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Convex Inequalities Without Constraint Qualification nor Closedness Condition, and Their Applications in Optimization

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Abstract Given two convex lower semicontinuous extended real valued functions F and h defined on locally convex spaces, we provide a dual transcription of the relation

$$F(0, \cdot) \geq h(\cdot). \qquad (\star)$$

Some results in this direction are obtained in the first part of the paper (Lemma 2, Theorem 1). These results then are applied to the case when the left-hand-side in ($\star$) is the sum of two convex functions with a convex composite one (Theorem 2). In the spirit of previous works (Hiriart-Urruty and Phelps, J Funct Anal 118:154–166, 1993; Penot, J Convex Anal 3:207–219, 1996, 2005; Thibault, 1995, SIAM J Control Optim 35:1434–1444, 1997, etc.) we give in Theorem 3 a formula for the subdifferential of such a function without any qualification condition. As a consequence of that, we extend to the nonreflexive setting a recent result (Jeyakumar et al., J Glob Optim 36:127–137 2006, Theorem 3.2) about subgradient optimality conditions without constraint qualifications. Finally, we apply Theorem 2 to obtain Farkas-type lemmas and new results on DC, convex, semi-definite, and linear optimization problems.

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1 Introduction

This paper deals with transcriptions of inequalities of the form

$$F(0, \cdot) \geq h(\cdot), \tag{$\star$}$$

where F and h are two convex and lower semicontinuous extended real valued functions defined on locally convex vector spaces, and their applications to optimization problems. With this purpose, we introduce dual characterizations of the inequality ($\star$) without constraint qualification (CQ) nor closedness condition (CC). The results are then applied to the case when the function F is the sum of two convex functions with a convex composite one. This, in turn, gives rise to a limiting formula for subdifferentials of such special type of functions. The rest of the paper is devoted to applications of the previous results to different settings. Firstly, we get various versions of generalized Farkas-type results without CQ nor CC which have their own interest. Secondly, several classes of optimization models are considered: DC problems with convex constraints (including semidefinite ones), convex and semidefinite problems, and infinite linear problems. For these classes of problems, optimality and duality theorems are given together with discussions on their connections with known results in the literature.

The paper is organized as follows. Section 2 contains the preliminary notions and notations. In Section 3 we give, using a dual approach, a simple characterization of the epigraph of certain marginal function defined on a dual space. This gives rise to another simple characterization of inequalities of the form ($\star$) which turns out to have fruitful applications, as shown in the rest of the paper. In Section 4, we give a transcription of a special, but important, case of ($\star$) where the function F is the sum of two convex functions with a convex composite one. An application of this result is given in Section 5, whose main result is the formula of subdifferential of the function of the form $f + g + k \circ \mathcal{H}$ without CQ, which covers the well-known one established by Hiriart-Urruty and Phelps in [14]. The last three sections, namely Sections 6, 7, and 8, present applications of the results obtained in previous sections to three optimization models: DC optimization problems with convex constraints, convex and semidefinite optimization, and infinite linear optimization, respectively. In each section, we firstly establish the Farkas lemma corresponding to the system associated with the problem, then we provide various forms of optimality conditions (such as dual and sequential Lagrange forms), and lastly, we give duality results. Throughout these last sections, discussions on the relation between the results obtained and the known ones in the literature are given.

2 Preliminary Notions

Let X be a locally convex Hausdorff topological vector space (l.c.H.t.v.s.) whose topological dual is denoted by X^*. The only topology we consider on X^* is the w^*-topology. Given $A \subset X$, we denote by co A, cone A and $\overline{A}$ the convex hull, the conical convex hull and the closure of A, respectively. We denote by $\overline{\mathbb{R}}$ the extended real line $\mathbb{R} \cup \{\pm\infty\}$. By convention, $(+\infty) - (+\infty) = +\infty$.

With any extended real-valued function $f : X \to \overline{\mathbb{R}}$ is associated the Legendre–Fenchel conjugate of f which is the function $f^* : X^* \to \overline{\mathbb{R}}$ defined by

$$f^*\left(x^*\right) = \sup_{x \in X} \left(\left\langle x^*, x\right\rangle - f\left(x\right)\right), \ \forall x^* \in X^*.$$

A similar notion holds for any $\varphi : X^* \to \overline{\mathbb{R}}$:

$$\varphi^*\left(x\right) = \sup_{x^* \in X^*} \left(\left\langle x^*, x\right\rangle - \varphi\left(x^*\right)\right), \ \forall x \in X.$$

We represent by dom $f := \{x \in X : f(x) < +\infty\}$ the effective domain of f and say that f is proper if dom $f \neq \emptyset$ and $f(x) > -\infty \ \forall x \in X$. We also use the notation

$$\left[f \leq \lambda\right] := \{x \in X : f(x) \leq \lambda\},$$

as well as the correspondingly defined sets $\left[f \geq \lambda\right]$, $\left[f < \lambda\right]$, and $\left[f > \lambda\right]$.

The set of proper lower semicontinuous (l.s.c.) convex functions on X is denoted by $\Gamma(X)$. For any proper function $f : X \to \overline{\mathbb{R}}$ one has

$$f \in \Gamma(X) \Leftrightarrow f = f^{**}.$$

The infimal convolution of two proper functions $f, g : X \to \overline{\mathbb{R}}$ is the function $f \Box g$ defined by

$$\left(f \Box g\right)(x) = \inf\left\{f\left(x'\right) + g\left(x''\right) : x' + x'' = x\right\}.$$

The operator $\Box$ is associative and if $h : X \to \overline{\mathbb{R}}$ is another proper function we set

$$f \Box g \Box h = \left(f \Box g\right) \Box h = f \Box \left(g \Box h\right).$$

Given $a \in f^{-1}(\mathbb{R})$ and $\varepsilon \geq 0$, the ε-subdifferential of f at the point a is defined by

$$\partial_\varepsilon f(a) = \left\{x^* \in X^* : f(x) - f(a) \geq \left\langle x^*, x - a\right\rangle - \varepsilon, \ \forall x \in X\right\}.$$

One has

$$\partial_\varepsilon f(a) = \left[f^* - \langle \cdot, a\rangle \leq \varepsilon - f(a)\right] = \left\{x^* \in X^* : \ f^*\left(x^*\right) - \left\langle x^*, a\right\rangle \leq \varepsilon - f(a)\right\}.$$

The Young–Fenchel inequality

$$f^*\left(x^*\right) \geq \left\langle x^*, a\right\rangle - f(a)$$

always holds. The equality holds if and only if $x^* \in \partial f(a) := \partial_0 f(a)$.

The indicator function of a set $A \subset X$ is given by $i_A(x) = 0$ if $x \in A$, $i_A(x) = +\infty$ if $x \in X \setminus A$. The conjugate of i_A is the support function of A, $i_A^* : X^* \to \mathbb{R} \cup \{+\infty\}$.

The ε-normal set to A at a point $a \in A$ is defined by

$$N_\varepsilon(A, a) = \partial_\varepsilon i_A(a).$$

The limit superior when $\eta \to 0_+$ of the family $(A_\eta)_{\eta>0}$ of subsets of a topological space is defined (in terms of generalized sequences or nets) by

$$\limsup_{\eta\to 0_+} A_\eta := \left\{\lim_i a_i : a_i \in A_{\eta_i}, \forall i \in I, \text{ and } \eta_i \to 0_+\right\},$$

where $\eta_i \to 0_+$ means that $(\eta_i)_{i\in I} \to 0$ and $\eta_i > 0$, $\forall i \in I$.

3 Dual Approach of Convex Inequalities

Let U be another l.c.H.t.v.s. whose topological dual we denote by U^*.

Given $G : U^* \times X^* \to \overline{\mathbb{R}}$, let us consider the marginal function on X^* associated with G, which is defined by

$$\gamma(x^*) = \inf_{u^*\in U^*} G(u^*, x^*), \ \forall x^* \in X^*. \tag{3.1}$$

The closure of γ, that is the greatest l.s.c. extended real-valued function minorizing γ, is given by

$$\overline{\gamma}(x^*) = \sup_{V\in N(x^*)} \inf_{\widetilde{x}^*\in V} \gamma(\widetilde{x}^*), \ \forall x^* \in X^*, \tag{3.2}$$

where $N(x^*)$ denotes a neighborhood basis of x^*. By using nets, one has

$$\overline{\gamma}(x^*) = \min_{x_i^*\to x^*} \liminf_{i\in I} \gamma(x_i^*), \ \forall x^* \in X^*. \tag{3.3}$$

In terms of epigraphs, $\operatorname{epi}\overline{\gamma} := \{(x^*, r) \in X^* \times \mathbb{R} : \overline{\gamma}(x^*) \le r\}$ coincides with the closure of epi γ with respect to the product topology on $X^* \times \mathbb{R}$. More precisely, one has:

Lemma 1 *Let γ be given by (3.1). For any $(x^*, r) \in X^* \times \mathbb{R}$, the following are equivalent*:

(a) $\overline{\gamma}(x^*) \le r$,
(b) *there exists* $(u_i^*, x_i^*, \varepsilon_i)_{i\in I} \subset U^* \times X^* \times \mathbb{R}$ *such that* $G(u_i^*, x_i^*) \le r + \varepsilon_i$ *for all* $i \in I$, *and* $(x_i^*, \varepsilon_i) \to (x^*, 0_+)$.

