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Metric Regularity of Mappings and Generalized Normals to Set Images

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Abstract The primary goal of this paper is to study some notions of normals to nonconvex sets in finite-dimensional and infinite-dimensional spaces and their images under single-valued and set-valued mappings. The main motivation for our study comes from variational analysis and optimization, where the problems under consideration play a crucial role in many important aspects of generalized differential calculus and applications. Our major results provide precise equality formulas (sometimes just efficient upper estimates) allowing us to compute generalized normals in various senses to direct and inverse images of nonconvex sets under single-valued and set-valued mappings between Banach spaces. The main tools of our analysis revolve around variational principles and the fundamental concept of metric regularity properly modified in this paper.

Keywords Variational analysis · Metric regularity · Generalized differentiation · Normal cones · Coderivatives · Calculus rules · Set images

Mathematics Subjects Classifications (2000) Primary 49J52 · 49J53; Secondary 46B20 · 58C20 · 90C31

Dedicated to Alex Ioffe in honor of his 70th birthday.

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1 Introduction

This paper primarily concerns applications of the concept and machinery of *metric regularity* to deriving new results of generalized differential calculus and to the study of some other related issues of variational analysis. The concept of metric regularity goes back to the seminal Lyusternik-Graves theorem of the classical nonlinear analysis and has been long recognized among the most fundamental tools in the modern stage of nonlinear analysis especially regarding its variational aspects; see, e.g., the books [5, 15, 17, 18, 22, 23], the extended surveys [2, 7, 12], the papers [1, 13, 14, 16, 20, 21], and the references therein for more details as well as for recent developments. Alex Ioffe is surely one of the major original contributors and nowadays leaders in this and related fields of modern set-valued and variational analysis. The range and depth of his results on metric regularity, starting with the now classical work [10] and including the very recent paper [13], are difficult to overstate. In particular, he was the first to apply variational principles into the area of metric regularity, to derive quantitative estimates of the regularity/surjection moduli, to apply metric regularity to subdifferential calculus, to introduce and deeply investigate new notions of relative metric regularity, etc.

This paper is mainly devoted to new applications of variational/extremal principles and metric regularity to generalized differential calculus in both settings of finite-dimensional spaces and infinite-dimensional Banach spaces.

Having a single-valued mapping $f\colon X \to Y$ between Banach spaces and a set $\Omega \subset X$ with $\bar{x} \in \Omega$, we pay the major attention to evaluating generalized normals (in various senses) to the *image set* $f(\Omega)$ at the point $f(\bar{x})$. Our primary goal is to derive *precise* (equality-type) formulas for computing generalized normals under appropriate differentiability assumptions on f at $\bar{x}$ in the framework of arbitrary Banach spaces with *no surjectivity* requirement on the derivative $\nabla f(\bar{x})$ as in the classical Lyusternik-Graves theorem.

It is well known that the surjectivity of $\nabla f(\bar{x})$ is a *characterization* of metric regularity for smooth (or strictly differentiable) mappings; see, e.g., [17, Theorem 1.57]. To deal with such mappings between Banach spaces that fail to have surjective derivatives at the points in question, we introduced in [20] the notion of *restrictive metric regularity* (RMR) for $f\colon X \to Y$ around the given point $\bar{x}$ by considering the usual (metrically defined) property of metric regularity for the mapping f from the original domain space X to the *nonlinear* metric space $f(X) \subset Y$ instead of the original range space Y with a linear structure. The RMR notion and verifiable conditions for its validity obtained in [20] allowed us to derive therein equality formulas for computing Fréchet-like and limiting normals to *inverse images* $f^{-1}(\Theta)$ of arbitrary sets $\Theta \subset Y$ under strictly differentiable mappings f between Banach spaces such that the derivative $\nabla f(\bar{x})$ may not be surjective.

In this paper we continue the line of development in [20] focusing mainly on computing generalized normals to *direct images* $f(\Omega)$ under differentiable (not always strictly) mappings $f\colon X \to Y$, with deriving new results for various normals (not only Fréchet-like and limiting ones) to inverse images as well. In the majority of our new results we relax the aforementioned RMR property requiring the *metric regularity* of the mapping $f\colon \Omega \to f(\Omega)$ between the both *metric spaces* Ω and $f(\Omega)$ (*around* and sometimes just *at* the reference point), verifiable conditions for the fulfillment of which are obtained by using advantages of the linear structure on the

Banach spaces X and Y and differentiability of f. We also present counterparts of these results for evaluating (as upper estimates) generalized normals to images of *set-valued* mappings $F\colon X \rightrightarrows Y$ between Asplund spaces.

The rest of the paper is organized as follows. Section 2 describes some major constructions of *generalized differentiation* and related notions of variational analysis widely used in formulating and proving the main results below. In Section 3 we introduce and discuss various modifications of the concept of *restrictive metric regularity* and establish some verifiable conditions for their fulfillment in terms of pointbased generalized differential constructions.

Section 4 concerns evaluating *Fréchet normals* and *ε-normals* to set images under *Fréchet differentiable* single-valued mappings between Banach spaces and presents also some related and auxiliary material of independent interest. In Section 5 we deal with *sequential weak* limits* of Fréchet-like normals to direct and inverse set images in general Banach spaces as well as in some of their remarkable subclasses. We derive *upper estimate* and *equality* types results under relaxed RMR assumptions on strictly differentiable mappings. We also obtain upper estimates for limiting normals to set images under *set-valued* mappings.

Section 6 is devoted to counterparts of the equality-type results from Section 4 for the so-called *Hölder s-normal cones* (with $s \in (0,1]$) to direct and inverse images of sets under Hölder s-differentiable mappings between Banach spaces. In the case of $s = 1$ the Hölder normal cone reduces to the *proximal normal cone*, which may be smaller than the Fréchet one even in finite dimensions. Further, in this section we establish equality-type results for computing the *convexified normal cone* to direct and inverse set images, which agrees with the Clarke normal cone in reflexive spaces.

Our notation is basically conventional in variational analysis; see, e.g., [17, 22]. Unless otherwise stated, all the spaces under consideration are *Banach* with their norms denoted by $\|\cdot\|$ and the canonical pairing $\langle\cdot,\cdot\rangle$ between the space X in question and its topological dual X^*. Recall that the symbol

$$\operatorname*{Lim\,sup}_{x\to\bar{x}} F(x) := \Big\{x^* \in X^* \Big|\ \exists \text{ sequences } x_k \to \bar{x} \text{ and } x_k^* \xrightarrow{w^*} x^* \text{ with } x_k^* \in F(x_k) \text{ for all } k \in I\!N\Big\} \tag{1.1}$$

stands for the *sequential Painlevé-Kuratowski upper/outer limit* of a set-valued mapping $F\colon X \rightrightarrows X^*$ as $x \to \bar{x}$ in the norm topology of X and the weak* topology w^* of X^*, where $I\!N := \{1, 2, \ldots\}$. Given a set $\Omega \subset X$, denote by $\operatorname{cl}\Omega$ and $\operatorname{co}\Omega$ the *closure* and *convex hull* of Ω, respectively; cl^* signifies the closure of a subset of the dual space in the *weak** topology. The symbol $x \xrightarrow{\Omega} \bar{x}$ means that $x \to \bar{x}$ with $x \in \Omega$. If no confusion arises, $I\!B$ and $I\!B^*$ stand for the *closed unit balls* of the space and dual space in question.

2 Constructions of Generalized Differentiation

In this section we present the underlying constructions of generalized differentiation and related properties of variational analysis widely used in the paper. We mainly follow the book [17] referring the reader also to [5, 12, 18, 22, 23] for associated and additional material.

Given a nonempty set $\Omega \subset X$ in a Banach space X and a number $\varepsilon \ge 0$, define the *collection of* ε*-normals* to Ω at $x \in \Omega$ by

$$\widehat{N}_\varepsilon(x;\Omega) := \left\{ x^* \in X^* \Big|\ \limsup_{u \xrightarrow{\Omega} x} \frac{\langle x^*, u - x\rangle}{\|u - x\|} \le \varepsilon \right\} \tag{2.1}$$

and let $\widehat{N}_\varepsilon(x;\Omega) := \emptyset$ if $x \notin \Omega$. When $\varepsilon = 0$ in Eq. 2.1, the set $\widehat{N}(x;\Omega) := \widehat{N}_0(x;\Omega)$ is a convex cone known as the *Fréchet or regular normal cone* to Ω at this point.

In general, we do not have satisfactory calculus and related properties for the normal collection Eq. 2.1 whenever $\varepsilon \ge 0$ even in finite dimensions; e.g., they may be trivial at boundary points of closed sets in $I\!R^2$. Much better calculus rules and other properties hold for the *sequential* outer limit Eq. 1.1 of the constructions $\widehat{N}_\varepsilon(x;\Omega)$ given by

$$N(\bar{x};\Omega) := \mathop{\mathrm{Lim\,sup}}_{\substack{x \to \bar{x} \\ \varepsilon \downarrow 0}} \widehat{N}_\varepsilon(x;\Omega) \tag{2.2}$$

and known as the *basic, or limiting, or Mordukhovich normal cone* to Ω at $\bar{x} \in \Omega$. It follows from Eqs. 2.2 and 1.1 that $x^* \in N(\bar{x};\Omega)$ if and only if there are sequences $\varepsilon_k \downarrow 0$, $x_k \xrightarrow{\Omega} \bar{x}$, and $x_k^* \xrightarrow{w^*} x^*$ as $k \to \infty$ such that $x_k^* \in \widehat{N}_{\varepsilon_k}(x_k;\Omega)$ for all $k \in I\!N$. By [17, Theorem 2.35], we can equivalently let $\varepsilon = 0$ in Eq. 2.2 if the set Ω is locally closed around $\bar{x}$ and if the space X is *Asplund*, i.e., every separable subspace of it has a separable dual. The class of Asplund spaces has been well investigated in geometric theory of Banach spaces and widely employed in variational analysis; see [5, 17, 18] for more details and references. Note, in particular, that any reflexive Banach space is Asplund while, e.g., the important classical spaces $C[0,1]$ and $L^\infty[0,1]$ are not.

Despite the *nonconvexity* of the limiting normal cone Eq. 2.2, or probably due to it, there is a fairly comprehensive amount of calculus rules available for this normal cone and the associated coderivative and subdifferential constructions, mainly in the Asplund space framework; see [17, Chapter 3] and the references therein. The list of calculus rules and related results for Eq. 2.2 known in general Banach spaces and largely presented in [17, Chapter 1] is by far less impressive. We extend this list in the paper.

Consider next a set-valued mapping $F\colon X \rightrightarrows Y$ between Banach spaces with the graph

$$\mathrm{gph}\, F := \big\{(x,y) \in X \times Y \big|\ y \in F(x)\big\},$$

we define the coderivative constructions generated by the above normal cones as follows: the *Fréchet coderivative* of F at $(\bar{x},\bar{y}) \in \mathrm{gph}\, F$ given by

$$\widehat{D}^* F(\bar{x},\bar{y})(y^*) := \big\{x^* \in X^* \big|\ (x^*, -y^*) \in \widehat{N}\big((\bar{x},\bar{y}); \mathrm{gph}\, F\big)\big\}, \quad y^* \in Y^*, \tag{2.3}$$

the *normal coderivative* of F at $(\bar{x},\bar{y}) \in \mathrm{gph}\, F$ given by

$$D_N^* F(\bar{x},\bar{y})(y^*) := \big\{x^* \in X^* \big|\ (x^*, -y^*) \in N\big((\bar{x},\bar{y}); \mathrm{gph}\, F\big)\big\}, \quad y^* \in Y^*, \tag{2.4}$$

and the *mixed coderivative* of F at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ given by

$$D^*_M F(\bar{x}, \bar{y})(y^*) := \Big\{ x^* \in X^* \Big| \exists \text{ sequences } \varepsilon_k \downarrow 0,\ (x_k, y_k) \overset{\operatorname{gph} F}{\to} (\bar{x}, \bar{y}),\ x^*_k \overset{w^*}{\to} x^*,$$
$$y^*_k \to y^* \text{ with } \big(x^*_k, -y^*_k\big) \in \widehat{N}_{\varepsilon_k}\big((x_k, y_k); \operatorname{gph} F\big) \Big\}. \quad (2.5)$$

As before, note that we can equivalently put $\varepsilon_k \equiv 0$ in Eq. 2.5 if both spaces X and Y (and hence their products) are Asplund and if the graph of F is locally closed around $(\bar{x}, \bar{y})$. Observe also that the normal coderivative Eq. 2.4 can be described in a similar limiting way as Eq. 2.5 with the replacement of the normal convergence $y^*_k \to y^*$ by the weak* one $y^*_k \overset{w^*}{\to} y^*$ in Y^*.

If $F = f\colon X \to Y$ is a single-valued mapping, we omit $\bar{y} = f(\bar{x})$ in the above coderivative notation. Note that $\widehat{D}^* f(\bar{x})(y^*) = \{\nabla f(\bar{x})^* y^*\}$ when f is Fréchet differentiable at $\bar{x}$ and

$$D^*_N f(\bar{x})(y^*) = D^*_M f(\bar{x})(y^*) = \big\{\nabla f(\bar{x})^* y^*\big\}, \quad y^* \in Y^*, \quad (2.6)$$

when f is strictly differentiable at $\bar{x}$ in the sense that

$$\lim_{x,u \to \bar{x}} \frac{f(u) - f(x) - \langle \nabla f(\bar{x}), u - x\rangle}{\|u - x\|} = 0,$$

which is automatic when $f \in C^1$ around $\bar{x}$.

