Set-Valued Anal (2008) 16:319–333
DOI 10.1007/s11228-008-0083-y

Regularization of Nonconvex Sweeping Process in Hilbert Space

Lionel Thibault

Received: 4 February 2008 / Accepted: 6 February 2008 /
Published online: 15 April 2008

Abstract This paper deals with the differential inclusion of sweeping process associated, on an interval I, with the normal cone to a moving set $C(t)$. Under a Lipschitzian variation of the set-valued mapping $C(\cdot)$ and under the prox-regularity assumption of $C(t)$, it is shown that one can regularize the sweeping process to obtain a family of classical differential equations whose solutions exist on a fixed appropriate interval and converge to the solution of the sweeping process on this interval. The general case where $C(t)$ moves in an absolutely continuous way is reduced to the previous one to obtain the existence and uniqueness of solution on all the interval I.

Keywords Moreau's sweeping process · Normal cone · Differential inclusion · Regularization · Existence · Uniqueness · Prox-regular set

Mathematics Subject Classifications (2000) Primary 34A60 · 49J52 · Secondary 49J24 · 49J53

1 Introduction

The paper deals with the differential inclusion

$$(E)\begin{cases} \dot{u}(t) \in -N_{C(t)}(u(t)) \\ u(T_0) = a \in C(T_0), \end{cases}$$

where $[T_0, T]$ is a fixed interval with $0 \leqslant T_0 < T$, $C(t)$ is a *closed* subset of a *Hilbert space* H for each $t \in [T_0, T]$, and $u(T_0) = a$ is the initial condition. Here $N_{C(t)}(\cdot)$

Dedicated to Boris Mordukhovich on the occasion of his 60th birthday.

L. Thibault (✉)
Département de Mathématiques,
Université Montpellier II,
Case courrier 051 Place Eugène Bataillon,
34095 Montpellier cedex 5, France
e-mail: thibault@math.univ-montp2.fr

denotes (see the next section) the Mordukhovich or basic normal cone to the closed set $C(t)$, see the book [22] by Mordukhovich. Throughout, we will consider only the concept of absolutely continuous solutions. So a solution of (E) will be an absolutely continuous mapping $u(\cdot)$ from $[T_0, T]$ into H such that $u(T_0) = a$, $u(t) \in C(t)$ for all $t \in [T_0, T]$, and $\dot{u}(t) \in -N_{C(t)}(u(t))$ for almost all $t \in [T_0, T]$. It is classical for the existence of absolutely continuous solutions to assume that the closed set $C(t)$ moves in an absolutely continuous way (see Eq. 2.8, in the next section, for the definition). With this absolute continuity assumption for $C(\cdot)$, it is easy to see that $u(t) \in C(t)$ for all $t \in [T_0, T]$ whenever this inclusion holds almost everywhere and hence, in particular, when the first inclusion in (E) holds. As formulated the differential inclusion (E) has been introduced and thoroughly studied in the 70s by Moreau [23–25] under the assumption that all the sets $C(t)$ are convex. The name of *Sweeping process* has been used in [23, 25]. Under a general closedness property (with respect to (t, x)) of the Clarke normal cone, the existence of solutions for (E) is proved in [30] in the finite dimensional setting with the Clarke normal cone in place of the Mordukhovich normal cone. Recently, again in the finite dimensional setting and with the Clarke normal cone, the authors in [2, 11, 29] showed that the differential inclusion (E) (with the Clarke normal cone) has always at least a solution provided that $C(t)$ moves in an absolutely continuous way. The mechanical motivation and interpretation of (E) can be found in [25, 26]. The differential inclusion (E) or some of its variants also appear in modelizations in several other fields as resource allocation mechanisms in economics (see, e.g., [14, 18]), complementarity systems (see [5]), dissipative systems in electrical circuits (see [19]), crowd motion modelization (see [21]), etc. See also [7–9] for other contributions.