Proof $[(a) \Rightarrow (b)]$ For any $V \in N(x^*)$ and any $\varepsilon > 0$ one has, from (3.2),

$$\inf_{\widetilde{x}^*\in V} \gamma(\widetilde{x}^*) < r + \varepsilon.$$

Hence there are $x^*_{V,\varepsilon} \in V$ and $u^*_{V,\varepsilon} \in U^*$ such that $G(u^*_{V,\varepsilon}, x^*_{V,\varepsilon}) \le r + \varepsilon$, and the net $(u^*_{V,\varepsilon}, x^*_{V,\varepsilon}, \varepsilon)_{(V,\varepsilon)\in N(x^*)\times]0,+\infty[}$ satisfies (b).

$[(b) \Rightarrow (a)]$ From (3.3) one has

$$\overline{\gamma}(x^*) \le \liminf_{i\in I} \gamma(x_i^*) \le \liminf_{i\in I} G(u_i^*, x_i^*) \le r.$$

□

Throughout this paper γ will be convex (i.e. epi γ is convex). This is in particular the case when G itself is convex [27, Theorem 2.1.3]. A classical argument allows us to express the Legendre–Fenchel conjugate γ^* of γ in terms of the one G^* of G. One has in fact [27, Theorem 2.6.1]

$$\gamma^*(x) = G^*(0, x), \ \forall x \in X. \tag{3.4}$$

Assuming that $\operatorname{dom}\gamma^* = \{x \in X : G^*(0, x) < +\infty\}$ is nonempty, we get the existence of a continuous minorant of the convex function γ, and so [9, Proposition 3.3]

$$\overline{\gamma} = \gamma^{**}. \tag{3.5}$$

The following lemma will be very useful in the sequel.

Lemma 2 *Assume that γ is convex and* $\operatorname{dom}\gamma^* \neq \emptyset$. *For any $h \in \Gamma(X)$, the following statements are equivalent*:

(a) $G^*(0, x) \geq h(x)$, *for all* $x \in X$,
(b) *for every* $x^* \in \operatorname{dom} h^*$, *there exists* $(u_i^*, x_i^*, \varepsilon_i)_{i\in I} \subset U^* \times X^* \times \mathbb{R}$ *such that* $G(u_i^*, x_i^*) \leq h^*(x^*) + \varepsilon_i$, *for all* $i \in I$, *and* $(x_i^*, \varepsilon_i) \to (x^*, 0_+)$.

Proof $[(a) \Rightarrow (b)]$ Since $G^*(0, \cdot) \geq h$ one has [27, Theorem 2.3.1(iii)] $(G^*(0, \cdot))^* \leq h^*$, which means, from (3.4) and (3.5), $\overline{\gamma} \leq h^*$.

Given $x^* \in \operatorname{dom} h^*$, we can apply Lemma 1 with $r = h^*(x^*)$, and (b) follows.

$[(b) \Rightarrow (a)]$ Let $x^* \in \operatorname{dom} h^*$ and $(u_i^*, x_i^*, \varepsilon_i)_{i\in I}$ as in (b). For any $i \in I$ and $x \in X$ one has

$$\begin{aligned}\langle x^*, x\rangle - h^*(x^*) &\leq \langle x^*, x\rangle - G(u_i^*, x_i^*) + \varepsilon_i \\ &= \langle x^* - x_i^*, x\rangle + \langle x_i^*, x\rangle - G(u_i^*, x_i^*) + \varepsilon_i \\ &\leq \langle x^* - x_i^*, x\rangle + G^*(0, x) + \varepsilon_i.\end{aligned}$$

Passing to the limit on i we get

$$\langle x^*, x\rangle - h^*(x^*) \leq G^*(0, x), \ \forall x^* \in \operatorname{dom} h^*.$$

Taking the supremum over $x^* \in \operatorname{dom} h^*$ we obtain

$$h(x) = h^{**}(x) \leq G^*(0, x), \ \forall x \in X.$$

□

Let us consider $F \in \Gamma(U \times X)$. Applying Lemma 2 with $G = F^*$, we can state:

Theorem 1 *Let* $F \in \Gamma(U \times X)$ *with* $\{x \in X : F(0, x) < +\infty\} \neq \emptyset$. *For any* $h \in \Gamma(X)$, *the following statements are equivalent*:

(a) $F(0, x) \geq h(x)$, *for all* $x \in X$,
(b) *for every* $x^* \in \operatorname{dom} h^*$, *there exists* $(u_i^*, x_i^*, \varepsilon_i)_{i\in I} \subset U^* \times X^* \times \mathbb{R}$ *such that* $F^*(u_i^*, x_i^*) \leq h^*(x^*) + \varepsilon_i$, *for all* $i \in I$, *and* $(x_i^*, \varepsilon_i) \to (x^*, 0_+)$.

4 Transcribing the Inequality $f + g + k \circ \mathcal{H} \geq h$

Let Z be another l.c.H.t.v.s., $f, g \in \Gamma(X)$, and $k \in \Gamma(Z)$. Let $\mathcal{H} : X \to Z$ be a mapping such that

$$z^* \circ \mathcal{H} \in \Gamma(X), \text{ for all } z^* \in \operatorname{dom} k^*. \tag{4.1}$$

Observe that condition (4.1) implies that $k \circ \mathcal{H} \in \Gamma(X)$, provided $\mathcal{H}(X) \cap \operatorname{dom} k \neq \emptyset$: one has in fact, for any $x \in X$,

$$(k \circ \mathcal{H})(x) = k^{**}(\mathcal{H}(X)) = \sup_{z^* \in \operatorname{dom} k^*} \left\{ \left(z^* \circ \mathcal{H} \right)(x) - k^*\left(z^*\right) \right\}$$

and, so,

$$k \circ \mathcal{H} = \sup_{z^* \in \operatorname{dom} k^*} \left\{ z^* \circ \mathcal{H} - k^*\left(z^*\right) \right\} \in \Gamma(X).$$

The following example shows that one may have $k \circ \mathcal{H} \in \Gamma(X)$ with $k \in \Gamma(Z)$, while (4.1) fails: take $X = Z = \mathbb{R}$, $\mathcal{H}(x) = 0$ if $x \leq 0$ and $\mathcal{H}(x) = -1/x$ otherwise, and $k(z) = \max\{z, 0\}$; we then have $0 \equiv k \circ \mathcal{H} \in \Gamma(X)$ and $z^* \circ \mathcal{H} = \mathcal{H} \notin \Gamma(X)$ for $z^* = \operatorname{id}_{\mathbb{R}} \in \operatorname{dom} k^*$.

Observe also that, in particular, (4.1) is satisfied when Z is equipped with a closed convex preordering cone S, k is nondecreasing with respect to S, $\mathcal{H}$ is convex with respect to (w.r.t.) S, and $\mathcal{H}$ is lower semicontinuous w.r.t. S, that means (see [23]):

$$\forall x \in X,\ \forall V \in N(\mathcal{H}(x)),\ \exists W \in N(x) \text{ such that } \mathcal{H}(W) \subset V + S.$$

For further investigation, assume that

$$(\operatorname{dom} f) \cap (\operatorname{dom} g) \cap \mathcal{H}^{-1}(\operatorname{dom} k) \neq \emptyset. \tag{4.2}$$

We are interested in transcribing the convex inequality of the form

$$f(x) + g(x) + k(\mathcal{H}(x)) \geq h(x), \text{ for all } x \in X.$$

The main result is given in the following theorem where Lemma 2 serves as a main tool for its proof.

Theorem 2 *Let $f, g \in \Gamma(X)$, $k \in \Gamma(Z)$, and $\mathcal{H} : X \to Z$ be such that* (4.1) *and* (4.2) *hold. Then, for any $h \in \Gamma(X)$, the following statements are equivalent*:

(a) $f(x) + g(x) + k(\mathcal{H}(x)) \geq h(x)$, *for all* $x \in X$,

(b) *for every $x^* \in \operatorname{dom} h^*$ there exists a net $(x_{1i}^*, x_{2i}^*, x_{3i}^*, z_i^*, \varepsilon_i)_{i \in I} \subset (X^*)^3 \times Z^* \times \mathbb{R}$ such that*

$$f^*(x_{1i}^*) + g^*(x_{2i}^*) + k^*(z_i^*) + (z_i^* \circ \mathcal{H})^*(x_{3i}^*) \leq h^*(x^*) + \varepsilon_i, \textit{ for all } i \in I,$$

and

$$(x_{1i}^* + x_{2i}^* + x_{3i}^*, \varepsilon_i) \to (x^*, 0_+).$$

Proof Let us consider the function

$$G : (X^* \times X^* \times Z^*) \times X^* \to \overline{\mathbb{R}}$$

defined as follows

$$G((x_1^*, x_2^*, z^*), x^*) = f^*(x_1^*) + g^*(x_2^*) + k^*(z^*) + (z^* \circ \mathcal{H})^*(x^* - x_1^* - x_2^*),$$

for any $((x_1^*, x_2^*, z^*), x^*) \in (X^* \times X^* \times Z^*) \times X^*$. Here $U = X \times X \times Z$.