Finally in this section, recall some normal compactness properties of sets and mappings used in the paper that are automatic in finite dimensions while playing a significant role in infinite-dimensional variational analysis and its applications. Given a set $\Omega \subset X$, we say that it is *sequentially normally compact* (SNC) at $\bar{x} \in \Omega$ if for all sequences $\varepsilon_k \downarrow 0$, $x_k \overset{\Omega}{\to} \bar{x}$, and $x^*_k \in \widehat{N}_{\varepsilon_k}(x_k; \Omega)$, $k \in \mathbb{N}$, we have the implication

$$\Big[x^*_k \overset{w^*}{\to} 0\Big] \Longrightarrow \big[\|x^*_k\| \to 0\big] \text{ as } k \to \infty,$$

which can be equivalently written with $\varepsilon_k \equiv 0$ when X is Asplund and Ω is locally closed around $\bar{x}$; see [17]. This property is always implied by, being closely related to, the *compactly epi-Lipschitzian* property of sets in the sense of Borwein and Strójwas [4], which is intrinsically defined in the primal space X while can be equivalently described in the dual space framework as a *topological* version of the SNC property; the reader can find all the details and comprehensive results in this direction in [9, 11].

A set-valued mapping $F\colon X \rightrightarrows Y$ is *SNC* at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ if its graph is SNC at this point. In the case of mappings (or sets in *product* spaces), a more delicate property of this type is important for variational theory and applications. We say that the mapping F is *partially SNC* (PSNC) at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ if for all sequences $\varepsilon_k \downarrow 0$, $(x_k, y_k) \overset{\operatorname{gph} F}{\to} (\bar{x}, \bar{y})$, and $(x^*_k, y^*_k) \in \widehat{N}_{\varepsilon_k}((x_k, y_k); \operatorname{gph} F)$ the implication

$$\Big[x^*_k \overset{w^*}{\to} 0,\ \ \|y^*_k\| \to 0\Big] \Longrightarrow \big[\|x^*_k\| \to 0\big] \text{ as } k \to \infty$$

holds. Further, F is *strongly PSNC* at $(\bar{x}, \bar{y})$ if the latter implication is replaced by

$$\Big[\big(x^*_k, y^*_k\big) \overset{w^*}{\to} (0, 0)\Big] \Longrightarrow \big[\|x^*_k\| \to 0\big] \text{ as } k \to \infty.$$

We refer the reader to [17, Subsections 1.1.4, 1.2.5, 3.1.1] and commentaries therein for the genesis and efficient conditions ensuring these properties. In particular, it follows from [17, Theorem 1.43] that the PSNC property of F at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ automatically holds for every mapping $F\colon X \rightrightarrows Y$ between Banach spaces satisfying the following *Lipschitz-like, or pseudo-Lipschitzian, or Aubin property* around $(\bar{x}, \bar{y})$: there are neighborhoods U of $\bar{x}$ and V of $\bar{y}$ and a constant $\ell \geq 0$ such that

$$F(u) \cap V \subset F(x) + \ell \|u - x\| I\!B \ \text{ whenever } \ x, u \in U. \tag{2.7}$$

The latter property is well known to be *equivalent* to the *metric regularity* of the *inverse* mapping F^{-1} around $(\bar{y}, \bar{x})$. Appropriate modifications of metric regularity for single-valued mappings are studied in the next section and are applied in what follows.

3 Versions of Restrictive Metric Regularity

Given a mapping $f\colon E_1 \to E_2$ between two *metric* spaces (E_1, d_1) and (E_2, d_2), recall that this mapping is *metrically regular around* $\bar{x}$ if there are neighborhoods U of $\bar{x}$ and V of $f(\bar{x})$ and a constant $\mu > 0$ such that

$$\operatorname{dist}\big(x; f^{-1}(y)\big) := \inf_{u \in f^{-1}(y)} d_1(x, u) \leq \mu \, d_2\big(f(x), y\big) \ \text{ for all } \ x \in U \ \text{ and } \ y \in V. \tag{3.1}$$

Furthermore, we say that f is metrically regular *at* $\bar{x}$ if Eq. 3.1 holds with $x = \bar{x}$, i.e., there exist a neighborhood V of $f(\bar{x})$ and a constant $\mu > 0$ such that

$$\operatorname{dist}\big(\bar{x}; f^{-1}(y)\big) \leq \mu \, d_2\big(f(\bar{x}), y\big) \ \text{ for all } \ \ y \in V.$$

There are some conditions ensuring metric regularity of mappings between metric spaces in terms of the so-called *strong slopes*; see, e.g., [2, 12] and the references therein. Considerably larger amount of efficient conditions for metric regularity is available for mappings between *Banach* spaces, where the notions of classical and generalized derivatives as well as tangent and normal constructions associated with them play a significant role in the study of metric regularity. We mention first of all the fundamental result known now as the Lyusternik-Graves theorem, which says that a mapping $f\colon X \to Y$ between Banach spaces strictly differentiable at $\bar{x}$ is metrically regular around this point *if and only if* the derivative operator $\nabla f(\bar{x})\colon X \to Y$ is surjective. A full *coderivative* analog of this result is given in [16] for set-valued mappings between finite-dimensional spaces and in [17, Theorem 4.18] for mappings between Asplund spaces. We refer the reader to [5, 7, 11, 12, 14, 16, 17, 22, 23] for the genesis of ideas and various approaches and for more results and discussions in this direction in finite-dimensional and infinite-dimensional Banach spaces.

It seems that the notion of *restrictive metric regularity* (RMR) introduced in [20] is the first one for mappings between Banach spaces, which combines advantages of Banach spaces with the general metric nature of metric regularity. Recall that $f\colon X \to Y$ has the *RMR property around* $\bar{x}$ if the mapping $f\colon X \to f(X) \subset Y$ between the Banach space X and the metric space $(E_2, d_2) := (f(X), \|\cdot\|_Y)$ is metrically regular around $\bar{x}$ in the sense of Eq. 3.1. A major result of [20] gives a full characterization of the RMR property of a mapping f strictly differentiable at $\bar{x}$

as follows: f is RMR around $\bar{x}$ *if and only if* the space $\nabla f(\bar{x})X$ is finite-codimensional in Y and

$$\widetilde{T}\big(\bar{y};\, f(X)\big) = \nabla f(\bar{x})X \ \text{ with } \ \bar{y} := f(\bar{x}),$$

where $\widetilde{T}(\bar{y};\Theta)$ stands for the so-called *paratingent cone* to $\Theta \subset Y$ defined by

$$\widetilde{T}(\bar{y};\Theta) := \Big\{ v \in Y \Big| \, \exists v_k \to v,\ t_k \downarrow 0,\ y_k \overset{\Theta}{\to} \bar{y} \ \text{ with } \ y_k + t_k v_k \in \Theta \Big\}.$$

In this paper we mainly study and employ another modification of the RMR notion, which deals with the metric regularity in the sense of Eq. 3.1 for the mapping $f\colon \Omega \to f(\Omega)$ between the two metric spaces $(E_1, d_1) := (\Omega, \|\cdot\|_X)$ and $(E_2, d_2) := (f(\Omega), \|\cdot\|_Y)$.

In this section we present some sufficient conditions for this property in the case of strictly differentiable mappings f between Asplund spaces and locally closed sets Ω. The proof of these result is heavily based on the pointbased *coderivative* and *PSNC characterizations* of the *Lipschitz-like* property of *set-valued* mappings between Asplund spaces and efficient rules of the *generalized differential* and *SNC calculi* established in [17, 19] for the limiting constructions involved.

Theorem 3.1 (sufficient conditions for metric regularity of restricted mappings) *Let $f\colon X \to Y$ be a mapping between Asplund spaces strictly differentiable at $\bar{x}$, and let Ω be a subset of X locally closed around $\bar{x} \in \Omega$. Impose further the qualification condition*

$$\big(\nabla f(\bar{x})^*\big)^{-1}\big(N(\bar{x};\Omega)\big) = \{0\}. \tag{3.2}$$

Then the restricted mapping $f\colon \Omega \to f(\Omega)$ between metric spaces is metrically regular around $\bar{x}$ in each of the following cases:

(a) *either the space Y is finite-dimensional,*
(b) *or the space $\nabla f(\bar{x})X \subset Y$ is closed and the set Ω is SNC at $\bar{x}$.*

Proof Denote $\bar{y} := f(\bar{x})$ and define a set-valued mapping $G\colon Y \rightrightarrows X$ by

$$G(y) := f^{-1}(y) \cap \Omega. \tag{3.3}$$

It follows directly from the definitions that the Lipschitz-like property Eq. 2.7 of the constructed mapping G around $(\bar{y}, \bar{x})$ implies the metric regularity of the restricted mapping $f\colon \Omega \to f(\Omega)$ around $\bar{x}$ in the metric sense Eq. 3.1. We intend in what follows to apply the aforementioned *coderivative characterization* of the Lipschitz-like property from [17, Theorem 4.10] to the mapping G from Eq. 3.3. To begin with, observe by Eq. 3.3 that

$$\operatorname{gph} G = \Omega_1 \cap \Omega_2 \subset Y \times X, \tag{3.4}$$

where the sets $\Omega_1 := \{(y, x) \in Y \times X \mid y = f(x)\}$ and $\Omega_2 := Y \times \Omega$ are locally closed around the point $(\bar{y}, \bar{x})$. Since both spaces X and Y are Asplund, the product space $Y \times X$ is Asplund as well. To estimate the limiting normal cone Eq. 2.2, we use the basic calculus result (normal cone *intersection rule*) result from [17, Theorem 3.4]. It is easy to see from the structures of Ω_1 and Ω_2 that

$$N\big((\bar{y},\bar{x});\Omega_1\big)=\big\{\big(y^*,-\nabla f(\bar{x})^*y^*\big)\big|\ y^*\in Y^*\big\}\ \text{ and }\ N\big((\bar{y},\bar{x});\Omega_2\big)=\{0\}\times N(\bar{x};\Omega).$$

This implies that the main qualification condition of [17, Theorem 3.4] formulated as

$$N\big((\bar{y},\bar{x});\Omega_1\big)\cap\big[-N\big((\bar{y},\bar{x});\Omega_2\big)\big]=\{0\} \tag{3.5}$$

is satisfied. To proceed, let us split the proof into the following two cases corresponding to the requirements in (a) and (b) in addition to the common assumptions of the theorem.

Case 1 *The range space Y is finite-dimensional.* It follows from the representation of Fréchet normal to the set Ω_1 with f strictly differentiable at $\bar{x}$ (cf. the proof of [17, Theorem 1.38]) and the finite dimensionality of Y that the set Ω_1 is SNC at $(\bar{y},\bar{x})$. Applying now [17, Corollary 3.5] (a consequence of the aforementioned basic intersection rule) to the set intersection Eq. 3.4, we get

$$N\big((\bar{y},\bar{x});\operatorname{gph}G\big)\subset N\big((\bar{y},\bar{x});\Omega_1\big)+N\big((\bar{y},\bar{x});\Omega_2\big). \tag{3.6}$$

This implies, by the above normal cone formulas for Ω_1 and Ω_2, that

$$N\big((\bar{y},\bar{x});\operatorname{gph}G)\subset\big\{\big(y^*,-\nabla f(\bar{x})^*y^*+N(\bar{x};\Omega)\big)\big|\ y^*\in Y^*\big\}. \tag{3.7}$$

The latter allows us to conclude, due to the qualification condition Eq. 3.2, that

$$D^*_M G(\bar{y},\bar{x})(0)\subset D^*_N G(\bar{y},\bar{x})(0)=\{0\}. \tag{3.8}$$

The inclusion in Eq. 3.8 is obvious; so it remains to observe by the normal coderivative definition Eq. 2.4 that the required implication

$$(y^*,0)\in N\big((\bar{y},\bar{x});\operatorname{gph}G\big)\Longrightarrow y^*=0$$

in Eq. 3.8 directly follows from Eq. 3.7 due to the imposed qualification condition Eq. 3.2.

To conclude that the mapping $G\colon Y\rightrightarrows X$ in Eq. 3.3 is Lipschitz-like around $(\bar{y},\bar{x})$ by using the coderivative criterion from [17, Theorem 4.10(c)], it is sufficient to observe that G is obviously *PSNC* at $(\bar{y},\bar{x})$, since Y is finite-dimensional in the case under consideration.

Case 2 *The image space $\nabla f(\bar{x})X$ is closed in Y and the set Ω is SNC at $\bar{x}$.* We begin with a simple observation that the qualification condition Eq. 3.2 implies the *injectivity* of the adjoint derivative operator, i.e., the validity of the inclusion

$$\nabla f(\bar{x})^*y^*=0\Longrightarrow y^*=0. \tag{3.9}$$

Indeed, assuming that Eq. 3.9 does not hold, we find $y^*\neq 0$ such that

$$y^*\in\big(\nabla f(\bar{x})^*\big)^{-1}(0)\subset\big(\nabla f(\bar{x})^*\big)^{-1}\big(N(\bar{x};\Omega)\big)=\{0\},$$

which contradicts the qualification condition Eq. 3.2. The injectivity of the adjoint operator $\nabla f(\bar{x})^*$ implies, by the closedness requirement on $\nabla f(\bar{x})X$, that in fact $\nabla f(\bar{x})X = Y$.

