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Codifferential Calculus

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Abstract In this paper, some exact calculus rules are obtained for calculating the coderivatives of the composition of two multivalued maps. Similar rules are displayed for sums. A crucial role is played by an intermediate set-valued map called the resolvent. We first establish inclusions for contingent, Fréchet and limiting coderivatives. Combining them, we get equality rules. The qualification conditions we present are natural and less exacting than classical conditions.

Keywords Allied sets · Coderivative · Lower semicontinuity · Normal cone · Openness with a linear rate · Synergetic sets

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1 Introduction

Set-valued mappings (or multivalued maps, multimaps, in short) are objects of fundamental importance, especially for optimization. Feasible sets of parametrized opti-

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mization problems, sets of solutions of such problems are instances of such multimaps [14, 19, 30, 33, 34, 36, 43, 65]. It has been well recognized that differential inclusions, which are certainly of independent interest, play a key role in optimal control theory ([1–3, 7–10, 13, 15, 20, 21, 37, 40, 70, 73]...). Many sophisticated control systems can be associated with differential inclusions, such as closed loop control systems, implicit control systems, systems with uncertainties and so on. Moreover, maximum principles are fundamental results of optimal control theories in which the adjoint inclusions are the basis of recent maximum principles [41, 50]. Naturally, funnels of solutions to differential inclusions and the adjoint inclusions are set-valued [42, 51]. Thus, many control systems are related to set-valued maps.

The analysis of sets and the analysis of functions can be incorporated in the analysis of multimaps and, conversely, the latter benefits from the study of the former objects. These passages explain the importance of codifferential analysis. The study of coderivatives of multimaps gives a rich collection of results about the behavior of the involved multimaps (see [41, 54, 71] for instance). However, coderivatives are not as intuitive as set-valued (graphical) derivatives. The reason stems from the dualization process which is a kind of reflection. Such a process reverses the directions of the maps, for instance, and, as in convex analysis, the dual of a property has features different from the ones of the primal property.

As with subdifferentials of functions (which also involve a form of dualization), the main advantage of coderivatives is represented by their calculus rules. A number of results have been obtained in the literature about such calculus rules ([22, 24, 28, 34, 41, 44–49, 71]...). In [44] subdifferential calculus is related to codifferential calculus. In [26] approximate (or fuzzy) rules are displayed. In [62] compactness conditions are introduced in order to provide openness criteria; see also [27] and [44]. Recent studies about error bounds also provide keys for calculus rules (see [25, 52, 53, 68, 72, 75–77]...). Applications to sensitivity analysis, variational inequalities, equilibrium problems are numerous ([4, 11, 12, 16–18, 29, 31–33, 63]...).

Some calculus rules for coderivatives are obvious. This is the case in particular for the coderivative of the product $F_1 \times F_2 : X_1 \times X_2 \rightrightarrows Y_1 \times Y_2$ of two multimaps $F_1 : X_1 \rightrightarrows Y_1$, $F_2 : X_2 \rightrightarrows Y_2$, as the graph of $F_1 \times F_2$ is isometric to the product of the graphs of F_1 and F_2. In some other cases, estimates are not so obvious. This is the case for the multimap $(F_1, F_2) : X \rightrightarrows Y_1 \times Y_2$ defined by $(F_1, F_2)(x) := F_1(x) \times F_2(x)$ when $X_1 := X_2 := X$. We also consider the case of composition and sum. For subdifferential calculus for functions, one may equivalently start from composition or sums; here we adopt the viewpoint that composition is an appropriate starting point for codifferential calculus. Since in [41] the starting step is a sum rule, we get that a similar situation prevails for coderivatives. In fact, composition involves two operations from the calculus of normal cones: projection and intersection. Both are rather simple and we deal with them from the beginning. As in [26] and [64] some metric estimates can ensure the expected inclusions for coderivatives. However, we aim at assumptions formulated in terms of coderivatives, as in [41]. The qualification conditions we present are slightly more general than the conditions in [41]; we show that through some examples and we make a thorough study of the relationships between various qualification conditions. We also stress the importance of giving symmetric qualification conditions. Such a viewpoint can be illustrated by the fact that a team may win because one of its member is very strong; but it may also win because the team is well coordinated and the weaknesses of some of its members is compensated by the quality of other members.

Although such a study is simpler when limited to finite dimensional spaces, we adopt an infinite dimensional framework in order to make the results available for some applications and for an easier comparison with previous results in [5, 26, 41, 45, 48, 49, 64]. It appears that some results remain valid without change in the infinite dimensional dimensional case; for some others one has to use compactness or metric qualification conditions. We strive to deal with elementary coderivatives as they inevitably play a role in the proofs, so that we consider it is useful to put them in full light, especially for some inclusions. Then, with some regularity assumptions on the given maps, we get regularity results for compositions or sums.

Applications to parametrized optimization problems and second-order analysis will be presented elsewhere.

2 Preliminaries

In the sequel, for the sake of simplicity, we identify a multimap F with its graph gph$F$. Such an identification cannot cause ambiguities because we do not use any operation on graphs but intersections. If $F : X \rightrightarrows Y$ is a multimap, F^{-1} denotes the multimap from Y to X given by $F^{-1}(y) := \{x \in X : y \in F(x)\}$. Thus its graph is given by $F^{-1} := \{(y, x) : (x, y) \in F\}$.

The closed unit ball of a normed vector space X is denoted by B_X. The notation $\overset{*}{\to}$ stands for weak* convergence in a dual space, while $(x_n) \overset{S}{\to} x$ means that the sequence (x_n) is contained in the subset S and converges to x.

Let us recall some classical concepts in order to fix the terminology. We start with openness properties.

Definition 1 A multimap $F : X \rightrightarrows Y$ between two metric spaces is said to be open at some $(\overline{x}, \overline{y}) \in F$ if for every neighborhood U of $\overline{x}$ the set $F(U)$ is a neighborhood of $\overline{y}$.

It is said to be open with a linear rate at some $(\overline{x}, \overline{y}) \in F$ if there exists some $c > 0$ such that $B(\overline{y}, r) \subset F(B(\overline{x}, cr))$ for $r > 0$ small enough.

It is said to be open around some $(\overline{x}, \overline{y}) \in F$ with a linear rate if there exist some $c > 0$ and some neighborhoods U of $\overline{x}$, V of $\overline{y}$ such that $B(y, r) \subset F(B(x, cr))$ for all $x \in U$, $y \in F(x) \cap V$ and $r > 0$ small enough.

Given subsets $C \subset X$, $D \subset Y$, a map $p : X \to Y$ is said to be open around some $\overline{x} \in C$ from C to D with a linear rate if the restriction $F : C \rightrightarrows D$ given by $F(x) := \{p(x)\}$ for $x \in C \cap p^{-1}(D)$, $F(x) := \varnothing$ otherwise is open around $(\overline{x}, p(\overline{x}))$ with a linear rate.

Now let us turn to some continuity/compactness properties.

Definition 2 [57] A multimap $G : Y \rightrightarrows Z$ between two metric spaces is said to be lower (or inner) semicontinuous (lsc) at $(\overline{y}, B)$ (on $E \subset Y$), where B is some subset of Z if for every sequence (y_n) (of E) converging to $\overline{y}$ there exist some $\overline{z} \in B$ and a sequence $(z_n) \to \overline{z}$ such that $z_n \in G(y_n)$ for n in an infinite subset N of $\mathbb{N}$.

Three special cases are of interest in the preceding definition: the case B is a singleton $\{\overline{z}\}$, the case $B = G(\overline{y})$ and the case $B := Z$. In the later case, the property has been renamed *semi-compactness of* G *at* $\overline{y}$ in [41] and elsewhere. It is of interest to combine such a property with closedness at $\overline{y}$: G is said to be *closed at* $\overline{y}$ if for any sequence $((y_n, z_n)) \overset{G}{\to} (\overline{y}, \overline{z})$ one has $(\overline{y}, \overline{z}) \in G$.

Let us note that when B is the singleton $\{\overline{z}\}$, the given definition of lower semicontinuity at $(\overline{y}, B)$ coincides with the classical one since then, for every sequence $(y_n) \to \overline{y}$ one has $(d(\overline{z}, G(y_n))) \to 0$. Let us also note that the preceding definition is related to a variant of the classical notion of *properness* [6, 55] described in the next definition.

Definition 3 A multimap $F : Z \rightrightarrows Y$ between two metric spaces is said to be proper at $(B, \overline{y})$ (with respect to $E \subset Y$), where B is some subset of Z if for every sequence (y_n) (of E) converging to $\overline{y}$ there exist some $\overline{z} \in B$ and a sequence $(z_n) \to \overline{z}$ such that $y_n \in F(z_n)$ for n in an infinite subset N of $\mathbb{N}$.

Clearly, G is lsc at $(\overline{y}, B)$ (on $E \subset Y$) if, and only if $F := G^{-1}$ is proper at $(B, \overline{y})$ (with respect to $E \subset Y$). In the classical literature, the multimap F is just a map.

Quantitative notions can be added to the preceding qualitative property.

Definition 4 A multimap $G : Y \rightrightarrows Z$ between two metric spaces is said to be lower semicontinuous at $(\overline{y}, \overline{z}) \in G$ (on $C \subset Y$) with a linear rate if there exist some $c > 0$ and some neighborhood V of $\overline{y}$ such that for all $y \in V(\cap C)$ one can find some $z \in G(y)$ satisfying $d(z, \overline{z}) \leq cd(y, \overline{y})$.

We observe that when $Z := X$, $\overline{z} := \overline{x}$ and $G := F^{-1}$, G is lsc at $(\overline{y}, \overline{z}) \in G$ with a linear rate if, and only if, F is open at $(\overline{x}, \overline{y})$ with linear rate (see [59]). A stronger property has been called Lipschitz-like property or Aubin property.

Definition 5 A multimap $G : Y \rightrightarrows Z$ between two metric spaces is said to be Lipschitz-like around $(\overline{y}, \overline{z})$ if there exist $c > 0$ and neighborhoods V of $\overline{y}$, W of $\overline{z}$ such that $d(z, G(y')) \leq cd(y, y')$ for all $y, y' \in V$, $z \in G(y) \cap W$.

We also observe that when $Z := X$, $\overline{z} := \overline{x}$ and $G := F^{-1}$, G is Lipschitz-like around $(\overline{y}, \overline{z}) \in G$, if and only if, F is open around $(\overline{x}, \overline{y})$ with a linear rate, if, and only if, F is metrically regular around $(\overline{x}, \overline{y}) \in F$ (see [59]).

Now let us turn to notions of infinitesimal analysis. In the sequel we say that a function $r : X \to \mathbb{R}_\infty := \mathbb{R} \cup \{+\infty\}$ on a normed vector space X is a *remainder* if $\lim_{\|x\| \to 0} r(x)/\|x\| = 0$.

Definition 6 The contingent cone (or just tangent cone) $T(S, \overline{x})$ to a subset S of a normed vector space X at $\overline{x} \in S$ is the set of limits of sequences $(t_n^{-1}(x_n - \overline{x}))$, where $(t_n) \to 0_+$, $x_n \in S$ for all n. The normal cone $N(S, \overline{x})$ to S at $\overline{x}$ is the polar cone of $T(S, \overline{x})$.

Given $\varepsilon \in \mathbb{R}_+$, the firm or Fréchet ε-normal cone to S at $\overline{x}$ is the set $N_F^{\varepsilon}(S, \overline{x})$ of $x^* \in X^*$ such that there exists a remainder r satisfying $\langle x^*, x - \overline{x} \rangle \leq r(x - \overline{x}) + \varepsilon \|x - \overline{x}\|$ for $x \in S$. For $\varepsilon = 0$, one sets $N_F(S, \overline{x}) := N_F^0(S, \overline{x})$.

The limiting normal cone to S at $\overline{x}$ is the set $N_L(S, \overline{x})$ of weak* limits of sequences (x_n^*) such that $x_n^* \in N_F^{\varepsilon_n}(S, x_n)$ for some sequences $(\varepsilon_n) \to 0_+$, $(x_n) \xrightarrow{S} \overline{x}$.

Let us recall the concepts of coderivative which form the core of our study.

Definition 7 The contingent coderivative $D^* F(\overline{x}, \overline{y})$ of a multimap $F : X \rightrightarrows Y$ between two normed vector spaces at $(\overline{x}, \overline{y}) \in F$ is the multimap $D^* F(\overline{x}, \overline{y}) : Y^* \rightrightarrows X^*$ whose graph is $\{(y^*, x^*) : (x^*, -y^*) \in N(F, (\overline{x}, \overline{y}))\}$.

The Fréchet coderivative $D_F^* F(\overline{x}, \overline{y})$ and the limiting (or normal) coderivative $D_L^* F(\overline{x}, \overline{y})$ are obtained similarly by replacing $N(F, (\overline{x}, \overline{y}))$ with $N_F(F, (\overline{x}, \overline{y}))$ and $N_L(F, (\overline{x}, \overline{y}))$ respectively.

It is well known that when X is an Asplund space one can replace $N_F^{\varepsilon_n}(S, x_n)$ with $N_F(S, x_n)$ in the definition of $N_L(S, \overline{x})$ (see [41, 44]). The concept of coderivative has been introduced by Mordukhovich in [39]. In that paper he mainly used the limiting normal cone introduced earlier in [38]. We refer to [41] and [69] for historical information about the concepts of limiting normal cones and coderivatives.

Definition 8 [41, Definition 1.32] The mixed coderivative $D_M^* F(\overline{x}, \overline{y})$ of a multimap $F : X \rightrightarrows Y$ between two normed vector spaces at $(\overline{x}, \overline{y}) \in F$ is the multimap $D_M^* F(\overline{x}, \overline{y}) : Y^* \rightrightarrows X^*$ defined by

$$D_M^* F(\overline{x}, \overline{y})(y^*) := \{x^* : \exists ((x_n, y_n)) \xrightarrow{F} (\overline{x}, \overline{y}),\ (\varepsilon_n) \to 0_+,\ (x_n^*) \xrightarrow{*} x^*,\ (y_n^*) \to y^*,$$
$$\forall n \in \mathbb{N}\ (x_n^*, -y_n^*) \in N_F^{\varepsilon_n}(F, (x_n, y_n))\}.$$

It is of interest to introduce a definition for the case these notions coincide at a given point.

Definition 9 [67] A multimap $F : X \rightrightarrows Y$ is said to be soft (resp. Fréchet soft or firmly soft, in short F-soft) at $(\overline{x}, \overline{y}) \in F$ if $D_L^* F(\overline{x}, \overline{y}) = D^* F(\overline{x}, \overline{y})$ (resp. $D_L^* F(\overline{x}, \overline{y}) = D_F^* F(\overline{x}, \overline{y})$). Then, one also has $D_M^* F(\overline{x}, \overline{y}) = D^* F(\overline{x}, \overline{y})$ (resp. $D_M^* F(\overline{x}, \overline{y}) = D_F^* F(\overline{x}, \overline{y})$).

When X and Y are Asplund spaces, F is F-soft at $(\overline{x}, \overline{y})$ if, and only if F is *F-regular at* $(\overline{x}, \overline{y})$ in the sense that $N_F(F, (\overline{x}, \overline{y}))$ coincides with the Clarke normal cone to F at $(\overline{x}, \overline{y})$.

Let us recall some compactness notions we shall use. They derive from a general concept of compactness for maps and multimaps [58].

Definition 10 [60, 62] A subset S of a normed vector space X is said to be (sequentially) normally compact at $\overline{x} \in S$ if for every sequences $(x_n) \xrightarrow{S} \overline{x}$, $(x_n^*) \xrightarrow{*} 0$ with $x_n^* \in N_F(S, x_n)$ for all $n \in \mathbb{N}$, one has $(x_n^*) \to 0$.

In [60] this condition is formulated in an equivalent way (for the case S is convex); it is shown in [60] that if S is a closed convex set with nonempty interior, then S is normally compact at each of its points. It is obviously also the case for an arbitrary subset of a finite dimensional space.

