^Z_{x^0} = H^{M'}_{x^0}$.

The following theorem, which is the main result of this section, establishes some relationships between the above CQ conditions for (P). Notice that, in the parts of this theorem which we prove the relations between the CQ conditions with respect to ∂_C and those with respect to ∂_M (Parts (ii), (iv), and (vi)), we assume that the universal space is the Asplund space Z.

Theorem 9

(i) *(SCCQ)* $\Longrightarrow$ *(CCCQ)* $\Longrightarrow$ *(CQ-1)*.
(ii) *(SCCQ)* $\Longrightarrow$ *(SMCQ)* $\Longrightarrow$ *(CMCQ)* $\Longrightarrow$ *(CQ-5)*.
(iii) *(LICCQ)* $\Longrightarrow$ *(CCCQ)*.
(iv) *(LICCQ)* $\Longrightarrow$ $(LIMCQ)$.

(v) *If* $conv\Big(\bigcup_{i\in I_{x^0}} \partial_M g_i(x^0)\Big)$ *is closed, then:* $(LIMCQ) \Longrightarrow (CMCQ)$.
(vi) *(CCCQ)* $\Longrightarrow$ *(CMCQ).*
(vii) *(CQ-2)* $\Longrightarrow$ *(CQ-3)* $\Longrightarrow$ *(CQ-4).*
(iix) *(CQ-6)* $\Longrightarrow$ *(CQ-7)* $\Longrightarrow$ *(CQ-8).*
(ix) *If* g_i *for each* $i \in I_{x^0}$ *is convex, then (CCCQ)* $\Longrightarrow$ *(ZCCQ)* $\Longrightarrow$ *(KTCCQ)* $\Longrightarrow$ *(ACCQ).*
(x) *If* g_i *for each* $i \in I_{x^0}$ *is convex, then (CMCQ)* $\Longrightarrow$ *(ZMCQ)* $\Longrightarrow$ *(KTMCQ)* $\Longrightarrow$ *(AMCQ).*

Proof

(i) Suppose that the SCCQ holds. Then C is open, g_i is CPC at x^0 for each $i \in I_{x^0}$, and there exists $\bar{x} \in S$ such that $g_i(\bar{x}) < 0$, for each $i \in I_{x^0}$. Considering $\zeta^* \in \bigcup_{i\in I_{x^0}} \partial_C g_i(x^0)$, there exists an $i \in I_{x^0}$ such that $\zeta^* \in \partial_C g_i(x^0)$ and

$$g_i(\bar{x}) < 0 = g_i\left(x^0\right).$$

By CPC assumption on g_i at x^0, we have

$$\left\langle \zeta^*, \bar{x} - x^0 \right\rangle < 0.$$

Therefore,

$$\left\langle \zeta^*, \bar{x} - x^0 \right\rangle < 0, \quad \forall \zeta^* \in \bigcup_{i\in I_{x^0}} \partial_C g_i\left(x^0\right).$$

This implies that $G^C_{x^0} \neq \emptyset$.
It is clear that $clG^C_{x^0} \subseteq G^{C'}_{x^0}$. Considering $d \in G^{C'}_{x^0}$ and $\bar{d} \in G^C_{x^0}$, we have $d + \lambda\bar{d} \in G^C_{x^0}$ for each $\lambda > 0$, and $\lim_{\lambda\downarrow 0}(d + \lambda\bar{d}) = d$. Hence, $d \in clG^C_{x^0}$. Therefore, $G^{C'}_{x^0} \subseteq clG^C_{x^0}$. Thus $clG^C_{x^0} = G^{C'}_{x^0}$ and hence (CCCQ) holds.
Now, if (CCCQ) holds, then $G^C_{x^0} \neq \emptyset$. Considering $\bar{d} \in G^C_{x^0}$ and $d \in H^{C'}_{x^0}$, we have $d + \lambda\bar{d} \in H^C_{x^0}$ for each $\lambda > 0$, and $\lim_{\lambda\downarrow 0}(d + \lambda\bar{d}) = d$. This implies that $d \in clH^C_{x^0}$. Hence $H^{C'}_{x^0} \subseteq clH^C_{x^0}$.
Now, we are going to show that $clH^C_{x^0} \subseteq H^{C'}_{x^0}$. To this end, assume that there exists a sequence $\{d_n\} \subseteq H^C_{x^0}$ such that $d_n \longrightarrow d$. It is sufficient to show that $d \in H^{C'}_{x^0}$. Let $i \in I_{x^0}$. For each $n \in \mathbb{N}$ there exists $\zeta^*_{i,n} \in \partial_C g_i(x^0)$ such that