To ensure that $D^*_M G(\bar{y}, \bar{x})(0) = \{0\}$, we proceed similarly to Case 1 observing that the calculus rule in Eq. 3.6 holds by [17, Corollary 3.5] under the qualification condition Eq. 3.5 due to the SNC property of Ω_2 at $(\bar{y}, \bar{x})$, which is obviously implied by the assumed SNC property of Ω at $\bar{x}$. To conclude now by [17, Theorem 4.10(c)] that the mapping $G\colon Y \rightrightarrows X$ in Eq. 3.3 is Lipschitz-like around $(\bar{y}, \bar{x})$, it remains to check that G is PSNC at this point. Since the qualification condition Eq. 3.5 and the SNC property of Ω_2 at $(\bar{y}, \bar{x})$ are satisfied and the product space $Y \times X$ is Asplund, we have from [17, Corollary 3.80] that the mapping G is PSNC at $(\bar{y}, \bar{x})$, which is the same as the PSNC property of the graph gph G with respect to Y, *provided* that the set Ω_1 in the intersection representation of this graph in Eq. 3.4 is PSNC with respect to Y at $(\bar{y}, \bar{x})$, i.e.,

$$\left[y_k^* \xrightarrow{w^*} 0,\ \|x_k^*\| \to 0,\ (y_k^*, x_k^*) \in \widehat{N}((y_k, x_k); \Omega_1)\right] \Longrightarrow \|y_k^*\| \to 0 \ \text{ as } \ k \to 0. \quad (3.10)$$

Taking into account that $\Omega_1 = \text{gph}\, f^{-1}$, observe that the implication in Eq. 3.10 means that the mapping $f^{-1}\colon Y \rightrightarrows X$ is PSNC at $(\bar{y}, \bar{x})$ in the sense defined in Section 1. Recall that, by [17, Theorem 1.43], the PSNC property of f^{-1} at $(\bar{y}, \bar{x})$ is implied by the Lipschitz-like property of f^{-1} around this point, which is equivalent to the metric regularity of f around $\bar{x}$. As shown above, $\nabla f(\bar{x})X = Y$ under the assumptions made in (b), which thus ensure the required metric regularity by the classical Lyusternik-Graves theorem. We have therefore the Lipschitz-like property of the mapping G in Eq. 3.3 around $(\bar{y}, \bar{x})$ in Case 2 and complete the proof of the theorem in both cases under consideration. □

4 Fréchet-Like Normals to Set Images

In this section we mainly concentrate on computing Fréchet normals and their ε-enlargements to images of sets in Banach spaces under Fréchet and strictly differentiable mappings satisfying the metric regularity requirements introduced and discussed in Section 3.

To proceed, we need the following tangential construction generated in duality by the Fréchet normal cone Eq. 2.1 as $\varepsilon = 0$. Given a set $\Omega \subset X$ and a point $\bar{x} \in \Omega$, define the *Fréchet tangent cone* to Ω at $\bar{x}$ by

$$\widehat{T}(\bar{x}; \Omega) := \left\{v \in X \,\middle|\, \langle x^*, v\rangle \le 0 \ \text{ for all } \ x^* \in \widehat{N}(\bar{x}; \Omega)\right\}. \quad (4.1)$$

A natural question that immediately arises is about relationships between the Fréchet tangent cone Eq. 4.1 and the widely spread in variational analysis tangential construction

$$T(\bar{x}; \Omega) := \left\{v \in X \,\middle|\, \exists v_k \to v,\ t_k \downarrow 0 \ \text{ with } \ \bar{x} + t_k v_k \in \Omega\right\} \quad (4.2)$$

known as the *Bouligand-Severi contingent cone* to Ω at $\bar{x}$; see [17, Subsection 1.1.2] and the commentaries therein. The following proposition establishes such relationships in the general Banach space setting.

Proposition 4.1 (relationships between the Fréchet tangent cone and Bouligand-Severi contingent cone to arbitrary sets) *Let Ω be a nonempty subset of a Banach space X, and let $\bar{x} \in \Omega$. Then we always have the inclusion*

$$\text{clco}\, T(\bar{x}; \Omega) \subset \widehat{T}(\bar{x}; \Omega), \tag{4.3}$$

which becomes an equality provided that

$$\widehat{N}\big(0; \text{clco}\, T(\bar{x}; \Omega)\big) \subset \widehat{N}(\bar{x}; \Omega). \tag{4.4}$$

Furthermore, condition (4.4) is satisfied in each of the following cases:

(a) *For any sequence $x_k \xrightarrow{\Omega} \bar{x}$ with $x_k \neq \bar{x}$ for all $k \in \mathbb{N}$, the normalized sequence*

$$\left\{ \frac{x_k - \bar{x}}{\|x_k - \bar{x}\|} \right\}, \quad k \in \mathbb{N},$$

contains a convergent subsequence as $k \to \infty$. This is true, in particular, when there is a neighborhood U of $\bar{x}$ such that the spanned space $\text{span}(\Omega \cap (U - \bar{x}))$ *is finite-dimensional, which is automatic if the original space X is finite-dimensional.*

(b) *There is a neighborhood U of $\bar{x}$ such that*

$$\Omega \cap U \subset \bar{x} + \text{clco}\, T(\bar{x}; \Omega),$$

which surely holds if $\Omega \cap U \subset \bar{x} + T(\bar{x}; \Omega)$. The latter is satisfied, in particular, when the set Ω is either convex or conic around $\bar{x}$.

Proof To justify Eq. 4.3, it is sufficient to show that $T(\bar{x}; \Omega) \subset \widehat{T}(\bar{x}; \Omega)$, since the Fréchet tangent cone $\widehat{T}(\bar{x}; \Omega)$ is obviously closed and convex. Pick any $v \in T(\bar{x}; \Omega)$ and by definition Eq. 4.2 find sequences $v_k \to v$ and $t_k \downarrow 0$ as $k \to \infty$ such that $\bar{x} + t_k v_k \in \Omega$ for all $k \in \mathbb{N}$. Given $x^* \in \widehat{N}(\bar{x}; \Omega)$ and $\varepsilon > 0$ and using definition Eq. 2.1 of Fréchet normals, we have

$$\langle x^*, t_k v_k \rangle \leq \varepsilon \|t_k v_k\| \ \text{ for all large } \ k \in \mathbb{N},$$

which implies that $\langle x^*, v \rangle \leq 0$ by passing to the limit as $k \to \infty$ and taking into account that $\varepsilon > 0$ was chosen arbitrarily. The latter inequality means that $v \in \widehat{T}(\bar{x}; \Omega)$, and thus we get the required inclusion Eq. 4.3.

Let us further prove that the *equality* holds in Eq. 4.3 under the assumption Eq. 4.4. To proceed, suppose the opposite and find an element $v \in \widehat{T}(\bar{x}; \Omega)$ with $v \notin \text{clco}\, T(\bar{x}; \Omega)$. By the classical *convex separation* theorem, there is $x^* \in X^*$ and $\gamma \in \mathbb{R}$ such that

$$\langle x^*, v \rangle > \gamma > \langle x^*, x \rangle \ \text{ for all } \ x \in \text{clco}\, T(\bar{x}; \Omega). \tag{4.5}$$

This implies that $\gamma > 0$, since $0 \in \text{clco}\, T(\bar{x}; \Omega)$. Taking now into account that $\text{clco}\, T(\bar{x}; \Omega)$ is a cone, we get from the second inequality in Eq. 4.5 that

$$\langle x^*, x \rangle \leq 0 \ \text{ for all } \ x \in \text{clco}\, T(\bar{x}; \Omega),$$

which gives $x^* \in \widehat{N}(0; \text{clco}\, T(\bar{x}; \Omega)) \subset \widehat{N}(\bar{x}; \Omega)$ by the assumption made in Eq. 4.4. On the other hand, it follows from the first inequality in Eq. 4.5 that $x^* \notin \widehat{N}(\bar{x}; \Omega)$, since otherwise we have $\langle x^*, v\rangle \le 0$. This contradiction justifies the equality in Eq. 4.3 under the validity of Eq. 4.4. The fulfillment of Eq. 4.4 under the conditions imposed in either (a) or (b) easily follows from the above definitions of the contingent cone and the Fréchet normal cone to the set in question. This completes the proof of the proposition. □

Observe that inclusion (4.3) is *strict* in general. To illustrate this, consider the set

$$\Omega := \big\{(u, v) \in I\!R^2 \big|\; v \ge -|u|\big\} \quad \text{and} \quad \bar{x} = (0, 0) \in \Omega.$$

Then it is easy to check that $T(\bar{x}; \Omega) = \Omega$ while $\widehat{T}(\bar{x}; \Omega) = I\!R^2$.

The next theorem contains two independent relationships between the Fréchet normal cone to a set Ω and to the set image $f(\Omega)$ under a Fréchet differentiable mapping f while imposing the *metric regularity* property of $f\colon \Omega \to f(\Omega)$ *at* the point $\bar{x} \in \Omega$ in question.

Theorem 4.2 (Fréchet normals to direct images of sets under differentiable mappings) *Let $f\colon X \to Y$ be a mapping between Banach spaces such that f is Fréchet differentiable at $\bar{x} \in \Omega$ and the restricted mapping $f\colon \Omega \to f(\Omega)$ is metrically regular at this point. Denote $\bar{y} := f(\bar{x})$. Then we have the equality*

$$\widehat{N}\big(\bar{y};\, f(\Omega)\big) = \big(\nabla f(\bar{x})^*\big)^{-1}\big(\widehat{N}(\bar{x}; \Omega)\big). \tag{4.6}$$

Furthermore, we have another equality

$$\nabla f(\bar{x})^* \widehat{N}\big(\bar{y};\, f(\Omega)\big) = \widehat{N}(\bar{x}; \Omega). \tag{4.7}$$

provided that the space $\nabla f(\bar{x})X$ is closed in Y and that

$$\ker \nabla f(\bar{x}) \subset \widehat{T}(\bar{x}; \Omega), \tag{4.8}$$

where the latter condition is necessary for the fulfillment of Eq. 4.7.

Proof Observe that the Fréchet differentiability of f at $\bar{x}$ implies the existence of a number $\ell > 0$ and a neighborhood U of $\bar{x}$ such that

$$\|f(x) - f(\bar{x})\| \le \ell\|x - \bar{x}\| \quad \text{for all} \quad x \in U.$$

Fix any $y^* \in \widehat{N}(\bar{y};\, f(\Omega))$ and get from the definition of Fréchet normals that

$$\limsup_{y \xrightarrow{f(\Omega)} \bar{y}} \frac{\langle y^*, y - \bar{y}\rangle}{\|y - \bar{y}\|} \le 0.$$

Applying the following transformations and estimates

$$\limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle \nabla f(\bar{x})^* y^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} = \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle y^*, \nabla f(\bar{x})(x - \bar{x}) \rangle}{\|x - \bar{x}\|}$$

$$\leq \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle y^*, \nabla f(\bar{x})(x - \bar{x}) - f(x) + f(\bar{x}) + f(x) - f(\bar{x}) \rangle}{\|x - \bar{x}\|}$$

$$\leq \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle y^*, f(x) - f(\bar{x}) \rangle}{\|x - \bar{x}\|} \leq \max \left\{ 0, \limsup_{y \xrightarrow{f(\Omega)} \bar{y}} \frac{\langle y^*, y - \bar{y} \rangle)}{\ell^{-1}\|y - \bar{y}\|} \right\} \leq 0,$$

we arrive at $\nabla f(\bar{x})^* y^* \in \widehat{N}(\bar{x}; \Omega)$, which justifies the inclusion "$\subset$" in Eq. 4.6.

To prove the opposite inclusion in Eq. 4.6, employ the *metric regularity* of $f : \Omega \to f(\Omega)$ *at* $\bar{x}$ and find a number $\mu > 0$ such that for any $y \in f(\Omega)$ close to $\bar{y} = f(\bar{x})$ we have

$$\operatorname{dist}\big(\bar{x}; f^{-1}(y) \cap \Omega\big) \leq \mu \|y - \bar{y}\|. \tag{4.9}$$

Fix any y^* satisfying $\nabla f(\bar{x})^* y^* \in \widehat{N}(\bar{x}; \Omega)$. Then for any $\varepsilon > 0$ there is $\eta > 0$ such that

$$\langle y^*, \nabla f(\bar{x})(x - \bar{x}) \rangle \leq \frac{\varepsilon}{2} \|x - \bar{x}\| \text{ whenever } \|x - \bar{x}\| < \eta \text{ and } x \in \Omega. \tag{4.10}$$

It follows from Eq. 4.10 and the Fréchet differentiability of f at $\bar{x}$ that for some $\nu < \eta$ we have

$$\langle y^*, f(x) - f(\bar{x}) \rangle \leq \varepsilon \|x - \bar{x}\| \text{ whenever } \|x - \bar{x}\| < \nu \text{ and } x \in \Omega.$$

Observe that estimate (4.9) ensures that for any $y \in f(\Omega)$ sufficiently close to $\bar{y}$ there is $x_y \in f^{-1}(y) \cap \Omega$ satisfying $\|x_y - \bar{x}\| \leq 2\mu \|y - \bar{y}\| < \nu$. For such x_y we have

$$\big\langle y^*, y - \bar{y} \big\rangle = \big\langle y^*, f(x_y) - f(\bar{x}) \big\rangle \leq \varepsilon \|x_y - \bar{x}\| \leq 2\mu\varepsilon \|y - \bar{y}\|,$$

which implies that $y^* \in \widehat{N}(\bar{y}; f(\Omega))$, since $\varepsilon > 0$ was chosen arbitrarily. This justifies the inclusion "$\supset$" in Eq. 4.6 and thus the equality therein.