The following definition introduced in [62, Definition 4.1] (for nets) is an adaptation to multimaps of the preceding compactness notion for sets. It is also considered in [23, 27, 41] and in numerous papers of Mordukhovich under the name partial sequential normal compactness (PSNC) (resp. strong partial sequential normal compactness) of the inverse.

Definition 11 [62] A multimap $F : X \rightrightarrows Y$ is said to be coderivatively compact (resp. strongly coderivatively compact) at $(\overline{x}, \overline{y}) \in F$ if for every sequences $((x_n, y_n)) \overset{F}{\to} (\overline{x}, \overline{y})$, $(x_n^*) \to 0$ (resp. $(x_n^*) \overset{*}{\to} 0$), $(y_n^*) \overset{*}{\to} 0$ with $x_n^* \in D_F^* F(x_n, y_n)(y_n^*)$ for all $n \in \mathbb{N}$ one has $(y_n^*) \to 0$.

Clearly, if the graph of F is normally compact at $(\overline{x}, \overline{y}) \in F$, then F is strongly coderivatively compact at $(\overline{x}, \overline{y})$. A subset S of Y is normally compact at $\overline{y} \in S$ if, and only if, for $\overline{x} \in C \subset X$, the multimap $F : X \rightrightarrows Y$ with graph $C \times S$ is coderivatively compact at $(\overline{x}, \overline{y})$. Coderivative compactness is obviously satisfied when Y is finite dimensional. In view of the following lemma, it is also satisfied when F^{-1} is Lipschitz-like around $(\overline{y}, \overline{x})$.

Lemma 12 [41, Theorem 1.43] *Suppose $F : X \rightrightarrows Y$ is Lipschitz-like around $(\overline{x}, \overline{y})$. Then there exists $c > 0$ such that $\|x^*\| \le c \|y^*\|$ for all $y^* \in Y^*$, all (x, y) near $(\overline{x}, \overline{y})$ and all $x^* \in D_M^* F(x, y)(y^*)$, hence for all $x^* \in D_F^* F(x, y)(y^*)$. If Y is finite dimensional, the same inequality holds when $x^* \in D_L^* F(\overline{x}, \overline{y})(y^*)$.*

3 Coderivatives of Intersections

Given multimaps $F : X \rightrightarrows Y$, $G : X \rightrightarrows Y$ between two normed vector spaces, the multimap $F \cap G$ is defined by $(F \cap G)(x) := F(x) \cap G(x)$, so that its graph is the intersection of the graphs of F and G and our notation is unambiguous. This remark allows to reduce the computation of the coderivative of $F \cap G$ to the calculus of the normal cone to an intersection.

We introduce the multimap $F \square G : X \rightrightarrows Y$ by

$$(F \square G)(x) := \{F(u) + G(v) : u, v \in X,\ u + v = x\}.$$

Thus the graph of $F \square G$ is simply the sum of the graphs of F and G.

We also use the following notions of joint behavior. The first one is automatically satisfied in finite dimensional spaces.

Definition 13 [64, Definition 3.2] Two closed subsets F and G of a normed vector space Z are said to be synergetic at $\overline{z} \in F \cap G$ if $(x_n) \overset{F}{\to} \overline{z}$, $(y_n) \overset{G}{\to} \overline{z}$, $(x_n^*) \overset{*}{\to} 0$, $(y_n^*) \overset{*}{\to} 0$ are such that $x_n^* \in N_F(F, x_n)$, $y_n^* \in N_F(G, y_n)$ for all $n \in \mathbb{N}$ and $(x_n^* + y_n^*) \to 0$, then one has $(x_n^*) \to 0$, $(y_n^*) \to 0$.

The sets F and G are synergetic at $\overline{z} \in F \cap G$ whenever one of them is normally compact at $\overline{z}$. It is easy to give examples showing that F and G can be synergetic at $\overline{z}$ while none of them is normally compact at $\overline{z}$.

Example Let X, Y be infinite dimensional normed vector spaces, let $Z := X \times Y$ and let $S \subset X$ and $T \subset Y$ be normally compact at $\overline{x} \in S$ and $\overline{y} \in T$, respectively. Then, for any subsets $S' \subset X$, $T' \subset Y$ containing $\overline{x}$ and $\overline{y}$ respectively, the sets $F := S \times T'$ and $G := S' \times T$ are synergetic at $\overline{z} := (\overline{x}, \overline{y})$.

Example [64] Two closed convex subsets C and D of Z are synergetic at $\overline{z} \in C \cap D$ whenever they are transverse (see [56]) in the sense that $\mathbb{R}_+ (C - \overline{z}) - \mathbb{R}_+ (D - \overline{z}) = Z$.

Definition 14 [41, Definition 3.2] Given two closed subsets F and G of a normed vector space Z, one says that they satisfy the limiting qualification condition (LQC) at $\overline{z} \in F \cap G$ if $(x_n) \overset{F}{\to} \overline{z}$, $(y_n) \overset{G}{\to} \overline{z}$, $(x_n^*) \overset{*}{\to} x^*$, $(y_n^*) \overset{*}{\to} y^*$ are such that $x_n^* \in N_F(F, x_n)$, $y_n^* \in N_F(G, y_n)$ for all $n \in \mathbb{N}$ and $(x_n^* + y_n^*) \to 0$, then one has $x^* = 0$, $y^* = 0$.

Such a condition is obviously a consequence of the so-called *normal qualification condition (NQC)*

$$(-N_L(F, \overline{z})) \cap N_L(G, \overline{z}) = \{0\}. \tag{1}$$

If Z is finite dimensional, (1) is equivalent to the (LQC) condition.

In [26] the following notion was introduced in a slightly different form. Clearly, it is a property stronger than the synergy and the (LQC) conditions.

Definition 15 [64] Given two closed subsets F and G of a normed vector space Z, one says that they are allied at $\overline{z} \in F \cap G$ (for the Fréchet normal cones) whenever $(x_n) \overset{F}{\to} \overline{z}$, $(y_n) \overset{G}{\to} \overline{z}$, $x_n^* \in N_F(F, x_n)$, $y_n^* \in N_F(G, y_n)$, the relation $\left(x_n^* + y_n^*\right) \to 0$ implies $\left(x_n^*\right) \to 0$, $\left(y_n^*\right) \to 0$.

The crucial proposition which follows reduces alliedness to a much easier requirement.

Proposition 16 *Two closed subsets F and G of Z are allied at $\overline{z} \in F \cap G$ whenever given $(x_n) \overset{F}{\to} \overline{z}$, $(y_n) \overset{G}{\to} \overline{z}$, $x_n^* \in N_F(F, x_n) \cap B_{Z^*}$, $y_n^* \in N_F(G, y_n) \cap B_{Z^*}$, the relation $\left(x_n^* + y_n^*\right) \to 0$ implies $\left(x_n^*\right) \to 0$, $\left(y_n^*\right) \to 0$.*

Proof Suppose F and G satisfy this condition. Let $(x_n) \overset{F}{\to} \overline{z}$, $(y_n) \overset{G}{\to} \overline{z}$, (x_n^*), (y_n^*) be sequences satisfying $\left(x_n^* + y_n^*\right) \to 0$ and $x_n^* \in N_F(F, x_n)$, $y_n^* \in N_F(G, y_n)$ for all $n \in \mathbb{N}$. Let $r_n := \max(\left\|x_n^*\right\|, \left\|y_n^*\right\|)$. If (r_n) is bounded, changing (x_n^*) and (y_n^*) into (x_n^*/r) and (y_n^*/r), with $r > \sup_n r_n$, we get that $(x_n^*) \to 0$ and $(y_n^*) \to 0$. It remains to discard the case (r_n) is unbounded. Taking a subsequence, we may suppose $(r_n) \to +\infty$. Setting $u_n^* := x_n^*/r_n$, $v_n^* := y_n^*/r_n$, so that $(\left\|u_n^* + v_n^*\right\|) \to 0$, we obtain from our assumption that $\left(\left\|u_n^*\right\|\right) \to 0$, $\left(\left\|v_n^*\right\|\right) \to 0$, a contradiction with $\max(\left\|u_n^*\right\|, \left\|v_n^*\right\|) = 1$. □

The following equivalence stems from the proposition via extraction of subsequences.

Corollary 17 *Suppose the dual unit ball of Z is weak* sequentially compact. Two closed subsets F and G of Z are allied at $\overline{z} \in F \cap G$ if, and only if, they are synergetic at $\overline{z}$ and satisfy the (LQC) condition at $\overline{z}$.*

Let us give criteria ensuring the limiting qualification condition (LQC) or the alliedness property. In order to formulate one of them, it is convenient to use the notion of apart cones.

Definition 18 [66] Two cones P, Q of a normed vector space Z are said to be apart if $\mathrm{gap}(P \cap S_Z, Q \cap S_Z) > 0$, where S_Z is the unit sphere in Z and for two subsets C, D of Z, $\mathrm{gap}(C, D)$ is defined by

$$\mathrm{gap}(C, D) := \inf\{\|x - y\| : x \in C, y \in D\}.$$

In [66] several characterizations of this property are displayed. In particular, P, Q are apart if, and only if, for some $\varepsilon > 0$, their enlargements (or plasterings) P_ε, Q_ε satisfy $P_\varepsilon \cap Q_\varepsilon = \{0\}$, if, and only if for some $\alpha > 0$ one has $P_\alpha \cap Q = \{0\}$, where

$$P_\varepsilon := \{z \in Z : d(z, P) < \varepsilon \|z\|\} \cup \{0\}.$$

Proposition 19 *The limiting qualification condition (LQC) at $\overline{z} \in F \cap G$ and the alliedness property are satisfied whenever the following local uniform alliedness property (LUA) holds: there exists $\varepsilon > 0$ such that for all $x \in F \cap B(\overline{z}, \varepsilon)$, $y \in G \cap B(\overline{z}, \varepsilon)$ one has $(-N_F(F, x)) \cap (N_F(G, y))_\varepsilon = \{0\}$.*

Proof Suppose the local uniform alliedness property holds while the alliedness property fails. Then there exist $r > 0$ and sequences $(x_n) \xrightarrow{F} \overline{z}$, $(y_n) \xrightarrow{G} \overline{z}$, (x_n^*), (y_n^*) such that $(x_n^* + y_n^*) \to 0$ and $x_n^* \in N_F(F, x_n)$, $y_n^* \in N_F(G, y_n)$, $\|x_n^*\| > r$ for all $n \in \mathbb{N}$. Picking $\varepsilon > 0$ as in the local uniform alliedness property, we get a contradiction with $(x_n^* + y_n^*) \to 0$:

$$\|x_n^* + y_n^*\| \geq d(-x_n^*, N_F(G, y_n)) \geq \varepsilon \|-x_n^*\| \geq \varepsilon r.$$

Even when the ball B_{Z^*} is not sequentially compact, the (LQC) condition follows from the local uniform alliedness property. In fact, if the (LQC) condition fails, one can find $x^*, y^* \in Z \backslash \{0\}$ and sequences $(x_n) \xrightarrow{F} \overline{z}$, $(y_n) \xrightarrow{G} \overline{z}$, $(x_n^*) \xrightarrow{*} x^*$, $(y_n^*) \xrightarrow{*} y^*$ such that $x_n^* \in N_F(F, x_n)$, $y_n^* \in N_F(G, y_n)$ for all $n \in \mathbb{N}$ and $(x_n^* + y_n^*) \to 0$. Taking $r \in (0, \|x^*\|)$ and using the weak* compactness of rB_{Z^*} one has $\|x_n^*\| > r$ for n large enough. Then, as above, one gets a contradiction with the local uniform alliedness property. □

Following the request of an anonymous referee of a preliminary version of the present paper, let us compare our assumptions with the *fuzzy qualification condition* of [41, p. 264] which is as follows: there exists $\gamma > 0$ such that

$$(N_F(F, x) + \gamma B_{Z^*}) \cap (-N_F(G, y) + \gamma B_{Z^*}) \cap B_{Z^*} \subset (1/2) B_{Z^*} \tag{2}$$

for all $x \in F \cap B(\overline{z}, \gamma)$, $y \in G \cap B(\overline{z}, \gamma)$.

Lemma 20 *The fuzzy qualification condition (FQC) is equivalent to the local uniform alliedness property (LUA).*

Proof Suppose the fuzzy qualification condition holds. Let $\gamma > 0$ be as in (2) and let us show that for $\varepsilon = \gamma$ we have $(-N_F(F, x)) \cap (N_F(G, y))_\varepsilon = \{0\}$ for all $x \in F \cap B(\overline{z}, \varepsilon)$, $y \in G \cap B(\overline{z}, \varepsilon)$. Suppose, on the contrary, that we can find $x \in F \cap B(\overline{z}, \gamma)$, $y \in G \cap B(\overline{z}, \gamma)$ and $x^* \in (-N_F(F, x)) \cap (N_F(G, y))_\varepsilon$ with $x^* \neq 0$. Replacing x^* with $x^* / \|x^*\|$ if necessary, we may suppose $-x^* \in S_{Z^*} \cap N_F(F, x)$ and $d(x^*, N_F(G, y)) < \varepsilon$. Then, we can find $y^* \in N_F(G, y)$ satisfying $\|x^* + y^*\| < \varepsilon$, so that, by (2) $x^* \in (1/2)B_{Z^*}$, a contradiction.

Suppose now that the fuzzy qualification condition does not hold: given a sequence $(\gamma_n) \to 0_+$, one can find sequences (x_n), (y_n) in F and G respectively, (u_n^*), (v_n^*), (w_n^*), (z_n^*) with $d(x_n, \overline{z}) < \gamma_n$, $d(y_n, \overline{z}) < \gamma_n$ $u_n^* \in N_F(F, x_n)$, $v_n^* \in N_F(G, y_n)$, $(w_n^*), (z_n^*) \in \gamma_n B_{Z^*}$, $u_n^* + w_n^* = -v_n^* + z_n^* \in B_{Z^*} \backslash (1/2) B_{Z^*}$ for all $n \in \mathbb{N}$. We may suppose $\gamma_n \leq 1/4$ for all $n \in \mathbb{N}$, so that $\|u_n^*\| \geq 1/4$ and

$$d(-u_n^*, N_F(G, y_n)) \leq \|u_n^* + v_n^*\| = \|z_n^* - w_n^*\| \leq 2\gamma_n \leq 8\gamma_n \|u_n^*\|.$$

Thus the local uniform alliedness property does not hold. □

The following result is inspired by a hint kindly given by the referee. It closes the circle of qualification conditions and shows the interest of giving symmetric roles to F and G.

Proposition 21 *Suppose the unit ball of Z^* is weak* sequentially compact, suppose F and G are synergetic at $\overline{z} \in F \cap G$ and the limiting qualification condition holds at $\overline{z}$. Then the local uniform alliedness property is satisfied at $\overline{z}$.*

Proof Suppose on the contrary that there exist sequences $(x_n) \overset{F}{\to} \overline{z}$, $(y_n) \overset{G}{\to} \overline{z}$ such that $(\text{gap}(N_F(F, x_n) \cap S_{Z^*}, (-N_F(G, y_n)) \cap S_{Z^*}))_n \to 0$. Then, there exist $x_n^* \in N_F(F, x_n) \cap S_{Z^*}$, $y_n^* \in N_F(G, y_n) \cap S_{Z^*}$ such that $(x_n^* + y_n^*) \to 0$. Since B_{Z^*} is sequentially compact, taking subsequences if necessary, we may suppose (x_n^*) and (y_n^*) have weak* limits x^* and y^* respectively. The limiting qualification condition ensures that $x^* = 0$, $y^* = 0$. Now, since F and G are synergetic at $\overline{z}$, we get $(x_n^*) \to 0$, a contradiction with $\|x_n^*\| = 1$ for all n. □

It is shown in [64, Thoerem 3.7] that when Z is an Asplund space and the normal qualification condition holds, the synergy condition implies the following linear estimate for some $r, c > 0$:

$$\forall z \in B(\overline{z}, r) \qquad d(z, F \cap G) \leq cd(z, F) + cd(z, G) \tag{3}$$

which in turn ([23, Proposition 6.2], [61, Proposition 5.2] and [64, Proposition 2.6] for instance) entails

$$N_L(F \cap G, \overline{z}) \subset N_L(F, \overline{z}) + N_L(G, \overline{z}). \tag{4}$$

We give here a direct proof of this inclusion while relaxing the qualification condition. It uses the following alternative.