$$\left\langle \zeta^*_{i,n}, d_n \right\rangle < 0.$$

The set $\partial_C g_i(x^0)$ is weak* compact, due to Theorem 2 and the Banach–Alaoglu theorem [4, 11]. Hence, $\{\zeta^*_{i,n}\}_{n=1}^{\infty}$ has a cluster point $\zeta^*_i \in \partial_C g_i(x^0)$. Therefore, the sequence $\{\langle \zeta^*_{i,n}, d\rangle\}_{n=1}^{\infty}$ has a subsequence $\{\langle \zeta^*_{i,n_k}, d\rangle\}_{k=1}^{\infty}$ convergent to $\langle \zeta^*_i, d\rangle$ in $\mathbb{R}$. Consequently, the boundedness of $\{\zeta^*_{i,n_k}\}_{k=1}^{\infty}$ implies that

$$\left\langle \zeta^*_i, d \right\rangle = \lim_{k\to+\infty} \left\langle \zeta^*_{i,n_k}, d_{n_k} \right\rangle \leq 0.$$

Thus

$$\forall i \in I_{x^0} \;\; \exists \zeta^*_i \in \partial_C g_i\left(x^0\right) \text{ such that } \left\langle \zeta^*_i, d \right\rangle \leq 0.$$

Hence $d \in H_{x^0}^{C'}$. This implies that $cl H_{x^0}^{C} \subseteq H_{x^0}^{C'}$.
Therefore, $cl H_{x^0}^{C} = H_{x^0}^{C'}$, and so, (CQ-1) holds.

(ii) There is the relation

$$\partial_C \varphi\left(x^0\right) = cl^* conv \partial_M \varphi\left(x^0\right) \tag{1}$$

between the Clarke and Mordukhovich subdifferential for locally Lipschitzian functions on Asplund spaces; see Theorem 3.57 in [9]. By equation (1), if a function is CPC, then it is MPC. Hence, (SCCQ)$\Longrightarrow$(SMCQ). The proof of (SMCQ) $\Longrightarrow$ (CMCQ) $\Longrightarrow$ (CQ-5) is similar to that of part (i) and is hence omitted.

(iii) Assume that the (LICCQ) holds. Then by Theorem 2 and Lemma 4, we have $0 \notin conv(\bigcup_{i \in I_{x^0}} \partial_C g_i(x^0))$. By Theorem 3.20 in [11] and Theorem 2, $conv(\bigcup_{i \in I_{x^0}} \partial_C g_i(x^0))$ is closed and hence, by alternative Theorem 6, there exists $d \in Y$, such that

$$\left\langle \zeta^*, d\right\rangle > 0, \quad \forall \zeta^* \in \bigcup_{i \in I_{x^0}} \partial_C g_i\left(x^0\right).$$

Hence, $-d \in G_{x^0}^{C}$.This implies that $G_{x^0}^{C} \neq \emptyset$. Therefore, $cl G_{x^0}^{C} = G_{x^0}^{C'}$. Thus, (CCCQ) holds.

(iv) Assume that (LICCQ) holds, while (LIMCQ) does not hold. This implies that $0 \in conv(\bigcup_{i \in I_{x^0}} \partial_M g_i(x^0))$. Hence, by equation (1), we have $0 \in conv(\bigcup_{i \in I_{x^0}} \partial_C g_i(x^0))$. And hence, by Theorem 2 and Lemma 4, we get $0 \in \sum_{i \in I_{x^0}} u_i \partial_C g_i(x^0)$ for some nonzero vector $(u_i;\ i \in I_{x^0})$. This contradicts (LICCQ). Therefore, (LICCQ)$\Longrightarrow$(LIMCQ).

(v) The proof of this part is similar to that of part (iii) and is hence omitted.

(vi) When C is open, then due to equation (1), we have: (CCCQ)$\Longrightarrow cl G_{x^0}^{C} = G_{x^0}^{C'} \Longrightarrow G_{x^0}^{C} \neq \emptyset \Longrightarrow G_{x^0}^{M} \neq \emptyset \Longrightarrow cl G_{x^0}^{M} = G_{x^0}^{M'} \Longrightarrow$(CMCQ).