Next we prove representation (4.7) under the additional assumptions made. By Eq. 4.6, it suffices to verify the inclusion

$$\widehat{N}(\bar{x}; \Omega) \subset \nabla f(\bar{x})^* \widehat{N}\big(\bar{y}; f(\Omega)\big). \tag{4.11}$$

To proceed, pick arbitrary $x^* \in \widehat{N}(\bar{x}; \Omega)$ and $v \in \widehat{T}(\bar{x}; \Omega)$ and get by definition (4.1) that $\langle x^*, v \rangle \leq 0$. This implies, since $-v \in \ker \nabla f(\bar{x})$ whenever $v \in \ker \nabla f(\bar{x})$, that

$$\langle x^*, v \rangle = 0 \text{ for all } v \in \ker \nabla f(\bar{x}) \tag{4.12}$$

due to the assumed inclusion Eq. 4.8. Define now a bounded linear functional y^* on the closed subspace $\nabla f(\bar{x}) X$ of Y by

$$\langle y^*, y \rangle = \langle x^*, x \rangle \text{ for some } x \in \nabla f(\bar{x})^{-1}(y).$$

It follows from Eq. 4.12 that y^* is well defined on $\nabla f(\bar{x})X$. Using the classical *Hahn-Banach theorem*, we can extend y^* to a bounded linear functional on the whole space Y, i.e., we can identify it with some $y^* \in Y^*$. The construction of y^* implies that $\nabla f(\bar{x})^* y^* = x^*$. Furthermore, by the above proof of the inclusion "$\supset$" in Eq. 4.6, we have $y^* \in \widehat{N}(\bar{y}; f(\Omega))$ and thus justify the equality in Eq. 4.7 under the kernel condition Eq. 4.8.

It remains to show that the kernel condition Eq. 4.8 is in fact *necessary* for the fulfillment of representation (4.7). To proceed, assume that Eq. 4.7 holds and take any $v \in \ker \nabla f(\bar{x})$. Then we obviously have the equality

$$\langle \nabla f(\bar{x})v, y^* \rangle = 0 \Longleftrightarrow \langle v, \nabla f(\bar{x})^* y^* \rangle = 0 \ \text{ for all } \ y^* \in \widehat{N}\big(\bar{y}; f(\Omega)\big).$$

By the assumed condition Eq. 4.7, the latter yields that $\langle v, x^* \rangle = 0$ for all $x^* \in \widehat{N}(\bar{x}; \Omega)$. Hence we get $v \in \widehat{T}(\bar{x}; \Omega)$ by definition (4.1) and complete the proof of the theorem. □

Next we study a special class of sets $\Omega \subset X$ in Theorem 4.2, which are representable as *inverse images* $f^{-1}(\Theta)$ of some subsets $\Theta \subset Y$ of the range space for the mappings $f\colon X \to Y$ from the theorem. We obtain efficient conditions that ensure the fulfillment of the kernel requirement (4.8) and of the equality in Eq. 4.7 for such sets. Let us first present a result that justifies the kernel condition Eq. 4.8 for inverse images and contains an additional information of certain independent interest.

Proposition 4.3 (kernel condition for inverse images) *Let $f\colon X \to Y$ be a mapping between Banach spaces that is Gâteaux differentiable at $\bar{x}$, and let $\bar{y} := f(\bar{x})$. Assume that the restricted mapping $f\colon X \to f(X)$ is metrically regular at $\bar{x}$. Then we have the inclusion*

$$\ker \nabla f(\bar{x}) \subset T\big(\bar{x}; f^{-1}(\bar{y})\big), \tag{4.13}$$

which holds as equality provided that f is Fréchet differentiable at $\bar{x}$.

Proof To justify Eq. 4.13, pick any $v \in \ker \nabla f(\bar{x})$ and observe, by the Gâteaux differentiability of f at $\bar{x}$, that

$$\frac{f(\bar{x} + t_k v) - \bar{y}}{t_k} \to 0 \ \text{ as } \ k \to \infty$$

whenever $t_k \downarrow 0$. Since $f\colon X \to f(X)$ is assumed to be metrically regular *at* $\bar{x}$, for any large $k \in \mathbb{N}$ we find $x_k \in f^{-1}(\bar{y})$ such that

$$w_k := \frac{\bar{x} + t_k v - x_k}{t_k} \to 0 \ \text{ as } \ k \to \infty.$$

Then we have $(x_k - \bar{x})/t_k = v - w_k \to v$ as $k \to \infty$, which gives $v \in T(\bar{x}; \Omega)$ by definition (4.2) of the contingent cone and thus justifies the required inclusion Eq. 4.13.

To prove the converse inclusion to Eq. 4.13, take any $v \in T(\bar{x}; f^{-1}(\bar{y}))$ and by definition (4.2) find sequences $x_k \to \bar{x}$ with $x_k \in f^{-1}(\bar{y})$ and $t_k \downarrow 0$ such that $(x_k - \bar{x})/t_k \to v$ as $k \to \infty$. By the the Fréchet differentiability of f at $\bar{x}$ we have

$$\frac{f(x_k) - f(\bar{x}) - \nabla f(\bar{x})(x_k - \bar{x})}{t_k} \to 0 \text{ as } k \to \infty,$$

which gives $v \in \ker \nabla f(\bar{x})$ and completes the proof of the proposition. □

The following result presents consequences of Theorem 4.2 for Fréchet normals to inverse images of sets. The second formula in this corollary is based on Proposition 4.3 and extends the corresponding one in [17, Theorem 1.14], where $\nabla f(\bar{x})$ is assumed to be surjective (and hence metrically regular at $\bar{x}$) that allows us to drop $f(X)$ in the latter formula.

Corollary 4.4 (Fréchet normals to inverse images of sets) *Let $f\colon X \to Y$ be a mapping between Banach spaces, and let Θ be a subset of Y such that f is Fréchet differentiable at $\bar{x}$ and that $\bar{y} := f(\bar{x}) \in \Theta$. Assume that the restricted mapping $f\colon f^{-1}(\Theta) \to \Theta \cap f(X)$ is metrically regular at $\bar{x}$. Then*

$$\big(\nabla f(\bar{x})^*\big)^{-1} \widehat{N}\big(\bar{x}; f^{-1}(\Theta)\big) = \widehat{N}\big(f(\bar{x}); \Theta \cap f(X)\big).$$

Furthermore, we have the equality

$$\widehat{N}\big(\bar{x}; f^{-1}(\Theta)\big) = \nabla f(\bar{x})^* \widehat{N}\big(f(\bar{x}); \Theta \cap f(X)\big)$$

provided that the mapping $f\colon X \to f(X)$ is metrically regular at $\bar{x}$.

Proof The first equality in the corollary follows directly from equality (4.6) of Theorem 4.2 with $\Omega = f^{-1}(\Theta)$. The second one follows from Eq. 4.7, Proposition 4.3, and the fact that the subspace $\nabla f(\bar{x})X$ is closed in Y if the restricted mapping $f\colon X \to f(X)$ is metrically regular at $\bar{x}$. The latter is proved in [20, Theorem 2.2] (see also [17, Lemma 1.56]), where the RMR property of f *at* $\bar{x}$ is actually used, although the formulation involves this property around $\bar{x}$. Note that the metric regularity of $f\colon X \to f(X)$ at $\bar{x}$ surely implies that of $f\colon f^{-1}(\Theta) \to \Theta \cap f(X)$ at this point. Thus we meet all the assumptions of Theorem 4.2 for $\Omega = f^{-1}(\Theta)$ and complete the proof of the corollary. □

Next we study some relations for ε-*normals* Eq. 2.1 to sets and set images (direct and inverse) under *strictly* differentiable mappings between Banach spaces. We derive certain *perturbed/fuzzy* counterparts of the results for Fréchet normals obtained above imposing similar metric regularity assumptions on restricted mappings that are required now *around* the reference points (not "at" as above). The results established here for ε-normals are of independent interest while their main role in this paper concerns applications to new formulas for *limiting normals* to set images developed in Section 5.

Our first result on ε-normals to set images gives a *uniform* fuzzy analog of formula (4.6) via the *rate of strict differentiability* of f at $\bar{x}$ introduced in [20] by

$$r_f(\bar{x}; \eta) := \sup_{\substack{x,u \in \bar{x} + \eta I\!B \\ u \neq x}} \frac{\| f(u) - f(x) - \nabla f(\bar{x})(u - x)\|}{\|u - x\|}, \tag{4.14}$$

where $\eta > 0$; see also [17, Subsection 1.1.3] for more details. It easily follows from Eq. 4.14 that $r_f(\bar{x};\eta) \downarrow 0$ as $\eta \downarrow 0$ if f is strictly differentiable at $\bar{x}$.

Theorem 4.5 (estimates of ε-normals to set images via the rate of strict differentiability) *Let $f\colon X \to Y$ be a mapping between Banach spaces, and let $\bar{x} \in \Omega \subset X$. Assume that f is strictly differentiable at $\bar{x} \in \Omega$ with the rate of strictly differentiability $r_f(\bar{x};\eta)$ defined in Eq. 4.14. Then the following hold:*

(i) *There are numbers $c > 0$ and $\bar{\eta} > 0$ such that for any $y^* \in \widehat{N}_\varepsilon(f(x); f(\Omega))$, $\varepsilon \geq 0$, $x \in (\bar{x} + \eta I\!B) \cap \Omega$, and $0 < \eta \leq \bar{\eta}$ we have the inclusion*

$$y^* \in \left(\nabla f(\bar{x})^*\right)^{-1}\left(\widehat{N}_\gamma(x;\Omega)\right) \ \text{ with } \ \gamma := c\varepsilon + \|y^*\| r_f(\bar{x};\eta). \tag{4.15}$$

(ii) *Assume in addition that the restricted mapping $f\colon \Omega \to f(\Omega)$ is metrically regular around $\bar{x}$. Then there are numbers $c > 0$ and $\bar{\eta} > 0$ such that for any $y^* \in (\nabla f(\bar{x})^*)^{-1}\widehat{N}_\varepsilon(x;\Omega)$, $\varepsilon \geq 0$, $x \in (\bar{x} + \eta I\!B) \cap \Omega$, and $0 < \eta \leq \bar{\eta}$ we have the inclusion*

$$y^* \in \widehat{N}_\nu\left(f(x); f(\Omega)\right) \ \text{ with } \ \nu := c\varepsilon + c\|y^*\| r_f(\bar{x};\eta). \tag{4.16}$$

Proof Since f is strictly differentiable at $\bar{x}$, it is locally Lipschitzian around this point, i.e., there are $\ell > 0$ and $\bar{\eta} > 0$ such that f is Lipschitz continuous on the set $\bar{x} + \bar{\eta} I\!B$ with constant ℓ. Then we have

$$\begin{aligned}
\limsup_{u \xrightarrow{\Omega} x} \frac{\langle \nabla f(\bar{x})^* y^*, u - x\rangle}{\|u - x\|} &= \limsup_{u \xrightarrow{\Omega} x} \frac{\langle y^*, \nabla f(\bar{x})(u - x)\rangle}{\|u - x\|} \\
&= \limsup_{u \xrightarrow{\Omega} x} \frac{\langle y^*, f(u) - f(x)\rangle}{\|u - x\|} + ||y^*|| r_f(\bar{x};\eta) \\
&\leq \limsup_{v \xrightarrow{f(\Omega)} f(x)} \max\left\{0, \frac{\langle y^*, v - f(x)\rangle}{\ell^{-1}\|v - f(x)\|}\right\} + \|y^*\| r_f(\bar{x};\eta) \\
&\leq \ell\varepsilon + \|y^*\| r_f(\bar{x};\eta) \ \text{ for all } \ \eta \in (0, \bar{\eta}].
\end{aligned}$$

This gives the inclusion $\nabla f(\bar{x})^* y^* \in \widehat{N}_\gamma(x;\Omega)$ with $c := \ell$ and γ defined in Eq. 4.15. Thus $y^* \in (\nabla f(\bar{x})^*)^{-1}(\widehat{N}_\gamma(x;\Omega))$, which justifies assertion (i).

