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Global Error Bounds for γ-paraconvex Multifunctions

Hui Huang · Runxin Li

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Abstract In this paper, error bounds for γ-paraconvex multifunctions are considered. A Robinson-Ursescu type Theorem is given in normed spaces. Some results on the existence of global error bounds are presented. Perturbation error bounds are also studied.

Keywords Error bound · γ-paraconvex multifunction · Recession cone · Normed space

Mathematics Subject Classifications (2010) 49J52 · 90C29 · 90C31

1 Introduction

Since the pioneering works of Hoffman [17], the notion of (global) error bounds plays an important role in variational analysis. Let X be a normed space, $f : X \to R \cup \{+\infty\}$ a proper function, and let $S = \{x \in X : f(x) \leq 0\}$. We always assume that $S \neq \emptyset$. Let $\tau > 0, \gamma > 0$. We say that f has an error bound τ of order γ if for each $x \in X$,

$$\mathrm{d}(x, S) \leq \tau([f(x)]_+)^\gamma,$$

where $\mathrm{d}(x, S) = \inf\{\|x - z\| : z \in S\}$, $[f(x)]_+ = \max\{f(x), 0\}$. Error bounds occur in many consistence or optimization problems, and have important applications in the convergence analysis of some algorithms and in the stability and sensitive analysis of mathematical programming [35]. For these reasons, the study of error bounds

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H. Huang (✉) · R. X. Li
Department of Mathematics, Yunnan University, Kunming 650091, People's Republic of China
e-mail: huanghui@ynu.edu.cn

has attracted the interest of many researchers [1–8, 10–44]. Today, there are vast literature on error bounds. For more details, see the survey paper of Pang [35], and a special issue of *Mathematical Programming* (Vol. 88, No. 2 (2000)) and the references therein.

Recently, some researchers [1, 15, 18, 25, 27, 30, 43, 44] considered error bounds for multifunctions. Let Y be a normed space and let $F : X \to 2^Y$ be a multifunction. For a given $b \in F(X)$, an inclusion problem is to find a point $\bar{x} \in X$ such that

$$b \in F(\bar{x}). \tag{1.1}$$

For $\tau > 0$ and $\gamma > 0$, we say that F has an error bound τ of order γ for the problem (1.1) if for each $x \in X$,

$$\mathrm{d}(x, F^{-1}(b)) \leq \tau[\mathrm{d}(b, F(x))]^\gamma,$$

where $\mathrm{d}(b, F(x))$ is understood as $+\infty$ if $F(x) = \emptyset$. Li and Singer [27] considered global error bounds of order 1 for convex multifunctions. They established some existence theorems of global error bounds for F when $F^{-1}(b)$ is bounded, and they also formulated the following conjecture when $F^{-1}(b)$ is unbounded.

Conjecture Let X and Y be Banach spaces, and $F : X \to 2^Y$ be a convex multifunction with closed graph. Suppose that $F^{-1}(b) = A + C$, where A is a bounded convex set and C is a closed convex cone. If $b \in \mathrm{int}(F(X))$, then there exists a $\tau > 0$ such that for each $x \in X$,

$$\mathrm{d}(x, F^{-1}(b)) \leq \tau \mathrm{d}(b, F(x)).$$

Zălinescu [43], Zheng [44] gave positive answer to this conjecture by using different methods, respectively.

Following [18, 37], we say that a multifunction F from a normed space X to a normed space Y is called γ-paraconvex ($\gamma > 0$) if there is a constant $r > 0$ such that for all $x, u \in X$ and all $\lambda \in [0, 1]$,

$$\lambda F(x) + (1 - \lambda)F(u) \subset F(\lambda x + (1 - \lambda)u) + r \min\{\lambda, 1 - \lambda\}\|x - u\|^\gamma B_Y,$$

where B_Y denotes the closed unit ball of Y. Clearly, a convex multifunction is a γ-paraconvex multifunction. However, the converse is not true. See Example 2.1 in [18].

In this paper, we consider global error bounds for γ-paraconvex multifunctions when $F^{-1}(b)$ is a unbounded convex set and both X and Y are infinite dimensional normed spaces. First, we give a Robinson-Ursescu type theorem for γ-paraconvex multifunctions in normed space setting. Using these results, we establish several existence theorems of error bounds of mixed order 1 and γ for γ-paraconvex multifunctions, where error bounds of mixed order 1 and γ means that there exist $\tau > 0$ and $r > 0$ such that for each $x \in X$,

$$\mathrm{d}(x, F^{-1}(b)) \leq \tau \mathrm{d}(b, F(x)) + r[\mathrm{d}(b, F(x))]^\gamma;$$

in particular, we give a positive answer to Li and Singer's conjecture for γ-paraconvex multifunction. Finally, when the original system (1.1) undergoes small perturbation, we also study the existence of uniform error bounds of the system. Our results extend the results in [30, 43, 44] from convex multifunctions to γ-paraconvex multifunctions.

Throughout this paper, unless stated otherwise, we let X and Y be normed spaces, and $F : X \to 2^Y$ be a multifunction. The following notions are needed in this paper. As usual, $\mathrm{Dom}(F) := \{x \in X : F(x) \neq \emptyset\}$ denotes the domain of F. The multifunction F is said to have closed values if $F(x)$ is a closed subset of Y for each $x \in X$. For $A \subseteq X$, $\partial(A)$ denotes the boundary of A, $\mathrm{diam}(A)$ denotes the diameter of A, where

$$\mathrm{diam}(A) := \sup\{\|x - y\| : x, y \in A\}.$$

We also use $\mathrm{aff}(A)$ to denote the affine subspace generalized by A, that is,

$$\mathrm{aff}(A) = \{ta_1 + (1 - t)a_2 : a_1, a_2 \in A \text{ and } t \geq 0\}.$$

Let $B \subseteq Y$, we define $\mathrm{e}(A, B) := \sup_{x \in A} \mathrm{d}(x, B)$. Note that when A, B are nonempty sets we have that

$$\mathrm{d}(x, B) \leq \mathrm{d}(x, A) + \mathrm{e}(A, B), \quad \forall\ x \in X.$$

Let $b \in Y$ and $\eta > 0$, we use $B(b, \eta)$ to denote the closed ball with center b and radius η in Y.

2 A Robinson-Ursescu Type Theorem for γ-paraconvex Multifunctions

The following theorem gives a sufficient condition for existence of a local error bound for γ-paraconvex multifunctions, which plays an important role in our main results.

Theorem 2.1 *Let F have closed values. Let $y_0 \in F(X)$, $x_0 \in X$, $\eta > 0$ and $\delta > 0$. Suppose that F^{-1} is γ-paraconvex and that $B(y_0, \eta) \cap \mathit{aff}(F(X)) \subseteq F(x_0 + \delta B_X)$. Then*

$$d(x, F^{-1}(y_0)) \leq \frac{d(y_0, F(x))}{\eta}(\delta + r\eta^\gamma + \|x - x_0\|), \quad \forall\, x \in X, \tag{2.1}$$

where r is as in the definition of γ-paraconvexity for F^{-1}.