In the papers [4, 11, 15, 16], existence (and uniqueness) results for (E) have been established in Hilbert space and when the closed sets $C(t)$ are merely prox-regular (see the next section for the definition) by proving the convergence of the Moreau catching-up algorithm in such a case. The prox-regularity concept is known to play a crucial role in the generalized interpretation or study of: curvature (see [17]), geodesics (see [6]), second order analysis of non differentiable functions (see [28]), behavior of some classes of nonsmooth functions (see [1] and the references therein), evolution equations associated with subdifferentials of functions (see, e.g., [20]), etc. The catching-up algorithm in [25] is constructed with a discretization (t_i) of time and taking $u_0 = a$ and $u(t_{i+1}) := \text{proj}_{C(t_{i+1})}(u_i)$. The well-posedness of such an algorithm in the convex case is clear. In [4, 15] it is proved that, under the prox-regularity of $C(t)$, appropriate choices of t_i make that the algorithm is still well-posed. The existence of a solution in those two papers is based on the convergence of that algorithm. In the present paper the approach is different. In the convex case (and more generally for differential inclusions associated with maximal monotone operators) it is known that one can regularize (see [23]) the normal cone of the convex set $C(t)$ (i.e., the subdifferential of its indicator function) to obtain a usual differential equation

$$(\mathcal{R}(E)) \begin{cases} \dot{u_\lambda}(t) = -(1/\lambda)\nabla d^2_{C(t)}(u_\lambda(t)) \\ u_\lambda(T_0)) = a \end{cases}$$

whose solution $u_\lambda(\cdot)$ on $[T_0, T]$ (recall that $\nabla d^2_{C(t)}(\cdot)$ is Lipschitzian on some appropriate bounded sets) verifies that the family $u_\lambda(\cdot)$ converges uniformly on $[T_0, T]$ to a solution of (E). One of the interests of such a process is that the solution of (E) is realized as the limit of solutions of appropriate classical differential equations and hence the machinery of classical differential equations can be used in the study of various questions about the solution. Our aim is to show that, when the set $C(t)$ is ρ-prox-regular and moves in a Lipschizian way with γ as Lipschitz modulus, there exists some $\theta > 0$ (independent of λ) such that the regularized differential equation $(\mathcal{R}(E))$ is well-posed on $[T_0, T_0 + \theta]$ with a unique solution $u_\lambda(\cdot)$ on $[T_0, T_0 + \theta]$ and that the family $(u_\lambda(\cdot))_\lambda$ still converges uniformly on $[T_0, T_0 + \theta]$ (when $\lambda \downarrow 0$) to a solution of (E) on $[T_0, T_0 + \theta]$. The existence and uniqueness of solution of (E) is then derived on all the interval $[T_0, T]$ under the weaker assumption of absolute continuity of $C(\cdot)$.

2 Preliminaries and Statement of the Main Result

Our main result in the paper concerns the sweeping process associated with prox-regular sets. Throughout H will be a real Hilbert space. As we will see below, the prox-regularity of a set is strongly connected with some properties of the normal cone associated with this set.

One among the normal cone notions that we will use in this paper is the one related to the Fréchet subdifferential. Let $f\colon H \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous (lsc, for short) function and let $x \in H$ be a point where $f(x)$ is finite. One says that a vector $\zeta \in H$ is a Fréchet subgradient of f at x when for any real number $\varepsilon > 0$ there exists some real number $\eta > 0$ such that

$$\langle \zeta, x' - x\rangle \leqslant f(x') - f(x) + \varepsilon\|x' - x\| \quad \text{for all } x' \in B(x, \eta), \tag{2.1}$$

where $B(x, \eta)$ denotes the open ball centered at x with radius η. The set of all Fréchet subgradients of f at x, denoted by $\widehat{\partial} f(x)$, is called the *Fréchet subdifferential* of f at x. If $f(x)$ is not finite, one puts $\widehat{\partial} f(x) = \emptyset$.