To prove (ii), we employ the *metric regularity* of $f\colon \Omega \to f(\Omega)$ *around* $\bar{x}$ and find numbers $\mu > 0$ and $\bar{\eta} > 0$ such that

$$\operatorname{dist}\left(x; f^{-1}(y) \cap \Omega\right) \leq \mu\|y - f(x)\| \ \text{ for any } \ x \in (\bar{x} + \bar{\eta} I\!B) \cap \Omega \ \text{ and } \ y \in f(\Omega) \tag{4.17}$$

with y close to $f(x)$. Pick any $y^* \in (\nabla f(\bar{x})^*)^{-1}\widehat{N}(x;\Omega)$ for such x with some $\varepsilon \geq 0$ and get by definition (2.1) the two equivalent inequalities

$$\left[\limsup_{u \xrightarrow{\Omega} x} \frac{\langle \nabla f(\bar{x})^* y^*, u - x\rangle}{\|u - x\|} \leq \varepsilon\right] \Longleftrightarrow \left[\limsup_{u \xrightarrow{\Omega} x} \frac{\langle y^*, \nabla f(\bar{x})(u - x)\rangle}{\|u - x\|} \leq \varepsilon\right].$$

The latter implies by construction (4.14) that

$$\limsup_{u \xrightarrow{\Omega} x} \frac{\langle y^*, f(u) - f(x) \rangle}{\|u - x\|} \le \varepsilon + \|y^*\| r_f(\bar{x}; \eta) \text{ for any } 0 < \eta \le \bar{\eta}.$$

Fix $x \in (\bar{x} + \eta I\!B) \cap \Omega$ and, by metric regularity (4.17), for each $y \in f(\Omega)$ close to $f(x)$ find $x_y \in f^{-1}(y) \cap \Omega$ satisfying $\|x - x_y\| \le 2\mu \|y - f(x)\|$. Thus we get

$$\begin{aligned}
\limsup_{y \xrightarrow{f(\Omega)} f(x)} \frac{\langle y^*, y - f(x) \rangle}{\|y - f(x)\|} &\le \limsup_{v \xrightarrow{f(\Omega)} f(x)} \frac{\langle y^*, f(x_y) - f(x) \rangle}{\|f(x_y) - f(x)\|} \\
&\le \limsup_{y \xrightarrow{f(\Omega)} f(x)} \max\left\{0, \frac{\langle y^*, f(x_y) - f(x) \rangle}{(2\mu)^{-1} \|x_y - x\|}\right\} \\
&\le 2\mu\varepsilon + 2\mu \|y^*\| r_f(\bar{x}; \eta) \text{ whenever } 0 < \eta \le \bar{\eta}.
\end{aligned}$$

This justifies the inclusion $y^* \in \widehat{N}_\nu(f(x); f(\Omega))$ with $c := 2\mu$ and ν defined in Eq. 4.16 and thus completes the proof of the theorem. □

The next result a fuzzy ε-normal counterpart of the inclusion "$\supset$" in Eq. 4.7; the opposite inclusion of this type is an immediate consequence of Theorem 4.5(ii). To proceed in this direction, we introduce a new concept used in what follows.

Definition 4.6 (tangential distance for sets) Let Ω be a subset of a Banach space X with $\bar{x} \in \Omega$, and let L be a linear subspace of X. The TANGENTIAL DISTANCE between Ω and L at $\bar{x}$ with accuracy $\eta \ge 0$ is defined by

$$\text{tand}_{\Omega, L}(\bar{x}; \eta) := \sup_{u \in L \setminus \{0\},\, x \in (\bar{x} + \eta I\!B) \cap \Omega} \ \liminf_{v \xrightarrow{\Omega} x} \left\| \frac{u}{\|u\|} - \frac{v - x}{\|v - x\|} \right\| \tag{4.18}$$

with the convention that $\text{tand}_{\Omega, L}(\bar{x}; \eta) := 0$ if $L = \{0\}$. For simplicity we denote

$$\text{tand}_{\Omega, L}(\bar{x}) := \text{tand}_{\Omega, L}(\bar{x}; 0).$$

It can be derived from Definition 4.6 and construction (4.2) of the contingent cone that

$$\text{tand}_{\Omega, L}(\bar{x}; \eta) \le 2 \sup_{\substack{u \in L \setminus \{0\} \\ x \in (\bar{x} + \eta I\!B) \cap \Omega}} \text{dist}\left(\frac{u}{\|u\|}; T(x; \Omega) \right) \tag{4.19}$$

provided that X is reflexive. We present more results on the tangential distance for image sets generated by strictly differentiable mappings in Section 5 and continue the study of these and related issues in our future research.

Theorem 4.7 (estimates of ε-normals via tangential distance) *Let $\Omega \subset X$ be a subset of a Banach space with $\bar{x} \in \Omega$, and let $\varepsilon \ge 0$. The following assertions hold:*

(i) *Given a linear bounded operator $A\colon X \to Y$ between Banach spaces with the closed range $AX \subset Y$ and given $x^* \in \widehat{N}_\varepsilon(\bar{x}; \Omega)$, there is $y^* \in Y^*$ such that*

$$A^* y^* \in x^* + \gamma I\!B^* \ \textit{with} \ \gamma := \varepsilon + \text{tand}_{\Omega, \ker A}(\bar{x}) \|x^*\|. \tag{4.20}$$

(ii) *Assume that a mapping* $f\colon X \to Y$ *between Banach spaces is strictly differentiable at* $\bar{x}$*, that the subspace* $\nabla f(\bar{x})X$ *is closed in* Y*, and that the restrictive mapping* $f\colon \Omega \to f(\Omega)$ *is metrically regular around* $\bar{x}$*. Then there is a number* $\bar{\eta} > 0$ *such that for any* $x^* \in \widehat{N}_\varepsilon(x;\Omega)$ *and* $x \in (\bar{x} + \eta I\!B) \cap \Omega$ *as* $0 < \eta \le \bar{\eta}$ *there exists a perturbation*

$$\widehat{x}^* \in x^* + \gamma I\!B^* \ \ \textit{with} \ \ \gamma := \varepsilon + \operatorname{tand}_{\Omega,\ker \nabla f(\bar{x})}(\bar{x})\|x^*\| \tag{4.21}$$

satisfying the conditions $(\nabla f(\bar{x})^*)^{-1}(\widehat{x}^*) \ne \emptyset$*. In addition, there is* $c > 0$ *such that for any* $y^* \in (\nabla f(\bar{x})^*)^{-1}(\widehat{x}^*)$ *we have the inclusion*

$$y^* \in \widehat{N}_\nu\big(f(x);\, f(\Omega)\big) \tag{4.22}$$

with $\nu := c\big(\varepsilon + \operatorname{tand}_{\Omega,\ker\nabla f(\bar{x})}(\bar{x};\eta)\|x^*\| + r_f(\bar{x};\eta)\|y^*\|\big)$.

Proof To justify assertion (i), take any $x^* \in \widehat{N}_\varepsilon(\bar{x};\Omega)$ and, by the definition of ε-normals, for every $\sigma > 0$ there is a neighborhood U of $\bar{x}$ such that

$$\langle x^*, x - \bar{x}\rangle \le (\varepsilon + \sigma)\|x - \bar{x}\| \ \ \text{whenever} \ \ x \in U. \tag{4.23}$$

Further, pick any $u \in \ker A \setminus \{0\}$ and, by Definition 4.18 with $L = \ker A$, find $v \in U$ satisfying the estimate

$$\left\| \frac{u}{\|u\|} - \frac{v - \bar{x}}{\|v - \bar{x}\|} \right\| < \operatorname{tand}_{\Omega,\ker A}(\bar{x}) + \sigma$$

with the same $\sigma > 0$ as in Eq. 4.23. Unifying the latter with Eq. 4.23, we get

$$\begin{aligned}\langle x^*, u\rangle &= \left(\left\langle x^*, \frac{u}{\|u\|} - \frac{v - \bar{x}}{\|v - \bar{x}\|}\right\rangle + \left\langle x^*, \frac{v - \bar{x}}{\|v - \bar{x}\|}\right\rangle\right)\|u\| \\ &\le \Big(\|x^*\|\big(\operatorname{tand}_{\Omega,\ker A}(\bar{x}) + \sigma\big) + (\varepsilon + \sigma)\Big)\|u\|.\end{aligned}$$

Taking into account that $\sigma > 0$ was chosen arbitrarily, this implies that

$$\langle x^*, u\rangle \le \gamma\|u\| \ \ \text{for all} \ \ u \in \ker A$$

with γ defined in Eq. 4.20. The Hahn-Banach theorem allows us to extend $x^*|_{\ker A}$ to a linear functional $\widetilde{x}^* \in X^*$ with $\|\widetilde{x}^*\| \le \gamma$. Denote now $\widehat{x}^* := x^* - \widetilde{x}^*$ and define $y^* \in (AX)^*$ by

$$\langle y^*, y\rangle = \langle \widehat{x}^*, x\rangle \ \ \text{for some} \ \ x \in A^{-1}(y).$$

Since $\widehat{x}^*|_{\ker A} = 0$, the linear functional y^* is well defined on AX and, furthermore, it is bounded on this subspace due to its assumed closedness in Y. Employing again the Hahn-Banach theorem, we extend y^* to a linear bounded functional $y^* \in Y^*$. Finally, it is easy to check that $A^*y^* = \widehat{x}^* \in x^* + \gamma I\!B^*$, which gives Eq. 4.20 and thus justifies assertion (i).

To prove assertion (ii) of the theorem, we first apply the result of assertion (i) with $A = \nabla f(\bar{x})$, which allows us to find $\widehat{x}^* \in X^*$ satisfying Eq. 4.21 and $(\nabla f(\bar{x})^*)^{-1}(\widehat{x}^*) \ne \emptyset$. Taking then any $y^* \in (\nabla f(\bar{x})^*)^{-1}(\widehat{x}^*)$, we easily have $\nabla f(\bar{x})^* y^* \in \widehat{N}_{\varepsilon+\gamma}(x;\Omega)$. Finally, assertion (ii) of Theorem 4.5 implies that $y^* \in \widehat{N}_\nu(f(x);\, f(\Omega))$ with ν defined in Eq. 4.22. This completes the proof of (ii) and of the whole theorem. □

5 The Limiting Normal Cone to Set Images

The primary goal of this section is to establish relationships between the *limiting normals* Eq. 2.2 to sets and to set images (direct and inverse) under strictly differentiable mappings acting in Banach spaces. We also obtain some counterparts (as upper estimates) of calculus results for limiting normals to direct images of sets under set-valued mappings.

A natural way to derive the corresponding formulas for limiting normals to set images under strictly differentiable mappings is to pass to the limit in the "fuzzy" results of Theorems 4.5 and 4.7 for ε-normals. To proceed carefully in this direction, we need to designate appropriate properties of the sets, mappings, and spaces under consideration. The following property of Banach spaces, introduced in [20] in a different framework and largely discussed in [17, Subsection 1.3.5], plays a significant role in justifying the limiting procedure.

Definition 5.1 (weak*-extensibility) Let L be a closed linear subspace of a Banach space X. We say that L is w^*–EXTENSIBLE in X if every sequence $\{v_k^*\} \subset L^*$, with $v_k^* \xrightarrow{w^*} 0$ as $k \to \infty$ contains a subsequence $\{v_{k_j}^*\}$ such that each $v_{k_j}^*$ can be extended to a linear bounded functional $x_j^* \in X^*$ with $x_j^* \xrightarrow{w^*} 0$ as $j \to \infty$.

As shown in [17, 20], the w^*-extensibility holds for every closed linear subspace of Banach spaces from fairly broad classes including all Asplund spaces, weakly compactly generated spaces (WCG), spaces admitting smooth renorms of any kinds, etc., but not in general.

Theorem 5.2 (limiting normals to direct images of sets under strictly differentiable mappings) *Let $f\colon X \to Y$ be a mapping between Banach spaces, and let $\Omega \subset X$ with $\bar{x} \in \Omega$. Assume that f is strictly differentiable at $\bar{x}$ and that the restricted mapping $f\colon \Omega \to f(\Omega)$ is metrically regular at this point. Then*

$$N\big(\bar{y};\, f(\Omega)\big) \subset (\nabla f(\bar{x})^*)^{-1}\big(N(\bar{x};\Omega)\big). \tag{5.1}$$

Furthermore, we have the equalities

$$N\big(\bar{y};\, f(\Omega)\big) = \big(\nabla f(\bar{x})^*\big)^{-1}\big(N(\bar{x};\Omega)\big), \tag{5.2}$$

$$\nabla f(\bar{x})^* N\big(\bar{y};\, f(\Omega)\big) = N(\bar{x};\Omega) \tag{5.3}$$

provided in addition that $f\colon \Omega \to f(\Omega)$ is metrically regular around $\bar{x}$ and the following assumptions hold:

(a) *the space $\nabla f(\bar{x})X$ is closed and w^*-extensible in Y;*
(b) $\operatorname{tand}_{\Omega,\ker \nabla f(\bar{x})}(\bar{x};\eta) \to 0$ *as* $\eta \downarrow 0$.

Proof To justify inclusion (5.1), pick $y^* \in N(\bar{y};\, f(\Omega))$ and, by definition of limiting normals, find sequences $\varepsilon_k \downarrow 0$, $y_k \xrightarrow{f(\Omega)} \bar{y}$, and $y_k^* \in \widehat{N}_{\varepsilon_k}(y_k;\, f(\Omega))$ such that $y_k^* \xrightarrow{w^*} y^*$ as $k \to \infty$. The metric regularity assumption on $f\colon \Omega \to f(\Omega)$ *at* $\bar{x}$ yields that

$$\operatorname{dist}\big(\bar{x},\, f^{-1}(y) \cap \Omega\big) \le \mu\|y - \bar{y}\|$$

for some $\mu > 0$ and all $y \in f(\Omega)$ sufficiently close to the reference point $\bar{y}$. This allows us to find $x_k \in f^{-1}(y_k) \cap \Omega$, $k \in I\!N$, with $x_k \to \bar{x}$ as $k \to \infty$. Then assertion (i) of Theorem 4.5 ensures the existence of a sequence $\gamma_k \downarrow 0$ such that

$$\nabla f(\bar{x})^* y_k^* \in \widehat{N}_{\gamma_k}(x_k; \Omega) \;\text{ for large }\; k \in I\!N,$$

which implies that $\nabla f(\bar{x})^* y^* \in N(\bar{x}; \Omega)$ by passing to the limit as $k \to \infty$. This gives Eq. 5.1 and also justifies the inclusion "$\subset$" in Eq. 5.3.