Proposition 22 *Let F and G be two closed subsets of an Asplund space Z and let $\overline{z} \in F \cap G$. Then, either $N_L(F \cap G, \overline{z}) \subset N_L(F, \overline{z}) + N_L(G, \overline{z})$ or there exist sequences $(x_n) \xrightarrow{F} \overline{z}$, $(y_n) \xrightarrow{G} \overline{z}$, (x_n^*), (y_n^*) in Z^* such that $(x_n^* + y_n^*) \to 0$, $x_n^* \in N_F(F, x_n)$, $y_n^* \in N_F(G, y_n)$, $(\|x_n^*\|) \to 1$, $(\|y_n^*\|) \to 1$.*

This result is a consequence of [26, Proposition 8.1] and of the sum rule for limiting subdifferentials of Lipschitzian functions. For the reader's convenience, we derive it from a fuzzy intersection rule.

Lemma 23 ([41, Lemma 3.1], [74]) *Let F and G be two closed subsets of an Asplund space Z and let $\overline{z} \in F \cap G$, $\overline{z}^* \in N_L(F \cap G, \overline{z})$ with $\|\overline{z}^*\| = 1$. Then, there exist sequences (t_n) in $[0, 1]$, $(x_n) \xrightarrow{F} \overline{z}$, $(y_n) \xrightarrow{G} \overline{z}$, (x_n^*), (y_n^*), (z_n^*) in Z^* such that $(z_n^*) \xrightarrow{*} \overline{z}^*$, $(x_n^* + y_n^* - t_n z_n^*) \to 0$, $x_n^* \in N_F(F, x_n)$, $y_n^* \in N_F(G, y_n)$, $\max(t_n, \|x_n^*\|) = 1$ for all $n \in \mathbb{N}$.*

Proof of Proposition 22 Let $\overline{z}^* \in N_L(F \cap G, \overline{z})$ with $\|\overline{z}^*\| = 1$. In view of Lemma 23, we can find sequences (t_n) in $[0, 1]$, $(x_n) \xrightarrow{F} \overline{z}$, $(y_n) \xrightarrow{G} \overline{z}$, (x_n^*), (y_n^*), (z_n^*) such that $(z_n^*) \xrightarrow{*} \overline{z}^*$, $(x_n^* + y_n^* - t_n z_n^*) \to 0$, $x_n^* \in N_F(F, x_n)$, $y_n^* \in N_F(G, y_n)$, $\max(t_n, \|x_n^*\|) = 1$ for all $n \in \mathbb{N}$. Taking subsequences if necessary, we may assume $(t_n) \to t$, $(x_n^*) \xrightarrow{*} x^*$, $(y_n^*) \xrightarrow{*} y^*$ for some $t \in [0, 1]$, $x^* \in B_{Z^*}$, $y^* \in Z^*$ with $x^* + y^* = t\overline{z}^*$. Suppose $t = 0$. Then, as (z_n^*) is bounded, we have $(x_n^* + y_n^*) \to 0$ and since $\|x_n^*\| = 1$ for n large enough, the second case of the alternative holds. When that case is excluded, we must have $t > 0$. Then, setting $\overline{x}^* := t^{-1}x^* \in N_L(F, \overline{z})$, $\overline{y}^* := t^{-1}y^* \in N_L(G, \overline{z})$, we have $\overline{z}^* = \overline{x}^* + \overline{y}^*$. □

The following result does not require any compactness assumption.

Proposition 24 *Given two closed subsets F and G of an Asplund space Z and $\overline{z} \in F \cap G$, the inclusion (4) holds provided the alliedness property holds at $\overline{z} \in F \cap G$.*

Proof The result follows from the fact that alliedness excludes the second case in the alternative of Proposition 22. □

Taking into account Proposition 21 and the fact that the closed unit ball of the dual of an Asplund space is weak* sequentially compact, we get the following consequence. In view of the preceding examples we thus get a generalization of [41, Theorem 3.4].

Corollary 25 *Let F and G be two closed subsets of an Asplund space Z and let $\overline{z} \in F \cap G$. Then inclusion (4) holds provided the sets F and G are synergetic at $\overline{z}$ and the limiting qualification condition (LQC) holds at $\overline{z} \in F \cap G$.*

The local uniform alliedness property is just a sufficient condition. Other conditions ensuring (4) can be given. In the following example, this relation is satisfied while the local uniform alliedness property does not hold.

Example Let $X = Y = \mathbb{R}$, $Z := X \times Y$, $F := \mathbb{R}_- \times \mathbb{R}_+$, $G := \mathbb{R}_- \times \mathbb{R}_-$, so that $F \cap G = \mathbb{R}_- \times \{0\}$ and for $\overline{z} := (0,0)$ one has $N_L(F, \overline{z}) = \mathbb{R}_+ \times \mathbb{R}_-$, $N_L(G, \overline{z}) = \mathbb{R}_+ \times \mathbb{R}_+$, $N_L(F \cap G, \overline{z}) = \mathbb{R}_+ \times \mathbb{R} = N_L(F, \overline{z}) + N_L(G, \overline{z})$ although for any $\varepsilon > 0$, $\{0\} \times \mathbb{R}_+ \subset (-N_F(F, \overline{z})) \cap N_F(G, \overline{z})_\varepsilon$.

Denoting by (S) the synergy condition at $\overline{z}$, by (A) the alliedness property, by (LUA) the local uniform alliedness property, by (FQC) the fuzzy qualification condition and by (I) the inclusion property (4), we summarize the revealed implications in the following diagram, Z being an Asplund space:

$$\begin{array}{c} \text{(S)} \wedge \text{(LQC)} \Leftrightarrow \text{(FQC)} \Leftrightarrow \text{(LUA)} \Leftrightarrow \text{(A)} \Rightarrow \text{(LQC)} \Leftarrow \text{(NQC)} \\ \Downarrow \\ \text{(I)} \end{array}$$

Now let us pass to multimaps. Since the graph of a multimap is a subset of a product space, the preceding concepts can be adapted to such a product structure.

Definition 26 Two multimaps $F, G : X \rightrightarrows Y$ are said to be range-allied (resp. source-allied) at $\overline{z} \in F \cap G$ if $(w_n) \overset{F}{\to} \overline{z}$, $(z_n) \overset{G}{\to} \overline{z}$, $(w_n^*) := ((u_n^*, v_n^*))$, $(z_n^*) := ((x_n^*, y_n^*))$ in $X^* \times Y^*$ are such that $w_n^* \in N_F(F, w_n)$, $z_n^* \in N_F(G, z_n)$ for all $n \in \mathbb{N}$ and $(w_n^* + z_n^*) \to 0$, then one has $(v_n^*) \to 0$ (resp. $(u_n^*) \to 0$).

Clearly, if $F := B \times C$, $G := D \times E$ where C and E are allied at $\overline{y}$, then F and G are range allied at $\overline{z} := (\overline{x}, \overline{y})$ for all $\overline{x} \in B \cap D$. A similar assertion holds for source-alliedness.

It is also easy to see that when F, G are source-allied at $\overline{z}$ and F or G is coderivatively compact at $\overline{z}$, then F, G are synergetic at $\overline{z}$. Similarly, if F, G are range-allied at $\overline{z}$ and F^{-1} or G^{-1} is coderivatively compact at $\overline{z}$, then F, G are synergetic at $\overline{z}$.

Calculus rules for the intersection of two multimaps are given in the next statement. It encompasses [41, Theorem 3.4] in its cases (f) and (g).

Proposition 27 *Let $F, G : X \rightrightarrows Y$ be two multimaps and let $\overline{z} := (\overline{x}, \overline{y}) \in F \cap G$. Then*

$$D^* F(\overline{x}, \overline{y}) \square D^* G(\overline{x}, \overline{y}) \subset D^*(F \cap G)(\overline{x}, \overline{y}),$$

$$D_F^* F(\overline{x}, \overline{y}) \square D_F^* G(\overline{x}, \overline{y}) \subset D_F^*(F \cap G)(\overline{x}, \overline{y}).$$

In order that the inclusion

$$D_L^*(F \cap G)(\overline{x}, \overline{y}) \subset D_L^* F(\overline{x}, \overline{y}) \square D_L^* G(\overline{x}, \overline{y}). \tag{5}$$

holds, it suffices that X and Y are Asplund spaces, the graphs of F and G are closed and one of the following assumptions is satisfied:

(a) *the graphs of F and G are allied at $\overline{z}$;*
(b) *they are synergetic at $\overline{z}$ and satisfy the (LQC) condition at $\overline{z}$;*
(c) *they are synergetic at $\overline{z}$ and satisfy the following condition*

$$u^* \in (-D_L^* F(\overline{x}, \overline{y})(v^*)) \cap D_L^* G(\overline{x}, \overline{y})(-v^*) \Longrightarrow u^* = 0, \; v^* = 0; \tag{6}$$

(d) *F^{-1} is coderivatively compact at $(\overline{y}, \overline{x})$, G is strongly coderivatively compact at $\overline{z}$ and (6) holds;*
(e) *F is coderivatively compact at $\overline{z}$, G^{-1} is strongly coderivatively compact at $\overline{z}$ and (6) holds.*
(f) *F and G are range-allied at $\overline{z}$, either F^{-1} or G^{-1} is coderivatively compact at $(\overline{y}, \overline{x})$ and the following condition holds*

$$(-D_M^* F(\overline{x}, \overline{y})(0)) \cap D_M^* G(\overline{x}, \overline{y})(0) = \{0\}; \tag{7}$$

(g) *F and G are source-allied at $\overline{z}$, either F or G is coderivatively compact at $(\overline{x}, \overline{y})$ and the following condition holds*

$$(-D_M^* F^{-1}(\overline{y}, \overline{x})(0)) \cap D_M^* G^{-1}(\overline{y}, \overline{x})(0) = \{0\}. \tag{8}$$

Proof The first assertion is an immediate consequence of an estimate for the normal cone to an intersection; here one uses the facts that the normal cones are convex and that the passage to the normal cone is antitone.

Under the assumptions (a), (b), (c), inclusion (5) is a consequence of Proposition 24 and of the preceding analysis, observing that condition (6) is equivalent to (1) and that $(y^*, x^*) \in D_L^* F(\overline{x}, \overline{y}) \Box D_L^* G(\overline{x}, \overline{y})$ if, and only if $(x^*, -y^*) \in N_L(F, \overline{z}) + N_L(G, \overline{z})$.

Let us prove case (d) by showing that the graphs of F and G are synergetic at $\overline{z} := (\overline{x}, \overline{y})$ whenever F is coderivatively compact at $\overline{z}$ and G^{-1} is strongly coderivatively compact at $(\overline{y}, \overline{x})$. In fact, if $(w_n) \overset{F}{\to} \overline{z}$, $(z_n) \overset{G}{\to} \overline{z}$, $(w_n^*) \overset{*}{\to} 0$, $(z_n^*) \overset{*}{\to} 0$ are such that $w_n^* := (u_n^*, v_n^*) \in N_F(F, w_n)$, $z_n^* := (x_n^*, y_n^*) \in N_F(G, z_n)$ for all $n \in \mathbb{N}$ and $(w_n^* + z_n^*) \to 0$, we have $(u_n^*) \to 0$ as F^{-1} is strongly coderivatively compact at $(\overline{y}, \overline{x})$, hence $(x_n^*) \to 0$ and $(y_n^*) \to 0$, $(v_n^*) \to 0$ as G is coderivatively compact at $\overline{z}$. Case (e) is similar.

Let us prove case (f). Suppose F and G are range-allied at $\overline{z}$, F^{-1} is coderivatively compact at $(\overline{y}, \overline{x})$ and relation (7) holds. If the inclusion (5) does not hold, by Proposition 22, we can find sequences $(w_n) \overset{F}{\to} \overline{z}$, $(z_n) \overset{G}{\to} \overline{z}$, $(w_n^*) \overset{*}{\to} w^*$, $(z_n^*) \overset{*}{\to} -w^*$ such that $(w_n^* + z_n^*) \to 0$, $(\|w_n^*\|) \to 1$ and $w_n^* := (u_n^*, v_n^*) \in N_F(F, w_n)$, $z_n^* := (x_n^*, y_n^*) \in N_F(G, z_n)$ for all $n \in \mathbb{N}$. Since F and G are range-allied, we have $(v_n^*) \to 0$, $(y_n^*) \to 0$. We may assume that (u_n^*) has a weak* limit u^*. Then $u^* \in (D_M^* F(\overline{x}, \overline{y})(0)) \cap (-D_M^* G(\overline{x}, \overline{y})(0))$, so that $u^* = 0$ by condition (7). Now, as F^{-1} is coderivatively compact at $(\overline{y}, \overline{x})$, we have $(u_n^*) \to 0$, a contradiction with $(\|w_n^*\|) \to 1$. Case (g) is similar, changing F and G into F^{-1} and G^{-1} respectively. □

Corollary 28 *Let X and Y be Asplund spaces, let $F, G : X \rightrightarrows Y$ be two closed multimaps which are soft (resp. F-soft) at $(\overline{x}, \overline{y}) \in F \cap G$ and satisfy one of the assumptions (a)–(g) of Proposition 27. Then $F \cap G$ is soft (resp. F-soft) at $(\overline{x}, \overline{y})$ and*

$$D_L^*(F \cap G)(\overline{x}, \overline{y}) = D_L^* F(\overline{x}, \overline{y}) \Box D_L^* G(\overline{x}, \overline{y}).$$

Given normed vector spaces X, Y_1, Y_2 and two multimaps $F_1 : X \rightrightarrows Y_1$, $F_2 : X \rightrightarrows Y_2$, in order to estimate the coderivative of the multimap $F := (F_1, F_2) : X \rightrightarrows Y := Y_1 \times Y_2$ defined by $F(x) := F_1(x) \times F_2(x)$, let us introduce the following definition in which $\overline{x} \in X$, $(\overline{y}_1, \overline{y}_2) \in F(\overline{x})$.

Definition 29 The multimaps $F_1 : X \rightrightarrows Y_1$, $F_2 : X \rightrightarrows Y_2$ are said to be cooperative *(resp. coordinated) at* $(\overline{x}, \overline{y}_1, \overline{y}_2)$ if for any sequences $((x'_n, y'_n)) \overset{F_1}{\to} (\overline{x}, \overline{y}_1)$, $((x''_n, y''_n)) \overset{F_2}{\to} (\overline{x}, \overline{y}_2)$, $(x'^*_n), (x''^*_n)$ in X^* (resp. $(x'^*_n), (x''^*_n) \overset{*}{\to} 0$), $(y'^*_n), (y''^*_n) \to 0$ such that $(x'^*_n + x''^*_n) \to 0$ and $x'^*_n \in D^*_F F_1(x'_n, y'_n)(y'^*_n)$, $x''^*_n \in D^*_F F_2(x''_n, y''_n)(y''^*_n)$ for all n, one has $(x'^*_n) \to 0$ (and $(x''^*_n) \to 0$).

Clearly, two subsets S_1, S_2 of a normed vector space X are allied (resp. synergetic) at $\overline{x} \in S_1 \cap S_2$ if, and only if, the multimaps $F_1, F_2 : X \rightrightarrows Y := \{0\}$ with graphs $S_1 \times \{0\}$ and $S_2 \times \{0\}$ respectively are cooperative (resp. coordinated) at $(\overline{x}, 0, 0)$.

If F_1^{-1} is coderivatively compact at $(\overline{y}_1, \overline{x})$ (or if F_2^{-1} is coderivatively compact at $(\overline{y}_2, \overline{x})$) , then F_1 and F_2 are coordinated at $(\overline{x}, \overline{y}_1, \overline{y}_2)$.

Introducing the multimaps $M_1, M_2 : X \rightrightarrows Y := Y_1 \times Y_2$ by setting $M_1(x) := F_1(x) \times Y_2$, $M_2(x) := Y_1 \times F_2(x)$, one can easily check the following equivalence.