Proof Let $x \in X$. Without loss of generality, we may assume that $F(x) \neq \emptyset$ and $\mathrm{d}(y_0, F(x)) > 0$ (otherwise Eq. 2.1 holds trivially). We divide x into two cases: the case $\mathrm{d}(y_0, F(x)) < \eta$ and the case $\mathrm{d}(y_0, F(x)) \geq \eta$ to consider. For the case $\mathrm{d}(y_0, F(x)) < \eta$, we let $\varepsilon \in (0, \eta - \mathrm{d}(y_0, F(x)))$. Then there exists $z \in F(x)$ such that

$$\|y_0 - z\| < \varepsilon + \mathrm{d}(y_0, F(x)) < \eta.$$

Let $y := y_0 + (\eta - \|y_0 - z\|)\frac{y_0 - z}{\|y_0 - z\|}$, then $y \in B(y_0, \eta) \cap \mathrm{aff}(F(X))$. By the assumption, there exists $\bar{a} \in X$ with $\|\bar{a}\| \le 1$ such that $y \in F(x_0 + \delta\bar{a})$. By the γ-paraconvexity of F^{-1},

$$\begin{aligned}
&\frac{\|y_0 - z\|}{\eta}(x_0 + \delta\bar{a}) + \left(1 - \frac{\|y_0 - z\|}{\eta}\right)x \\
&\quad \in \frac{\|y_0 - z\|}{\eta}F^{-1}(y) + \left(1 - \frac{\|y_0 - z\|}{\eta}\right)F^{-1}(z) \\
&\quad \subseteq F^{-1}\left(\frac{\|y_0 - z\|}{\eta}y + \left(1 - \frac{\|y_0 - z\|}{\eta}\right)z\right) + r\frac{\|y_0 - z\|}{\eta}\|y - z\|^{\gamma}B_X.
\end{aligned}$$

Noting that $\frac{\|y_0 - z\|}{\eta}y + \left(1 - \frac{\|y_0 - z\|}{\eta}\right)z = y_0$, then there exists $u \in B_X$ such that

$$\frac{\|y_0 - z\|}{\eta}(x_0 + \delta\bar{a}) + \left(1 - \frac{\|y_0 - z\|}{\eta}\right)x - r\frac{\|y_0 - z\|}{\eta}\|y - z\|^{\gamma}u \in F^{-1}(y_0).$$

Therefore,

$$\begin{aligned}
&\mathrm{d}(x, F^{-1}(y_0)) \\
&\quad \le \left\|x - \frac{\|y_0 - z\|}{\eta}(x_0 + \delta\bar{a}) - \left(1 - \frac{\|y_0 - z\|}{\eta}\right)x + r\frac{\|y_0 - z\|}{\eta}\|y - z\|^{\gamma}u\right\| \\
&\quad \le \frac{\|y_0 - z\|}{\eta}(\|x - (x_0 + \delta\bar{a})\| + r\|y - z\|^{\gamma}) \\
&\quad \le \frac{\|y_0 - z\|}{\eta}(\delta + \|x - x_0\| + r(\|y - y_0\| + \|y_0 - z\|)^{\gamma}) \\
&\quad = \frac{\|y_0 - z\|}{\eta}(\delta + \|x - x_0\| + r(\eta - \|y_0 - z\| + \|y_0 - z\|)^{\gamma}) \\
&\quad = \frac{\|y_0 - z\|}{\eta}(\delta + \|x - x_0\| + r\eta^{\gamma}) \\
&\quad < \frac{\varepsilon + \mathrm{d}(y_0, F(x))}{\eta}(\delta + \|x - x_0\| + r\eta^{\gamma}).
\end{aligned}$$

Letting $\varepsilon \to 0^+$, we have

$$\mathrm{d}(x, F^{-1}(y_0)) \le \frac{\mathrm{d}(y_0, F(x))}{\eta}(\delta + r\eta^{\gamma} + \|x - x_0\|).$$

Now, we consider the case when $\mathrm{d}(y_0, F(x)) \ge \eta$. Since $y_0 \in B(y_0, \eta) \cap \mathrm{aff}(F(X)) \subseteq F(x_0 + \delta B_X)$, there exists $v \in X$ with $\|v\| \le 1$ such that $y_0 \in F(x_0 + \delta v)$. Therefore, $x_0 + \delta v \in F^{-1}(y_0)$, and

$$\mathrm{d}(x, F^{-1}(y_0)) \le \|x - (x_0 + \delta v)\| \le \|x - x_0\| + \delta.$$

Since $\frac{\mathrm{d}(y_0, F(x))}{\eta} \ge 1$, it follows that

$$\mathrm{d}(x, F^{-1}(y_0)) \le \frac{\mathrm{d}(y_0, F(x))}{\eta}(\|x - x_0\| + \delta) \le \frac{\mathrm{d}(y_0, F(x))}{\eta}(\delta + r\eta^{\gamma} + \|x - x_0\|).$$

□

Theorem 2.2 *Let F have closed values, $y_0 \in F(X)$, $x_0 \in X$, $\eta > 0$ and $\delta > 0$. Suppose that F^{-1} is γ-paraconvex and that $B(y_0, \eta) \cap aff(F(X)) \subseteq F(x_0 + \delta B_X)$. Let $\eta_1 > 0$, $\eta_2 > 0$ with $\eta_1 + \eta_2 = \eta$. Then for any $y \in B(y_0, \eta_1) \cap aff(F(X))$,*

$$d(x, F^{-1}(y)) \leq \frac{d(y, F(x))}{\eta_2}(\delta + r\eta_2^{\gamma} + \|x - x_0\|), \quad \forall\, x \in X, \tag{2.2}$$

where r is as in the definition of γ-paraconvexity for F^{-1}.

Proof Let $y \in B(y_0, \eta_1) \cap \mathrm{aff}(F(X))$. Then, by assumption,

$$B(y, \eta_2) \cap \mathrm{aff}(F(X)) \subseteq B(y_0, \eta) \cap \mathrm{aff}(F(X)) \subseteq F(x_0 + \delta B_X).$$

Therefore Eq. 2.2 holds by Theorem 2.1 (replaced y_0 by y). □

Remark 2.1 Theorem 2.1 and 2.2 are motivated by Robinson-Ursescu Theorem [36, 39] and [18, 25, 44]. In [18], under similar condition as Theorem 2.1, Huang conclude that

$$\mathrm{d}\big(x, F^{-1}(y_0)\big) \leq \frac{\mathrm{d}(y_0, F(x))}{\eta + \mathrm{d}(y_0, F(x))}(\delta + r(\eta + \mathrm{d}(y_0, F(x)))^{\gamma} + \|x - x_0\|). \tag{2.3}$$

Therefore, Eq. 2.1 can be viewed as a variation of Eq. 2.3. However, Eq. 2.1 can not be deduced from Eq. 2.3 directly.