The marginal function γ associated with G by (3.1) is

$$\gamma(x^*) = \inf_{z^* \in \operatorname{dom} k^*} \left\{ k^*(z^*) + \left(f^* \square g^* \square (z^* \circ \mathcal{H})^* \right)(x^*) \right\}, \text{ for all } x^* \in X^*.$$

It is worth observing first that γ is convex. This is due to the fact that G is convex because it is the sum of the convex function

$$((x_1^*, x_2^*, z^*), x^*) \mapsto f^*(x_1^*) + g^*(x_2^*) + k^*(z^*),$$

and the supremum over $x \in X$ of the affine functions

$$((x_1^*, x_2^*, z^*), x^*) \mapsto \langle x^* - x_1^* - x_2^*, x \rangle - \langle z^*, \mathcal{H}(x) \rangle.$$

Let us now calculate the conjugate, γ^*, of the function γ. Thanks to (4.1) we can write

$$\begin{aligned}
\gamma^*(x) &= \sup_{x^* \in X^*} \left\{ \langle x^*, x \rangle - \inf_{z^* \in \operatorname{dom} k^*} \left\{ k^*(z^*) + \left(f^* \square g^* \square (z^* \circ \mathcal{H})^* \right)(x^*) \right\} \right\} \\
&= \sup_{x^* \in X^*} \sup_{z^* \in \operatorname{dom} k^*} \left\{ \langle x^*, x \rangle - k^*(z^*) - \left(f^* \square g^* \square (z^* \circ \mathcal{H})^* \right)(x^*) \right\} \\
&= \sup_{z^* \in \operatorname{dom} k^*} \left\{ -k^*(z^*) + \sup_{x^* \in X^*} \left\{ \langle x^*, x \rangle - \left(f^* \square g^* \square (z^* \circ \mathcal{H})^* \right)(x^*) \right\} \right\} \\
&= \sup_{z^* \in \operatorname{dom} k^*} \left\{ -k^*(z^*) + \left(f^* \square g^* \square (z^* \circ \mathcal{H})^* \right)^*(x) \right\} \\
&= \sup_{z^* \in \operatorname{dom} k^*} \left\{ -k^*(z^*) + \left(f + g + (z^* \circ \mathcal{H}) \right)(x) \right\} \\
&= f(x) + g(x) + \sup_{z^* \in \operatorname{dom} k^*} \left\{ \langle z^*, \mathcal{H}(x) \rangle - k^*(z^*) \right\},
\end{aligned}$$

so that

$$\gamma^*(x) = f(x) + g(x) + k(\mathcal{H}(x)). \tag{4.3}$$

Thus, γ is convex and, by (4.2) and (4.3), $\operatorname{dom} \gamma^* \neq \emptyset$. The conclusion of the theorem follows from Lemma 2. □

In the next sections we give relevant applications of Theorem 2. The first one concerns the subdifferential of the function $f + g + k \circ \mathcal{H}$. In the spirit of previous works [13, 14, 21, 24, 25] and [27], we derive a formula without any CQ in terms of ε-subdifferentials. The remaining applications are Farkas–Minkowski inequality systems and containments, without CQ nor CC, which provide optimality and duality results for different optimization models.

5 Subdifferential of $f + g + k \circ \mathcal{H}$

Let $f, g \in \Gamma(X)$, $k \in \Gamma(Z)$, and $\mathcal{H} : X \to Z$ be as in Theorem 2, and let $a \in (\operatorname{dom} f) \cap (\operatorname{dom} g) \cap \mathcal{H}^{-1}(\operatorname{dom} k)$.

We are now in position to establish a "limiting form" of the subdifferential of $f + g + k \circ \mathcal{H}$ which is an extension of the well-known one given in [14] (see Corollary 1 below).

Theorem 3 *Let $f, g \in \Gamma(X)$, $k \in \Gamma(Z)$, and $\mathcal{H} : X \to Z$ be such that (4.1) holds. Then, for every $a \in X$ such that $f(a) + g(a) + k(\mathcal{H}(a)) \in \mathbb{R}$, it holds*

$$\partial(f + g + k \circ \mathcal{H})(a) = \limsup_{\eta \to 0_+} \left(\bigcup_{z^* \in \partial_\eta k(\mathcal{H}(a))} \left\{ \partial_\eta f(a) + \partial_\eta g(a) + \partial_\eta (z^* \circ \mathcal{H})(a) \right\} \right).$$

Proof Let $x^* \in X^*$.

Assume that $x^* \in \partial(f + g + k \circ \mathcal{H})(a)$. Observe that $x^* \in \partial(f + g + k \circ \mathcal{H})(a)$ if and only if the statement (a) in Theorem 2 holds with

$$h(x) := \langle x^*, x - a \rangle + f(a) + g(a) + k(\mathcal{H}(a)).$$

In order to apply Theorem 2 let us first quote that

$$h^*(\cdot) = i_{\{x^*\}}(\cdot) + \langle x^*, a \rangle - f(a) - g(a) - k(\mathcal{H}(a)).$$

Note that $\operatorname{dom} \gamma^* \neq \emptyset$ since $a \in \operatorname{dom} \gamma^* = (\operatorname{dom} f) \cap (\operatorname{dom} g) \cap \mathcal{H}^{-1}(\operatorname{dom} k)$. It follows, from the previous arguments and Theorem 2, that $x^* \in \partial(f + g + k \circ \mathcal{H})(a)$ if and only if there exists a net

$$(x_{1i}^*, x_{2i}^*, x_{3i}^*, z_i^*, \varepsilon_i)_{i \in I} \subset (X^*)^3 \times Z^* \times \mathbb{R}$$

such that

$$\begin{aligned} f^*(x_{1i}^*) + g^*(x_{2i}^*) + (z_i^* \circ \mathcal{H})^*(x_{3i}^*) + k^*(z_i^*) &\leq \langle x^*, a \rangle \\ - (f + g + k \circ \mathcal{H})(a) + \varepsilon_i, \ \forall i \in I, \end{aligned} \tag{5.1}$$

and

$$(x_{1i}^* + x_{2i}^* + x_{3i}^*, \varepsilon_i) \to (x^*, 0_+).$$

By the Young–Fenchel inequality, we can rewrite (5.1) as follows

$$\begin{aligned} &\left[f^*(x_{1i}^*) + f(a) - \langle x_{1i}^*, a \rangle \right] + \left[g^*(x_{2i}^*) + g(a) - \langle x_{2i}^*, a \rangle \right] \\ &\quad + \left[(z_i^* \circ \mathcal{H})^*(x_{3i}^*) + (z_i^* \circ \mathcal{H})(a) - \langle x_{3i}^*, a \rangle \right] \\ &\quad + \left[k^*(z_i^*) + k(\mathcal{H}(a)) - \langle z_i^*, \mathcal{H}(a) \rangle \right] \\ &\qquad \leq \langle x^* - x_{1i}^* - x_{2i}^* - x_{3i}^*, a \rangle + \varepsilon_i, \ \forall i \in I. \end{aligned}$$

Setting $\eta_i := \langle x^* - x_{1i}^* - x_{2i}^* - x_{3i}^*, a\rangle + \varepsilon_i$, we get $\eta_i \to 0_+$. Moreover, since the four brackets above are nonnegative, each of them is less or equal than η_i, for any $i \in I$. Therefore we have

$$\left\{\begin{array}{l} x^* = \lim_i (x_{1i}^* + x_{2i}^* + x_{3i}^*) \text{ with} \\ x_{1i}^* \in \partial_{\eta_i} f(a),\ x_{2i}^* \in \partial_{\eta_i} g(a),\ x_{3i}^* \in \partial_{\eta_i}(z_i^* \circ \mathcal{H})(a), \\ z_i^* \in \partial_{\eta_i} k(\mathcal{H}(a)),\ \text{and } \eta_i \to 0_+ \end{array}\right. \tag{5.2}$$

or, equivalently,

$$x^* \in \limsup_{\eta \to 0_+} \left(\bigcup_{z^* \in \partial_\eta k(\mathcal{H}(a))} \left\{ \partial_\eta f(a) + \partial_\eta g(a) + \partial_\eta (z^* \circ \mathcal{H})(a) \right\} \right).$$

It is worth emphasizing here that $\partial_\eta k(\mathcal{H}(a))$ (the same for $\partial_{\eta_i} k(\mathcal{H}(a))$) represents the η -subdifferential of the function k at the point $\mathcal{H}(a)$.

Conversely, assume now that $x^* \in X^*$ satisfies (5.2). It follows from (3.3), (3.5), (5.2), and (4.3) that

$$\begin{aligned} (f + g + k \circ \mathcal{H})^* (x^*) &= \overline{\gamma}(x^*) \leq \liminf_{i \in I} \gamma(x_{1i}^* + x_{2i}^* + x_{3i}^*) \\ &\leq \liminf_{i \in I} \left[k^*(z_i^*) + f^*(x_{1i}^*) + g^*(x_{2i}^*) + (z_i^* \circ \mathcal{H})^*(x_{3i}^*) \right] \\ &\leq \liminf_{i \in I} \left[\langle x_{1i}^* + x_{2i}^* + x_{3i}^*, a \rangle - f(a) - g(a) - k(\mathcal{H}(a)) + 4\eta_i \right] \\ &= \langle x^*, a \rangle - f(a) - g(a) - k(\mathcal{H}(a)), \end{aligned}$$

and hence, $x^* \in \partial(f + g + k \circ \mathcal{H})(a)$. The proof is complete. □

In Theorem 3, if we take $k \equiv 0$, then the subdifferential formula in this theorem collapses to the well-known one established by Hiriart-Urruty and Phelps in [14], as it is stated in the following corollary.

Corollary 1 *Let $f, g \in \Gamma(X)$. Then*

$$\partial(f + g)(a) = \bigcap_{\varepsilon > 0} cl\ (\partial_\varepsilon f(a) + \partial_\varepsilon g(a))$$

for any $a \in (dom\ f) \cap (dom\ g)$.