Let us next prove the opposite inclusions in Eqs. 5.2 and 5.3 under the additional assumptions made. They surely follow from the fact that for any $x^* \in N(\bar{x}; \Omega)$ we have

$$\emptyset \neq \big(\nabla f(\bar{x})^*\big)^{-1}(x^*) \subset N\big(\bar{y};\, f(\Omega)\big). \tag{5.4}$$

To justify Eq. 5.4, pick a limiting normal $x^* \in N(\bar{x}; \Omega)$ and find by definition (2.2) sequences $\varepsilon_k \downarrow 0$, $x_k \overset{\Omega}{\to} \bar{x}$, and $x_k^* \in \widehat{N}_{\varepsilon_k}(x_k; \Omega)$ with $x_k^* \overset{w^*}{\to} x^*$ as $k \to \infty$. Applying now, for each $k \in I\!N$, Theorem 4.7(ii) requiring the metric regularity of $f\colon \Omega \to f(\Omega)$ *around* $\bar{x}$ and then using assumption (b) as well as the boundedness of the sequence $\{x_k^*\} \subset X^*$ by the uniform boundedness principle, we find from Eq. 4.15 sequences $\gamma_k \downarrow 0$ and $\{\widehat{x}_k^*\} \subset X^*$ such that

$$\widehat{x}_k^* \in x_k^* + \gamma_k I\!B^* \;\text{ and }\; \big(\nabla f(\bar{x})^*\big)^{-1}\big(\widehat{x}_k^*\big) \neq \emptyset, \quad k \in I\!N. \tag{5.5}$$

It follows from the first relationship in Eq. 5.5 that $x_k^* \overset{w^*}{\to} x^*$ as $k \to \infty$. Furthermore, we get $x^* \in \nabla f(\bar{x})^* Y^*$ due the well-known fact that the assumed closedness of the image subspace $\nabla f(\bar{x}) X$ in Y implies the w^*-closedness of the adjoint image $\nabla f(\bar{x})^* Y^*$ in X^*.

Thus $(\nabla f(\bar{x})^*)^{-1}(x^*) \neq \emptyset$, and it remains to show for Eq. 5.4 that $y^* \in N(\bar{y};\, f(\Omega))$ whenever $y^* \in Y^*$ satisfies $\nabla f(\bar{x})^* y^* = x^*$. To proceed, we use both relationships in Eq. 5.5 and the w^*-extensibility assumption in (b) to derive, as in the proof of [17, Proposition 1.125] and [20, Proposition 3.7], that there is a sequence $\{y_k^*\} \subset Y^*$ with $\nabla f(\bar{x})^* y_k^* = \widehat{x}_k^*$, which contains a subsequence w^*-convergent to y^*; assume with no loss of generality that the whole sequence converges as $k \to \infty$. Applying again Theorem 4.7(ii), we get

$$y_k^* \in \widehat{N}_{\nu_k}\big(f\big(x_k\big),\, f(\Omega)\big) \;\text{ with some }\; \nu_k \downarrow 0 \;\text{ as }\; k \to \infty, \tag{5.6}$$

where the latter convergence follows from the expression for ν in Eq. 4.16 due to the strict differentiability of f at $\bar{x}$, the tangential distance assumption (b) of the theorem, and the boundedness of $\{x_k^*\} \subset X^*$. By passing to the limit in Eq. 5.6 as $k \to \infty$, we conclude that $y^* \in N(\bar{y};\, f(\Omega)$ and thus complete the proof of the theorem. □

As discussed above, both assumptions in (a) of Theorem 5.2 do not provide serious limitations. By Definition (4.18) and estimate (4.19), condition (b) of the theorem signifies that the contingent cone to Ω around $\bar{x}$ is close enough to the unit sphere of the kernel subspace $\ker \nabla f(\bar{x})$. Note also that all the assumptions in (a) and (b) of Theorem 5.2 are satisfied when the derivative operator $\nabla f(\bar{x})$ is *isomorphic*. The next result implies, in particular, that condition (b) always holds if Ω is the *inverse image* of some set generated by the mapping f under consideration enjoying the RMR property around $\bar{x}$. In this way the equality relations in Eqs. 5.2 and 5.3

in Theorem 5.2 extend the corresponding results of [20, Theorem 3.8] for limiting normals to inverse images.

Proposition 5.3 (relationship between the tangential distance to inverse images and the rate of strict differentiability) *Let $f\colon X \to Y$ be a mapping between Banach spaces, let $\Theta \subset Y$, and let $\Omega := f^{-1}(\Theta) \subset X$. Assume that f is strictly differentiable at $\bar{x} \in \Omega$ and the restrictive mapping $f\colon X \to f(X)$ is metrically regular around this point. Then there are numbers $\alpha > 0$ and $\bar{\eta} > 0$ such that for all $x \in (\bar{x} + \eta I\!B) \cap \Omega$ with $\eta \in (0, \bar{\eta}]$ we have the relationship*

$$\operatorname{tand}_{\Omega, \ker \nabla f(\bar{x})}(x) \le \frac{\alpha\, r_f(\bar{x}; \eta)}{1 - \alpha[r_f(\bar{x}; \eta)]^2}. \tag{5.7}$$

between the tangential distance and the rate of strict differentiability. In particular,

$$\operatorname{tand}_{\Omega, \ker \nabla f(\bar{x})}(\bar{x}; \eta) \to 0 \ \text{ as } \ \eta \downarrow 0. \tag{5.8}$$

Proof It follows directly from the strict differentiability of f at $\bar{x}$ that Eq. 5.7 implies Eq. 5.8. To justify Eq. 5.8, take a modulus $\mu > 0$ of the metric regularity of $f\colon X \to f(X)$ around $\bar{x}$ and choose $\bar{\eta} > 0$ such that

$$r_f(\bar{x}; \bar{\eta}) < \min\left\{1/\mu, 1/(2\sqrt{\mu})\right\}. \tag{5.9}$$

For any $v \in \ker \nabla f(\bar{x})$ with $\|v\| = 1$ and $x \in (\bar{x} + \eta I\!B) \cap \Omega$ with $0 < \eta < \bar{\eta}$ we have

$$\left\| \frac{f(x + tv) - f(x)}{t} \right\| \le r_f(\bar{x}; \eta) \tag{5.10}$$

whenever $t > 0$ is sufficiently small. By the metric regularity of $f\colon X \to f(X)$ around $\bar{x}$ with modulus μ, for any small $t > 0$ find $x_t \in X$ satisfying

$$f(x_t) = f(x) \ \text{ and } \ \|x + tv - x_t\| \le \mu \| f(x + tv) - f(x)\|.$$

This implies that $x_t \in \Omega$ and, by using Eq. 5.10, that

$$\left\| v - \frac{x_t - x}{t} \right\| = \left\| \frac{x + tv - x_t}{t} \right\| \le \mu r_f(\bar{x}; \eta).$$

Consequently we have the following relationships:

$$\begin{aligned}
-\mu r_f(\bar{x}; \eta) &\le 1 - \left\| \frac{x_t - x}{t} \right\| \le \mu r_f(\bar{x}; \eta),\\
1 - \mu r_f(\bar{x}; \eta) &\le \left\| \frac{x_t - x}{t} \right\| \le 1 + \mu r_f(\bar{x}; \eta),\\
\frac{1}{1 + \mu r_f(\bar{x}; \eta)} &\le \frac{t}{\|x_t - x\|} \le \frac{1}{1 - \mu r_f(\bar{x}; \eta)},\\
\left| 1 - \frac{t}{\|x_t - x\|} \right| &\le \frac{1}{1 - \mu r_f(\bar{x}; \eta)} - \frac{1}{1 + \mu r_f(\bar{x}; \eta)} = \frac{2\mu r_f(\bar{x}; \eta)}{1 - \mu^2[r_f(\bar{x}; \eta)]^2},
\end{aligned}$$

which yield the further estimates:

$$\left\| v - \frac{x_t - x}{\|x_t - x\|} \right\| = \left\| \left(v - \frac{x_t - x}{t} \right) \cdot \frac{t}{\|x_t - x\|} + \left(1 - \frac{t}{\|x_t - x\|} \right) v \right\|$$

$$\le \mu r_f(\bar{x}; \eta) \cdot \frac{1}{1 - \mu r_f(\bar{x}; \eta)} + \frac{2\mu r_f(\bar{x}; \eta)}{1 - \mu^2 [r_f(\bar{x}; \eta)]^2}$$

$$= \frac{(3 + \mu r_f(\bar{x}; \eta))\mu r_f(\bar{x}; \eta)}{1 - \mu^2 [r_f(\bar{x}; \eta)]^2} \le \frac{4\mu r_f(\bar{x}; \eta)}{1 - \mu^2 [r_f(\bar{x}; \eta)]^2}.$$

It allows us to finally arrive at the inequality

$$\left\| v - \frac{x_t - x}{\|x_t - x\|} \right\| \le \frac{\alpha\, r_f(\bar{x}; \eta)}{1 - \alpha [r_f(\bar{x}; \eta)]^2} \quad \text{with } \alpha := \max\left\{4\mu, \mu^2\right\},$$

which implies Eq. 5.7 by construction (4.18) and completes the proof of the proposition. □

We conclude this section with an *upper estimate* of the limiting normal cone to direct set images under *set-valued* mappings. The results obtained in what follows are generally independent of the corresponding one from Theorem 5.2 in the common setting for both theorems; see more discussions below in Remark 5.5.

To formulate the next theorem, we need to recall two useful properties of set-valued mappings between Banach spaces; see [17, Definition 1.63]. A mapping $F\colon X \rightrightarrows Y$ is said to be *inner semicontinuous* at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ if for every sequence $x_k \to \bar{x}$ with $F(x_k) \ne \emptyset$ there is a subsequence of $y_k \in F(x_k)$ converging to $\bar{y}$ as $k \to \infty$. We say that F is *inner semicompact* at $\bar{x}$ if for every sequence $x_k \to \bar{x}$ with $F(x_k) \ne \emptyset$ there is a sequence of $y_k \in F(x_k)$ containing a convergent subsequence.

It is easy to see that the inner semicompactness property of F always holds if $\dim Y < \infty$ and the mapping F is uniformly bounded around $\bar{x}$ (or, more generally, if F is locally compact around $\bar{x}$ in infinite dimensions), which is not the case for the inner semicontinuity. On the other hand, the inner semicontinuity for *inverse* mappings is implied by the appropriate *metric regularity* of the mapping in question *at* the corresponding point. In particular, it is proved in Theorem 5.2 that the metric regularity of $f\colon \Omega \to f(\Omega)$ at $\bar{x}$ ensures the inner semicontinuity of $f^{-1} \cap \Omega$ at $(\bar{y}, \bar{x})$ with $\bar{y} := f(\bar{x})$, which is a requirement of Theorem 5.4 for general set-valued mappings.

Having a set $\Omega \subset X$, we define its image under a set-valued mapping $F\colon X \rightrightarrows Y$ by

$$F(\Omega) := \bigcup_{x \in \Omega} F(x).$$

Theorem 5.4 (limiting normals to set images under set-valued mappings) *Let $F\colon X \rightrightarrows Y$ be a closed-graph mapping between Asplund spaces, let Ω be a closed subset of X, and let $\bar{y} \in F(\Omega)$. The following assertions hold:*

(i) *Given $\bar{x} \in \Omega$ with $(\bar{x}, \bar{y}) \in \operatorname{gph} F$, assume that the mapping $y \rightrightarrows F^{-1}(y) \cap \Omega$ is inner semicontinuous at $(\bar{y}, \bar{x})$, that either Ω is SNC at $\bar{x}$ or F is PSNC at $(\bar{x}, \bar{y})$, and that the qualification condition*

$$D^*_M F(\bar{x}, \bar{y})(0) \cap \big[-N(\bar{x}; \Omega) \big] = \{0\} \tag{5.11}$$

is satisfied; the latter is automatic together with the PSNC property of F at $(\bar{x}, \bar{y})$ *if F is Lipschitz-like around this point. Then we have the inclusion*

$$N\big(\bar{y};\, F(\Omega)\big) \subset D_N^* F^{-1}(\bar{y}, \bar{x})\big(N(\bar{x};\, \Omega)\big). \tag{5.12}$$

(ii) *Assume that the mapping* $y \rightrightarrows F^{-1}(y) \cap \Omega$ *is inner semicompact at* $\bar{y}$ *and that all the other assumptions in* (i) *hold for every* $\bar{x} \in F^{-1}(\bar{y}) \cap \Omega$. *Then we have the inclusion*

$$N\big(\bar{y};\, F(\Omega)\big) \subset \bigcup_{\bar{x} \in F^{-1}(\bar{y}) \cap \Omega} D_N^* F^{-1}(\bar{y}, \bar{x})\big(N(\bar{x};\, \Omega)\big). \tag{5.13}$$

Proof It is sufficient to justify assertion (i); the reader can check that the proof of assertion (ii) is similar. To prove Eq. 5.12, pick any $y^* \in N(\bar{y};\, F(\Omega))$ and find by definition sequences $y_k \xrightarrow{F(\Omega)} \bar{y}$ and $y_k^* \xrightarrow{w^*} y^*$ as $k \to \infty$ such that $y_k^* \in \widehat{N}(y_k;\, F(\Omega))$ for all $k \in I\!N$. By the inner semicontinuity of the mapping $y \rightrightarrows F^{-1}(y) \cap \Omega$ at $(\bar{y}, \bar{x})$ there is a sequence of $x_k \in F^{-1}(y_k) \cap \Omega$ such that $x_k \to \bar{x}$ as $k \to \infty$. Define the closed subsets

$$\Omega_1 := \operatorname{gph} F \;\text{ and }\; \Omega_2 := \Omega \times Y. \tag{5.14}$$

of the space $X \times Y$, which is Asplund as a product of Asplund spaces. It is easy to see from the structures of Ω_1 and Ω_2 in Eq. 5.14 that

$$\big(0, y_k^*\big) \in \widehat{N}\big((x_k, y_k);\, \Omega_1 \cap \Omega_2\big) \;\text{ for all }\; k \in I\!N,$$

which yields by passing to the limit as $k \to \infty$ that

$$(0, y^*) \in N\big((\bar{x}, \bar{y});\, \Omega_1 \cap \Omega_2\big). \tag{5.15}$$

Now we apply to the set intersection in Eq. 5.15 the fundamental intersection rule for limiting normals in Asplund spaces from [17, Theorem 3.4]. It is not hard to check that the structures of the sets Ω_1 and Ω_2 in Eq. 5.14 and the mixed coderivative construction (2.5) ensure the fulfillment of the limiting qualification condition required in the aforementioned theorem by the assumed qualification condition Eq. 5.11.