Lemma 30 *The multimaps* F_1 *and* F_2 *are cooperative (resp. coordinated) at* $(\overline{x}, \overline{y}_1, \overline{y}_2)$ *if, and only if,* M_1 *and* M_2 *are allied (resp. synergetic) at* $(\overline{x}, (\overline{y}_1, \overline{y}_2))$.

Also, F_1 and F_2 are coordinated at $(\overline{x}, \overline{y}_1, \overline{y}_2)$ if, and only if, M_1 and M_2 are source-allied at $(\overline{x}, (\overline{y}_1, \overline{y}_2))$.

Corollary 31 *Let* X, Y_1, Y_2 *be Asplund spaces and let the multimaps* $F_1 : X \rightrightarrows Y_1$, $F_2 : X \rightrightarrows Y_2$ *have closed graphs. If they are cooperative at* $(\overline{x}, \overline{y}_1, \overline{y}_2)$ *then, for every* $(\overline{y}^*_1, \overline{y}^*_2) \in Y^*_1 \times Y^*_2$*, one has*

$$D^*_L(F_1, F_2)(\overline{x}, \overline{y}_1, \overline{y}_2)(\overline{y}^*_1, \overline{y}^*_2) \subset D^*_L F_1(\overline{x}, \overline{y}_1)(\overline{y}^*_1) + D^*_L F_2(\overline{x}, \overline{y}_2)(\overline{y}^*_2). \tag{9}$$

The same relation holds if they are coordinated at $(\overline{x}, \overline{y}_1, \overline{y}_2)$ *and if*

$$(-D^*_M F_1(\overline{x}, \overline{y}_1)(0)) \cap D^*_M F_2(\overline{x}, \overline{y}_2)(0) = \{0\}. \tag{10}$$

Proof Let $F := (F_1, F_2)$ and let M_1 and M_2 be defined as above, so that $F = M_1 \cap M_2$ and one has the relations

$$D^*_L F_1(\overline{x}, \overline{y}_1)(\overline{y}^*_1) = D^*_L M_1(\overline{x}, \overline{y}_1, 0)(\overline{y}^*_1, 0),$$
$$D^*_L F_2(\overline{x}, \overline{y}_2)(\overline{y}^*_2) = D^*_L M_2(\overline{x}, 0, \overline{y}_1)(0, \overline{y}^*_2)$$

and similar ones in which the limiting coderivatives are replaced with mixed coderivatives. In view of the preceding lemma, the first assertion is a consequence of Proposition 24.

The proof of the second one is similar to the proof of case (f) of Proposition 27, observing that here we can dispense with the condition that M_1^{-1} or M_2^{-1} is coderivatively compact at $\overline{z} := (\overline{x}, \overline{y}_1, \overline{y}_2) \in M_1 \cap M_2$. The details are as follows. We have to prove that the second case of the alternative of Proposition 22 does not occur under the assumptions of the second assertion. Suppose $((x'_n, y'_n, v'_n)) \overset{M_1}{\to} \overline{z}$, $((x''_n, v''_n, y''_n)) \overset{M_2}{\to} \overline{z}$, $((x'^*_n, y'^*_n, v'^*_n)) \overset{*}{\to} (x'^*, y'^*, v'^*)$, $((x''^*_n, v''^*_n, y''^*_n)) \overset{*}{\to} (x''^*, v''^*, y''^*)$ and $(x'^*_n, y'^*_n, v'^*_n) \in N_F(M_1, (x'_n, y'_n, v'_n))$, $(x''^*_n, v''^*_n, y''^*_n) \in N_F(M_2, (x''_n, v''_n, y''_n))$ are such that $((x'^*_n, y'^*_n, v'^*_n) + (x''^*_n, v''^*_n, y''^*_n)) \to 0$. Since $v'^*_n = 0$, $v''^*_n = 0$ for all $n \in \mathbb{N}$, we get $(y'^*_n) \to 0$, $(y''^*_n) \to 0$. Since $x'^*_n \in D^*_F F_1(x'_n, y'_n)(-y'^*_n)$, $x''^*_n \in D^*_F F_2(x''_n, y''_n)(-y''^*_n)$ for all n, we have $x'^* \in D^*_M F_1(\overline{x}, \overline{y}_1)(0)$, $x''^* \in D^*_M F_2(\overline{x}, \overline{y}_2)(0)$ and $x'^* + x''^* = 0$.

Relation (10) ensures that $x'^* = 0 = x''^*$. Since F_1 and F_2 are coordinated at $\overline{z}$, we have $(x'^*_n) \to 0$ and $(x''^*_n) \to 0$, so that the requirement $(\|(x'^*_n, y'^*_n, v'^*_n)\|) \to 1$ is impossible. □

Definition 29 and Corollary 31 can be generalized to a finite family of multimaps in an obvious way. We just state an application to the case of a map with values in $\mathbb{R}^k$.

Corollary 32 *Let X be an Asplund space and let $f := (f_1, ..., f_k) : X \to \mathbb{R}^k$. Suppose $(f_1, ..., f_k)$ is cooperative at $(\overline{x}, \overline{y}) := (\overline{x}, \overline{y}_1, ..., \overline{y}_k) := (\overline{x}, f_1(\overline{x}), ..., f_k(\overline{x}))$. Then, for all $(\overline{y}^*_1, ..., \overline{y}^*_k) \in \mathbb{R}^k$ one has*

$$D^*_L f(\overline{x}, \overline{y})(\overline{y}^*_1, ..., \overline{y}^*_k) \subset D^*_L f_1(\overline{x}, \overline{y}_1)(\overline{y}^*_1) + ... + D^*_L f(\overline{x}, \overline{y}_k)(\overline{y}^*_k).$$

The versatility of set-valued analysis can be experienced through the following statement whose proof consists in taking inverses.

Corollary 33 *Let X_1, X_2, Y be Asplund spaces, let $G_1 : X_1 \rightrightarrows Y$, $G_2 : X_2 \rightrightarrows Y$ be multimaps with closed graphs and let $C : X_1 \times X_2 \rightrightarrows Y$ be defined by $C(x_1, x_2) := G_1(x_1) \cap G_2(x_2)$ for $(x_1, x_2) \in X := X_1 \times X_2$. If G_1^{-1} and G_2^{-1} are cooperative at $(\overline{y}, \overline{x}_1, \overline{x}_2)$ then, for every $y^* \in Y^*$ one has*

$$D^*_L C(\overline{x}_1, \overline{x}_2, \overline{y})(y^*) \subset \bigcup_{v^* \in Y^*} D^*_L G_1(\overline{x}_1, \overline{y})(v^*) \times D^*_L G_2(\overline{x}_2, \overline{y})(y^* - v^*).$$

The same conclusion holds when G_1^{-1} and G_2^{-1} are coordinated at $(\overline{y}, \overline{x}_1, \overline{x}_2)$ and

$$(-D^*_M G_1^{-1}(\overline{y}, \overline{x}_1)(0)) \cap D^*_M G_2^{-1}(\overline{y}, \overline{x}_2)(0) = \{0\}. \tag{11}$$

Proof One has $y \in C(x_1, x_2)$ if, and only if $(x_1, x_2) \in F_1(y) \times F_2(y)$ for $F_1 := G_1^{-1}$, $F_2 := G_2^{-1}$. Thus, the result stems from Corollary 31 when rewriting coderivatives in terms of normal cones. □

4 Coderivatives of Compositions

In the present section we study the coderivatives of $H := G \circ F$, where $F : X \rightrightarrows Y$, $G : Y \rightrightarrows Z$ are multimaps between two normed vector spaces. We set

$$C := \{(x, z, y) : (x, y) \in F, (y, z) \in G\},$$

$$F_Z := \{((x, z), y) : (x, y) \in F,\ z \in Z\}, \quad G_X := \{(y, (x, z)) : x \in X, (y, z) \in G\},$$

so that $G_X^{-1} = \{(x, z, y) : x \in X, (y, z) \in G\} = X \times G^{-1}$,

$$C = F_Z \cap G_X^{-1} \tag{12}$$

and, denoting by $p_{X \times Z}$ the canonical projection $X \times Z \times Y \to X \times Z$,

$$H := G \circ F = p_{X \times Z}(C).$$

An easy dualization enables to pass from a composition result for contingent derivatives to a result for contingent coderivatives. In that statement, various choices for B can be adopted, for instance a singleton, or $F(\overline{x}) \cap G^{-1}(\overline{z})$, or any intermediate choice. In [35], criteria ensuring the assumption are provided.

Proposition 34 *Suppose that for a subset B of Y one has*

$$DH(\overline{x}, \overline{z}) \subset \bigcup_{y \in B} DG(y, \overline{z}) \circ DF(\overline{x}, y).$$

Then

$$\bigcap_{y \in B} D^*F(\overline{x}, y) \circ D^*G(y, \overline{z}) \subset D^*H(\overline{x}, \overline{z}). \tag{13}$$

Proof Let $z^* \in Z^*$ and $x^* \in (D^*F(\overline{x}, y) \circ D^*G(y, \overline{z}))(z^*)$ for all $y \in B$. Let us prove that $x^* \in D^*H(\overline{x}, \overline{z})(z^*)$, i.e. for all $(u, w) \in T(H, (\overline{x}, \overline{z}))$ one has $\langle (x^*, -z^*), (u, w) \rangle \leq 0$. Since $w \in DH(\overline{x}, \overline{z})(u)$, by assumption, there exist some $y \in B$ and some $v \in DF(\overline{x}, y)(u)$ such that $w \in DG(y, \overline{z})(v)$. Since $x^* \in (D^*F(\overline{x}, y) \circ D^*G(y, \overline{z}))(z^*)$, there exists $y^* \in D^*G(y, \overline{z})(z^*)$ such that $x^* \in D^*F(\overline{x}, y)(y^*)$. Then one has $\langle (x^*, -y^*), (u, v) \rangle \leq 0$ and $\langle (y^*, -z^*), (v, w) \rangle \leq 0$, hence, by addition $\langle x^*, u \rangle - \langle z^*, w \rangle \leq 0$ or $\langle (x^*, -z^*), (u, w) \rangle \leq 0$. □

Now we directly tackle the dual viewpoint. We first establish an easy inclusion. It relies on the following result of independent interest. Again, various choices for A and B can be adopted.

Lemma 35 *Let V, W be normed vector spaces, $C \subset V$, $E \subset W$, $p : V \to W$ be linear and continuous and such that $p(C) \subset E$. Let $e \in E$, $A \subset p^{-1}(\{e\}) \cap C$ and $c \in A$. Then*

(a) *one has $N(E, e) \subset (p^*)^{-1}(N(C, c))$; if $T(E, e) \subset p(T(C, c))$, in particular if there exists a map $q : W \to V$ which is Hadamard differentiable at e and such that $q(e) = c$, $q(E) \subset C$, $p \circ q\mid_E = I_E$, then $N(E, e) = (p^*)^{-1}(N(C, c))$;*
(b) *one has $N_F(E, e) \subset (p^*)^{-1}(N_F(C, c))$ and for all $\varepsilon \in \mathbb{R}_+$, $N_F^{\varepsilon}(E, e) \subset (p^*)^{-1}(N_F^{\|p\|\varepsilon}(C, c))$; if p is open at c from C to E with a linear rate κ, then $(p^*)^{-1}(N_F^{\gamma}(C, c)) \subset N_F^{\kappa\gamma}(E, e)$, in particular $N_F(E, e) = (p^*)^{-1}(N_F(C, c))$;*
(c) *if p is open from C to E at c, one has $N_L(E, e) \subset (p^*)^{-1}(N_L(C, c))$; more generally, if $p\mid_C$ is proper at (A, e) with respect to E, one has $N_L(E, e) \subset \bigcup_{c \in A} (p^*)^{-1}(N_L(C, c))$;*
(d) *if $V = W \times U$, and C is (the graph of) a multimap from W to U with domain E, which is lsc at (e, B) for some $B \subset U$, one has $N_L(E, e) \subset \bigcup_{b \in B} D_M^*C(e, b)(0)$; if C is Lipschitz-like around $c := (e, u)$ on E, then $N_L(E, e) = D_M^*C(e, u)(0)$.*

Proof

(a) The first assertion is a consequence of the inclusion $p(T(C, c)) \subset T(E, e)$ which is immediate. When $T(E, e) \subset p(T(C, c))$, given $e^* \in W^*$ such that $c^* := p^*(e^*) \in N(C, c)$, for all $w \in T(E, e)$, picking $v \in T(C, c)$ such that $p(v) = w$, we get that $\langle e^*, w \rangle = \langle e^*, p(v) \rangle = \langle c^*, v \rangle \leq 0$, so that $e^* \in N(E, e)$. The inclusion $T(E, e) \subset p(T(C, c))$ is clearly satisfied when there exists a Hadamard differentiable right inverse q of p such that $q(e) = c$, $q(E) \subset C$ since then $q'(e)(w) \in T(C, c)$ and $p(q'(e)w) = w$ for all $w \in T(E, e)$.

(b) Let $e^* \in N_F^{\varepsilon}(E, e)$ with $\varepsilon \in \mathbb{R}_+$. There exists a remainder r such that $\langle e^*, w - e\rangle \leq r(w - e) + \varepsilon \|w - e\|$ for all $w \in E$. Then $r \circ p$ is a remainder and for $v \in C$ one has $p(v) \in E$, hence $\langle p^*(e^*), v - c\rangle = \langle e^*, p(v) - p(c)\rangle \leq r(p(v - c)) + \varepsilon \|p\| \|v - c\|$, so that $p^*(e^*) \in N_F^{\|p\|\varepsilon}(C, c)$.
Suppose p is open at c with rate $\kappa > 0$ from C to E and $e^* \in W^*$ is such that $c^* := p^*(e^*) \in N_F^{\gamma}(C, c)$. Let o be a remainder such that $\langle c^*, v - c\rangle \leq o(v - c) + \gamma \|v - c\|$ for $v \in C$. Then, given $\varepsilon > 0$, $\lambda > \kappa$, for all $w \in E$ close enough to e one can find some $v \in C$ such that $w = p(v)$, $\|v - c\| \leq \lambda \|w - e\|$ and $o(v - c) \leq \varepsilon \|v - c\|$, so that

$$\langle e^*, w - e\rangle = \langle c^*, v - c\rangle \leq (\varepsilon + \gamma) \|v - c\| \leq (\varepsilon + \gamma)\lambda \|w - e\| .$$

Since $(\varepsilon + \gamma)\lambda$ can be chosen arbitrarily close to $\gamma\kappa$, we get $e^* \in N_F^{\gamma\kappa}(E, e)$. In particular, if $\gamma = 0$, we have $e^* \in N_F(E, e)$.

(c) Let $e^* \in N_L(E, e)$. There exist sequences $(\varepsilon_n) \to 0_+$, $(e_n) \overset{E}{\to} e$, $(e_n^*) \overset{*}{\to} e^*$ such that $e_n^* \in N_F^{\varepsilon_n}(E, e_n)$ for all n. When $p\,|_C$ is proper at (A, e) with respect to E there exist $c \in A$ and a sequence $(c_n) \overset{C}{\to} c$ such that $p(c_n) = e_n$ for n large enough. By (b) we have $p^*(e_n^*) \in N_F^{\alpha_n}(C, c_n)$ with $\alpha_n := \|p\| \varepsilon_n$. Since $(p^*(e_n^*)) \overset{*}{\to} p^*(e^*)$, we conclude that $p^*(e^*) \in N_L(C, c)$. Taking $A := \{c\}$, we get the first assertion of (c).

(d) Suppose $V = W \times U$ and C is (the graph of) a multimap from W to U with domain E, which is lsc at (e, B). Since in what precedes $p^*(e_n^*) = (e_n^*, 0) \in N_F^{\alpha_n}(C, c_n)$, we see that $e^* \in D_M^*C(e, b)(0)$, where $c := (e, b) \in A := \{e\} \times B$.