6 DC Optimization with Convex Constraints in the Absence of CQ's

Let $f, h \in \Gamma(X)$, C be a closed convex set in X, S a preordering closed convex cone in Z, with its positive dual cone S^+ which is defined as

$$S^+ := \{z^* \in Z^* \ :\ \langle z^*, s\rangle \geq 0,\ \forall s \in S\},$$

and let $\mathcal{H} : X \to Z$ be a mapping. Remember that we use the notation

$$[f - h \geq 0] := \{x \in X : f(x) - h(x) \geq 0\},$$

and observe that $[f - h \geq 0] = [f \geq h]$.

The main result of this section is the generalized Farkas lemma (in dual form) without any constraint qualification which is given in Theorem 4 below. This, at the same time, gives a characterization of set containment of a convex set defined by a cone constraint, $C \cap \mathcal{H}^{-1}(-S)$, in a DC set (i.e., a set defined by a DC inequality) which, in certain sense, extends the ones involving convex and DC sets in earlier works [7, 10, 16].

Theorem 4 (Farkas lemma involving DC functions) *Let $f, h \in \Gamma(X)$, C be a closed convex set in X, S a preordering closed convex cone in Z, and $\mathcal{H} : X \to Z$ a mapping. Assume that for all $z^* \in S^+$, $z^* \circ \mathcal{H} \in \Gamma(X)$, and $C \cap \operatorname{dom} f \cap \mathcal{H}^{-1}(-S) \neq \emptyset$. Then the following statements are equivalent*:

(a) $C \cap \mathcal{H}^{-1}(-S) \subset [f - h \geq 0]$,
(b) *for all $x^* \in \operatorname{dom} h^*$, there exists a net $(x_{1i}^*, x_{2i}^*, x_{3i}^*, z_i^*, \varepsilon_i)_{i \in I} \subset (X^*)^3 \times Z^* \times \mathbb{R}$ such that $\left(z_i^*\right)_{i \in I} \subset S^+$,*

$$f^*(x_{1i}^*) + i_C^*(x_{2i}^*) + (z_i^* \circ \mathcal{H})^*(x_{3i}^*) \leq h^*(x^*) + \varepsilon_i, \text{ for all } i \in I,$$

and

$$(x_{1i}^* + x_{2i}^* + x_{3i}^*, \varepsilon_i) \to (x^*, 0_+).$$

Proof We are going to apply Theorem 2, with $g = i_C$ and $k = i_{-S}$. Then $k^* = i_{-S}^*$, and we can easily observe that dom $k^* = S^+$, which entails the fulfilment of (4.1). Moreover, if $z^* \in S^+$ then $k^*(z^*) = 0$.

We are assuming that $C \cap \operatorname{dom} f \cap \mathcal{H}^{-1}(-S) \neq \emptyset$, which is equivalent to condition (4.2) in our particular setting. Hence, we can apply Theorem 2, and the rest of the proof is devoted to verify that statements (a) and (b) here are equivalent to the corresponding ones in Theorem 2.

Since the equivalence between both statements (a) is straightforward, let us prove the equivalence of both (b)'s. In fact, statement (b) in Theorem 2 now reads:

For all $x^* \in \operatorname{dom} h^*$, there exists an associated net $(x_{1i}^*, x_{2i}^*, x_{3i}^*, z_i^*, \varepsilon_i)_{i \in I} \subset (X^*)^3 \times Z^* \times \mathbb{R}$ such that

$$f^*(x_{1i}^*) + i_C^*(x_{2i}^*) + k^*(z_i^*) + (z_i^* \circ \mathcal{H})^*(x_{3i}^*) \leq h^*(x^*) + \varepsilon_i, \text{ for all } i \in I, \tag{6.1}$$

and

$$(x_{1i}^* + x_{2i}^* + x_{3i}^*, \varepsilon_i) \to (x^*, 0_+).$$

Since $x^* \in \operatorname{dom} h^*$, we have $h^*(x^*) < +\infty$, and this entails that $z_i^* \in S^+$, and so $k^*(z_i^*) = 0$, for all $i \in I$. In this way, we get statement (b) in our theorem. □

Theorem 4 can be applied in various situations: convex and reverse convex containment (see, e.g., [7, 10, 15, 16]), approximate Farkas lemma for systems with DC functions [3, 4, 19], and DC optimization problems under convex constraints [17].

Now we consider the following model of DC problems:

$$(\text{DC}) \quad \begin{cases} \text{minimize } [f(x) - h(x)] \\ \text{s.t.} \qquad x \in C,\ \mathcal{H}(x) \in -S, \end{cases}$$

where f, h, $\mathcal{H}$, C, and S are as in Theorem 4.

Let $k = i_{-S}$ and $g = i_C$. Then the relation $\mathcal{H}(x) \in -S$ is equivalent to $(k \circ \mathcal{H})(x) = k(\mathcal{H}(x)) = 0$ and the (DC) problem is equivalent to the following one:

$$\text{(DC1)} \quad \inf_{x \in X} [f(x) + i_C(x) + i_{-S}(\mathcal{H}(x)) - h(x)].$$

Let us denote the feasible set of (DC) by $A := C \cap \mathcal{H}^{-1}(-S)$.

It is worth mentioning that in the case where the mapping $\mathcal{H}$ is convex w.r.t. S and continuous the problem (DC) collapses to the problem considered in [7]. If in addition, $h \equiv 0$ then it is the cone-constrained convex problems in [8, 20], or in [18] when $C = X$.

Proposition 1 (Characterization of global optimality for (DC)) *Let f, h, $\mathcal{H}$, C, and S be as in Theorem 4. Then a point $a \in A \cap \operatorname{dom} f \cap \operatorname{dom} h$ is a global minimum of* (DC) *if and only if for every $x^* \in \operatorname{dom} h^*$ there exists a net $(x^*_{1i}, x^*_{2i}, x^*_{3i}, z^*_i, \varepsilon_i)_{i \in I} \subset (X^*)^3 \times Z^* \times \mathbb{R}$ satisfying $\left(z^*_i\right)_{i \in I} \subset S^+$,*

$$f^*(x^*_{1i}) + i^*_C(x^*_{2i}) + (z^*_i \circ \mathcal{H})^*(x^*_{3i}) \leq h^*(x^*) + h(a) - f(a) + \varepsilon_i, \ \textit{for all } i \in I,$$

and

$$(x^*_{1i} + x^*_{2i} + x^*_{3i}, \varepsilon_i) \to (x^*, 0_+).$$

Proof It is worth observing that $a \in A \cap \operatorname{dom} f \cap \operatorname{dom} h$ is a global minimum of (DC) if and only if a is a global optimal solution of (DC1), if and only if

$$f(x) + i_C(x) + i_{-S}(\mathcal{H}(x)) - h(x) \geq f(a) - h(a), \ \forall x \in X,$$

if and only if

$$f(x) + i_C(x) + i_{-S}(\mathcal{H}(x)) \geq h(x) + f(a) - h(a), \ \forall x \in X.$$

Applying Theorem 2 with i_C, i_{-S}, and $\tilde{h}(\cdot) := h(\cdot) + f(a) - h(a)$ playing the roles of g, k, and h, respectively, the last inequality is equivalent to the following fact: for every $x^* \in \operatorname{dom} h^*$ there exists a net $(x^*_{1i}, x^*_{2i}, x^*_{3i}, z^*_i, \varepsilon_i)_{i \in I} \subset (X^*)^3 \times Z^* \times \mathbb{R}$ satisfying

$$f^*(x^*_{1i}) + i^*_C(x^*_{2i}) + i^*_{-S}(z^*_i) + (z^*_i \circ \mathcal{H})^*(x^*_{3i}) \leq h^*(x^*) + h(a) - f(a) + \varepsilon_i, \ \text{for all } i \in I,$$

and

$$(x^*_{1i} + x^*_{2i} + x^*_{3i}, \varepsilon_i) \to (x^*, 0_+).$$

The argument used in the last part of the proof of Theorem 4 ensures that $z^*_i \in S^+$ for all $i \in I$, and hence, $i^*_{-S}(z^*_i) = 0$ for all $i \in I$. The proof is complete. □

Necessary conditions for local optimality of (DC) without qualification condition can be derived directly from previous results and the following lemma.

Lemma 3 *Let $f, h : X \to \overline{\mathbb{R}}$, $a \in f^{-1}(\mathbb{R}) \cap h^{-1}(\mathbb{R})$, and assume that f is convex. If a is a local minimum of $f - h$, then*

$$\partial h(a) \subset \partial f(a).$$

Proof By assumption, there is $V \in N(a)$ such that

$$f(x) - h(x) \geq f(a) - h(a),\ \forall x \in V.$$

For any $x^* \in \partial h(a)$ one thus has

$$\begin{aligned} f(x) &\geq h(x) + f(a) - h(a) \\ &\geq \langle x^*, x - a\rangle + f(a),\ \forall x \in V, \end{aligned}$$

and

$$f(x) - \langle x^*, x\rangle \geq f(a) - \langle x^*, a\rangle,\ \forall x \in V,$$

which implies that a is a local minimum (hence, global) of the convex function $f - \langle x^*, \cdot\rangle$. Thus,

$$0 \in \partial(f - \langle x^*, \cdot\rangle)(a) = \partial f(a) - \{x^*\},$$

entailing $x^* \in \partial f(a)$. □

Proposition 2 (Necessary condition for local optimality for (DC)) *Let f, h, $\mathcal{H}$, C, and S be as in Theorem* 4*. If $a \in A \cap \operatorname{dom} f \cap \operatorname{dom} h$ is a local minimum of* (DC) *then*

$$\partial h(a) \subset \limsup_{\eta \to 0_+} \left(\bigcup_{z^* \in \partial_\eta i_{-S}(\mathcal{H}(a))} \left\{ \partial_\eta f(a) + \partial_\eta (z^* \circ \mathcal{H})(a) + N_\eta(C, a) \right\} \right),$$

or, equivalently, for any $x^* \in \partial h(a)$, there exists a net $(x^*_{1i}, x^*_{2i}, x^*_{3i}, z^*_i, \eta_i)_{i \in I} \subset (X^*)^3 \times Z^* \times \mathbb{R}$ such that

$$\begin{aligned} &x^*_{1i} \in \partial_{\eta_i} f(a),\ x^*_{2i} \in N_{\eta_i}(C, a),\ x^*_{3i} \in \partial_{\eta_i}(z^*_i \circ \mathcal{H})(a), \\ &z^*_i \in S^+,\ 0 \leq \langle z^*_i, -\mathcal{H}(a)\rangle \leq \eta_i,\ \textit{and} \\ &(x^*_{1i} + x^*_{2i} + x^*_{3i}, \eta_i) \longrightarrow (x^*, 0_+). \end{aligned}$$