Furthermore, the PSNC condition imposed on F at $(\bar{x}, \bar{y})$ in the assumptions of the theorem means in fact that the set Ω_1 in Eq. 5.14 is PSNC at $(\bar{x}, \bar{y})$ with respect to X while the other set Ω_2 is automatically strongly *PSNC* at this point $(\bar{x}, \bar{y})$ with respect to Y, which is required in [17, Theorem 3.4]. On the other hand, if Ω is assumed to be *SNC* at $\bar{x}$, then Ω_2 in Eq. 5.14 is obviously *SNC* at $(\bar{x}, \bar{y})$, which meets the alternative requirements of [17, Theorem 3.4]. By the latter result we thus have

$$N\big((\bar{x}, \bar{y});\, \Omega_1 \cap \Omega_2\big) \subset N\big((\bar{x}, \bar{y});\, \Omega_1\big) + N\big((\bar{x}, \bar{y});\, \Omega_2\big)$$

that allows us to represent the pair $(0, y^*)$ in Eq. 5.15 as

$$(0, y^*) = (-x^*, y^*) + (x^*, 0).$$

with some $(-x^*, y^*) \in N((\bar{x}, \bar{y}); \operatorname{gph} F)$ and $x^* \in N(\bar{x};\, \Omega)$. Taking into account definition (2.4) of the normal coderivative for the case of the inverse mapping

$F^{-1}: Y \rightrightarrows X$, we finally arrive at Eq. 5.12 under the assumptions in (i). To complete the proof of the theorem, it remains to observe that $D^*_M F(\bar{x}, \bar{y})(0) = \{0\}$ if F is Lipschitz-like around $(\bar{x}, \bar{y})$ by [17, Theorem 4.10] and that we automatically have the PSNC property at $(\bar{x}, \bar{y})$ for such mappings F by [17, Theorem 1.43]. □

Remark 5.5 (discussions on the results for limiting normals to set images) It is easy to check the relationship

$$y^* \in D^*_N F^{-1}(\bar{y}, \bar{x})(x^*) \Longleftrightarrow -x^* \in D^*_N F(\bar{x}, \bar{y})(-y^*)$$

between the normal coderivative of an arbitrary mapping $F: X \rightrightarrows Y$ at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ and the one for its inverse F^{-1}. Using this relationship and expression (4.14) for the normal coderivative of single-valued and strictly differentiable mappings, we conclude that inclusion (5.1) of Theorem 5.2 reduces to Eq. 5.13 in the case of strict differentiability. Observe that all the assumptions of Theorem 5.4(i) hold under those needed for Eq. 5.1 in Theorem 5.2 (see the discussion on inner semicontinuity before the formulation of Theorem 5.4) except the general *Banach* space setting of Theorem 5.2 versus the *Asplund* space setting of Theorem 5.4. Note also that Theorem 5.2 contains the *equality* relationships (5.2) and (5.3) under the additional assumptions imposed therein, which do not have any counterparts in the general nonsmooth and set-valued setting of Theorem 5.4 even in finite dimensions.

6 Other Normal Cones to Set Images

In this section we obtain some analogs of the results established in Section 4 for Fréchet normals in the new case of *Hölder normals* (including *proximal* ones) to set images under differentiable mappings between Banach spaces. Similar results are also derived from those in Section 5 for the *convexified normal cone* to set images in Asplund spaces.

Given a set $\Omega \subset X$ and a number $s \in (0, 1]$, the *Hölder s-normal cone* to Ω at $\bar{x} \in \Omega$ is

$$\widehat{N}^s(\bar{x}; \Omega) := \Big\{ x^* \in X \,\Big|\, \exists \sigma \geq 0,\ \eta > 0 \ \text{such that} \ \langle x^*, x - \bar{x} \rangle \leq \sigma \|x - \bar{x}\|^{1+s} \ \text{whenever} \ x \in (\bar{x} + \eta I\!B) \cap \Omega \Big\}. \tag{6.1}$$

For $s = 1$, the set Eq. 6.1 is known as the *proximal normal cone* to Ω at $\bar{x}$. Obviously

$$\widehat{N}^s(\bar{x}; \Omega) \subset \widehat{N}(\bar{x}; \Omega) \ \text{for all} \ 0 < s \leq 1. \tag{6.2}$$

Introducing further the *Hölder s-tangent cone* to Ω at $\bar{x} \in \Omega$ by the duality/polarity correspondence

$$\widehat{T}^s(\bar{x}; \Omega) := \big\{ v \in X \,\big|\, \langle x^*, v \rangle \leq 0 \ \text{for all} \ x^* \in \widehat{N}^s(\bar{x}; \Omega) \big\}, \tag{6.3}$$

we observe the relationship between Eq. 6.3 and the contingent cone Eq. 4.2:

$$\operatorname{clco} T(\bar{x}; \Omega) \subset \widehat{T}^s(\bar{x}; \Omega) \ \text{for each} \ 0 < s \leq 1,$$

which follows from inclusion (4.3) by using the polarity in Eq. 6.2.

Given now a mapping $f\colon X \to Y$ between Banach spaces, we say that f is *Hölder s-differentiable* at $\bar{x}$ with $s \in (0, 1]$ if there exist a neighborhood U of $\bar{x}$, a constant $\gamma > 0$, and a bounded linear operator $A\colon X \to Y$ such that

$$\frac{\|f(x) - f(\bar{x}) - A(x - \bar{x})\|}{\|x - \bar{x}\|^{1+s}} \le \gamma \quad \text{for all} \quad x \in U \setminus \{\bar{x}\}. \tag{6.4}$$

It is not hard to check that the operator A in Eq. 6.4 is unique if exists; we call it the *Hölder s-derivative* of f at $\bar{x}$ and denote for simplicity by $\nabla f(\bar{x})$ if no confusion arises. It is easy to check that Hölder differentiability implies Fréchet differentiability at the reference point with the same derivative operator.

Observe that the notions of proximal normals and subgradients have been known from the very beginning of nonsmooth analysis, widely studied and applied, in particular, to various optimization-related and control problems; see, e.g., the books [6, 17, 22] and the references therein. Note also that, although the interest to Hölder s-differentiation and s-subdifferentiation in variational analysis goes back to the seminal paper [3] for $0 < s < 1$, these notions have not been well investigated yet. We intend to apply the new calculus results obtained below for the Hölder-type constructions to the study of stability and optimality issues in our future research.

The next theorem gives (independent) Hölder counterparts, whenever $s \in (0, 1]$, of Theorem 4.2 for Fréchet normal to direct images.

Theorem 6.1 (Hölder normals to direct images of sets under differentiable mappings) *Let $f\colon X \to Y$ be a mapping between Banach spaces, let Ω be a subset of X with $\bar{x} \in \Omega$, and let $s \in (0, 1]$. Assume that f is Hölder s-differentiable at $\bar{x}$ and that the restricted mapping $f\colon \Omega \to f(\Omega)$ is metrically regular at this point. Then we have the equality*

$$\widehat{N}^s\big(\bar{y};\, f(\Omega)\big) = \big(\nabla f(\bar{x})^*\big)^{-1}\big(\widehat{N}^s(\bar{x}; \Omega)\big) \;\; \textit{with} \;\; \bar{y} := f(\bar{x}). \tag{6.5}$$

If furthermore the subspace $\nabla f(\bar{x})X$ is closed in Y and if the kernel condition

$$\ker \nabla f(\bar{x}) \subset T^s(\bar{x}; \Omega)$$

is satisfied, then we also have

$$\nabla f(\bar{x})^* \widehat{N}^s\big(\bar{y};\, f(\Omega)\big) = \widehat{N}^s(\bar{x}; \Omega). \tag{6.6}$$

Proof It easily follows from the Fréchet differentiability of $\bar{x}$ (which is a consequence of its Hölder s-differentiability) that there are constants $\ell > 0$ and $\eta > 0$ such that

$$\|f(x) - f(\bar{x})\| \le \ell \|x - \bar{x}\| \quad \text{whenever} \quad \|x - \bar{x}\| \le \eta.$$

Fix any $y^* \in \widehat{N}^s(\bar{y};\, f(\Omega))$ and observe by Eq. 6.1 that there are $\sigma > 0$ and $\eta_1 > 0$ for which

$$\langle y^*, y - \bar{y}\rangle \le \sigma \|y - \bar{y}\|^{1+s} \quad \text{whenever} \quad y \in (\bar{y} + \eta_1 I\!B) \cap f(\Omega).$$

Therefore we get for $x \in \Omega$ sufficiently close to $\bar{x}$ that

$$\begin{aligned}\langle \nabla f(\bar{x})^*(y^*), x-\bar{x}\rangle &= \langle y^*, \nabla f(\bar{x})(x-\bar{x})\rangle \le \langle y^*, \nabla f(\bar{x})(x-\bar{x}) + f(\bar{x}) - f(x)\rangle \\ &\quad + \langle y^*, f(x) - f(\bar{x})\rangle \le \|y^*\|\gamma\|x-\bar{x}\|^{1+s} + \sigma\ell^{1+s}\|x-\bar{x}\|^{1+s} \\ &\le \sigma_1\|x-\bar{x}\|^{1+s} \text{ with } \sigma_1 := \|y^*\|\gamma + \sigma\ell^{1+s},\end{aligned}$$

where $\gamma > 0$ is the constant taken from Eq. 6.4. This shows that $y^* \in (\nabla f(\bar{x})^*)^{-1}(\widehat{N}^s(\bar{x}; \Omega))$ and thus justifies the inclusion "$\subset$" in Eq. 6.5.

To prove the opposite inclusion in Eq. 6.5, fix any $y^* \in (\nabla f(\bar{x})^*)^{-1}(N^s(\bar{x}; \Omega))$ and immediately conclude that $\nabla f(\bar{x})^* y^* \in \widehat{N}^s(\bar{x}; \Omega)$. We need to show that $y^* \in \widehat{N}^s(\bar{y}; f(\Omega))$. To proceed, find by definition (6.1) numbers $\sigma > 0$ and $\eta > 0$ such that

$$\langle \nabla f(\bar{x})^* y^*, x-\bar{x}\rangle \le \sigma\|x-\bar{x}\|^{1+s} \text{ for all } x \in (\bar{x} + \eta I\!B) \cap \Omega \tag{6.7}$$

and derive from this the estimate

$$\langle y^*, \nabla f(\bar{x})(x-\bar{x})\rangle \le \sigma\|x-\bar{x}\|^{1+s}, \quad x \in (\bar{x} + \eta I\!B) \cap \Omega.$$

The latter implies by definition (6.4) of Hölder differentiability the existence of $\gamma > 0$ with

$$\begin{aligned}\langle y^*, f(x) - f(\bar{x})\rangle &= \langle y^*, f(x) - f(\bar{x}) - \nabla f(\bar{x})(x-\bar{x})\rangle + \langle y^*, \nabla f(\bar{x})(x-\bar{x})\rangle \\ &\le (\gamma\|y^*\| + \sigma)\|x-\bar{x}\|^{1+s}, \quad x \in (\bar{x} + \eta I\!B) \cap \Omega.\end{aligned}$$

Using now the metric regularity of $f\colon \Omega \to f(\Omega)$ at $\bar{x}$ similarly to the proof of Theorem 4.2, for any $y \in f(\Omega)$ near $\bar{y}$ we find $\mu > 0$ and $x_y \in f^{-1}(y) \cap \Omega$ satisfying the estimate

$$\|x_y - \bar{x}\| \le \mu\|y - \bar{y}\| \le \eta,$$

where $\eta > 0$ is taken from Eq. 6.7. Combining all the above allows us to conclude that

$$\begin{aligned}\langle y^*, y-\bar{y}\rangle &= \langle y^*, f(x_y) - f(\bar{x})\rangle \le (\gamma\|y^*\| + \sigma)\|x_y - \bar{x}\|^{1+s} \\ &\le (\gamma\|y^*\| + \sigma)\mu^{1+s}\|y-\bar{y}\|^{1+s}\end{aligned}$$

whenever y is close to $\bar{y}$. This implies that $y^* \in \widehat{N}^s(\bar{y}, f(\Omega))$ and thus completes the proof of equality (6.5). The justification of the other equality (6.6) under the kernel condition imposed is similar to the corresponding proof in Theorem 4.2. □

Similarly to the case of Fréchet normals we derive consequences of Theorem 6.1 for Hölder normals to *inverse images* of sets in Banach spaces.