3 Global Error Bounds for γ-paraconvex Multifunctions

Let X be a Banach space. Let $K \subset X$ be a closed convex set. Following [30, 44], we say that a subset A of K has the property (R) if $K = A + K^{\infty}$, where $K^{\infty} := \{h \in X : K + th \subseteq K \text{ for all } t > 0\}$, the recession cone of K. If $K = A + C$, where C is a convex cone, then A is a subset of K with property (R) since $C \subseteq K^{\infty}$.

The following lemma is cited from [30], which can be easily deduced from [30, Lemma 2.1 and the proof of Theorem 3.1]. For completeness, we give a sketch proof for it.

Lemma 3.1 *Let K be a nonempty closed convex subset of a Banach space X and let A be a subset of K with property (R). Let $\theta \in (0, 1)$. Then for each $x \in X \setminus K$ there exists $a \in A \cap \partial(K)$ such that*

$$d(\lambda x + (1 - \lambda)a, K) \geq \theta\lambda d(x, K), \quad \forall\, \lambda \in [0, +\infty).$$

Proof Let $x \in X \setminus K$. By [30, Lemma 2.1], there exist $a \in A$ and $c \in K^{\infty}$ such that $a + \lambda c \in \partial(K)$ and

$$\frac{x - a - c}{\|x - a - c\|} \in N_k^1(a + \lambda c, \theta)$$
$$:= \{h \in X : \|h\| = 1, \mathrm{d}(a + \lambda c + sh, K) \geq \theta s, \forall\, s \geq 0\}, \ \forall \lambda \in [0, +\infty).$$

It follows that

$$\mathrm{d}\left(a + \lambda c + s\frac{x - a - c}{\|x - a - c\|}, K\right) \geq \theta s, \quad \forall\, \lambda, s \in [0, +\infty).$$

Letting $s = \lambda\|x - a - c\|$, we have

$$d(a + \lambda(x - a), K) \geq \theta\lambda\|x - a - c\| \geq \theta\lambda d(x, K), \quad \forall\, \lambda \in [0, +\infty).$$

It remains to show that $a \in \partial(K)$. Indeed, if $a \notin \partial(K)$ then $a \in \mathrm{int}(K)$, whence $a + c \in \mathrm{int}K + K^\infty \subseteq \mathrm{int}K$, a contradiction. □

The following corollary is a direct consequence of Lemma 3.1, since K itself has the property (R) (applied to K in place of A).

Corollary 3.1 *Let K be a nonempty closed convex subset of a Banach space X. Let $\theta \in (0, 1)$. Then for each $x \in X \setminus K$ there exist $u \in \partial(K)$ such that*

$$d(\lambda x + (1 - \lambda)u, K) \geq \theta\lambda d(x, K), \quad \forall\, \lambda \in [0, +\infty).$$

For the remainder of this paper, we always assume that X is a Banach space and Y is a normed space, and $F : X \to 2^Y$ is a multifunction with closed values. we need the following blanket assumptions:

Assumption 1 $b \in F(X)$, F^{-1} is a γ-paraconvex multifunction, namely, for all y_1, $y_2 \in Y$, $\lambda \in (0, 1)$,

$$\lambda F^{-1}(y_1) + (1 - \lambda)F^{-1}(y_2) \subseteq F^{-1}(\lambda y_1 + (1 - \lambda)y_2) + r\min\{\lambda, 1 - \lambda\}\|y_1 - y_2\|^\gamma B_X,$$

where $r \geq 0$.

Assumption 2 $F^{-1}(b)$ is a closed set, and A is a subset of $F^{-1}(b)$ with property (R), that is,

$$F^{-1}(b) = A + (F^{-1}(b))^\infty.$$

From Assumption 1, we easily see that $F^{-1}(y)$ is a convex set for all $y \in Y$ by taking $y_1 = y_2 = y$, and for all $\lambda \in (0, 1)$,

$$\lambda F^{-1}(y_1) + (1 - \lambda)F^{-1}(y_2) \subseteq F^{-1}(\lambda y_1 + (1 - \lambda)y_2) + r\lambda\|y_1 - y_2\|^\gamma B_X,$$

$$\lambda F^{-1}(y_1) + (1 - \lambda)F^{-1}(y_2) \subseteq F^{-1}(\lambda y_1 + (1 - \lambda)y_2) + r(1 - \lambda)\|y_1 - y_2\|^\gamma B_X.$$

We first show that if F^{-1} is γ-paraconvex, then local error bounds for F imply global error bounds for F.

Theorem 3.1 *Let $\tau > 0$ and Assumption 1 and 2 hold. Suppose that for each $a \in A \cap \partial(F^{-1}(b))$ there exists $\delta_a > 0$ such that*

$$d(x, F^{-1}(b)) \leq \tau d(b, F(x)), \quad \forall\, x \in a + \delta_a B_X,$$

Then

$$d(x, F^{-1}(b)) \leq \tau d(b, F(x)) + r[d(b, F(x))]^\gamma, \quad \forall\, x \in X. \tag{3.1}$$

Proof Let $x \in X$. Without loss of generality, we may assume that $F(x) \neq \emptyset$ and $\mathrm{d}(x, F^{-1}(b)) > 0$. Let $\theta \in (0, 1)$. Noting that $F^{-1}(b)$ is a closed convex subset of X, by Lemma 3.1 there exists $a \in A \cap \partial(F^{-1}(b))$ such that

$$\mathrm{d}(\lambda x + (1-\lambda)a, F^{-1}(b)) \geq \theta\lambda \mathrm{d}(x, F^{-1}(b)), \quad \forall\, \lambda \in [0, +\infty). \tag{3.2}$$

Let $\varepsilon \in (0, 1)$ and take $y \in F(x)$ such that

$$\|b - y\| < \mathrm{d}(b, F(x)) + \varepsilon. \tag{3.3}$$

Clearly, $x \in F^{-1}(y)$. By Assumption 1, for all $\lambda \in (0, 1)$, we have

$$\lambda x + (1-\lambda)a \in \lambda F^{-1}(y) + (1-\lambda)F^{-1}(b) \subseteq F^{-1}(\lambda y + (1-\lambda)b) + r\lambda\|y - b\|^{\gamma} B_X.$$

Then there exists $u_\lambda \in B_X$ such that

$$\lambda x + (1-\lambda)a - r\lambda\|y - b\|^{\gamma} u_\lambda \in F^{-1}(\lambda y + (1-\lambda)b),$$

and so

$$\lambda y + (1-\lambda)b \in F(\lambda x + (1-\lambda)a - r\lambda\|y - b\|^{\gamma} u_\lambda).$$

Since $a \in A \cap \partial(F^{-1}(b))$, by the assumption, there exists $\delta_a > 0$ such that

$$\mathrm{d}(v, F^{-1}(b)) \leq \tau \mathrm{d}(b, F(v)), \quad \forall\, v \in a + \delta_a B_X.$$