Proof If $a \in A \cap \operatorname{dom} f \cap \operatorname{dom} h$ is a local minimum of (DC), then it is also a local solution of the DC program (DC1). Since $f + i_C + i_{-S} \circ \mathcal{H}$ is convex, it follows from Lemma 3 that

$$\partial h(a) \ \subset \ \partial(f + i_C + i_{-S} \circ \mathcal{H})(a).$$

Combining this inclusion and the formula of subdifferentials of the function $f + i_C + i_{-S} \circ \mathcal{H}$ in Theorem 3, we get

$$\partial h(a) \subset \limsup_{\eta \to 0_+} \left(\bigcup_{z^* \in \partial_\eta i_{-S}(\mathcal{H}(a))} \left\{ \partial_\eta f(a) + \partial_\eta (z^* \circ \mathcal{H})(a) + N_\eta(C, a) \right\} \right).$$

The first assertion is proved.

The second assertion is just another representation of the first one if we observe that $z^*_i \in \partial_\eta i_{-S}(\mathcal{H}(a))$ is equivalent to $z^*_i \in S^+$ and $0 \leq \langle z^*_i, -\mathcal{H}(a)\rangle \leq \eta_i$. The proof is complete. □

Since $z^* \in \partial_\eta i_{-S}(\mathcal{H}(a))$ implies $z^* \in S^+$, the following result is a direct consequence of Proposition 2.

Corollary 2 (Necessary condition for local optimality for (DC)) *Let f, h, $\mathcal{H}$, C, and S be as in Theorem* 4. *If $a \in A \cap \mathrm{dom}\, f \cap \mathrm{dom}\, h$ is a local minimum of* (DC) *then*

$$\partial h(a) \subset \limsup_{\eta \to 0_+} \left(\bigcup_{z^* \in S^+} \left\{ \partial_\eta f(a) + \partial_\eta (z^* \circ \mathcal{H})(a) + N_\eta(C, a) \right\} \right).$$

We now consider a special case of (DC) where $X = \mathbb{R}^m$, $Z = S_n$ is the space of symmetric $(n \times n)$-matrices, and $\mathcal{H}(x) := -F_0 - \sum_{j=1}^m x_i F_j$ for all $x = (x_1, \cdots, x_m) \in \mathbb{R}^m$, where $F_0, F_j \in S_n$. Denote by $\succeq$ the L öwer partial order of S_n, that is, for $M, N \in S_n$, $M \succeq N$ means that $M - N$ is a positive semidefinite matrix. S_n will be considered as a vector space with the trace inner product defined by $\langle M, N \rangle := \mathrm{Tr}\,[MN]$ where $\mathrm{Tr}\,[.]$ is the trace operation.

Let S be the cone of all positive semidefinite matrices of S_n. Then $S^+ = S$ and $M \in S$ if and only if $\mathrm{Tr}\,[ZM] \geq 0$ for all $Z \in S$. Given $F_0, F_j \in S_n$, $j = 1, \cdots, m$, we are interested in the inclusion involving a semidefinite inequality and a DC inequality of the following form:

$$\{x \in \mathbb{R}^m : x \in C,\ F_0 + \sum_{j=1}^m x_j F_j \succeq 0\} \ \subset\ [f - h \geq 0].$$

Recall that $\mathcal{H}(x) = -F_0 - \sum_{j=1}^m x_j F_j$. Let $\hat{\mathcal{H}}(x) = \sum_{j=1}^m x_j F_j$. Then $\hat{\mathcal{H}} : \mathbb{R}^m \to S_n$ is a linear operator and its dual operator $\hat{\mathcal{H}}^*$ is

$$\hat{\mathcal{H}}^*(Z) = (\mathrm{Tr}\,[F_1 Z], \ldots, \mathrm{Tr}\,[F_m Z]),\ Z \in S_n.$$

The proof of the next result is based upon Theorem 4.

Proposition 3 (Farkas lemma involving semidefinite and DC inequalities) *Let $X = \mathbb{R}^m$, $f, h \in \Gamma(\mathbb{R}^m)$, and $C \subset \mathbb{R}^m$ be a closed convex set. Assume that $C \cap \mathrm{dom}\, f \cap \mathcal{H}^{-1}(-S) \neq \emptyset$. Then the following statements are equivalent*:

(a) $\{x \in \mathbb{R}^m : x \in C,\ F_0 + \sum_{j=1}^m x_j F_j \succeq 0\} \subset [f - h \geq 0]$,

(b) *for all $x^* \in \mathrm{dom}\, h^*$, there exists a net $(x_{1i}^*, x_{2i}^*, Z_i, \varepsilon_i)_{i \in I} \subset (\mathbb{R}^m)^2 \times S_n \times \mathbb{R}$ such that $Z_i \succeq 0$, for all $i \in I$,*

$$f^*(x_{1i}^*) + i_C^*(x_{2i}^*) + Tr\,[F_0 Z_i] \leq h^*(x^*) + \varepsilon_i,\ \textit{for all } i \in I,$$

and

$$(x_{1i}^* + x_{2i}^* - \hat{\mathcal{H}}^*(Z_i), \varepsilon_i) \to (x^*, 0_+).$$

Proof We observe first that the inequality in (a) can be rewritten as follows:

$$C \cap \mathcal{H}^{-1}(-S) \subset [f - h \geq 0].$$

Moreover, for each $Z \in S$ and $u \in \mathbb{R}^m$, we have

$$\begin{aligned}(Z \circ \mathcal{H})^*(u) &= \sup_{x \in \mathbb{R}^m} \{\langle u, x\rangle - \langle Z, \mathcal{H}(x)\rangle\} \\ &= \sup_{x \in \mathbb{R}^m} \left\{ \langle u, x\rangle + \sum_{j=1}^{m} x_j \mathrm{Tr}\,[Z F_j] + \mathrm{Tr}\,[Z F_0] \right\} \\ &= \mathrm{Tr}\,[Z F_0] + \sup_{x \in \mathbb{R}^m} \langle u + \hat{\mathcal{H}}^*(Z), x\rangle.\end{aligned}$$

Therefore,

$$(Z \circ \mathcal{H})^*(u) = \begin{cases} \mathrm{Tr}\,[Z F_0], & \text{if } u = -\hat{\mathcal{H}}^*(Z), \\ +\infty, & \text{otherwise.} \end{cases}$$

The conclusion now follows directly from Theorem 4. □

7 Convex and Semidefinite Optimization without CQ's

Taking $h \equiv 0$ in Theorem 4 we get a generalized version of Farkas lemma for convex system without constraint qualification as shown in the next result.

Proposition 4 (Farkas lemma for convex systems) *Assume that f, C, $\mathcal{H}$, and S satisfy the conditions in Theorem* 4. *Then the following statements are equivalent*:

(a) $C \cap \mathcal{H}^{-1}(-S) \subset [f \geq 0]$,

(b) *there exists a net* $(x^*_{1i}, x^*_{2i}, x^*_{3i}, z^*_i, \varepsilon_i)_{i \in I} \subset (X^*)^3 \times Z^* \times \mathbb{R}$ *such that* $\left(z^*_i\right)_{i \in I} \subset S^+$,

$$f^*(x^*_{1i}) + i^*_C(x^*_{2i}) + (z^*_i \circ \mathcal{H})^*(x^*_{3i}) \leq \varepsilon_i, \text{ for all } i \in I,$$

and

$$(x^*_{1i} + x^*_{2i} + x^*_{3i}, \varepsilon_i) \to (0, 0_+).$$

(c) *there exists a net* $(z^*_i)_{i \in I} \subset S^+$ *such that*

$$f(x) + \liminf_{i \in I} (z^*_i \circ \mathcal{H})(x) \geq 0, \ \forall x \in C.$$

□

Proof The equivalence between (a) and (b) follows directly from Theorem 4 with $h \equiv 0$ (and hence, $\mathrm{dom}\, h^* = \{0\}$). Next, we prove $[(b) \Rightarrow (c)]$ and $[(c) \Rightarrow (a)]$.