(vii) To prove this part, it is sufficient to show that

$$cl D_{x^0}^{Y} \subseteq cl A_{x^0}^{Y} \subseteq T_{x^0}^{Y} \subseteq H_{x^0}^{C'}. \tag{2}$$

Considering $d \in D_{x^0}^{Y}$, there exists a $\delta > 0$ such that $x^0 + \lambda d \in S,\ \forall \lambda \in (0, \delta)$. Now, we define $\alpha : \mathbb{R} \to Y$ by $\alpha(\lambda) = x^0 + \lambda d$. Then we have $\alpha(\lambda) \in S\ \ \forall \lambda \in (0, \delta),\ \alpha(0) = x^0$, and $d = \frac{\alpha(\lambda) - \alpha(0)}{\lambda}$. Hence $d \in A_{x^0}^{Y}$. This implies that $D_{x^0}^{Y} \subseteq A_{x^0}^{Y}$. Therefore, $cl D_{x^0}^{Y} \subseteq cl A_{x^0}^{Y}$.
To prove $cl A_{x^0}^{Y} \subseteq T_{x^0}^{Y}$, we assume that $0 \neq d \in A_{x^0}^{Y}$. Thus, there exist $\delta > 0,\ \alpha : \mathbb{R} \to Y$ such that $\alpha(\lambda) \in S\ \forall \lambda \in (0, \delta)$, $\alpha(0) = x^0$, and $d = \lim_{\lambda \downarrow 0} \dfrac{\alpha(\lambda) - \alpha(0)}{\lambda}$. For each $k \in \mathbb{N}$, we define

$$\lambda_k = \frac{2k}{\delta}, \quad x_k = \alpha\left(\frac{\delta}{2k}\right).$$

Then $\frac{\delta}{2k} \in (0, \delta)$ and so, $x_k \in S$ for each $k \in \mathbb{N}$. Since, α is right differentiable at 0, it is right continuous at 0. Thus

$$\lim_{k\to+\infty} x_k = \lim_{k\to+\infty} \alpha\left(\frac{\delta}{2k}\right) = \alpha(0) = x^0.$$

Furthermore,

$$d = \lim_{\lambda\downarrow 0} \frac{\alpha(\lambda) - \alpha(0)}{\lambda} = \lim_{k\to+\infty} \frac{\alpha\left(\frac{\delta}{2k}\right) - \alpha(0)}{\frac{\delta}{2k}} = \lim_{k\to+\infty} \lambda_k \left(x_k - x^0\right).$$

Thus, $d \in A^Y_{x^0}$ which implies $A^Y_{x^0} \subseteq T^Y_{x^0}$. Since $T^Y_{x^0}$ is a closed set, we have $cl A^Y_{x^0} \subseteq T^Y_{x^0}$.

Now, for completing the proof of (2), we assume that $d \in T^Y_{x^0}$. Thus, there exist $\{\lambda_k\} \subseteq (0, +\infty)$ and $\{x_k\} \subseteq S$ such that $x_k \to x^0$ and $d = \lim_{k\to+\infty} \lambda_k(x_k - x^0)$. Considering $i \in I_{x^0}$, for each k, by mean value Theorem 3, there exists $c_k \in (x^0, x_k)$ such that

$$0 \geq g_i(x_k) = g_i(x_k) - g_i\left(x^0\right) = \left\langle \zeta^*_k, x_k - x^0\right\rangle, \quad \text{for some } \zeta^*_k \in \partial_C g_i\left(c_k\right).$$

This implies that

$$\left\langle \zeta^*_k, \lambda_k\left(x_k - x^0\right)\right\rangle \leq 0.$$

Since g_i is Lipschitz near x^0, there exist a neighborhood U of x^0 and a constant $K > 0$ such that g_i is Lipschitz on U with Lipschitz rank K. On the other hand, since $c_k \to x^0$, there exists $k_1 \in \mathbb{N}$ such that $c_k \in U$ for each $k \geq k_1$. Thus, by Theorem 2,

$$\|\zeta^*_k\| \leq K, \quad \forall k \geq k_1.$$

By the Banach–Alaoglu theorem [4, 11], the set $\{y^* \in Y^* \ : \ \|y^*\| \leq K\}$ is weak* compact. Hence, $\{\zeta^*_k\}_{k=k_1}^{\infty}$ has a cluster point ζ^*. Therefore, the sequence $\{\langle\zeta^*_k, d\rangle\}_{k=1}^{\infty}$ has a subsequence $\{\langle\zeta^*_{k_n}, d\rangle\}_{n=1}^{\infty}$ convergent to $\langle\zeta^*, d\rangle$ in $\mathbb{R}$. Consequently, the boundedness of $\{\zeta^*_{k_n}\}_{n=1}^{\infty}$ implies that

$$\left\langle \zeta^*, d\right\rangle = \lim_{n\to+\infty} \left\langle \zeta^*_{k_n}, \lambda_{k_n}\left(x_{k_n} - x^0\right)\right\rangle \leq 0.$$