It follows that there exists $\lambda_a \in (0, 1)$ such that for all $\lambda \in (0, \lambda_a)$,

$$\begin{aligned} &\mathrm{d}(\lambda x + (1-\lambda)a - r\lambda\|y - b\|^{\gamma} u_\lambda, F^{-1}(b)) \\ &\quad \leq \tau \mathrm{d}(b, F(\lambda x + (1-\lambda)a - r\lambda\|y - b\|^{\gamma} u_\lambda)) \\ &\quad \leq \tau\|b - (\lambda y + (1-\lambda)b)\| = \tau\lambda\|b - y\|. \end{aligned} \tag{3.4}$$

Since $\mathrm{d}(\cdot, F^{-1}(b))$ is 1-Lipschitz on X [9], we have

$$\begin{aligned} &\mathrm{d}(\lambda x + (1-\lambda)a, F^{-1}(b)) \\ &\quad \leq \mathrm{d}(\lambda x + (1-\lambda)a - r\lambda\|y - b\|^{\gamma} u_\lambda, F^{-1}(b)) \\ &\qquad + \|\lambda x + (1-\lambda)a - (\lambda x + (1-\lambda)a - r\lambda\|y - b\|^{\gamma} u_\lambda)\| \\ &\quad \leq \mathrm{d}(\lambda x + (1-\lambda)a - r\lambda\|y - b\|^{\gamma} u_\lambda, F^{-1}(b)) + r\lambda\|y - b\|^{\gamma}. \end{aligned} \tag{3.5}$$

It follows from Eqs. 3.1–3.7 that for all $\lambda \in (0, \lambda_a)$,

$$\lambda\theta \mathrm{d}(x, F^{-1}(b)) \leq \lambda\tau(\mathrm{d}(b, F(x)) + \varepsilon) + r\lambda(\mathrm{d}(b, F(x)) + \varepsilon)^{\gamma},$$

that is,

$$\theta \mathrm{d}(x, F^{-1}(b)) \leq \tau(\mathrm{d}(b, F(x)) + \varepsilon) + r(\mathrm{d}(b, F(x)) + \varepsilon)^{\gamma}.$$

Letting $\varepsilon \to 0^+$ and then letting $\theta \to 1^-$, Eq. 3.1 is seen to hold. □

Theorem 3.2 *Let* $\eta > 0$, $\delta > 0$ *and Assumption* 1 *and* 2 *hold. Suppose that*

$$B(b, \eta) \cap aff(F(X)) \subseteq F(a + \delta B_X), \quad \forall\, a \in A \cap \partial(F^{-1}(b)). \tag{3.6}$$

Then

$$d(x, F^{-1}(b)) \leq \frac{\delta + r\eta^{\gamma}}{\eta}\, d(b, F(x)) + r[d(b, F(x))]^{\gamma}, \quad \forall\, x \in X. \tag{3.7}$$

Proof In virtue of Eq. 3.6, Theorem 2.1 implies that for all $x \in X$,

$$\mathrm{d}(x, F^{-1}(b)) \leq \frac{\mathrm{d}(b, F(x))}{\eta}(\delta + r\eta^{\gamma} + \|x - a\|). \tag{3.8}$$

Let $\varepsilon > 0$ be given. It follows from Eq. 3.8 that for all $x \in X$ with $x \in a + \varepsilon B_X$, one has

$$\mathrm{d}(x, F^{-1}(b)) \leq \frac{\mathrm{d}(b, F(x))}{\eta}(\delta + r\eta^{\gamma} + \varepsilon).$$

By Theorem 3.1, for all $x \in X$, one has

$$\mathrm{d}(x, F^{-1}(b)) \leq \frac{\delta + r\eta^{\gamma} + \varepsilon}{\eta}\mathrm{d}(b, F(x)) + r[\mathrm{d}(b, F(x))]^{\gamma}.$$

Letting $\varepsilon \to 0^+$, Eq. 3.7 is seen to hold. □

In the special case when $\gamma = 1$, we do not require that η and δ in Eq. 3.6 are fixed.

Theorem 3.3 *Let $\tau > 0$ and Assumption 1 (for $\gamma = 1$) and 2 hold. Suppose that for each $a \in A \cap \partial(F^{-1}(b))$ there exists $\delta_a > 0$ such that*

$$B(b, \delta_a) \cap aff(F(X)) \subseteq F(a + \tau\delta_a B_X). \tag{3.9}$$

Then

$$d(x, F^{-1}(b)) \leq (\tau + 2r)d(b, F(x)), \quad \forall\, x \in X. \tag{3.10}$$

Proof In virtue of Eq. 3.9, Theorem 2.1 implies that for all $x \in X$,

$$\mathrm{d}(x, F^{-1}(b)) \leq \frac{\mathrm{d}(b, F(x))}{\delta_a}(\tau\delta_a + r\delta_a + \|x - a\|). \tag{3.11}$$

Let $\varepsilon > 0$ be given. It follows from Eq. 3.11 that for all $x \in X$ with $x \in a + \delta_a\varepsilon B_X$, one has

$$\mathrm{d}(x, F^{-1}(b)) \leq \frac{\mathrm{d}(b, F(x))}{\delta_a}(\tau\delta_a + r\delta_a + \delta_a\varepsilon) = (\tau + r + \varepsilon)\mathrm{d}(b, F(x)).$$

By Theorem 3.1, for all $x \in X$, one has

$$\mathrm{d}(x, F^{-1}(b)) \leq (\tau + r + \varepsilon)\mathrm{d}(b, F(x)) + r\mathrm{d}(b, F(x)).$$

Letting $\varepsilon \to 0^+$, Eq. 3.10 is seen to hold. □

Theorem 3.4 *Let $\eta > 0, \delta > 0$ and Assumption 1 and 2 hold. Suppose that A is bounded and there exists $x_0 \in X$ such that*

$$B(b, \eta) \cap aff(F(X)) \subseteq F(x_0 + \delta B_X). \tag{3.12}$$

Then

$$d(x, F^{-1}(b)) \leq \frac{\delta + r\eta^{\gamma} + diam(A) + d(x_0, A)}{\eta} d(b, F(x)) + r[d(b, F(x))]^{\gamma}, \quad \forall\, x \in X. \tag{3.13}$$

Proof In virtue of Eq. 3.12, Theorem 2.1 implies that for all $x \in X$,

$$\mathrm{d}(x, F^{-1}(b)) \leq \frac{\mathrm{d}(b, F(x))}{\eta}(\delta + r\eta^{\gamma} + \|x - x_0\|). \tag{3.14}$$

Let $\varepsilon > 0$ be given. Noting that A is bounded, then for each $a \in A \cap \partial(F^{-1}(b))$ and all $x \in X$ with $x \in a + \varepsilon B_X$, we have

$$\|x - x_0\| \leq \|x - a\| + \|a - v\| + \|v - x_0\|, \quad \forall\, u, v \in A,$$

and so,

$$\|x - x_0\| \leq \varepsilon + \mathrm{diam}(A) + \mathrm{d}(x_0, A).$$

It follows from Eq. 3.14 that

$$\mathrm{d}(x, F^{-1}(b)) \leq \frac{\mathrm{d}(b, F(x))}{\eta}(\delta + r\eta^{\gamma} + \varepsilon + \mathrm{diam}(A) + \mathrm{d}(x_0, A)), \quad \forall\, x \in a + \varepsilon B_X.$$

By Theorem 3.1, we have

$$\mathrm{d}(x, F^{-1}(b)) \leq \frac{\mathrm{d}(b, F(x))}{\eta}(\delta + r\eta^{\gamma} + \varepsilon + \mathrm{diam}(A) + \mathrm{d}(x_0, A)) + r[\mathrm{d}(b, F(x))]^{\gamma}, \quad \forall\, x \in X.$$

Letting $\varepsilon \to 0^+$, Eq. 3.13 is seen to hold. □

The following corollary gives a positive answer to Li and Singer's conjecture for γ-paraconvex multifunctions.