$[(b) \Rightarrow (c)]$ Assume that (b) holds. By the definition of conjugate functions, we get, for any $x \in C$, any $i \in I$,

$$\begin{aligned} f^*(x^*_{1i}) &\geq \langle x^*_{1i}, x\rangle - f(x), \\ i^*_C(x^*_{2i}) &\geq \langle x^*_{2i}, x\rangle, \\ (z^*_i \circ \mathcal{H})^*(x^*_{3i}) &\geq \langle x^*_{3i}, x\rangle - (z^*_i \circ \mathcal{H})(x), \end{aligned}$$

where x_{1i}^*, x_{2i}^*, x_{3i}^*, and z_i^*, $i \in I$, are the elements in the net whose existence is assumed in (b). Then the inequality in (b) yields

$$f(x) + (z_i^* \circ \mathcal{H})(x) \geq -\varepsilon_i + \langle x_{1i}^* + x_{2i}^* + x_{3i}^*, x\rangle, \ \forall i \in I.$$

We get (c) by taking the $\liminf_{i \in I}$ in both sides of the last inequality.

$[(c) \Rightarrow (a)]$ Assume (c) holds. If $x \in C \cap \mathcal{H}^{-1}(-S)$ then $(z_i^* \circ \mathcal{H})(x) \leq 0$ for all $i \in I$ (note that $z_i^* \in S^+$ and $\mathcal{H}(x) \in -S$). Hence, since $x \in C$,

$$f(x) \geq f(x) + \liminf_{i \in I}(z_i^* \circ \mathcal{H})(x) \geq 0.$$

Thus, (a) holds. □

It is worth observing that the equivalence between statements (a) and (c) in Proposition 4 was established in [7] and [18] in the case where X is a reflexive Banach space and $\mathcal{H}$ is continuous, S-convex (i.e., convex w.r.t. the cone S), while the other equivalences, to our knowledge, are new. The generalized version of Farkas lemma in Proposition 4 and its counterpart for the system involving semidefinite functions given below are the key tools for establishing limiting Lagrangian conditions for convex and semidefinite programs (see [8, 20]).

Corollary 3 (Farkas lemma for convex systems with semidefinite constraints) *Assume that f, C, and $\mathcal{H}$ satisfy the conditions in Proposition* 3. *Then the following statements are equivalent*:

(a) $\{x \in \mathbb{R}^m : x \in C, \ F_0 + \sum_{j=1}^m x_j F_j \succeq 0\} \subset [f \geq 0]$,

(b) *there exists a net* $(x_{1i}^*, x_{2i}^*, Z_i, \varepsilon_i)_{i \in I} \subset (\mathbb{R}^m)^2 \times S_n \times \mathbb{R}$ *such that* $Z_i \succeq 0$, *for all* $i \in I$,

$$f^*(x_{1i}^*) + i_C^*(x_{2i}^*) + Tr[F_0 Z_i] \leq \varepsilon_i, \ \textit{for all } i \in I,$$

and

$$(x_{1i}^* + x_{2i}^* - \hat{\mathcal{H}}^*(Z_i), \varepsilon_i) \to (x^*, 0_+).$$

(c) *there exists a net* $(Z_i)_{i \in I} \subset S_n$ *such that* $Z_i \succeq 0$, *for all* $i \in I$, *and*

$$f(x) + \liminf_{i \in I} Tr[Z_i \mathcal{H}(x)] \geq 0, \ \forall x \in C.$$

Proof It is a direct consequence of Proposition 4 with $\mathcal{H}(x) = -F_0 - \sum_{j=1}^m x_j F_j$. The equivalence between the first two statements comes from Proposition 3. □

Taking $h \equiv 0$ in the problem (DC), we come back to the classical convex optimization problem of the following form

$$\text{(PC)} \qquad \text{minimize } f(x) \text{ s.t. } x \in C \text{ and } \mathcal{H}(x) \in -S,$$

which was considered in many recent works (see, for instance, [2, 20, 25, 26]). We now give some consequences of the previous results for this class of problems. More precisely, we will give a result about sequential optimality conditions and Lagrange duality for (PC), which improves those established in [8, 18, 20] and [25].

Proposition 5 (Optimality characterization for (PC)) *Let* $f \in \Gamma(X)$, $\mathcal{H} : X \to Z$ *satisfying* $z^* \circ \mathcal{H} \in \Gamma(X)$ *for all* $z^* \in S^+$. *For any* $a \in C \cap (dom\ f) \cap \mathcal{H}^{-1}(-S)$ *the following assertions are equivalent* :

(a) *a is optimal for* (PC),
(b) *there exist* $(\eta_i)_{i\in I} \to 0_+$, *and for every* $i \in I$, *there also exist* $x^*_{1i} \in \partial_{\eta_i} f(a)$, $x^*_{2i} \in N_{\eta_i}(C, a)$, $z^*_i \in S^+$, *and* $x^*_{3i} \in \partial_{\eta_i}(z^*_i \circ \mathcal{H})(a)$ *such that* $0 \leq \langle z^*_i, -\mathcal{H}(a)\rangle \leq \eta_i$ *and* $\lim_i(x^*_{1i} + x^*_{2i} + x^*_{3i}) = 0$.

Proof Observe that the local minima of (PC) are global because this problem is convex.

For the implication $[(a) \Rightarrow (b)]$, apply Proposition 2 with $h \equiv 0$, and hence, $\partial h(a) = \{0\}$.

The converse implication can be proved directly, using definitions of η -subdifferentials as in [20, Theorem 3.2]. □

We now give a direct application of Proposition 5 to a class of simple semidefinite programming problems which have received much attention in the last decades (see, e.g., [2, 5], and references therein). For the sake of simplicity, we consider the case where $C = X = \mathbb{R}^m$ and $f(x) = \langle c, x\rangle$, $x \in \mathbb{R}^m$, where c is a given vector in $\mathbb{R}^m$. Specifically, we consider the linear semidefinite programming problem:

$$\text{(SDP)} \qquad \text{minimize } \langle c, x\rangle \text{ s.t. } F_0 + \sum_{j=1}^{m} x_j F_j \succeq 0.$$

Here $F_0, F_1, \cdots, F_m$ are given matrices of S_n (we maintain the notation of Proposition 3). We get the following result from Proposition 5.

Corollary 4 (Optimality characterization for (SDP)) *Let* $c \in X = \mathbb{R}^m$. *Assume that a is a feasible solution of* (SDP). *Then a is an optimal solution of* (SDP) *if and only if there exists a net* $(Z_i)_{i\in I} \subset S_n$ *such that* $Z_i \succeq 0$, *for all* $i \in I$, *and*

$$\hat{\mathcal{H}}^*(Z_i) \to c, \quad Tr[Z_i\mathcal{H}(a)] \to 0.$$

Proof It is worth observing that for any $\eta > 0$, and any $Z \in S_n$, $\partial_\eta(Z \circ \mathcal{H})(a) = -\hat{\mathcal{H}}^*(Z)$. The conclusion now follows directly from Proposition 5. □

Farkas lemmas for convex/semidefinite systems (Proposition 4 and Corollary 3) may be used to derive limiting Lagrangian conditions for convex problem (PC), which recover the ones given recently in [20] and [8] as shown in the next result. But first, let us denote by

$$L(x, z^*) := f(x) + (z^* \circ \mathcal{H})(x)$$

the Lagrange function of (PC). Sometimes we write (z^*_i) instead of $(z^*_i)_{i\in I}$.

Proposition 6 (Duality theorem for (PC)) *Let* $f \in \Gamma(X)$ *and* $\mathcal{H} : X \to Z$ *satisfying* $z^* \circ \mathcal{H} \in \Gamma(X)$ *for all* $z^* \in S^+$. *If dom* $f \cap C \cap \mathcal{H}^{-1}(-S) \neq \emptyset$ *then there exists a net* $(\bar{z}_i^*) \subset S^+$ *such that*

$$\sup_{(z_i^*)\subset S^+} \inf_{x\in C} \liminf_{i\in I} L(x, z_i^*) = \inf_{x\in C} \liminf_{i\in I} L(x, \bar{z}_i^*) = \inf(\text{PC}).$$

Moreover,

$$\inf_{x\in C} \sup_{(z_i^*)\subset S^+} \liminf_{i\in I} L(x, z_i^*) = \sup_{(z_i^*)\subset S^+} \inf_{x\in C} \liminf_{i\in I} L(x, z_i^*).$$

Proof When $\inf(\text{PC}) = -\infty$ the equalities hold trivially (the net $(\bar{z}_i^*)_{i\in I} \subset S^+$ can be an arbitrarily chosen). Assume that $\inf(\text{PC}) \in \mathbb{R}$. Then we have

$$C \cap \mathcal{H}^{-1}(-S) \subset [f \geq \inf(\text{PC})].$$

By Proposition 4, applied to $f - \inf(\text{PC})$ instead of f, there exists $(\bar{z}_i^*)_{i\in I} \subset S^+$ such that

$$f(x) + \liminf_{i\in I}(\bar{z}_i^* \circ \mathcal{H})(x) \geq \inf(\text{PC}), \ \forall x \in C,$$

which yields

$$\inf_{x\in C} \liminf_{i\in I} L(x, \bar{z}_i^*) = \inf_{x\in C}[f(x) + \liminf_{i\in I}(\bar{z}_i^* \circ \mathcal{H})(x)] \geq \inf(\text{PC}). \tag{7.1}$$

On the other hand, note that if $z^* \in S^+$ and $x \in \mathcal{H}^{-1}(-S)$ then $(z^* \circ \mathcal{H})(x) \leq 0$. Therefore,

$$\inf(\text{PC}) \geq \inf_{x\in C\cap\mathcal{H}^{-1}(-S)} \sup_{(z_i^*)\subset S^+} \liminf_{i\in I} L(x, z_i^*) \geq \inf_{x\in C} \sup_{(z_i^*)\subset S^+} \liminf_{i\in I} L(x, z_i^*). \tag{7.2}$$

The statement follows by combining (7.1), (7.2), and the following straightforward inequalities:

$$\inf_{x\in C} \sup_{(z_i^*)\subset S^+} \liminf_{i\in I} L(x, z_i^*) \geq \sup_{(z_i^*)\subset S^+} \inf_{x\in C} \liminf_{i\in I} L(x, z_i^*) \geq \inf_{x\in C} \liminf_{i\in I} L(x, \bar{z}_i^*).$$

□

Remark 1 When X and Z are reflexive Banach spaces, and $\mathcal{H}$ is an S -convex and continuous mapping, Proposition 5 coincides with [20, Theorem 3.2] (see also [18, 25]) while, under the additional condition $C = X$, Proposition 6 coincides with [8, Theorem 3.1]. In the same manner, using the Farkas lemma for semidefinite systems (Corollary 3), we can establish the limiting Lagrangian condition for (SDP) that covers the one given in [8].