Also, by Theorem 2, $\zeta^* \in \partial_C g_i(x^0)$, because $c_k \longrightarrow x^0$ when $k \to +\infty$. Thus

$$\forall i \in I_{x^0} \ \ \exists \zeta^*_i \in \partial_C g_i\left(x^0\right) \text{ such that } \left\langle \zeta^*_i, d\right\rangle \leq 0.$$

Hence $d \in H^{C'}_{x^0}$, and (2) is proved. This completes the proof of part (vii).

(iix) To prove this part, it is sufficient to show that

$$cl D^Z_{x^0} \subseteq cl A^Z_{x^0} \subseteq T^Z_{x^0} \subseteq H^{M'}_{x^0}. \tag{3}$$

The relation $cl D^Z_{x^0} \subseteq cl A^Z_{x^0} \subseteq T^Z_{x^0}$ is resulted from the proof of the previous part. To prove $T^Z_{x^0} \subseteq H^{M'}_{x^0}$, we assume that $d \in T_{x^0}$. Thus, there exist $\{\lambda_k\} \subseteq (0, +\infty)$ and $\{x_k\} \subseteq S$ such that $x_k \to x^0$ and $d = \lim_{k\to+\infty} \lambda_k(x_k - x^0)$. Considering $i \in I_{x^0}$, for each k, by mean value Theorem 1, there exists $c_k \in [x_k, x^0)$

such that

$$0 \leq -g_i(x_k) = g_i\left(x^0\right) - g_i(x_k) \leq \left\langle \zeta_k^*, x^0 - x_k \right\rangle, \quad \text{for some } \zeta_k^* \in \partial_M g_i(c_k).$$

This implies that

$$\left\langle \zeta_k^*, \lambda_k(x_k - x^0) \right\rangle \leq 0.$$

Since g_i is locally Lipschitz, the sequence $\{\zeta_k^*\}$ is bounded due to Proposition 1.85 in [9]. Hence, when $k \to +\infty$, remembering that Z is an Asplund space, $\{\zeta_k^*\}$ has a weak* convergent subsequence. For simplicity we denote this subsequence too by $\{\zeta_k^*\}$, and assume that $\zeta_k^* \xrightarrow{w^*} \zeta_i^*$, where w^* stands for convergence in the weak*-topology. On the other hand, if $k \to +\infty$, then $x_k \to x^0$, and hence $c_k \to x^0$. Hence, because of $\zeta_k^* \in \partial_M g_i(c_k)$, and remembering that Z is an Asplund space, we have $\zeta_i^* \in \partial_M g_i(x^0)$. Thus, due to the boundedness of $\{\zeta_k^*\}_{k=1}^{\infty}$, we have

$$0 \geq \lim_{k \to +\infty} \left\langle \zeta_k^*, \lambda_k(x_k - x^0) \right\rangle = \left\langle \zeta_i^*, d \right\rangle,$$

for some $\zeta_i^* \in \partial_M g_i(x^0)$. Thus

$$\forall i \in I_{x^0}\ \exists \zeta_i^* \in \partial_M g_i\left(x^0\right) \text{ such that } \left\langle \zeta_i^*, d \right\rangle \leq 0.$$

Hence, we have $d \in H_{x^0}^{M'}$, which implies that $T_{x^0}^Z \subseteq H_{x^0}^{M'}$. This completes the proof of (3), and so the proof of part (iix) is completed.

(ix) To establish this part of the theorem, it is sufficient to prove that

$$cl G_{x^0}^C \subseteq cl D_{x^0}^Y \subseteq cl A_{x^0}^Y \subseteq T_{x^0}^Y \subseteq G_{x^0}^{C'}. \tag{4}$$

Suppose that $d \in G_{x^0}^C$. Considering $i \in I_{x^0}$, assume that

$$\forall k \in \mathbb{N}\ \exists \lambda_k \in \left(0, \frac{1}{k}\right) \text{ such that } g_i\left(x^0 + \lambda_k d\right) > 0. \tag{5}$$