Corollary 3.2 *Let X and Y be Banach spaces, and let Assumption* 1 *and* 2 *hold. Suppose that A is bounded and $b \in int(F(X))$. Then there exists $\tau > 0$ such that*

$$d(x, F^{-1}(b)) \leq \tau d(b, F(x)) + r[d(b, F(x))]^{\gamma}, \quad \forall x \in X. \tag{3.15}$$

Proof Since X and Y are Banach spaces, and $b \in \mathrm{int}(F(X))$, by [25, Theorem 2.2 and Remark], there exists $x_0 \in X$, $\eta > 0$ and $\delta > 0$ such that

$$B(b, \eta) \subseteq F(x_0 + \delta B_X).$$

By Theorem 3.4, we have

$$\mathrm{d}(x, F^{-1}(b)) \leq \frac{\delta + r\eta^{\gamma} + \mathrm{diam}(A) + \mathrm{d}(x_0, A)}{\eta} \mathrm{d}(b, F(x)) + r[\mathrm{d}(b, F(x))]^{\gamma}, \quad \forall x \in X.$$

Letting $\tau := \dfrac{\delta + r\eta^\gamma + \mathrm{diam}(A) + \mathrm{d}(x_0, A)}{\eta}$, then Eq. 3.15 is seen to hold. □

It is well known that a convex function with a bounded sub-level set at a height greater than its infimum has all sub-level sets bounded. The following proposition generalizes this result to γ-paraconvex multifunctions.

Proposition 3.1 *Let Assumption* 1 *hold. Let* $\eta > 0$, $\delta > 0$ *and* $x_0 \in X$. *Suppose that* $B(b, \eta) \cap aff(F(X)) \subseteq F(x_0 + \delta B_X)$ *and* $F^{-1}(b)$ *is a closed bounded set. Then* F^{-1} *is bounded on bounded set.*

Proof Let $\rho > 0$. Let $y \in b + \rho B_Y$. If $F^{-1}(y) = \emptyset$, then $F^{-1}(y)$ is bounded. Now, we assume that $F^{-1}(y) \neq \emptyset$. Let $x \in F^{-1}(y)$. By Theorem 3.4 (applied to $F^{-1}(b)$ in place of A),

$$\begin{aligned}
&\mathrm{d}(x, F^{-1}(b)) \\
&\quad\le \frac{\delta + r\eta^\gamma + \mathrm{diam}(F^{-1}(b)) + \mathrm{d}(x_0, F^{-1}(b))}{\eta}\mathrm{d}(b, F(x)) + r[\mathrm{d}(b, F(x))]^\gamma \\
&\quad\le \frac{\delta + r\eta^\gamma + \mathrm{diam}(F^{-1}(b)) + \mathrm{d}(x_0, F^{-1}(b))}{\eta}\|b - y\| + r\|b - y\|^\gamma \\
&\quad\le \frac{\delta + r\eta^\gamma + \mathrm{diam}(F^{-1}(b)) + \mathrm{d}(x_0, F^{-1}(b))}{\eta}\rho + r\rho^\gamma .
\end{aligned} \tag{3.16}$$

Since for all $u, v \in F^{-1}(b)$,

$$\|x\| \le \|x - u\| + \|u - v\| + \|v\| \le \|x - u\| + \mathrm{diam}(F^{-1}(b)) + \|v\|,$$

we have

$$\|x\| \le \mathrm{d}(x, F^{-1}(b)) + \mathrm{diam}(F^{-1}(b)) + \mathrm{d}(0, F^{-1}(b)).$$

It follows from Eq. 3.16 that

$$\begin{aligned}
\|x\| \le\ & \frac{\delta + r\eta^\gamma + \mathrm{diam}(F^{-1}(b)) + \mathrm{d}(x_0, F^{-1}(b))}{\eta}\rho \\
& + r\rho^\gamma + \mathrm{diam}(F^{-1}(b)) + \mathrm{d}(0, F^{-1}(b)),
\end{aligned}$$

which implies that $F^{-1}(y)$ is bounded. □

Theorem 3.5 *Let* $\eta > 0$ *and* $\delta > 0$. *Let assumption* 1 *hold. Suppose that* $e(F^{-1}(b), F^{-1}(y)) \le \eta$ *for every* $y \in b + \delta B_Y$. *Then*

$$d(x, F^{-1}(b)) \le \frac{\eta + r2^\gamma\delta^\gamma}{\delta}\, d(b, F(x)) + \frac{r2^\gamma}{\delta}[d(b, F(x))]^{\gamma+1}, \quad \forall\, x \in X.$$

Proof We first note that our hypothesis implies that $b + \delta B_Y \subset F(X)$. Let $x \in X$. If $x \in F^{-1}(b)$ or $F(x) = \emptyset$ the conclusion is obvious. Now, suppose that neither of these situation holds. Let $\varepsilon > 0$ be given. Take $y \in F(x)$ such that

$$\|b - y\| < \mathrm{d}(b, F(x)) + \varepsilon.$$

Let

$$\bar{y} := \left(1 + \frac{\delta}{\|b - y\|}\right) b - \frac{\delta}{\|b - y\|} y.$$

Obviously, $\|\bar{y} - b\| = \delta$. Let $\bar{x} \in F^{-1}(\bar{y})$. It follows from the γ-paraconvexity of F^{-1} that

$$\begin{aligned}
&\frac{\|b - y\|}{\delta + \|b - y\|}\bar{x} + \frac{\delta}{\delta + \|b - y\|}x \\
&\quad \in \frac{\|b - y\|}{\delta + \|b - y\|}F^{-1}(\bar{y}) + \frac{\delta}{\delta + \|b - y\|}F^{-1}(y) \\
&\quad \subseteq F^{-1}\left(\frac{\|b - y\|}{\delta + \|b - y\|}\bar{y} + \frac{\delta}{\delta + \|b - y\|}y\right) + r\frac{\|b - y\|}{\delta + \|b - y\|}\|\bar{y} - y\|^{\gamma} B_X \\
&\quad \subseteq F^{-1}(b) + r\frac{\|b - y\|}{\delta + \|b - y\|}\|\bar{y} - y\|^{\gamma} B_X,
\end{aligned}$$

and hence, there exists $u \in B_X$ such that

$$\frac{\|b - y\|}{\delta + \|b - y\|}\bar{x} + \frac{\delta}{\delta + \|b - y\|}x - r\frac{\|b - y\|}{\delta + \|b - y\|}\|\bar{y} - y\|^{\gamma} u \in F^{-1}(b).$$