Suppose now that C is Lipschitz-like around c on E with rate λ. Let $e^* \in D_M^*C(e, u)(0)$. There exist sequences $(\gamma_n) \to 0_+$, $(c_n) := (e_n, u_n) \overset{C}{\to} c$, $(c_n^*) := (e_n^*, u_n^*) \overset{*}{\to} c^* := (e^*, 0)$ such that $c_n^* \in N_F^{\gamma_n}(C, c_n)$ for all n and $(u_n^*) \to 0$. Let $\delta_n > 0$ be such that $\langle c_n^*, c' - c_n\rangle \leq 2\gamma_n \left\|c' - c_n\right\|$ for $c' \in C \cap B(c_n, \delta_n)$. Taking $\delta_n > 0$ small enough, for every $e' \in E \cap B(e_n, \delta_n/(\lambda + 1))$ one can find some $u' \in C(e')$ such that $\left\|u' - u_n\right\| \leq \lambda \left\|e' - e_n\right\|$. Let $c' := (e', u')$. Then one has $c' \in C \cap B(c_n, \delta_n)$ and

$$\begin{aligned}\langle e_n^*, e' - e_n\rangle &= \langle c_n^*, c' - c_n\rangle - \langle u_n^*, u' - u_n\rangle \\ &\leq 2\gamma_n \left\|c' - c_n\right\| + \left\|u_n^*\right\| . \left\|u' - u_n\right\| \leq (2\gamma_n(\lambda + 1) + \lambda \left\|u_n^*\right\|) \left\|e' - e_n\right\| ,\end{aligned}$$

so that $e_n^* \in N_F^{\alpha_n}(E, e_n)$ with $\alpha_n := 2\gamma_n(\lambda + 1) + \lambda \left\|u_n^*\right\|$ and $(\alpha_n) \to 0$ since $(\left\|u_n^*\right\|) \to 0$. Thus $e^* \in N_L(E, e)$. □

Remark When $V = W \times U$, where U is a finite dimensional normed vector space and when p is the canonical projection $p_W : W \times U \to W$, the inclusion $T(E, e) \subset p(T(C, c))$ holds whenever p is open at $c := (e, u) \in C$ from C to E with a linear rate β. In fact, given $\gamma > \beta$, $w \in T(E, e)$ and sequences $(w_n) \to w$, $(t_n) \to 0_+$ such that $e_n := e + t_n w_n \in E$ for all n, one can find $c_n \in C$ such that $p(c_n) = e_n$ and $\|c_n - c\| \leq \gamma \|e_n - e\|$. Setting $c_n := (e_n, u_n)$, we have $\|u_n - u\| \leq \gamma t_n \|w_n\|$. Since U is finite dimensional, taking a subsequence if necessary, we may suppose $(t_n^{-1}(u_n - u))$ has a limit y. Then $(w, y) \in T(C, c)$ and $w = p(w, y)$.

Example Let $f : \mathbb{R} \to \mathbb{R}$ be a function which is differentiable on $\mathbb{R}\backslash\{0\}$, stable at 0 (i.e. such that for some $c > 0$ one has $|f(x) - f(0)| \leq c\,|x|$ for x near 0) and such that there exists a sequence $(r_n) \to 0_+$ with $(f'(r_n)) \to +\infty$, $f(\mathbb{R}) = \mathbb{R}$. Let C be the graph of f in $V := \mathbb{R}^2$, let $E := W := \mathbb{R}$ and let $p : (r, s) \mapsto r$. Then p is open from C to E with a linear rate at $(0, f(0))$ and $(1, 0) \in N_L(C, (0, f(0)))$. However $N_L(E, 0) = \{0\}$, so that $(1, 0) \notin p^*(N_L(E, 0))$. Therefore, the inclusion $N_L(E, e) \subset (p^*)^{-1}(N_L(C, c))$ is strict.

Let us first draw a direct application of the preceding lemma to composition.

Proposition 36

(a) *Suppose the multimap $C : X \times Z \rightrightarrows Y$ of* (12) *is lsc at $((\overline{x}, \overline{z}), \overline{y})$ on H with a linear rate and Y is finite dimensional. Then*

$$D^* F(\overline{x}, \overline{y}) \circ D^* G(\overline{y}, \overline{z}) \subset D^* H(\overline{x}, \overline{z}). \tag{14}$$

(b) *Suppose $C : X \times Z \rightrightarrows Y$ is lsc at $((\overline{x}, \overline{z}), \overline{y})$ on H with a linear rate. Then*

$$D^*_F F(\overline{x}, \overline{y}) \circ D^*_F G(\overline{y}, \overline{z}) \subset D^*_F H(\overline{x}, \overline{z}). \tag{15}$$

(c) *Suppose the multimap $C : X \times Z \rightrightarrows Y$ is lsc at $((\overline{x}, \overline{z}), \overline{y})$ on H with a linear rate and*

$$\forall y^* \in Y^* \qquad D^*_L F(\overline{x}, \overline{y})(y^*) \times D^*_L G^{-1}(\overline{z}, \overline{y})(-y^*) \subset D^*_F C(\overline{x}, \overline{z}, \overline{y})(0) \tag{16}$$

or C is Lipschitz-like around $((\overline{x}, \overline{z}), \overline{y})$ on H and

$$\forall y^* \in Y^* \qquad D^*_L F(\overline{x}, \overline{y})(y^*) \times D^*_L G^{-1}(\overline{z}, \overline{y})(-y^*) \subset D^*_M C(\overline{x}, \overline{z}, \overline{y})(0), \tag{17}$$

Then

$$D^*_L F(\overline{x}, \overline{y}) \circ D^*_L G(\overline{y}, \overline{z}) \subset D^*_L H(\overline{x}, \overline{z}).$$

Proof We use the fact that $H := p_{X\times Z}(C)$ with $C = F_Z \cap G_X^{-1}$ and that the transpose of the canonical projection $p := p_{X\times Z} : V := X \times Z \times Y \to W := X \times Z$ is given by $p^*(u^*, w^*) = (u^*, w^*, 0)$ for all $(u^*, w^*) \in X^* \times Z^*$.

(a), (b) Given $z^* \in Z^*$, $y^* \in D^* G(\overline{y}, \overline{z})(z^*)$ and $x^* \in D^* F(\overline{x}, \overline{y})(y^*)$, the definition of the coderivative ensures that $(x^*, -y^*) \in N(F, (\overline{x}, \overline{y}))$, $(y^*, -z^*) \in N(G, (\overline{y}, \overline{z}))$, hence $(-z^*, y^*) \in N(G^{-1}, (\overline{z}, \overline{y}))$, and

$$(x^*, 0, -y^*) \in N(F_Z, (\overline{x}, \overline{z}, \overline{y})), \qquad (0, -z^*, y^*) \in N(G_X^{-1}, (\overline{x}, \overline{z}, \overline{y})),$$

by the construction of F_Z and G_X^{-1}. Since $C \subset F_Z$, $C \subset G_X^{-1}$ and since taking the normal cone is an antitone process, by convexity of the normal cone, we have

$$N(F_Z, (\overline{x}, \overline{z}, \overline{y})) + N(G_X^{-1}, (\overline{x}, \overline{z}, \overline{y})) \subset N(C, (\overline{x}, \overline{z}, \overline{y})). \tag{18}$$

Thus, we get $(x^*, -z^*, 0) \in N(C, (\overline{x}, \overline{z}, \overline{y}))$. Then, the preceding remark and assertion (a) of the preceding lemma ensure that $(x^*, -z^*) \in N(H, (\overline{x}, \overline{z}))$ and $x^* \in D^* H(\overline{x}, \overline{z})(z^*)$. The proof with the Fréchet coderivatives is similar since

$$N_F(F_Z, (\overline{x}, \overline{z}, \overline{y})) + N_F(G_X^{-1}, (\overline{x}, \overline{z}, \overline{y})) \subset N_F(C, (\overline{x}, \overline{z}, \overline{y})).$$

(c) Now, given $z^* \in Z^*$, $y^* \in D_L^* G(\overline{y}, \overline{z})(z^*)$ and $x^* \in D_L^* F(\overline{x}, \overline{y})(y^*)$, it follows from $-z^* \in D_L^* G^{-1}(\overline{z}, \overline{y})(-y^*)$ that condition (16) ensures that $(x^*, -z^*) \in D_F^* C(\overline{x}, \overline{z}, \overline{y})(0)$. Then Lemma 35(b) ensures that $(x^*, -z^*) \in N_F(H, (\overline{x}, \overline{z}))$, hence $x^* \in D_F^* H(\overline{x}, \overline{z})(z^*) \subset D_L^* H(\overline{x}, \overline{z})(z^*)$.
The case C is Lipschitz-like around $((\overline{x}, \overline{z}), \overline{y})$ on H follows from Lemma 35(d).

□

Remarks

(a) In view of Lemma 12, it is crucial to assume that C is Lipschitz-like on H rather than Lipschitz-like since the latter assumption would make condition (17) very restrictive.
(b) Let us observe that (16) is a consequence of the condition

$$N_L(F_Z, (\overline{x}, \overline{z}, \overline{y})) + N_L(G_X^{-1}, (\overline{x}, \overline{z}, \overline{y})) \subset N_F(C, (\overline{x}, \overline{z}, \overline{y})) \tag{19}$$

In fact, given $y^* \in Y^*$ and $(x^*, z^*) \in D_L^* F(\overline{x}, \overline{y})(y^*) \times D_L^* G^{-1}(\overline{z}, \overline{y})(-y^*)$, the definition of the coderivative ensures that $(x^*, -y^*) \in N_L(F, (\overline{x}, \overline{y}))$, $(z^*, y^*) \in N_L(G^{-1}, (\overline{z}, \overline{y}))$, hence

$$(x^*, 0, -y^*) \in N_L(F_Z, (\overline{x}, \overline{z}, \overline{y})), \qquad (0, z^*, y^*) \in N_L(G_X^{-1}, (\overline{x}, \overline{z}, \overline{y})),$$

by the construction of F_Z and G_X^{-1}. Then, by addition, using (19), we get $(x^*, z^*, 0) \in N_F(C, (\overline{x}, \overline{z}, \overline{y}))$, i.e. $(x^*, z^*) \in D_F^* C(\overline{x}, \overline{z}, \overline{y})(0)$ and (16) holds.

Let us compare (16) with other conditions.

Proposition 37 *Among the following assertions one has the implications*

$$\text{(a)} \Longrightarrow \text{(b)} \Longrightarrow \text{(c)} \Longrightarrow \text{(f)}, \quad \text{(b)} \Longrightarrow \text{(d)} \Longrightarrow \text{(e)} \Longrightarrow \text{(f)}$$

(a) *F and G are F-soft at $(\overline{x}, \overline{y})$ and $(\overline{y}, \overline{z})$ respectively;*
(b) $N_L(F_Z, (\overline{x}, \overline{z}, \overline{y})) + N_L(G_X^{-1}, (\overline{x}, \overline{z}, \overline{y})) \subset N_F(C, (\overline{x}, \overline{z}, \overline{y}))$;
(c) $N_L(F_Z, (\overline{x}, \overline{z}, \overline{y})) + N_L(G_X^{-1}, (\overline{x}, \overline{z}, \overline{y})) \subset N_L(C, (\overline{x}, \overline{z}, \overline{y}))$;
(d) $D_L^* F(\overline{x}, \overline{y})(y^*) \times D_L^* G^{-1}(\overline{z}, \overline{y})(-y^*) \subset D_F^* C(\overline{x}, \overline{z}, \overline{y})(0)$ *for all* $y^* \in Y^*$;
(e) $D_L^* F(\overline{x}, \overline{y})(y^*) \times D_L^* G^{-1}(\overline{z}, \overline{y})(-y^*) \subset D_M^* C(\overline{x}, \overline{z}, \overline{y})(0)$ *for all* $y^* \in Y^*$.
(f) $D_L^* F(\overline{x}, \overline{y})(y^*) \times D_L^* G^{-1}(\overline{z}, \overline{y})(-y^*) \subset D_L^* C(\overline{x}, \overline{z}, \overline{y})(0)$ *for all* $y^* \in Y^*$.

Proof

(a)⟹(b) As already observed, in view of (12), we have $N_F(F_Z, (\overline{x}, \overline{z}, \overline{y})) \subset N_F(C, (\overline{x}, \overline{z}, \overline{y}))$ and $N_F(G_X^{-1}, (\overline{x}, \overline{z}, \overline{y})) \subset N_F(C, (\overline{x}, \overline{z}, \overline{y}))$, so that when (a) holds, the inclusion in (b) stems from the fact that $N_F(C, (\overline{x}, \overline{z}, \overline{y}))$ is a convex cone.
(b)⟹(c), (d)⟹(e)⟹(f) are obvious.
(c)⟹(f) has been proved in the preceding remark.
(b)⟹(d) is similar. □

Recall that a map $f : X \to Y$ between two metric spaces is said to be *stable at* $\overline{x} \in X$ (or Stepanovian at $\overline{x}$) if there exist some $\sigma > 0$ and some neighborhood U of

$\overline{x}$ such that $d(f(x), f(\overline{x})) \leq \sigma d(x, \overline{x})$ for all $x \in U$. Such a property is clearly satisfied when X and Y are normed vector spaces and f is (Fréchet) differentiable at $\overline{x}$.

Corollary 38 *Suppose F is a single-valued map which is stable at $\overline{x}$ or G^{-1} is a single-valued map which is stable at $\overline{z}$. Then (15) holds and, if Y is finite dimensional, (14) holds.*

In particular, when F is a single-valued map which is Fréchet differentiable (resp. Hadamard differentiable and stable) at $\overline{x}$, one has $F'(\overline{x})^* \circ D^*_F G(\overline{y}, \overline{z}) \subset D^*_F H(\overline{x}, \overline{z})$ (resp. $F'(\overline{x})^* \circ D^* G(\overline{y}, \overline{z}) \subset D^* H(\overline{x}, \overline{z})$) and when G^{-1} is a single-valued map which is Fréchet differentiable (resp. Hadamard differentiable and stable) at $\overline{z}$ one has $D^*_F F(\overline{x}, \overline{y}) \circ [(\left(G^{-1}\right)'(\overline{z}))^*]^{-1} \subset D^*_F H(\overline{x}, \overline{z})$ (resp. $D^* F(\overline{x}, \overline{y}) \circ [(\left(G^{-1}\right)'(\overline{z}))^*]^{-1} \subset D^* H(\overline{x}, \overline{z})$).

Proof Assuming that F is single-valued and stable at $\overline{x}$, setting $\overline{y} := F(\overline{x})$, taking $\sigma > 0$ and U as in the definition of stability, for all $(x, z) \in (U \times Z) \cap H$ one has $\|F(x) - \overline{y}\| \leq \sigma \|x - \overline{x}\| \leq \sigma \|(x, z) - (\overline{x}, \overline{z})\|$ and $F(x) \in C(x, z)$ since $z \in G(F(x))$, so that C is lsc on H at $(\overline{x}, \overline{z}, \overline{y})$ with linear rate σ.

When G^{-1} is single-valued and stable at $\overline{z}$, taking $\sigma > 0$ and U as in the definition of stability, for all $(x, z) \in (X \times U) \cap H$ one has $\left\|G^{-1}(z) - \overline{y}\right\| \leq \sigma \|z - \overline{z}\| \leq \sigma \|(x, z) - (\overline{x}, \overline{z})\|$ with $\overline{y} := G^{-1}(\overline{z})$ and $G^{-1}(z) \in C(x, z)$, so that again, C is lsc at $(\overline{x}, \overline{z}, \overline{y})$ with linear rate σ.