8 Infinite Linear Optimization without CQ's

In this section we consider different kinds of linear systems and linear optimization problems with an arbitrary number of constraints.

Proposition 7 (Farkas lemma for linear systems I) *Consider two l.c.H.t.v.s.'s* X *and* Z, *let* S *be a preordering closed convex cone in* Z, *let* $\mathcal{A} : X \to Z$ *be a linear mapping* *such that for all* $z^* \in S^+$ *we have* $\mathcal{A}^* z^* \in X^*$, *where* $\mathcal{A}^*$ *is the adjoint operator of* $\mathcal{A}$, *and let* $b \in Z$ *be such that the linear system* $\mathcal{A}x \leq b$ *(i.e.* $\mathcal{A}x - b \in -S$*) is consistent. Then, for any* $x^* \in X^*$, $r \in \mathbb{R}$, *the following statements are equivalent*:

(a) $x \in X$ *and* $\mathcal{A}x \leq b \Longrightarrow \langle x^*, x\rangle \leq r$,
(b) *there exists a net* $(z_i^*, \varepsilon_i)_{i\in I} \subset S^+ \times \mathbb{R}$ *such that*

$$\langle z_i^*, b\rangle \leq r + \varepsilon_i, \ \forall i \in I, \ \textit{and} \ (\mathcal{A}^* z_i^*, \varepsilon_i) \to (x^*, 0_+).$$

Proof This is a direct consequence of the generalized Farkas lemma, Theorem 4, with $f \equiv 0$, $C = X$, $\mathcal{H}(x) = \mathcal{A}x - b$, and $h(x) = \langle x^*, x\rangle - r$. □

Remark 2 If $\mathcal{A}$ is continuous, for all $z^* \in Z^*$, $\mathcal{A}^* z^*$ is continuous since $\langle \mathcal{A}^* z^*, \cdot\rangle = \langle z^*, \mathcal{A}(\cdot)\rangle$. Therefore, the assumption in Proposition 7 holds.

Given an arbitrary set T, consider the space $\mathbb{R}^T$ equipped with the product topology and the space

$$\mathbb{R}^{(T)} = \{\lambda \in \mathbb{R}^T : \text{finitely many } \lambda_t \text{ are different from } 0\},$$

equipped with the direct sum topology. It is well-known that $(\mathbb{R}^T, \mathbb{R}^{(T)})$ is a dual pair through the bilinear form given by

$$\langle \gamma, \lambda\rangle = \sum_{t\in T} \gamma_t \lambda_t, \ \text{for all } \gamma \in \mathbb{R}^{(T)}, \lambda \in \mathbb{R}^T,$$

and according to this fact, $(\mathbb{R}^T)^* = \mathbb{R}^{(T)}$ and $(\mathbb{R}^{(T)})^* = \mathbb{R}^T$.

By means of this notation, the convex conical hull of a set $\{x_t, t \in T\} \subset X$ can be expressed as cone $\{x_t, t \in T\} = \left\{ \sum_{t\in T} \lambda_t x_t : \lambda \in \mathbb{R}_+^{(T)} \right\}$, where $\lambda_t := \lambda(t)$, $t \in T$.

Proposition 8 (Farkas lemma for linear systems II) *Let* X *be an l.c.H.t.v.s., let* T *be an arbitrary (possibly infinite) index set, and let* $x_t^* \in X^*$, $r_t \in \mathbb{R}$, *for all* $t \in T$, *such that the linear inequality system* $\{\langle x_t^*, x\rangle \leq r_t,\ t \in T\}$ *is consistent. Then, for any pair* $x^* \in X^*$, $r \in \mathbb{R}$, *the following statements are equivalent*:

(a) $x \in X$ *and* $\langle x_t^*, x\rangle \leq r_t$, *for all* $t \in T \Longrightarrow \langle x^*, x\rangle \leq r$,
(b) *there is a net* $\left(\lambda^i, \varepsilon_i\right)_{i\in I} \subset \mathbb{R}_+^{(T)} \times \mathbb{R}$ *such that*

$$\sum_{t\in T} \lambda_t^i r_t \leq r + \varepsilon_i, \ \forall i \in I, \ \textit{and} \ \left(\sum_{t\in T} \lambda_t^i x_t^*, \varepsilon_i \right) \to (x^*, 0_+).$$

(c) $(x^*, r) \in \overline{\text{cone}} \left\{ \left(x_t^*, r_t\right), t \in T; (0, 1) \right\}$.

Proof The equivalence between (a) and (b) follows directly from Proposition 7, just by taking $Z = \mathbb{R}^T$, $S = \mathbb{R}_+^T$, $\mathcal{A}x = (\langle x_t^*, x\rangle)_{t\in T}$, $b = (r_t)_{t\in T}$, $Z^* = \mathbb{R}^{(T)}$, and $S^+ = \mathbb{R}_+^{(T)}$. Here, if $\gamma = (\gamma_t)_{t\in T} \in \mathbb{R}_+^{(T)}$ we have $\mathcal{A}^*\gamma = \sum_{t\in T} \gamma_t x_t^* \in X^*$. The equivalence between (b) and (c) follows by standard arguments. However, for easy reading and for the completeness of the proof, we present the implication $[(b) \Rightarrow (c)]$.

Assume that (b) holds. Let $\mu_i := r + \varepsilon_i - \sum_{t\in T} \lambda_t^i r_t \geq 0, \ \forall i \in I$. Then (c) holds because

$$\left(x^*, r\right) = \lim_{i\in I} \left\{ \sum_{t\in T} \lambda_t^i \left(x_t^*, r_t\right) + \mu_i \left(0, 1\right) \right\}$$

and $\sum_{t\in T} \lambda_t^i \left(x_t^*, r_t\right) + \mu_i \left(0, 1\right) \in \text{cone} \left\{ \left(x_t^*, r_t\right), t \in T; (0, 1) \right\}$ for all $i \in I$.

To complete the proof it is sufficient to prove the implication $[(c) \Rightarrow (a)]$. Assume that (c) holds. Let $(\lambda_i)_{i\in I} \subset \mathbb{R}_+^{(T)}$ and $(\mu_i)_{i\in I}$ be such that $(x^*, r) = \lim_{i\in I} \left\{ \sum_{t\in T} \lambda_t^i \left(x_t^*, r_t\right) + \mu_i \left(0, 1\right) \right\}$. Then, for any $x \in X$ such that $\langle x_t^*, x\rangle \leq r_t, \ t \in T$, we have

$$\begin{aligned} \langle x^*, x\rangle - r &= \langle (x^*, r), (x, -1)\rangle \\ &= \lim_{i\in I} \left\{ \sum_{t\in T} \lambda_t^i \left\langle \left(x_t^*, r_t\right), (x, -1)\right\rangle + \mu_i \left\langle (0, 1), (x, -1)\right\rangle \right\} \\ &= \lim_{i\in I} \left\{ \sum_{t\in T} \lambda_t^i \left(\left\langle x_t^*, x\right\rangle - r_t\right) - \mu_i \right\} \leq 0. \end{aligned}$$

Thus (a) holds. □

The equivalence between (a) and (c) was proved in [6, Theorem 2]. The finite dimensional version of this result ($X = \mathbb{R}^n$) is a basic theoretical tool in linear semi-infinite programming (LSIP in brief). Next we consider the infinite linear programming problem

$$(\text{LIP}) \quad \begin{cases} \text{minimize} & \langle c^*, x\rangle \\ \text{s.t.} & x \in A, \end{cases}$$

$A := \left\{ x \in X : \langle x_t^*, x\rangle \leq r_t, t \in T \right\}$.

Proposition 9 (Primal optimal value of (LIP)) *Let $X, \ T, \ x_t^*$, and $r_t, t \in T$, be as in Proposition 8, and let $c^* \in X^*$. Then, one has*

$$\inf(\text{LIP}) = \sup \left\{ s \in \mathbb{R} : (c^*, s) \in -\overline{\text{cone}} \left\{ \left(x_t^*, r_t\right), \ t \in T; (0, 1)\right\}\right\} \in \mathbb{R} \cup \{-\infty\}.$$

Proof Let us denote

$$\begin{aligned} \alpha &: = \inf(\text{LIP}), \\ \beta &: = \sup \left\{ s \in \mathbb{R} : (c^*, s) \in -\overline{\text{cone}} \left\{ \left(x_t^*, r_t\right), \ t \in T; (0, 1)\right\}\right\}. \end{aligned}$$

We first prove that $\beta \geq \alpha$.