Therefore,

$$\begin{aligned}
&\mathrm{d}(x, F^{-1}(b)) \\
&\quad \leq \left\|x - \left(\frac{\|b - y\|}{\delta + \|b - y\|}\bar{x} + \frac{\delta}{\delta + \|b - y\|}x - r\frac{\|b - y\|}{\delta + \|b - y\|}\|\bar{y} - y\|^{\gamma} u\right)\right\| \\
&\quad \leq \frac{\|b - y\|}{\delta + \|b - y\|}\|\bar{x} - x\| + r\frac{\|b - y\|}{\delta + \|b - y\|}\|\bar{y} - y\|^{\gamma} \\
&\quad \leq \frac{\mathrm{d}(b, F(x)) + \varepsilon}{\delta + \mathrm{d}(b, F(x)) + \varepsilon}(\|\bar{x} - x\| + r\|\bar{y} - y\|^{\gamma}),
\end{aligned}$$

where the last inequality holds since $\varphi(t) = \frac{t}{\delta + t}$ is increasing on $[0, +\infty)$. Since

$$\begin{aligned}
\|\bar{y} - y\|^{\gamma} &= \left\|\left(1 + \frac{\delta}{\|b - y\|}\right) b - \frac{\delta}{\|b - y\|} y - y\right\|^{\gamma} = (\delta + \|b - y\|)^{\gamma} \\
&\leq (2\max\{\delta, \|b - y\|\})^{\gamma} = 2^{\gamma}\max\{\delta^{\gamma}, \|b - y\|^{\gamma}\} \leq 2^{\gamma}(\delta^{\gamma} + \|b - y\|^{\gamma}) \\
&\leq 2^{\gamma}\delta^{\gamma} + 2^{\gamma}(\mathrm{d}(b, F(x)) + \varepsilon)^{\gamma},
\end{aligned}$$

it follows that

$$\mathrm{d}(x, F^{-1}(b)) \leq \frac{\mathrm{d}(b, F(x)) + \varepsilon}{\delta + \mathrm{d}(b, F(x)) + \varepsilon}\left(\|\bar{x} - x\| + r2^{\gamma}\delta^{\gamma} + r2^{\gamma}(\mathrm{d}(b, F(x)) + \varepsilon)^{\gamma}\right).$$

Taking the infimum with respect to $\bar{x} \in F^{-1}(\bar{y})$, we obtain that

$$\begin{aligned}
&\mathrm{d}(x, F^{-1}(b)) \\
&\quad \le \frac{\mathrm{d}(b, F(x)) + \varepsilon}{\delta + \mathrm{d}(b, F(x)) + \varepsilon} \left(\mathrm{d}(x, F^{-1}(\bar{y})) + r2^{\gamma}\delta^{\gamma} + r2^{\gamma}(\mathrm{d}(b, F(x)) + \varepsilon)^{\gamma}\right) \\
&\quad \le \frac{\mathrm{d}(b, F(x)) + \varepsilon}{\delta + \mathrm{d}(b, F(x)) + \varepsilon} \\
&\quad \times \left(\mathrm{d}(x, F^{-1}(b)) + \mathrm{e}(F^{-1}(b), F^{-1}(\bar{y})) + r2^{\gamma}\delta^{\gamma} + r2^{\gamma}(\mathrm{d}(b, F(x)) + \varepsilon)^{\gamma}\right) \\
&\quad \le \frac{\mathrm{d}(b, F(x)) + \varepsilon}{\delta + \mathrm{d}(b, F(x)) + \varepsilon} \left(\mathrm{d}(x, F^{-1}(b)) + \eta + r2^{\gamma}\delta^{\gamma} + r2^{\gamma}(\mathrm{d}(b, F(x)) + \varepsilon)^{\gamma}\right).
\end{aligned}$$

Letting $\varepsilon \to 0^+$, we have

$$\mathrm{d}(x, F^{-1}(b)) \le \frac{\mathrm{d}(b, F(x))}{\delta + \mathrm{d}(b, F(x))} \left(\mathrm{d}(x, F^{-1}(b)) + \eta + r2^{\gamma}\delta^{\gamma} + r2^{\gamma}(\mathrm{d}(b, F(x)))^{\gamma}\right),$$

that is,

$$\mathrm{d}(x, F^{-1}(b)) \le \frac{\eta + r2^{\gamma}\delta^{\gamma}}{\delta} \mathrm{d}(b, F(x)) + \frac{r2^{\gamma}}{\delta}[\mathrm{d}(b, F(x))]^{\gamma+1}.$$

□

4 Perturbation Analysis of Error Bounds

In this section, we provide sufficient conditions to ensure the existence of uniform error bounds when original system (1.1) undergoes small perturbation.

Lemma 4.1 *Let Assumption* 1 *and* 2 *hold. Let* $\eta > 0$ *and* $\delta > 0$. *Suppose that*

$$B(b, \eta) \subseteq F(a + \delta B_X), \quad \forall\, a \in A \cap \partial(F^{-1}(b)). \tag{4.1}$$

Then for each $\varepsilon \in (0, 1)$,

$$B(b, (1-\varepsilon)\eta) \subseteq F(z + (1+\varepsilon)\delta B_X), \quad \forall\, z \in \partial(F^{-1}(b)).$$

Proof Let $z \in \partial(F^{-1}(b))$ and choose $a \in A$ and $c \in (F^{-1}(b))^{\infty}$ such that $z = a + c$. It is easy to see that $a \in \partial(F^{-1}(b))$, and so $a \in A \cap \partial(F^{-1}(b))$. Let $y \in B(b, \eta)$. By Eq. 4.1, there exists $u \in B_X$ such that $y \in F(a + \delta u)$, and so $a + \delta u \in F^{-1}(y)$. Let $\varepsilon \in (0, 1)$ be given and choose $\lambda \in (0, +\infty)$ such that

$$\frac{\lambda}{1+\lambda} > 1 - \varepsilon \quad \text{and} \quad \frac{\lambda}{1+\lambda}\delta + \frac{r\eta^{\gamma}}{1+\lambda} < \delta(1+\varepsilon). \tag{4.2}$$

Since $a + (1+\lambda)c \in F^{-1}(b)$, it follows from the γ-paraconvexity of F^{-1} that

$$\begin{aligned} z + \frac{\lambda}{1+\lambda}\delta u &= \frac{\lambda}{1+\lambda}(a+\delta u) + \frac{1}{1+\lambda}(a+(1+\lambda)c) \\ &\in \frac{\lambda}{1+\lambda}F^{-1}(y) + \frac{1}{1+\lambda}F^{-1}(b) \\ &\subseteq F^{-1}\left(\frac{\lambda}{1+\lambda}y + \frac{1}{1+\lambda}b\right) + r\frac{1}{1+\lambda}\|b-y\|^{\gamma}B_X \\ &\subseteq F^{-1}\left(\frac{\lambda}{1+\lambda}y + \frac{1}{1+\lambda}b\right) + r\frac{1}{1+\lambda}\eta^{\gamma}B_X. \end{aligned}$$