The last inclusion follows from the equivalences

$$\begin{aligned} y^* \in D^* G(\overline{y}, \overline{z})(z^*) &\Leftrightarrow -z^* \in D^* G^{-1}(\overline{z}, \overline{y})(-y^*) \\ &\Leftrightarrow z^* = (\left(G^{-1}\right)'(\overline{z}))^*(y^*) \Leftrightarrow y^* \in [(\left(G^{-1}\right)'(\overline{z}))^*]^{-1}(z^*). \end{aligned}$$

Let us note that in the last assertion we do not require Y be finite dimensional in the Hadamard differentiable case because we dispose of a Hadamard differentiable right inverse q of the projection $p_{X \times Z}$ given by $q(x, z) = (x, z, F(x))$ or $q(x, z) = (x, z, G^{-1}(z))$ for $(x, z) \in X \times Z$ which is such that $q(\overline{x}, \overline{z}) = (\overline{x}, \overline{z}, \overline{y})$, $q(H) \subset C$, as required by the assumptions of Lemma 35(a). □

Now, let us turn to the reverse of inclusion (14) and its variants. In order to get some versatility, we introduce a subset B of $C(\overline{x}, \overline{z})$. The extreme cases $B = C(\overline{x}, \overline{z})$ and B a singleton, $B = \{\overline{y}\}$, are the most remarkable cases, but intermediate situations may occur.

Theorem 39

(a) *Suppose that for some subset B of $C(\overline{x}, \overline{z})$ one has*

$$\bigcap_{\overline{y} \in C(\overline{x}, \overline{z})} D^* C(\overline{x}, \overline{z}, \overline{y})(0) \subset \bigcup_{\overline{y} \in B} \bigcup_{y^* \in Y^*} D^* F(\overline{x}, \overline{y})(y^*) \times D^* G^{-1}(\overline{z}, \overline{y})(-y^*). \quad (20)$$

Then one has

$$D^* H(\overline{x}, \overline{z}) \subset \bigcup_{\overline{y} \in B} D^* F(\overline{x}, \overline{y}) \circ D^* G(\overline{y}, \overline{z}). \quad (21)$$

(b) *Suppose that for some subset B of* $C(\overline{x}, \overline{z})$ *one has*

$$\bigcap_{\overline{y}\in C(\overline{x},\overline{z})} D_F^*C(\overline{x}, \overline{z}, \overline{y})(0) \subset \bigcup_{\overline{y}\in B} \bigcup_{y^*\in Y^*} D_F^*F(\overline{x}, \overline{y})(y^*) \times D_F^*G^{-1}(\overline{z}, \overline{y})(-y^*). \quad (22)$$

Then

$$D_F^*H(\overline{x}, \overline{z}) \subset \bigcup_{\overline{y}\in B} D_F^*F(\overline{x}, \overline{y}) \circ D_F^*G(\overline{y}, \overline{z}). \quad (23)$$

(c) *Suppose C is lsc at* $((\overline{x}, \overline{z}), B)$ *on H for some subset B of* $C(\overline{x}, \overline{z})$ *and that*

$$\bigcup_{\overline{y}\in B} D_M^*C(\overline{x}, \overline{z}, \overline{y})(0) \subset \bigcup_{\overline{y}\in B} \bigcup_{y^*\in Y^*} D_L^*F(\overline{x}, \overline{y})(y^*) \times D_L^*G^{-1}(\overline{z}, \overline{y})(-y^*). \quad (24)$$

Then

$$D_L^*H(\overline{x}, \overline{z}) \subset \bigcup_{\overline{y}\in B} D_L^*F(\overline{x}, \overline{y}) \circ D_L^*G(\overline{y}, \overline{z}). \quad (25)$$

Proof

(a) Let $z^* \in Z^*$ and let $x^* \in D^*H(\overline{x}, \overline{z})(z^*)$, i.e. $(x^*, -z^*) \in N(H, (\overline{x}, \overline{z}))$. For all $\overline{y} \in C(\overline{x}, \overline{z})$, the first assertion of Lemma 35 ensures that $(x^*, -z^*, 0) = p^*(x^*, -z^*) \in N(C, (\overline{x}, \overline{z}, \overline{y}))$, i.e. $(x^*, -z^*) \in D^*C(\overline{x}, \overline{z}, \overline{y})(0)$. Then (20) implies that there exists some $\overline{y}' \in B$, $y^* \in Y^*$ such that $x^* \in D^*F(\overline{x}, \overline{y}')(y^*)$, $-z^* \in D^*G^{-1}(\overline{z}, \overline{y}')(-y^*)$. As already observed, the last relation means that $y^* \in D^*G(\overline{y}', \overline{z})(z^*)$. Therefore, we have $x^* \in \left(D^*F(\overline{x}, \overline{y}') \circ D^*G(\overline{y}', \overline{z})\right)(z^*)$ and (21) holds.

(b) The proof is similar.

(c) Let $z^* \in Z^*$ and let $x^* \in D_L^*H(\overline{x}, \overline{z})(z^*)$. Then $(x^*, -z^*) \in N_L(H, (\overline{x}, \overline{z}))$ and since C is lsc at $(\overline{x}, \overline{z}, B)$ on H, Lemma 35(d) yields some $\overline{y} \in B$ such that $(x^*, -z^*) \in D_M^*C((\overline{x}, \overline{z}), \overline{y})(0)$, hence, by (24) there exists some $\overline{y}' \in B$, $y^* \in Y^*$ such that $x^* \in D_L^*F(\overline{x}, \overline{y}')(y^*)$, $-z^* \in D_L^*G^{-1}(\overline{z}, \overline{y}')(-y^*)$. Thus $x^* \in \left(D_L^*F(\overline{x}, \overline{y}') \circ D_L^*G(\overline{y}', \overline{z})\right)(z^*)$.

□

We give an example to illustrate the conditions and conclusions of Theorem 39.

Example Let $X = Y = Z = \mathbb{R}$ and let F, G be defined by $F(x) = \{0, x\}$ for $x \in X$ and $G(y) = |y - 1|$ for $y \in Y$, so that $H(x) := G(F(x)) = \{1, |x - 1|\}$ for $x \in X$ and

$$C = \mathbb{R} \times \{(1, 0)\} \cup \{(x, |x - 1|, x) \mid x \in \mathbb{R}\}.$$

We see that $C(x, z) = \{x\}$ for $z = |1 - x|$ and $z \neq 1$, $C(x, 1) = \{0\}$ for $x \neq 2$, $C(2, 1) = \{0, 2\}$ and $C(x, z) = \varnothing$ else. Let $(\overline{x}, \overline{z}) = (2, 1)$, $\overline{y}_1 = 0$ and $\overline{y}_2 = 2$. Clearly, $\overline{z} \in (G \circ F)(\overline{x})$ and $C(\overline{x}, \overline{z}) = \{\overline{y}_1, \overline{y}_2\}$. Easy computations show that, for any $y^* \in \mathbb{R}$, $z^* \in \mathbb{R}$, $D^*F(\overline{x}, \overline{y}_1)(y^*) = 0$, $D^*F(\overline{x}, \overline{y}_2)(y^*) = y^*$, $D^*G(\overline{y}_1, \overline{z})(z^*) = -z^*$, $D^*G(\overline{y}_2, \overline{z})(z^*) = z^*$. Similar relations hold for the Fréchet and the limiting coderivatives. On the other hand, $D^*C(\overline{x}, \overline{z}, \overline{y}_1)(0) = D_F^*C(\overline{x}, \overline{z}, \overline{y}_1)(0) = D_L^*C(\overline{x}, \overline{z}, \overline{y}_1)(0) = \{0\} \times \mathbb{R}$ and $D^*C(\overline{x}, \overline{z}, \overline{y}_2)(0) = D_F^*C(\overline{x}, \overline{z}, \overline{y}_2)(0) = D_M^*C(\overline{x}, \overline{z}, \overline{y}_2)(0) = \{(x^*, z^*) \in \mathbb{R} \times \mathbb{R} : x^* + z^* = 0\}$. Therefore, (20), (22) and (24) are satisfied, so that (21), (23) and (25) hold. That can be checked directly as $D^*H(\overline{x},\overline{z}) = D^*_F H(\overline{x},\overline{z}) = \{(0,0)\}$ and $D^*_L H(\overline{x},\overline{z})(z^*) = \{0, z^*\}$ for any $z^* \in \mathbb{R}$.

Remark Let us observe that the condition

$$\forall \overline{y} \in B \qquad N_L(C,(\overline{x},\overline{z},\overline{y})) \subset N_L(F_Z,(\overline{x},\overline{z},\overline{y})) + N_L(G_X^{-1},(\overline{x},\overline{z},\overline{y})) \tag{26}$$

implies (24) and similarly

$$\forall \overline{y} \in B \qquad N(C,(\overline{x},\overline{z},\overline{y})) \subset N(F_Z,(\overline{x},\overline{z},\overline{y})) + N(G_X^{-1},(\overline{x},\overline{z},\overline{y})), \tag{27}$$

$$\forall \overline{y} \in B \qquad N_F(C,(\overline{x},\overline{z},\overline{y})) \subset N_F(F_Z,(\overline{x},\overline{z},\overline{y})) + N_F(G_X^{-1},(\overline{x},\overline{z},\overline{y})) \tag{28}$$

imply (20) and (22) respectively. In fact, given $(x^*, z^*) \in D^*_L C(\overline{x},\overline{z},\overline{y})(0)$, so that $(x^*, z^*, 0) \in N_L(C,(\overline{x},\overline{z},\overline{y}))$, (26) asserts that one can find $y^* \in Y^*$ such that

$$(x^*, 0, -y^*) \in N_L(F_Z,(\overline{x},\overline{z},\overline{y})), \qquad (0, z^*, y^*) \in N_L(G_X^{-1},(\overline{x},\overline{z},\overline{y})).$$

Since the projections $p_{X\times Y} : F_Z \to F$ and $p_{Z\times Y} : G_X^{-1} \to G^{-1}$ are open with a linear rate, Lemma 35(c) entails that $(x^*, -y^*) \in N_L(F,(\overline{x},\overline{y}))$, $(z^*, y^*) \in N_L(G^{-1},(\overline{z},\overline{y}))$, or $(x^*, z^*) \in D^*_L F(\overline{x},\overline{y})(y^*) \times D^*_L G^{-1}(\overline{z},\overline{y})(-y^*)$. □

More sufficient conditions for (24) will be drawn from Section 3.

In the next corollary we point out the links with an approach using metric conditions, as in [26] and [64].

Corollary 40 *Suppose X, Y, Z are Asplund spaces, F and G have closed graphs, C is lsc at $((\overline{x},\overline{z}), B)$ on H and for every $\overline{y} \in B$ there are $c > 0$ and a neighborhood U of $(\overline{x},\overline{z},\overline{y})$ such that for all $(x, y, z) \in U$ one has*

$$d((x,z,y),C) \leq cd((x,y),F) + cd((y,z),G). \tag{29}$$

Then (25) holds.

Proof Relation (29) ensures that for all $\overline{y} \in B$ condition (26) is satisfied by well known rules for computing limiting subdifferentials in Asplund spaces and the fact that the limiting normal cone to a subset C at some $w \in C$ is the cone generated by $\partial_L d_C(w)$. □

In the following corollary we make a comparison with [64, Corollary 5.6] and [41, Theorem 3.13].

Definition 41 [64, Definition 5.3] The multimaps $F : X \rightrightarrows Y$, $G : Y \rightrightarrows Z$ are said to be *allied* at (x, y, z) with $y \in R(x,z) := F(x) \cap G^{-1}(z)$, if for any sequences $(x_n, y_n) \overset{F}{\to} (x, y)$, $(w_n, z_n) \overset{G}{\to} (y, z)$, $(x_n^*) \to 0$, (w_n^*),(y_n^*) in Y^*, $(z_n^*) \to 0$ with $(w_n^* - y_n^*) \to 0$, $x_n^* \in D^*_F F(x_n, y_n)(y_n^*)$, $w_n^* \in D^*_F G(w_n, z_n)(z_n^*)$ one has $(y_n^*) \to 0$. They are said to be *synergetic* at (x, y, z) if the conditions $(y_n^*) \overset{*}{\to} 0$, $(w_n^*) \overset{*}{\to} 0$ are added to the preceding assumptions.

Thus the multimaps F and G are allied (resp. synergetic) at $(\overline{x},\overline{y},\overline{z})$ if F^{-1} and G are cooperative (resp. coordinated) at $(\overline{y},\overline{x},\overline{z})$. Synergy is obviously fulfilled if Y is finite dimensional. It is also satisfied if F is coderivatively compact at $(\overline{x}, \overline{y})$ or if G^{-1} is coderivatively compact at $(\overline{z}, \overline{y})$.

Let us note that the multimaps F and G are synergetic (resp. allied) at $(\overline{x}, \overline{y}, \overline{z})$ if, and only if the associated sets F_Z and G_X^{-1} are synergetic (resp. allied) subsets of $X \times Z \times Y$. Therefore, to get an estimate of the normal cone to $C = F_Z \cap G_X^{-1}$ we can use Proposition 24.

Theorem 42 *Suppose X, Y, Z are Asplund spaces, F and G have closed graphs and are allied at $(\overline{x}, \overline{y}, \overline{z})$. Then (24) holds with $B := \{\overline{y}\}$. If, for some subset B of $C(\overline{x}, \overline{z})$, C is lsc at $((\overline{x}, \overline{z}), B)$ on H and F and G are allied at $(\overline{x}, \overline{y}, \overline{z})$ for all $\overline{y} \in B$, then (25) holds.*

We can also use Proposition 27. However, the qualification condition we present in the next corollary is weaker than the condition obtained from (1) or (6) for F_Z and G_X^{-1}.

Corollary 43 *Suppose X, Y, Z are Asplund spaces, F and G have closed graphs and are synergetic at $(\overline{x}, \overline{y}, \overline{z})$ for all $\overline{y} \in B$. Then (24) holds whenever the following condition is satisfied for all $\overline{y} \in B$:*

$$(-D_M^* F^{-1}(\overline{y}, \overline{x})(0)) \cap D_M^* G(\overline{y}, \overline{z})(0) = \{0\}. \tag{30}$$

When Y is finite dimensional, one can replace condition (30) with

$$\left(D_L^* F(\overline{x}, \overline{y})\right)^{-1}(0) \cap D_L^* G(\overline{y}, \overline{z})(0) = \{0\}. \tag{31}$$

Proof We apply Corollary 33 with $X_1 := X$, $X_2 := Z$, $G_1 := F$, $G_2 := G^{-1}$, as $C(x, z) = F(x) \cap G^{-1}(z)$ for all $(x, z) \in X \times Z$ and as (30) coincides with (11). □

Using Lemma 12, we get the following consequence.

Corollary 44 *Suppose X, Y, Z are Asplund spaces, F and G have closed graphs, and, for every $\overline{y} \in B$, either G is Lipschitz-like around $(\overline{y}, \overline{z})$ or F^{-1} is Lipschitz-like around $(\overline{y}, \overline{x})$. Then (24) holds.*

Combining Proposition 36 with Theorem 39, we get an exact expression for the coderivative of the composition H. Corollary 43 can also be used here.

Corollary 45 *Suppose $C : X \times Z \rightrightarrows Y$ is lsc at $(\overline{x}, \overline{z}, \overline{y})$ on H with a linear rate and (22) holds (resp. (20) holds, Y being finite dimensional) with $B := \{\overline{y}\}$. Then*

$$D_F^* H(\overline{x}, \overline{z}) = D_F^* F(\overline{x}, \overline{y}) \circ D_F^* G(\overline{y}, \overline{z}) \tag{32}$$

$$(\text{resp. } D^* H(\overline{x}, \overline{z}) = D^* F(\overline{x}, \overline{y}) \circ D^* G(\overline{y}, \overline{z})). \tag{33}$$

If moreover F and G have closed graphs, F is F-soft (resp. soft) at $(\overline{x}, \overline{y})$, G is F-soft (resp. soft) at $(\overline{y}, \overline{z})$ and (24) holds, then H is F-soft (resp. soft) at $(\overline{x}, \overline{y})$ and

$$D_L^* H(\overline{x}, \overline{z}) = D_L^* F(\overline{x}, \overline{y}) \circ D_L^* G(\overline{y}, \overline{z}).$$

Proof The first assertions are consequences of Proposition 36(b) (resp. (a)) and Theorem 39(b) (resp. (a)). The last one follows from Proposition 36(b) (resp. (a)), Theorem 39(c) and the inclusions

$$D_L^*H(\overline{x},\overline{z}) \subset D_L^*F(\overline{x},\overline{y}) \circ D_L^*G(\overline{y},\overline{z}) = D_F^*F(\overline{x},\overline{y}) \circ D_F^*G(\overline{y},\overline{z}) \subset D_F^*H(\overline{x},\overline{z})$$

and their analogues with directional coderivatives. □

A simple case in which condition (20) is satisfied is given in the next corollary with some variants; thus, we recover the Fréchet and limiting cases obtained in [47, Theorem 4.6] and [41, Theorem 3.13]. Here a map $G : Y \to Z$ is said to be *strictly differentiable* at $\overline{y} \in Y$ if there exist a continuous linear map $A : Y \to Z$ and a modulus μ such that $\|G(y) - G(y') - A(y - y')\| \leq \mu(\|y - \overline{y}\| + \|y' - \overline{y}\|)\,\|y - y'\|$.