Corollary 6.2 (Hölder normal cones to inverse images) *Let $f\colon X \to Y$ be a mapping between Banach spaces, and let Θ be a subset of Y such that f is Hölder s-differentiable at $\bar{x}$ with $s \in (0,1]$ and that $\bar{y} := f(\bar{x}) \in \Theta$. The following assertions hold:*

(i) *Assume that the restricted mapping $f\colon f^{-1}(\Theta) \to \Theta \cap f(X)$ is metrically regular at $\bar{x}$. Then the equality*

$$\big(\nabla f(\bar{x})^*\big)^{-1}\widehat{N}^s\big(\bar{x};\, f^{-1}(\Theta)\big) = \widehat{N}^s\big(\bar{y};\Theta \cap f(X)\big)$$

is fulfilled. Furthermore, we have the equality

$$\widehat{N}^s(\bar{x};\, f^{-1}(\Theta)\big) = \nabla f(\bar{x})^*\widehat{N}^s\big(\bar{y};\Theta \cap f(X)\big)$$

provided that the mapping f is RMR at $\bar{x}$.

(ii) *Assume that the original mapping $f\colon X \to Y$ is metrically regular at $\bar{x}$. Then we have the equalities*

$$\big(\nabla f(\bar{x})^*\big)^{-1}\widehat{N}^s\big(\bar{x};\, f^{-1}(\Theta)\big) = \widehat{N}^s(\bar{y};\Theta),$$

$$\widehat{N}^s\big(\bar{x};\, f^{-1}(\Theta)\big) = \nabla f(\bar{x})^*\widehat{N}^s(\bar{y};\Theta).$$

Proof Assertion (i) is justified similarly to the proof of Corollary 4.4 with the use of Theorem 6.1 instead of Theorem 4.2. Asserting (ii) follows from (i) by a simple observation that $\bar{y} \in \operatorname{int} f(X)$ if $f\colon X \to Y$ is metrically regular at $\bar{x}$, and hence the set $f(X)$ can be removed from the corresponding formulas in (i). □

Next we consider the so-called *convexified normal cone* to $\Omega \subset X$ at $\bar{x} \in \Omega$ defined as the norm closure "cl" in X^* of the convex hull "co" of the limiting normal cone Eq. 2.2 by

$$\overline{N}(\bar{x};\Omega) := \operatorname{clco} N(\bar{x};\Omega). \tag{6.8}$$

To compare Eq. 6.8 with the *Clarke normal cone* $N_C(\bar{x};\Omega)$ defined in general Banach spaces [6], we use the relationship

$$N_C(\bar{x};\Omega) = \operatorname{cl}^*\operatorname{co} N(\bar{x};\Omega) \tag{6.9}$$

established in [17, Theorem 3.57] provided that X is Asplund and Ω is locally closed around $\bar{x}$, where cl^* stands for the weak* closure in X^*. It follows from Eqs. 6.8, 6.9, and the Mazur weak closure theorem that $\overline{N}(\bar{x};\Omega) = N_C(\bar{x};\Omega)$ for closed sets in reflexive spaces.

Employing now the results of Section 5 on limiting normals to direct and inverse images of sets, we derive the corresponding results for the convexified normal cone Eq. 6.8 in general Banach spaces and hence for its Clarke counterpart for locally closed subsets of reflexive spaces. We start with the following simple proposition.

Proposition 6.3 (convexified normal cone to direct images) *Let $f\colon X \to Y$ be a mapping between Banach spaces, and let Ω be a nonempty subset of X. Having $\bar{x} \in \Omega$ with $\bar{y} := f(\bar{x})$, assume that f is strictly differentiable at $\bar{x}$ and that the restricted mapping $f\colon \Omega \to f(\Omega)$ is metrically regular at $\bar{x}$. Then we have the inclusion*

$$\overline{N}\big(\bar{y};\, f(\Omega)\big) \subset \big(\nabla f(\bar{x})^*\big)^{-1}\big(\overline{N}(\bar{x};\Omega)\big), \tag{6.10}$$

which holds as equality provided that either $\nabla f(\bar{x})$ *is isomorphic or* $\overline{N}(\bar{x};\Omega)=\widehat{N}(\bar{x};\Omega)$.

Proof Using definition (6.8) and inclusion (5.1) of Theorem 5.2, which holds under the assumptions made, we get

$$\overline{N}\,(\bar{y};\, f(\Omega))\subset \operatorname{clco}\Big[\big(\nabla f(\bar{x})^*\big)^{-1}\,(N(\bar{x};\,\Omega))\Big]\subset \operatorname{clco}\Big[\big(\nabla f(\bar{x})^*\big)^{-1}\Big(\overline{N}(\bar{x};\,\Omega)\Big)\Big]. \tag{6.11}$$

It easy follows from the linearity and continuity of the operator $\nabla f(\bar{x})^*\colon Y^*\to X^*$ that the set $(\nabla f(\bar{x})^*)^{-1}(\overline{N}(\bar{x};\Omega))$ is closed and convex. Thus Eq. 6.11 implies Eq. 6.10.

The converse inclusion to Eq. 6.10 obviously holds if the operator $\nabla f(\bar{x})\colon X\to Y$ is an isomorphism. On the other hand, the assumption $\overline{N}(\bar{x};\Omega)=\widehat{N}(\bar{x};\Omega)$ and equality (4.6) in Theorem 4.2 yield that

$$\big(\nabla f(\bar{x})^*\big)^{-1}\big(\overline{N}(\bar{x};\,\Omega)\big)=\big(\nabla f(\bar{x})^*\big)^{-1}\big(\widehat{N}(\bar{x};\,\Omega)\big)=\widehat{N}\big(\bar{y};\, f(\Omega)\big)\subset\overline{N}\big(\bar{y};\, f(\Omega)\big)$$

justifying the equality in Eq. 6.10 and thus completing the proof of the proposition. □

The next result concerning the convexified normal cone Eq. 6.8 to inverse images of sets under strictly differentiable mappings is significantly more involved.

Theorem 6.4 (convexified normals to inverse images) *Let* $f\colon X\to Y$ *be a mapping between Banach spaces, and let* Θ *be a subset of* Y *such that* $\bar{y}:=f(\bar{x})\in\Theta$ *for some* $\bar{x}\in X$ *at which* f *is strictly differentiable. The following assertions hold:*

(i) *If* $\nabla f(\bar{x})\colon X\to Y$ *is surjective, then*

$$\overline{N}\big(\bar{x};\, f^{-1}(\Theta)\big)=\nabla f(\bar{x})^*\overline{N}(\bar{y};\,\Theta). \tag{6.12}$$

(ii) *Let* X *be Asplund, let* $Y=I\!R^n$*, and let* Θ *be locally closed around* $\bar{y}$*. Then we have*

$$\overline{N}\big(\bar{x};\, f^{-1}(\Theta)\big)\subset\nabla f(\bar{x})^*\overline{N}(\bar{y};\,\Theta) \tag{6.13}$$

provided that the qualification condition

$$\ker\nabla f(\bar{x})^*\cap\overline{N}(\bar{y};\,\Theta)=\{0\} \tag{6.14}$$

is satisfied. The equality holds in Eq. 6.13 *if in addition* $\overline{N}(\bar{y};\Theta)=\widehat{N}(\bar{y};\Theta)$.

Proof Under the assumptions in (i) we have from [17, Theorem 1.17] that

$$N\big(\bar{x};\, f^{-1}(\Theta)\big)=\nabla f(\bar{x})^*N(\bar{y};\,\Theta)\subset\nabla f(\bar{x})^*\overline{N}(\bar{y};\,\Theta). \tag{6.15}$$

The set $\nabla f(\bar{x})^*\overline{N}(\bar{y};\Theta)$ in Eq. 6.15 is obviously convex. Let us show that it is closed in the norm topology of X^* under the imposed surjectivity assumption on $\nabla f(\bar{x})$. Indeed, pick any $x^*\in\operatorname{cl}\big[\nabla f(\bar{x})^*\overline{N}(\bar{y};\Theta)\big]$ and find sequences of $y_k^*\in\overline{N}(\bar{y};\Theta)$ and

$x_k^* = \nabla f(\bar{x})^* y_k^*$ for $k \in I\!N$ such that $x_k^* \to x^*$ as $k \to \infty$. Taking any (x_m^*, y_m^*) and (x_l^*, y_l^*) from the above sequences and using the surjectivity of $\nabla f(\bar{x})$, we get

$$\|x_m^* - x_l^*\| = \|\nabla f(\bar{x})^* y_m^* - \nabla f(\bar{x})^* y_l^*\| \geq \mu \|y_m^* - y_l^*\|$$

with some constant $\mu > 0$; see [17, Lemma 1.18]. This implies that $\{y_k^*\}$ is a Cauchy sequence in the norm topology of Y^*, and hence it converges to an element $y^* \in Y^*$, which surely belongs to the cone $\overline{N}(\bar{y}; \Theta)$ due to the norm-closedness on the latter in Y^*. By this and the continuity of $\nabla f(\bar{x})^*\colon Y^* \to X^*$, we have $x^* = \nabla f(\bar{x})^* y^* \in \overline{N}(\bar{y}; \Theta)$, which justifies the norm-closedness of the set $\nabla f(\bar{x})^* \overline{N}(\bar{y}; \Theta)$ in X^*. Thus taking the convex closure of the sets in Eq. 6.15, we arrive at the inclusion "$\subset$" in Eq. 6.12. The converse inclusion "$\supset$" in Eq. 6.12 easily follows from the one

$$N\big(\bar{x};\ f^{-1}(\Theta)\big) \supset \nabla f(\bar{x})^* N(\bar{y}; \Theta)$$

in Eq. 6.15 by taking the convex closure on both sides therein and using the linearity and continuity of the operator $\nabla f(\bar{x})^*$. This completes the proof of assertion (i) in the theorem.

To justify assertion (ii), we employ the well-developed calculus of limiting normals in Asplund spaces [17] that do not require restrictive assumptions of the surjectivity type as in (i). By [17, Theorem 3.8], which holds under the assumptions imposed in (ii) for the fulfillment of Eq. 6.13, we get the inclusion

$$N\big(\bar{x};\ f^{-1}(\Theta)\big) \subset \nabla f(\bar{x})^* N(\bar{y}; \Theta)$$

that implies due to definition (6.8) that

$$N\big(\bar{x};\ f^{-1}(\Theta)\big) \subset \nabla f(\bar{x})^* \overline{N}(\bar{y}; \Theta). \tag{6.16}$$

To get Eq. 6.13 from Eq. 6.16, it is sufficient to prove that the (convex) set

$$\Lambda := \nabla f(\bar{x})^* \overline{N}(\bar{y}; \Omega) \tag{6.17}$$

is closed in the norm topology of X^*. Take a sequence $\{x_k^*\} \subset \Lambda$ that converges to some $x^* \in X^*$ as $k \to \infty$ and find by Eq. 6.17 a sequence $\{y_k^*\} \subset Y^*$ such that

$$x_k^* = \nabla f(\bar{x})^* y_k^* \ \text{ and } \ y_k^* \in \overline{N}(\bar{y}; \Theta), \quad k \in I\!N. \tag{6.18}$$

Let us show that the sequence $\{y_k^*\}$ in Eq. 6.18 is bounded under the assumed qualification condition (6.14). If it is not the case, then there is a subsequence of $\{y_k^*\}$ such that (without relabeling) $\|y_k^*\| \to \infty$ as $k \to \infty$. We have from the equality in Eq. 6.18 that

$$\frac{x_k^*}{\|y_k^*\|} = \nabla f(\bar{x})^* \left(\frac{y_k^*}{\|y_k^*\|}\right), \quad k \in I\!N. \tag{6.19}$$

and can assume due to the finite dimensionality of Y that $\widetilde{y}_k^* := y_k^*/\|y_k^*\| \to y^*$ as $k \to \infty$ for some y^* with $\|y^*\| = 1$. It follows that $y^* \in \overline{N}(\bar{y}; \Theta)$, since $\overline{N}(\bar{y}; \Theta)$ is a closed cone. By passing to the limit in Eq. 6.19 as $k \to \infty$ and using the continuity of the operator $\nabla f(\bar{x})^*$, we get $y^* \in \ker \nabla f(\bar{y})^*$ and arrive therefore at a contradiction with the qualification condition (6.14). Thus the sequence $\{y_k^*\}$ is bounded in $Y^* = I\!R^n$, which implies that

$$x^* \in \nabla f(\bar{x})^* y^* \subset \overline{N}(\bar{y}; \Theta)$$

by passing to the limit in Eq. 6.18 as $k \to \infty$. This justifies inclusion (6.13). The converse inclusion to Eq. 6.13 follows under the condition $\overline{N}(\bar{y}; \Theta) = \widehat{N}(\bar{y}; \Theta)$ from the one

$$\widehat{N}\big(\bar{x};\ f^{-1}(\Theta)\big) \supset \nabla f(\bar{x})^* \widehat{N}(\bar{y}; \Theta)$$

established in [17, Theorem 1.14(i)] for mappings $f\colon\ X \to Y$ between Banach spaces that are merely Fréchet differentiable at $\bar{x}$. This completes the proof of the theorem. □

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