Then there exists $v \in B_X$ such that

$$z + \frac{\lambda}{1+\lambda}\delta u - r\frac{1}{1+\lambda}\eta^{\gamma}v \in F^{-1}\left(\frac{\lambda}{1+\lambda}y + \frac{1}{1+\lambda}b\right),$$

and so

$$\begin{aligned} \frac{\lambda}{1+\lambda}y + \frac{1}{1+\lambda}b &\in F\left(z + \frac{\lambda}{1+\lambda}\delta u - r\frac{1}{1+\lambda}\eta^{\gamma}v\right) \\ &\subseteq F\left(z + \left(\frac{\lambda}{1+\lambda}\delta + r\frac{1}{1+\lambda}\eta^{\gamma}\right)B_X\right). \end{aligned}$$

Since $y \in B(b, \eta)$ is arbitrary, we obtain that

$$B\left(b, \frac{\lambda}{1+\lambda}\eta\right) = \frac{\lambda}{1+\lambda}B(b,\eta) + \frac{1}{1+\lambda}b \subseteq F\left(z + \left(\frac{\lambda}{1+\lambda}\delta + r\frac{1}{1+\lambda}\eta^{\gamma}\right)B_X\right).$$

This and Eq. 4.2 imply that

$$B(b, (1-\varepsilon)\eta) \subseteq F(z + (1+\varepsilon)\delta B_X).$$

□

Theorem 4.1 *Let Assumption* 1 *and* 2 *hold. Let* F^{-1} *have closed values. Let* $\eta > 0$ *and* $\delta > 0$. *Suppose that*

$$B(b, \eta) \subseteq F(a + \delta B_X), \quad \forall\, a \in A \cap \partial(F^{-1}(b)). \tag{4.3}$$

Then for each $b' \in Y$ *with* $\|b' - b\| < \eta$,

$$\begin{aligned} &d(x, F^{-1}(b')) \\ &\le \frac{2\delta + 2r\eta^{\gamma} + r(\eta - \|b'-b\|)^{\gamma}}{\eta - \|b'-b\|}d(b', F(x)) + r[d(b', F(x))]^{\gamma}, \quad \forall\, x \in X. \end{aligned} \tag{4.4}$$

In particular, for each $b' \in b + \frac{\eta}{2}B_Y$,

$$d(x, F^{-1}(b')) \le \frac{4\delta + 6r\eta^{\gamma}}{\eta}\, d(b', F(x)) + r[d(b', F(x))]^{\gamma}, \quad \forall\, x \in X. \tag{4.5}$$

Proof We need only prove Eq. 4.4 as Eq. 4.5 follows from Eq. 4.4 directly. Let $b' \in Y$ with $\|b' - b\| < \eta$. Let $x \in X$. Without loss of generality, we may assume that $x \notin$

$F^{-1}(b')$ and $F(x) \neq \emptyset$. Let $\varepsilon \in (0, 1 - \frac{\|b'-b\|}{\eta})$. Noting that $F^{-1}(b')$ is a closed convex set, by Corollary 3.1, there exists $z \in \partial(F^{-1}(b'))$ such that for each $\lambda \in (0, 1)$,

$$\lambda(1-\varepsilon)\mathrm{d}(x, F^{-1}(b')) \leq \mathrm{d}(\lambda x + (1-\lambda)z, F^{-1}(b')). \tag{4.6}$$

By Eq. 4.3 and Theorem 3.2, we have

$$\begin{aligned}\mathrm{d}(z, F^{-1}(b)) &\leq \frac{\delta + r\eta^{\gamma}}{\eta}\mathrm{d}(b, F(z)) + r[\mathrm{d}(b, F(z))]^{\gamma}\\ &\leq \frac{\delta + r\eta^{\gamma}}{\eta}\|b - b'\| + r\|b - b'\|^{\gamma}\\ &< (\delta + r\eta^{\gamma}) + r\eta^{\gamma} = \delta + 2r\eta^{\gamma}.\end{aligned}$$

Then there exists $v \in \partial(F^{-1}(b))$ such that

$$\|z - v\| < \delta + 2r\eta^{\gamma}.$$

For the above ε, by Lemma 4.1 and Eq. 4.3, we have

$$\begin{aligned}B(b, (1-\varepsilon)\eta) &\subseteq F(v + (1+\varepsilon)\delta B_X) = F(z + (v - z) + (1+\varepsilon)\delta B_X)\\ &\subseteq F(z + (\delta + 2r\eta^{\gamma} + (1+\varepsilon)\delta)B_X).\end{aligned}$$

Let $\beta_\varepsilon := \delta + 2r\eta^{\gamma} + (1+\varepsilon)\delta$, then

$$B(b', (1-\varepsilon)\eta - \|b - b'\|) \subseteq B(b, (1-\varepsilon)\eta) \subseteq F(z + \beta_\varepsilon B_X).$$

It follows from Theorem 2.1 that for each $w \in X$,

$$\mathrm{d}(w, F^{-1}(b')) \leq \frac{\mathrm{d}(b', F(w))}{(1-\varepsilon)\eta - \|b - b'\|}(\beta_\varepsilon + r((1-\varepsilon)\eta - \|b - b'\|)^{\gamma} + \|w - z\|). \tag{4.7}$$

Take $y \in F(x)$ such that

$$\|b' - y\| < \mathrm{d}(b', F(x)) + \varepsilon. \tag{4.8}$$

Clearly, $x \in F^{-1}(y)$. By Assumption 1, for all $\lambda \in (0, 1)$, we have

$$\begin{aligned}\lambda x + (1-\lambda)z &\in \lambda F^{-1}(y) + (1-\lambda)F^{-1}(b')\\ &\subseteq F^{-1}(\lambda y + (1-\lambda)b') + r\lambda\|y - b'\|^{\gamma}B_X.\end{aligned}$$

Then there exists $u_\lambda \in B_X$ such that

$$\lambda x + (1-\lambda)z - r\lambda\|y - b'\|^{\gamma}u_\lambda \in F^{-1}(\lambda y + (1-\lambda)b'),$$

and so

$$\lambda y + (1-\lambda)b' \in F(\lambda x + (1-\lambda)z - r\lambda\|y - b'\|^{\gamma}u_\lambda).$$