Corollary 46 *Suppose G is a single-valued map which is Hadamard differentiable at $\overline{y} \in F(\overline{x})$ (resp. Fréchet differentiable at $\overline{y}$, resp. strictly differentiable at $\overline{y}$, C being lsc at $((\overline{x},\overline{z}),\overline{y})$ on H, with $\overline{z} := G(\overline{y})$). Then*

$$D^*H(\overline{x},\overline{z}) \subset D^*F(\overline{x},\overline{y}) \circ \left(G'(\overline{y})\right)^*$$

$$(\textit{resp. } D_F^*H(\overline{x},\overline{z}) \subset D_F^*F(\overline{x},\overline{y}) \circ \left(G'(\overline{y})\right)^*,$$

$$\textit{resp. } D_L^*H(\overline{x},\overline{z}) \subset D_L^*F(\overline{x},\overline{y}) \circ \left(G'(\overline{y})\right)^*).$$

Proof Let us observe that for every $(u, v) \in T(F, (\overline{x},\overline{y}))$ we have $(u, G'(\overline{y})v, v) \in T(C, (\overline{x},\overline{z},\overline{y}))$ since we can find sequences $(t_n) \to 0_+$, $((u_n, v_n)) \to (u, v)$ such that $(\overline{x} + t_n u_n, G(\overline{y} + t_n v_n), \overline{y} + t_n v_n) \in C$ for all n. Thus, for every $(x^*, z^*) \in D^*C(\overline{x},\overline{z},\overline{y})(0)$, we have

$$\langle (x^*, \left(G'(\overline{y})\right)^* z^*), (u, v)\rangle = \langle (x^*, z^*, 0), (u, G'(\overline{y})v, v)\rangle \leq 0,$$

hence, for $y^* := -\left(G'(\overline{y})\right)^*(z^*)$, we obtain $x^* \in D^*F(\overline{x},\overline{y})(y^*)$, $z^* \in D^*G^{-1}(\overline{z},\overline{y})(-y^*)$, so that (20) is satisfied with $B := \{\overline{y}\}$.

Suppose $(x^*, z^*) \in D_F^*C(\overline{x},\overline{z},\overline{y})(0)$. Let $y^* := -\left(G'(\overline{y})\right)^*(z^*)$ and let r be a remainder such that $\langle (x^*, z^*, 0), (x - \overline{x}, z - \overline{z}, y - \overline{y})\rangle \leq r(x - \overline{x}, z - \overline{z}, y - \overline{y})$ for all $(x, z, y) \in C$. Then, for all $(x, y) \in F$, taking $z := G(y)$, we get

$$\begin{aligned}
&\langle (x^*, -y^*), (x - \overline{x}, y - \overline{y})\rangle \\
&\quad = \langle (x^*, z^*, 0), (x - \overline{x}, G'(\overline{y})(y - \overline{y}), y - \overline{y})\rangle \\
&\quad \leq r(x - \overline{x}, z - \overline{z}, y - \overline{y}) + \|z^*\| . \|G(y) - G(\overline{y}) - G'(\overline{y})(y - \overline{y})\| \\
&\quad = r(x - \overline{x}, G'(\overline{y})(y - \overline{y}) + r_1(y - \overline{y}), y - \overline{y}) + \|z^*\| . \|G(y) - G(\overline{y}) - G'(\overline{y})(y - \overline{y})\|,
\end{aligned}$$

where r_1 is another remainder. Since this last term is a remainder in (x, y), we obtain $x^* \in D_F^*F(\overline{x},\overline{y})(y^*)$. Thus (22) is satisfied for $B := \{\overline{y}\}$.

When G is strictly differentiable at $\overline{y}$, using similar arguments, one can show that (24) holds. The last inclusion follows from Theorem 39(c) and the expression of the coderivative of G. □

Relaxing the lower semicontinuity condition on C, we get the following variant.

Corollary 47 *Suppose F has a closed graph, C is lsc at $((\overline{x}, \overline{z}), B)$ on H and, for every $\overline{y} \in B$, G is single-valued around $\overline{y}$ and strictly differentiable at $\overline{y}$. Then* (24) *holds and*

$$D_L^* H(\overline{x}, \overline{z}) \subset \bigcup_{\overline{y} \in B} D_L^* F(\overline{x}, \overline{y}) \circ \left(G'(\overline{y})\right)^* . \tag{34}$$

Replacing Y by $Y \times Y$ and taking for G a continuously differentiable operation, one gets numerous calculus rules as in [41, Theorem 3.18, Corollary 3.19]. In the next section we shall consider the case of the sum.

Corollary 48 *Suppose* (22) *(resp.* (20)*) holds and F is a single-valued map which is stable at $\overline{x}$ or G^{-1} is a single-valued map which is stable at $\overline{z}$. Then* (32) *holds (resp.* (33) *holds if Y is finite dimensional).*

In particular, when (22) *(resp.* (20)*) holds and F is a single-valued map which is Fréchet differentiable (resp. Hadamard differentiable and stable) at $\overline{x}$ one has $D_F^* H(\overline{x}, \overline{z}) = F'(\overline{x})^* \circ D_F^* G(\overline{y}, \overline{z})$ (resp. $D^* H(\overline{x}, \overline{z}) = F'(\overline{x})^* \circ D^* G(\overline{y}, \overline{z})$) and when G^{-1} is a single-valued map which is Fréchet differentiable (resp. Hadamard differentiable and stable) at $\overline{z}$ one has $D_F^* H(\overline{x}, \overline{z}) = D_F^* F(\overline{x}, \overline{y}) \circ ((G^{-1})'(\overline{z})^*)^{-1}$ (resp. $D_F^* H(\overline{x}, \overline{z}) = D_F^* F(\overline{x}, \overline{y}) \circ ((G^{-1})'(\overline{z})^*)^{-1}$).*

Under strict differentiability of F or G^{-1}, one gets an analogous assertion about $D_L^* H(\overline{x}, \overline{z})$.

Proof As already observed, the assumptions on F or G^{-1} ensure that C is lsc at $(\overline{x}, \overline{z}, \overline{y})$ on H with a linear rate.

The second assertions are consequences of calculations made in the proof of Corollary 38. □

Let us give two examples to show that the qualification conditions we present are slightly more general than the conditions in [41, Theorem 3.13].

Example Let $X = Y = Z = \mathbb{R}$ and let F, G be given by $F(x) = \{0\}$ for $x \in X$, $G(y) := [0, y]$ for $y \in \mathbb{R}_+$, $G(y) := \varnothing$ else. Then $H := G \circ F = X \times \{0\}$. For $\overline{x} = 0$, $\overline{y} = 0$, $\overline{z} = 0$ one has $D^* F(\overline{x}, \overline{y})(y^*) = \{0\}$ for all $y^* \in \mathbb{R}$, $D^* G(\overline{y}, \overline{z})(z^*) = (-\infty, 0]$ for $z^* \in \mathbb{R}_+$, $D^* G(\overline{y}, \overline{z})(z^*) = (-\infty, -z^*]$ else and $D^* H(\overline{x}, \overline{z})(z^*) = \{0\}$. Similar relations hold for the Fréchet, the limiting and the mixed coderivatives. The qualification condition (31) does not hold as

$$\left(D_M^* F(\overline{x}, \overline{y})\right)^{-1}(0) \cap D_M^* G(\overline{y}, \overline{z})(0) = (-\infty, 0].$$

On the other hand, $C = (\mathbb{R} \times \{0\}) \times \{0\}$, so that $N_L(C, (\overline{x}, \overline{z}, \overline{y})) = \{0\} \times \mathbb{R} \times \mathbb{R}$, while $F_Z = \mathbb{R} \times \mathbb{R} \times \{0\}$, $G_X^{-1} = \mathbb{R} \times G^{-1}$ so that $N_L(F_Z, (0, 0, 0)) = \{0\} \times \{0\} \times \mathbb{R}$, $N_L(G_X^{-1}, (0, 0, 0)) = \{0\} \times N_L(G^{-1}, (0, 0)) = \{(0, z^*, y^*) : y^* \leq 0, z^* \leq -y^*\}$ and (19) and (26) are satisfied. Note that here F, G, H are simple convex processes, not sophisticated multimaps.

Example Suppose that $X = Y = Z = \mathbb{R}$, $F(x) = \{x\}$ for $x \in \{0\} \cup \{a_n : n \in \mathbb{N}\}$, $F(x) = \varnothing$ else, where (a_n) is a decreasing sequence of positive numbers with limit 0 and $G(y) = \mathbb{R}_+$ for $x \in \mathbb{R}_+$, $G(y) = \varnothing$ else. Let $\overline{x} = 0$, $\overline{y} = 0$, $\overline{z} = 0$. The definition of the limiting coderivative shows that $D_L^* F(\overline{x}, \overline{y})(y^*) = \mathbb{R}$ for all $y^* \in \mathbb{R}$, $D_L^* G(\overline{y}, \overline{z})(z^*) = \mathbb{R}_-$ for $z^* \in \mathbb{R}_+$, $D_L^* G(\overline{y}, \overline{z})(z^*) = \varnothing$ else. Then, $D_M^* F(\overline{x}, \overline{y})^{-1}(0) \cap D_M^* G(\overline{y}, \overline{z})(0) = \mathbb{R}_- \neq \{0\}$. However, since $C(x, z) = F(x)$ for $(x, z) \in \mathbb{R} \times \mathbb{R}_+$, $C(x, z) = \varnothing$ else and $N_L(C, (\overline{x}, \overline{z}, \overline{y})) = \mathbb{R} \times \mathbb{R}_- \times \mathbb{R}$, we have $D_L^* C(\overline{x}, \overline{z}, \overline{y})(0) = \mathbb{R} \times \mathbb{R}_-$ and thus conditions (16) and (24) hold with $B := \{0\}$.

5 Coderivatives of Sums

Now we turn to the case of the sum $S := F_1 + F_2$ of two multimaps $F_1, F_2 : X \rightrightarrows Y$. There are several ways of reducing the computation of the coderivatives of a sum to the case of a composition. One can decompose $F_1 + F_2$ into $F_1 + F_2 = G \circ F$ or $F_1 + F_2 = Q \circ P$, where

$$F := (F_1, F_2) : x \rightrightarrows F_1(x) \times F_2(x), \quad G : (y_1, y_2) \mapsto y_1 + y_2,$$
$$P : x \rightrightarrows \{x\} \times F_1(x), \quad Q : (x, y) \rightrightarrows y + F_2(x).$$

For the first decomposition the intermediate space is Y^2 while for the second one it is $X \times Y$. Both ways are interesting: the first one is symmetric but the second one allows asymmetric assumptions such as differentiability of one of the maps F_1, F_2. Moreover, the codifferential properties of F_1 and F_2 are easily transferred to P and Q. These close relationships allow to easily translate the results of the preceding sections. On the other hand, while G is a map easy to deal with, the coderivative of $F := (F_1, F_2)$ cannot be obtained from the coderivatives of F_1 and F_2 without some qualification condition. Thus we shall use both decompositions and set for $(x, z) \in X \times Y$

$$R_1(x, z) := F_1(x) \cap (z - F_2(x)), \quad R_2(x, z) := F_2(x) \cap (z - F_1(x)),$$
$$C(x, z) := \{(y_1, y_2) \in F_1(x) \times F_2(x) : y_1 + y_2 = z\}.$$

Thus, the resultant multimap C_1 associated to P and Q as in the preceding section is given by

$$C_1(x, z) := P(x) \cap Q^{-1}(z) = \{x\} \times R_1(x, z),$$

while interchanging the roles of F_1 and F_2 would lead to consider R_2 and C_2 given by $C_2(x, z) = \{x\} \times R_2(x, z)$. Because of the symmetric character of the multimap C associated with the first decomposition, we shall use it in most statements. Clearly C is lsc (resp. lsc with a linear rate) at $(\overline{x}, \overline{z}, (\overline{y}_1, \overline{y}_2))$ if, and only if, R_1 is lsc (resp. lsc with a linear rate) at $(\overline{x}, \overline{z}, \overline{y}_1)$, if, and only if, R_2 is lsc (resp. lsc with a linear rate) at $(\overline{x}, \overline{z}, \overline{y}_2)$. Here and in the sequel, $\overline{x} \in X$, $\overline{y}_1 \in F_1(\overline{x})$, $\overline{y}_2 \in F_2(\overline{x})$, $\overline{z} := \overline{y}_1 + \overline{y}_2$.

Lemma 49 *The coderivatives of* $P : x \mapsto \{x\} \times F_1(x)$ *and* $Q : (x, y) \mapsto y + F_2(x)$ *are given by*

$$D^* P(\overline{x}, (\overline{x}, \overline{y}_1))(x^*, y^*) = x^* + D^* F_1(\overline{x}, \overline{y}_1)(y^*),$$
$$D^* Q((\overline{x}, \overline{y}_1), \overline{z})(z^*) = D^* F_2(\overline{x}, \overline{y}_2)(z^*) \times \{z^*\},$$

with similar relations when D^* is replaced with D^*_F or D^*_L. Moreover, for every $\overline{y}_1 \in R_1(\overline{x}, \overline{z})$

$$D^*P(\overline{x}, (\overline{x}, \overline{y}_1)) \circ D^*Q((\overline{x}, \overline{y}_1), \overline{z}) = D^*F_1(\overline{x}, \overline{y}_1) + D^*F_2(\overline{x}, \overline{y}_2)$$

and similar relations in which D^* is replaced with D^*_F or D^*_L.

Proof

(a) Clearly, $T(P, (\overline{x}, \overline{x}, \overline{y}_1)) = \{(u, u, v) : (u, v) \in T(F_1, (\overline{x}, \overline{y}_1))\}$, so that $u^* \in D^*P(\overline{x}, (\overline{x}, \overline{y}_1))(x^*, y^*)$ if, and only if $u^* - x^* \in D^*F_1(\overline{x}, \overline{y}_1)(y^*)$ or $u^* \in x^* + D^*F_1(\overline{x}, \overline{y}_1)(y^*)$.
Now $(u, v, w) \in T(Q, (\overline{x}, \overline{y}_1, \overline{z}))$ if, and only if, there exist sequences $(t_n) \to 0_+$, $((u_n, v_n, w_n)) \to (u, v, w)$ such that $\overline{z} + t_n w_n \in \overline{y}_1 + t_n v_n + F_2(\overline{x} + t_n u_n)$ for all n, if, and only if $(u, w - v) \in T(F_2, (\overline{x}, \overline{z} - \overline{y}_1))$. Therefore, $(x^*, y^*) \in D^*Q((\overline{x}, \overline{y}_1), \overline{z})(z^*)$ if, and only if, $y^* = z^*$ and $x^* \in D^*F_2(\overline{x}, \overline{y}_2)(z^*)$.