If $\alpha = -\infty$, the inequality trivially holds. If $\alpha > -\infty$, one has $\alpha \in \mathbb{R}$ because the feasible set of (LIP) is nonempty by assumption. Observe that $(x^*, r) := -(c^*, \alpha)$ satisfies the condition (a) in Proposition 8, which is equivalent to (c), i.e. to

$$(c^*, \alpha) \in -\overline{\text{cone}} \left\{ \left(x_t^*, r_t\right), \ t \in T; (0, 1)\right\}$$

and, so, by the own definition of β, $\beta \geq \alpha$.

We now prove the opposite inequality $\alpha \geq \beta$. Let $s \in \mathbb{R}$ be such that

$$(c^*, s) \in -\overline{\text{cone}} \left\{ \left(x_t^*, r_t\right), \ t \in T; (0, 1)\right\}.$$

By Proposition 8, for any feasible point x of (LIP) we have $\langle c^*, x\rangle \geq s$. Taking the supremum over s and the infimum over $x \in A$, we get $\alpha \geq \beta$. □

Corollary 5 (Optimality characterization for (LIP)) *Let X, T, x_t^*, and r_t, $t \in T$, be as in Proposition* 8. *Let $c^* \in X^*$ and consider $a \in A$. Then the following statements are equivalent*:

(a) *a is an optimal solution of* (LIP),
(b) *there is a net $\left(\lambda^i, \varepsilon_i\right)_{i\in I} \subset \mathbb{R}_+^{(T)} \times \mathbb{R}$ such that*

$$\sum_{t\in T} \lambda_t^i r_t \leq \varepsilon_i - \left\langle c^*, a\right\rangle, \ \forall i \in I, \ \textit{and} \ \left(\sum_{t\in T} \lambda_t^i x_t^*, \varepsilon_i\right) \to (-c^*, 0_+).$$

In that case, the optimal value of (LIP) *is*

$$\inf(\text{LIP}) = \max\left\{s \in \mathbb{R} : (c^*, s) \in -\overline{\mathit{cone}}\left\{\left(x_t^*, r_t\right), t \in T; (0, 1)\right\}\right\} \in \mathbb{R}. \tag{8.1}$$

Proof Taking $x^* = -c^*$ and $r = -\langle c^*, a\rangle$, the equivalence between (a) and (b) follows from the corresponding equivalence in Proposition 8, whose statement (c) becomes here

$$\left(c^*, \left\langle c^*, a\right\rangle\right) \in -\overline{\text{cone}}\left\{\left(x_t^*, r_t\right), t \in T; (0, 1)\right\},$$

and this, together with Proposition 9, implies (8.1). □

To the authors' knowledge the above characterization of optimality in (LIP) is new. Even in finite dimensions, no characterization of the optimal solution in (LSIP) without CQ is available (the KKT condition is sufficient, but not necessary, and the same is true for stronger conditions as those obtained in [12] by means of the concept of extended active constraints). In the same framework, the finite dimensional version of (8.1) is the well-known geometric interpretation of the optimal value of the primal (LSIP) problem (see, e.g., [11, (8.5)]). If one considers (LIP) as a parametric optimization problem with parameter c^*, then (8.1) can be interpreted in terms of the hypograph of the optimal value function of (LIP), $-i_A^*\left(-c^*\right)$:

$$-\text{epi}\, i_A^*(\cdot) = \text{hypo}\ -i_A^*\left(-(\cdot)\right) = -\overline{\text{cone}}\left\{\left(x_t^*, r_t\right), t \in T; (0, 1)\right\}.$$

Next we extend from (LSIP) to (LIP) the notion of Haar's dual problem:

$$(\text{DLIP}) \quad \begin{cases} \text{maximize} & \sum_{t\in T} \lambda_t r_t \\ \text{s.t.} & \sum_{t\in T} \lambda_t x_t^* = c^*, \ \lambda \in -\mathbb{R}_+^{(T)}. \end{cases}$$

It is easy to check that, adopting the standard conventions $\sup\emptyset = -\infty$ and $\inf\emptyset = +\infty$, one has

$$\begin{aligned} -\infty &\leq \sup(\text{DLIP}) \\ &= \sup\left\{s \in \mathbb{R} : \left(c^*, s\right) \in -\text{cone}\left\{\left(x_t^*, r_t\right), t \in T; (0, 1)\right\}\right\} \\ &\leq \inf(\text{LIP}) \leq +\infty, \end{aligned} \tag{8.2}$$

and so, the *weak duality* holds. If (LIP) is feasible and cone $\left\{\left(x_t^*, r_t\right), t \in T; (0, 1)\right\}$ is w^*-closed, it comes from Proposition 9 and (8.2) that

$$-\infty \leq \sup(\text{DLIP}) = \inf(\text{LIP}) < +\infty.$$

Moreover if (DLIP) is feasible, then

$$-\infty < \max(\text{DLIP}) = \inf(\text{LIP}) < +\infty,$$

i.e. *strong duality* holds in the sense that there is no duality gap and the dual problem has at least an optimal solution.

It is worth observing that the constraints of (DLIP) constitute a linear system in the decision space $\mathbb{R}^{(T)}$. The following corollary is a Farkas lemma for linear systems posed in $\mathbb{R}^{(T)}$, whose general form is $\left\{\sum_{t\in T} \lambda_t a_t^j \leq s_j,\ \ j \in J\right\}$, with $a^j \in \mathbb{R}^T$ and $s_j \in \mathbb{R}$, for all $j \in J$.

Corollary 6 (Farkas lemma for linear systems III) *Let* $\{\sum_{t\in T} \lambda_t a_t^j \leq s_j,\ \ j \in J\}$ *be a consistent system in* $\mathbb{R}^{(T)}$. *Then, for any pair* $a \in \mathbb{R}^T,\ \ s \in \mathbb{R}$, *the following statements are equivalent*:

(a) $\lambda \in \mathbb{R}^{(T)}$ *and* $\sum_{t\in T} \lambda_t a_t^j \leq s_j,\ \ j \in J \implies \sum_{t\in T} \lambda_t a_t \leq s$,

(b) *there exists a net* $(\gamma^i, \varepsilon_i)_{i\in I} \subset \mathbb{R}_+^{(J)} \times \mathbb{R}$ *such that*

$$\sum_{j\in J} \gamma_j^i r_j \leq r + \varepsilon,\ \ \forall i \in I,\ \ \text{and}\ (\mathcal{A}^*\gamma^i, \varepsilon_i) \to (a, 0_+),$$

where $\mathcal{A}^*\gamma^i = \sum_{j\in J} \gamma_j^i a^j$.

Proof It is a direct consequence of Proposition 7 taking $X = \mathbb{R}^{(T)}$, $\mathcal{A} : \mathbb{R}^{(T)} \to Z = \mathbb{R}^J$ (equipped with the product topology) such that $(\mathcal{A}\lambda)_j = \sum_{t\in T} \lambda_t a_t^j$, $\forall j \in J$, $S = \mathbb{R}_+^J$ (so that $S^+ = \mathbb{R}_+^{(J)}$), $b = \left(s_j\right)_{j\in J}$, and $x^* = a$. □

Corollary 7 (Optimality characterization for (DLIP)) *Let X be an l.c.H.t.v.s., let T be an arbitrary (possibly infinite) index set, and let* $c^*, x_t^* \in X^*$, $r_t \in \mathbb{R}$, *for all* $t \in T$, *such that the linear inequality system* $\{\sum_{t\in T} \lambda_t x_t^* = c^*,\ \ \lambda \in -\mathbb{R}_+^{(T)}\}$ *is consistent. Let* $\alpha \in \mathbb{R}^{(T)}$ *be a feasible solution of* (DLIP). *Then the following statements are equivalent*:

(a) α *is an optimal solution of* (DLIP),

(b) *there exists a net* $(\mu^i, \varepsilon_i)_{i\in I} \subset \left(\mathbb{R}^{(X)} \times \mathbb{R}_+^{(T)}\right) \times \mathbb{R}$ *such that*

$$\sum_{x\in X} \left\langle c^*, x\right\rangle \mu_x^i \leq \sum_{t\in T} \alpha_t r_t + \varepsilon_i,\ \ \forall i \in I,\ \ \text{and}\ \left(\mathcal{A}^*\mu^i, \varepsilon_i\right) \to (r, 0_+),$$

where $r = (r_t)_{t\in T}$ *and* $\left(\mathcal{A}^*\mu^i\right)_t = \sum_{x\in X} \left\langle x_t^*, x\right\rangle \mu_x^i + \mu_t^i$, $\forall i \in I$.

Proof (a) can be reformulated as

$$\lambda \in \mathbb{R}^{(T)} \text{and} \sum_{t\in T} \lambda_t a_t^j \leq s_j,\ j \in J \Longrightarrow \sum_{t\in T} \lambda_t a_t \leq s,$$

just taking $a_t = r_t$, for all $t \in T$, $s = \sum_{t\in T} \alpha_t r_t$, $J = (X \times \{0, 1\}) \cup T$, with $a_t^{(x,k)} = (-1)^k \left\langle x_t^*, x \right\rangle$, $s_{(x,k)} = (-1)^k \langle c^*, x\rangle$, for all $(x, k) \in X \times \{0, 1\}$, $a_t^u = 1$, if $t = u$, and $a_t^u = 0$, otherwise, and $s_u = 0$, for all $u \in T$. Applying Corollary 6 we get $(a) \Leftrightarrow (b)$ by defining $\mu_x^i := \gamma_{(x,0)}^i - \gamma_{(x,1)}^i$ for all $x \in X$ and $\mu_t^i = \gamma_t^i$ for all $t \in T$. □

The last two results are new even in finite dimensions (compare, e.g., with [1] and [11]).

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