Replacing w in Eq. 4.7 by $\lambda x + (1-\lambda)z - r\lambda\|y-b'\|^{\gamma}u_{\lambda}$, we have

$$\begin{aligned}
&\mathrm{d}(\lambda x + (1-\lambda)z - r\lambda\|y-b'\|^{\gamma}u_{\lambda}, F^{-1}(b'))\\
&\quad\le \frac{\mathrm{d}(b', F(\lambda x + (1-\lambda)z - r\lambda\|y-b'\|^{\gamma}u_{\lambda}))}{(1-\varepsilon)\eta - \|b-b'\|}\\
&\quad\times\left[\beta_{\varepsilon} + r((1-\varepsilon)\eta - \|b'-b\|)^{\gamma} + \|\lambda x + (1-\lambda)z - r\lambda\|y-b'\|^{\gamma}u_{\lambda} - z\|\right]\\
&\quad\le \frac{\|b' - (\lambda y + (1-\lambda)b')\|}{(1-\varepsilon)\eta - \|b-b'\|}\\
&\quad\times\left[\beta_{\varepsilon} + r((1-\varepsilon)\eta - \|b'-b\|)^{\gamma} + \lambda(\|x-z\| + r\|y-b'\|^{\gamma})\right]\\
&\quad= \frac{\lambda\|b'-y\|}{(1-\varepsilon)\eta - \|b-b'\|}\left[\beta_{\varepsilon} + r((1-\varepsilon)\eta - \|b'-b\|)^{\gamma} + \lambda(\|x-z\| + r\|y-b'\|^{\gamma})\right].
\end{aligned} \tag{4.9}$$

Since $\mathrm{d}(\cdot, F^{-1}(b))$ is 1-Lipschitz on X [9], we have

$$\begin{aligned}
&\mathrm{d}(\lambda x + (1-\lambda)z, F^{-1}(b'))\\
&\quad\le \mathrm{d}(\lambda x + (1-\lambda)z - r\lambda\|y-b'\|^{\gamma}u_{\lambda}, F^{-1}(b'))\\
&\quad+ \|\lambda x + (1-\lambda)z - (\lambda x + (1-\lambda)z - r\lambda\|y-b'\|^{\gamma}u_{\lambda})\|\\
&\quad\le \mathrm{d}(\lambda x + (1-\lambda)z - r\lambda\|y-b'\|^{\gamma}u_{\lambda}, F^{-1}(b')) + r\lambda\|y-b'\|^{\gamma}.
\end{aligned} \tag{4.10}$$

It follows from Eqs. 4.6, 4.8, 4.9 and 4.10 that for all $\lambda \in (0, 1)$,

$$\begin{aligned}
\lambda(1-\varepsilon)\mathrm{d}(x, F^{-1}(b')) &\le \frac{\lambda(\mathrm{d}(b', F(x)) + \varepsilon)}{(1-\varepsilon)\eta - \|b-b'\|}\\
&\times\left[\beta_{\varepsilon} + r((1-\varepsilon)\eta - \|b'-b\|)^{\gamma} + \lambda(\|x-z\| + r(\mathrm{d}(b', F(x)) + \varepsilon)^{\gamma})\right]\\
&+ r\lambda(\mathrm{d}(b', F(x)) + \varepsilon)^{\gamma}.
\end{aligned}$$

Dividing the above inequality by λ and then letting $\lambda \to 0^{+}$, we have

$$\begin{aligned}
&(1-\varepsilon)\mathrm{d}(x, F^{-1}(b'))\\
&\quad\le \frac{\mathrm{d}(b', F(x)) + \varepsilon}{(1-\varepsilon)\eta - \|b-b'\|}\times\left[\beta_{\varepsilon} + r((1-\varepsilon)\eta - \|b'-b\|)^{\gamma}\right] + r(\mathrm{d}(b', F(x)) + \varepsilon)^{\gamma}.
\end{aligned}$$

Letting $\varepsilon \to 0^{+}$, Eq. 4.4 is seen to hold. □

The following proposition proves that the converse of Theorem 4.1 is true.

Proposition 4.1 *Let $\tau > 0$, $\eta > 0$ and $\gamma > 0$. Suppose that for any $b' \in b + \eta B_Y$,*

$$d(x, F^{-1}(b')) \le \tau d(b', F(x)) + \tau[d(b', F(x))]^{\gamma}, \quad \forall\, x \in X.$$

Then for each $\varepsilon \in (0, 1)$,

$$b + \eta B_Y \subseteq F(a + (1+\varepsilon)\tau(\eta + \eta^{\gamma})B_X), \quad \forall\, a \in F^{-1}(b).$$

Proof Let $\varepsilon \in (0, 1)$ be given. Let $a \in F^{-1}(b)$. Then $b \in F(a)$. Let $b' \in b + \eta B_Y$, by assumption, we have

$$\mathrm{d}(a, F^{-1}(b')) \leq \tau \mathrm{d}(b', F(a)) + \tau[\mathrm{d}(b', F(a))]^{\gamma} \leq \tau(\|b'-b\| + \|b'-b\|^{\gamma}) \leq \tau(\eta+\eta^{\gamma}),$$

and so there exists $x \in F^{-1}(b')$ such that

$$\|a - x\| < (1+\varepsilon)\tau(\eta+\eta^{\gamma}).$$

This implies that

$$b' \in F(a + (1+\varepsilon)\tau(\eta+\eta^{\gamma})B_X),$$

and hence

$$b + \eta B_Y \subseteq F(a + (1+\varepsilon)\tau(\eta+\eta^{\gamma})B_X).$$

□

Corollary 4.1 *Let Assumption* 1 *and* 2 *hold. Let F^{-1} have closed values. Let $\eta > 0$, $\delta > 0$ and $x_0 \in X$. Suppose that A is bounded and that*

$$B(b, \eta) \subseteq F(x_0 + \delta B_X). \tag{4.11}$$

Then for each $b' \in Y$ with $\|b' - b\| < \eta$,

$$d(x, F^{-1}(b')) \leq \frac{2d(x_0, A) + 2diam(A) + 2\delta + 2r\eta^{\gamma} + r(\eta - \|b' - b\|)^{\gamma}}{\eta - \|b - b'\|} d(b', F(x)) + r[d(b', F(x))]^{\gamma}, \quad \forall x \in X. \tag{4.12}$$

In particular, for each $b' \in b + \frac{\eta}{2} B_Y$,

$$d(x, F^{-1}(b')) \leq \frac{4d(x_0, A) + 4diam(A) + 4\delta + 6r\eta^{\gamma}}{\eta} d(b', F(x)) + r[d(b', F(x))]^{\gamma}, \quad \forall x \in X. \tag{4.13}$$

Proof Let $a \in A \cap \partial(F^{-1}(b))$. Then for all $u \in A$,

$$\|x_0 - a\| \leq \|x_0 - u\| + \|u - a\| \leq \|x_0 - u\| + \mathrm{diam}(A),$$

and so,

$$\|x_0 - a\| \leq \mathrm{d}(x_0, A) + \mathrm{diam}(A).$$

It follows that

$$x_0 \in a + (\mathrm{d}(x_0, A) + \mathrm{diam}(A))B_X.$$

Combined with Eq. 4.11, we have

$$B(b, \eta) \subseteq F(a + (\mathrm{d}(x_0, A) + \mathrm{diam}(A) + \delta)B_X), \quad \forall a \in A \cap \partial(F^{-1}(b)).$$

Equations 4.12 and 4.13 now follows from Theorem 4.1. □

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