(b) Let $x^* \in X^*$, $y^* \in Y^*$, $x_1^* \in D^*_F F_1(\overline{x}, \overline{y}_1)(y^*)$ and let r_1 be a remainder such that $\langle (x_1^*, -y^*), (x, y) - (\overline{x}, \overline{y}_1)\rangle \le r_1(x - \overline{x}, y - \overline{y}_1)$ for $(x, y) \in F_1$. Then, whenever $(x, x, y) \in P$, we have $(x, y) \in F_1$ hence

$$\begin{aligned}\langle (x^* + x_1^*, -x^*, -y^*), (x, x, y) - (\overline{x}, \overline{x}, \overline{y}_1)\rangle &= \langle (x_1^*, -y^*), (x, y) - (\overline{x}, \overline{y}_1)\rangle \\ &\le r_1(x - \overline{x}, y - \overline{y}_1),\end{aligned}$$

so that $x^* + x_1^* \in D^*_F P(\overline{x}, (\overline{x}, \overline{y}_1))(x^*, y^*)$. Conversely, given $x^*, x_1^* \in X^*$, $y^* \in Y^*$ such that $x^* + x_1^* \in D^*_F P(\overline{x}, (\overline{x}, \overline{y}_1))(x^*, y^*)$, let r be a remainder such that for all $(x, y) \in F_1$

$$\langle (x^* + x_1^*, -x^*, -y^*), (x, x, y) - (\overline{x}, \overline{x}, \overline{y}_1)\rangle \le r(x - \overline{x}, y - \overline{y}_1).$$

Then we have $\langle (x_1^*, -y^*), (x, y) - (\overline{x}, \overline{y}_1)\rangle \le r(x - \overline{x}, y - \overline{y}_1)$ and $x_1^* \in D^*_F F_1(\overline{x}, \overline{y}_1)(y^*)$.
Now, let $z^* \in Y^*$ and let $(x^*, y^*) \in D^*_F Q((\overline{x}, \overline{y}_1), \overline{z})(z^*)$. Let r be a remainder such that $\langle (x^*, y^*, -z^*), (x, y, z) - (\overline{x}, \overline{y}_1, \overline{z})\rangle \le r(x - \overline{x}, y - \overline{y}_1, z - \overline{z})$ for all $(x, y, z) \in Q$. Taking $y = \overline{y}_1$ we see that $x^* \in D^*F_2(\overline{x}, \overline{y}_2)(z^*)$. Taking $x = \overline{x}$ and $z = \overline{y}_2 + y$, we get $y^* = z^*$. Conversely, it is easy to see that for every (x^*, z^*) such that $x^* \in D^*F_2(\overline{x}, \overline{y}_2)(z^*)$ one has $(x^*, z^*) \in D^*Q((\overline{x}, \overline{y}_1), \overline{z})(z^*)$.

(c) The case of the limiting coderivatives is obtained by easy passages to the limit. Now given $z^* \in Z^*$, $(u^*, y^*) \in D^*Q((\overline{x}, \overline{y}_1), \overline{z})(z^*)$ and $x^* \in D^*P(\overline{x}, (\overline{x}, \overline{y}_1))(u^*, y^*)$ one has $u^* \in D^*F_2(\overline{x}, \overline{y}_1)(z^*)$, $y^* = z^*$ and $x^* - u^* \in D^*F_1(\overline{x}, \overline{y}_1)(y^*)$, hence $x^* \in D^*F_1(\overline{x}, \overline{y}_1)(z^*) + D^*F_2(\overline{x}, \overline{y}_2)(z^*)$. Conversely, if $x^* \in D^*F_1(\overline{x}, \overline{y}_1)(z^*) + D^*F_2(\overline{x}, \overline{y}_2)(z^*)$, taking $u^* \in D^*F_2(\overline{x}, \overline{y}_1)(z^*)$ such that $x^* - u^* \in D^*F_1(\overline{x}, \overline{y}_1)(z^*)$, we see that $(u^*, z^*) \in D^*Q((\overline{x}, \overline{y}_1), \overline{z})(z^*)$ and $x^* \in D^*P(\overline{x}, (\overline{x}, \overline{y}_1))(u^*, z^*)$, so that $x^* \in (D^*P(\overline{x}, (\overline{x}, \overline{y}_1)) \circ D^*Q((\overline{x}, \overline{y}_1), \overline{z}))(z^*)$. A similar analysis holds for D^*_F and D^*_L. □

Similar computations yield the following formula for the coderivative of C_1.

Lemma 50 *The coderivative of C_1 at $((\overline{x}, \overline{z}), (\overline{x}, \overline{y}_1)) \in C_1$ satisfies*

$$D^*C_1((\overline{x}, \overline{z}), (\overline{x}, \overline{y}_1))(x^*, z^*) = (x^*, 0) + D^*R_1((\overline{x}, \overline{z}), \overline{y}_1)(z^*),$$

for all $(x^*, z^*) \in X^* \times Y^*$, *with similar expressions for the Fréchet and the limiting coderivatives.*

Proposition 51

(a) *Suppose* Y *is finite dimensional and the multimap* $R_1 : X \times Y \rightrightarrows Y$ *is lsc at* $(\overline{x}, \overline{z}, \overline{y}_1)$ *on* S *with a linear rate. Then,*

$$D^* F_1(\overline{x}, \overline{y}_1) + D^* F_2(\overline{x}, \overline{y}_2) \subset D^* S(\overline{x}, \overline{y}_1 + \overline{y}_2). \tag{35}$$

(b) *Suppose the multimap* $R_1 : X \times Z \rightrightarrows Y$ *is lsc at* $(\overline{x}, \overline{z}, \overline{y}_1)$ *on* S *with a linear rate. Then*

$$D^*_F F_1(\overline{x}, \overline{y}_1) + D^*_F F_2(\overline{x}, \overline{y}_2) \subset D^*_F S(\overline{x}, \overline{y}_1 + \overline{y}_2). \tag{36}$$

(c) *If the multimap* $R_1 : X \times Y \rightrightarrows Y$ *is lsc at* $((\overline{x}, \overline{z}), \overline{y}_1)$ *on* S *with a linear rate and if*

$$\forall y^* \in Y^* \quad (D^*_L F_1(\overline{x}, \overline{y}_1)(y^*) + D^*_L F_2(\overline{x}, \overline{y}_2)(y^*), -y^*) \subset D^*_F R_1((\overline{x}, \overline{z}), \overline{y}_1)(0) \tag{37}$$

then

$$D^*_L F_1(\overline{x}, \overline{y}_1) + D^*_L F_2(\overline{x}, \overline{y}_2) \subset D^*_L S(\overline{x}, \overline{y}_1 + \overline{y}_2). \tag{38}$$

Proof (a) and (b) follow from Proposition 36 and the last part of Lemma 49. For assertion (c), let us show that (37) implies (16) with C, F, G replaced with C_1, P, Q. Let $(x^*, y^*) \in X^* \times Y^*$ and let $u^* \in D^*_L P(\overline{x}, (\overline{x}, \overline{y}_1))(x^*, y^*)$, $z^* \in D^*_L Q^{-1}(\overline{z}, (\overline{x}, \overline{y}_1))(-x^*, -y^*)$. Then by Lemma 49 $u^* - x^* \in D^*_L F_1(\overline{x}, \overline{y}_1)(y^*)$ and $(x^*, y^*) \in D^*_L Q((\overline{x}, \overline{y}_1), \overline{z})(-z^*) = D^*_L F_2(\overline{x}, \overline{y}_2)(-z^*) \times \{-z^*\}$, so that $z^* = -y^*$ and $x^* \in D^*_L F_2(\overline{x}, \overline{y}_2)(y^*)$, hence, by (37) and Lemma 50, $(u^*, z^*) \in D^*_F R_1((\overline{x}, \overline{z}), \overline{y}_1)(0) = D^*_F C_1((\overline{x}, \overline{z}), (\overline{x}, \overline{y}_1))(0, 0)$: (16) holds with C replaced with C_1. □

Now, let us turn to the reverse inclusion. We first use the multimap $F := (F_1, F_2)$.

Theorem 52 *Suppose that* F_1 *and* F_2 *have closed graphs,* C *is lsc at* $((\overline{x}, \overline{z}), B)$ *on* S *for some subset* B *of* $C(\overline{x}, \overline{z})$ *and, for every* $\overline{y} := (\overline{y}_1, \overline{y}_2) \in B$, *one has*

$$D^*_L (F_1, F_2)(\overline{x}, (\overline{y}_1, \overline{y}_2))(y^*_1, y^*_2) \subset D^*_L F_1(\overline{x}, \overline{y}_1)(y^*_1) + D^*_L F_2(\overline{x}, \overline{y}_2)(y^*_2). \tag{39}$$

Then, for all $z^* \in Y^*$, *one has the inclusion*

$$D^*_L (F_1 + F_2)(\overline{x}, \overline{z})(z^*) \subset \bigcup_{(\overline{y}_1, \overline{y}_2) \in B} D^*_L F_1(\overline{x}, \overline{y}_1)(z^*) + D^*_L F_2(\overline{x}, \overline{y}_2)(z^*). \tag{40}$$

Proof This is an immediate consequence of Corollary 47, as G is linear continuous, hence strictly differentiable, with $\left(G'(\overline{y})\right)^* (z^*) = (z^*, z^*)$, so that relation (34) yields for all $z^* \in Y^*$

$$D^*_L S(\overline{x}, \overline{z})(z^*) \subset \bigcup_{\overline{y} \in B} \left(D^*_L F(\overline{x}, \overline{y}) \circ \left(G'(\overline{y})\right)^*\right)(z^*) = \bigcup_{\overline{y} \in B} D^*_L F(\overline{x}, \overline{y})(z^*, z^*).$$

Then, (39) entails (40). □

Let us observe that (39) is a rather abstract assumption. It can be replaced by the following one in which we set $M_1(x) := F_1(x) \times Y$, $M_2(x) := Y \times F_2(x)$: for all $(\overline{y}_1, \overline{y}_2) \in B$

$$N_L((F_1, F_2), (\overline{x}, \overline{y}_1, \overline{y}_2)) \subset N_L(M_1, (\overline{x}, \overline{y}_1, \overline{y}_2)) + N_L(M_2, (\overline{x}, \overline{y}_1, \overline{y}_2)).$$

When F_1 and F_2 are coordinated at $(\overline{x}, \overline{y}_1, \overline{y}_2)$, such a relation is ensured by the condition

$$(-D_M^* F_1(\overline{x}, \overline{y}_1))(0) \cap D_M^* F_2(\overline{x}, \overline{y}_2)(0) = \{0\}, \tag{41}$$

as seen in Corollary 31. Thus, assumption (39) is a consequence of the assumptions of [41, Thm 3.10]. Other conditions can be found in [25] and [68].

The following example shows that the qualification condition (39) is weaker than the corresponding qualification condition (41) in [41, Theorem 3.10].

Example Let $X = Y = Z = \mathbb{R}$, $F_1 = \mathbb{R}_- \times \mathbb{R}_-$, $F_2 = \mathbb{R}_+ \times \mathbb{R}_+$ and $(\overline{x}, \overline{z}) = (0, 0)$, $(\overline{y}_1, \overline{y}_2) = (0, 0)$. Then we have $D_L^* F_1(\overline{x}, \overline{y}_1) = \mathbb{R}_- \times \mathbb{R}_+$ and $D_L^* F_2(\overline{x}, \overline{y}_2) = \mathbb{R}_+ \times \mathbb{R}_-$, hence $D_L^* F_1(\overline{x}, \overline{y}_1) + D_L^* F_2(\overline{x}, \overline{y}_2) = \mathbb{R} \times \mathbb{R} = D_L^*(F_1 + F_2)(\overline{x}, \overline{z})$. The qualification condition (41) does not hold as

$$(-D_M^* F_1(\overline{x}, \overline{y}_1)(0)) \cap D_M^* F_2(\overline{x}, \overline{y}_2)(0) = \mathbb{R}_- \neq \{0\}.$$

On the other hand, since $(F_1, F_2) = \{0\} \times \mathbb{R}_- \times \mathbb{R}_+$, we have $D_L^*(F_1, F_2)(\overline{x}, \overline{y}_1, \overline{y}_2) = \mathbb{R}_- \times \mathbb{R}_+ \times \mathbb{R}$ and condition (39) holds. □

Taking into account Corollary 31, we get the following corollary.

Corollary 53 *Let X, Y_1, Y_2 be Asplund spaces and let the multimaps $F_1 : X \rightrightarrows Y_1$, $F_2 : X \rightrightarrows Y_2$ have closed graphs. If C is lsc at $((\overline{x}, \overline{z}), B)$ on S for some subset B of $C(\overline{x}, \overline{z})$ and, for every $\overline{y} := (\overline{y}_1, \overline{y}_2) \in B$, F_1 and F_2 are cooperative at $(\overline{x}, \overline{y}_1, \overline{y}_2)$ then, for all $z^* \in Y^*$ relation (40) holds.*

Let us show that the second decomposition of S which reduces the sum to a composition gives similar results and again includes [47, Corollary 3.3] and [41, Theorem 3.10].

Theorem 54 *Suppose X and Y are Asplund spaces, F_1 and F_2 have closed graphs, C is lsc at $((\overline{x}, \overline{z}), B)$ on S for some subset B of $C(\overline{x}, \overline{z})$ and, for every $(\overline{y}_1, \overline{y}_2) \in B$, either the multimaps F_1, F_2 are cooperative at $(\overline{x}, \overline{y}_1, \overline{y}_2)$ or coordinated at $(\overline{x}, \overline{y}_1, \overline{y}_2)$ and (41) holds. Then, for all $z^* \in Y^*$, inclusion (40) holds.*

Proof Lemma 49 shows that P, Q are allied (resp. synergetic) at $(\overline{x}, (\overline{x}, \overline{y}_1), \overline{y}_1 + \overline{y}_2)$ whenever F_1 and F_2 are cooperative (resp. coordinated) at $(\overline{x}, \overline{y}_1, \overline{y}_2)$. Thus, the cooperative case follows from Theorem 42, since inclusion (25) can be transformed into inclusion (40) by using Lemma 49. The coordinated case ensues from Corollary 43 provided we check the analogue of condition (30) which reads as follows: for every $(\overline{y}_1, \overline{y}_2) \in B$

$$(-D_M^* P^{-1}((\overline{x}, \overline{y}_1), \overline{x})(0)) \cap D_M^* Q((\overline{x}, \overline{y}_1), \overline{z})(0) = \{(0, 0)\}.$$

A calculation similar to the one in Lemma 49 shows that for all $w^* \in X^*$, $z^* \in Y^*$ one has

$$-D_M^* P^{-1}((\overline{x}, \overline{y}_1), \overline{x})(-w^*) = \{(x^*, y^*) : x^* \in w^* - D_M^* F_1(\overline{x}, \overline{y}_1)(y^*)\},$$

$$D_M^* Q((\overline{x}, \overline{y}_1), \overline{z})(z^*) = D_M^* F_2(\overline{x}, \overline{y}_2)(z^*) \times \{z^*\}.$$

Let $(x^*, y^*) \in D_M^* Q((\overline{x}, \overline{y}_1), \overline{z})(0)$, so that $x^* \in D_M^* F_2(\overline{x}, \overline{y}_1)(0)$, $y^* = 0$. If moreover $(-x^*, -y^*) \in D_M^* P^{-1}((\overline{x}, \overline{y}_1), \overline{x})(0)$, we have $x^* \in -D_M^* F_1(\overline{x}, \overline{y}_1)(0)$, hence $x^* = 0$ by (41). □

Along with Proposition 51, this theorem and the preceding one yield sum rules in equality form by combining the assumptions. We leave this task to the reader, but we focus on a case of special interest as in [41, Theorem 1.62].

Corollary 55 *Suppose F_2 is a single-valued map which is strictly differentiable at $\overline{x}$. Then*

$$D_L^* S(\overline{x}, \overline{y}_1 + \overline{y}_2) = D_L^* F_1(\overline{x}, \overline{y}_1) + F_2'(\overline{x})^*.$$

Proof It is easy to see that the multimaps F_1, F_2 are coordinated at $(\overline{x}, \overline{y}_1, \overline{y}_2)$ and (41) holds. Moreover C is lsc at $((\overline{x}, \overline{z}), (\overline{y}_1, \overline{y}_2))$ on S. Theorem 54 yields

$$D_L^*(F_1 + F_2)(\overline{x}, \overline{z})(z^*) \subset D_L^* F_1(\overline{x}, \overline{y}_1)(z^*) + F_2'(\overline{x})^*(z^*).$$

Since $F_1(x) = S(x) - F_2(x)$, a similar argument proves the reverse inclusion. □

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