Therefore, $g_i(x^0 + \lambda_k d) > g_i(x^0)$. Hence, by mean value Theorem 3,

$$\begin{aligned} \exists c_k \in \left(x^0, x^0 + \lambda_k d\right);\ \ 0 &< g_i\left(x^0 + \lambda_k d\right) - g_i\left(x^0\right) \\ &= \lambda_k \left\langle \zeta_k^*, d \right\rangle, \quad \text{for some } \zeta_k^* \in \partial_C g_i(c_k). \end{aligned}$$

Hence

$$\forall k \in \mathbb{N}\ \exists \left(\lambda_k \in \left(0, \frac{1}{k}\right),\ c_k \in \left(x^0, x^0 + \lambda_k d\right)\right);$$
$$\left\langle \zeta_k^*, d \right\rangle > 0, \quad \text{for some } \zeta_k^* \in \partial_C g_i(c_k).$$

Now, if $k \longrightarrow +\infty$, then $c_k \longrightarrow x^0$, and hence, regarding Theorems 2 and the Banach–Alaoglu theorem [4, 11], similar to the proof of Part (vii), it can be shown that

$$\left\langle \zeta^*, d \right\rangle \geq 0, \quad \text{for some } \zeta^* \in \partial_C g_i\left(x^0\right).$$

This implies that $d \notin G_{x^0}^C$ and contradicts the assumption. Hence, (5) does not hold and so

$$\forall i \in I_{x^0}, \quad \exists \delta_i > 0;\ \ g_i\left(x^0 + \lambda d\right) \leq 0, \ \ \forall \lambda \in (0, \delta_i).$$

On the other hand, g_i functions are continuous, and for each $i \notin I_{x^0}$ we have $g_i(x^0) < 0$. Therefore,

$$\forall i \notin I_{x^0}, \quad \exists \delta_i > 0; \quad g_i\left(x^0 + \lambda d\right) < 0, \quad \forall \lambda \in (0, \delta_i).$$

Also, since C is open and $x^0 \in C$, there exists $\delta_0 > 0$ such that $x^0 + \lambda d \in C$ for $\lambda \in (0, \delta_0)$. Thus setting $\delta = \min_{0 \le i \le m} \{\delta_i\}$, we have $\delta > 0$ and $x^0 + \lambda d \in S$ for each $\lambda \in (0, \delta)$. Therefore, $d \in D^Y_{x^0}$. Thus $G^Y_{x^0} \subseteq D^Y_{x^0}$, and this implies that $clG^Y_{x^0} \subseteq clD^Y_{x^0}$.

Furthermore, by (2), we have $clD^Y_{x^0} \subseteq clA^Y_{x^0} \subseteq T^Y_{x^0}$. Hence, for completing the proof of this part of the theorem, it is sufficient to show that $T^Y_{x^0} \subseteq G^{C'}_{x^0}$. To this end, we consider $d \in T^Y_{x^0}$. Thus, there exist $\{\lambda_k\} \subseteq (0, +\infty)$ and $\{x_k\} \subseteq S$ such that $x_k \to x^0$ and $d = \lim_{k \to +\infty} \lambda_k(x_k - x^0)$. By convexity of g_i functions and because of Theorem 4.3 in [3], for each $i \in I_{x^0}$ and $k \in \mathbb{N}$, we have

$$0 \ge g_i(x_k) = g_i(x_k) - g_i\left(x^0\right) \ge \left\langle \zeta_i^*, x_k - x^0 \right\rangle, \quad \forall \zeta_i^* \in \partial_C g_i\left(x^0\right).$$

Hence,

$$\left\langle \zeta_i^*, d \right\rangle = \lim_{k \to +\infty} \left\langle \zeta_i^*, \lambda_k\left(x_k - x^0\right) \right\rangle \le 0, \quad \forall \zeta_i^* \in \partial_C g_i\left(x^0\right).$$

Therefore,

$$\left\langle \zeta^*, d \right\rangle \le 0, \quad \forall \zeta^* \in \bigcup_{i \in I_{x^0}} \partial_C g_i\left(x^0\right).$$

Thus, $d \in G^{C'}_{x^0}$. This implies that $T^Y_{x^0} \subseteq G^{C'}_{x^0}$, and completes the proof of part (ix).

(x) The proof of this part is similar to that of part (ix), due to Theorem 1.93 in [9] and mean value Theorem 1.

□

In part (v) of the above theorem, in a finite dimensional context the assumption "$conv(\bigcup_{i \in I_{x^0}} \partial_M g_i(x^0))$ is closed" automatically holds.

The results established in this paper can be useful in sketching numerical algorithms, establishing duality results and deriving optimality conditions in single objective programming as well as multi-objective programming [18].

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