The Fréchet subdifferential enjoys only fuzzy calculus rules (see, e.g., [22]). A limiting process is hence needed for a robust rich calculus. A vector $\zeta \in H$ (see [22]) is a basic (or limiting) subgradient of f at x (with $f(x) < +\infty$) when there are a sequence $(x_n, f(x_n))_n$ converging strongly in $H \times \mathbb{R}$ to $(x, f(x))$ and a sequence $(\zeta_n)_n$ converging *weakly* to ζ with $\zeta_n \in \widehat{\partial} f(x_n)$. As above the set of all basic subgradients of f at x is called the Mordukhovich or *basic subdifferential* of f at x and it is denoted by $\partial f(x)$. For the exact statements of the calculus rules of the basic subdifferential in Hilbert space (and more general infinite dimensional spaces) and for several applications in variational analysis, optimal control theory, economics etc., we refer the reader to the book [22] by Mordukhovich.

When f is the indicator function ψ_S of a closed subset $S \subset H$, that is, $\psi_S(x) = 0$ if $x \in S$ and $\psi_S(x) = +\infty$ otherwise, the Fréchet subdifferential of ψ_S at x is the *Fréchet normal cone* to S at x and one denotes $\widehat{N}(S, x)$. Translating Eq. 2.1 we see that a vector ζ is then in $\widehat{N}(S, x)$ when for any positive number ε there is a positive number η such that

$$\langle \zeta, x' - x\rangle \leqslant \varepsilon\|x' - x\| \quad \text{for all } x' \in S \cap B(x, \eta). \tag{2.2}$$

Also, $\widehat{N}(S, x) = \emptyset$ when $x \notin S$.

In the same way, a basic subgradient of ψ_S gives a basic normal vector to S. So, a vector $\zeta \in H$ belongs to the Mordukhovich or *basic normal cone* $N(S, x)$ when there exist a sequence $(x_n)_n$ converging strongly to x with $x_n \in S$ and a sequence $(\zeta_n)_n$

converging weakly to ζ with $\zeta_n \in \widehat{N}(S, x_n)$. One also has $N(S, x) = \emptyset$ when $x \notin S$. Sometimes it may be convenient to write $N_S(x)$ in place of $N(S, x)$.

Let $\rho \in]0, +\infty]$. Extending slightly the finite dimensional definition of positively reached sets of Federer (see [17]), the closed set S is (uniformly) *ρ-prox-regular* (see [27]) when any point x in the open ρ-enlargement of S

$$U_\rho(S) := \{u \in H\colon d_S(u) < \rho\}$$

has a unique nearest point $\operatorname{proj}_S(x)$ in S and the mapping proj_S is continuous over $U_S(\rho)$. Here $d_S(x)$, also denoted by $d(x, S)$, is the distance from the point x to the set S. In [27] the expression of prox-regularity is used since it is proved there that the corresponding local property is equivalent to the concept of prox-regularity of the indicator function ψ_S, concept introduced for functions by Poliquin and Rockafellar (see [28]). In fact the paper [27] was motivated by a geometrical characterization, for a closed set of a Hilbert space, of the prox-regularity near a point of the indicator function of the set. A first characterization in terms of the normal cone (see [27]) is that for any nonzero vector $\zeta \in N_S(x)$ one has $x \in \operatorname{Proj}_S(x + \frac{\rho}{\|\zeta\|}\zeta)$, where $\operatorname{Proj}_S(y)$ is the set (eventually empty) of all nearest points of y in S. Translating this in the fact that for all $x' \in S$

$$\left\| x - \left(x + \frac{\rho}{\|\zeta\|}\zeta \right) \right\|^2 \leqslant \left\| x + \frac{\rho}{\|\zeta\|}\zeta - x' \right\|^2$$

we see that this is equivalent to

$$\langle \zeta, x' - x \rangle \leqslant \frac{\|\zeta\|}{2\rho}\|x' - x\|^2 \quad \text{for all } x' \in S. \tag{2.3}$$

The latter inequality, in particular, implies that

$$\langle \zeta' - \zeta, x' - x \rangle \geqslant -\|x' - x\|^2 \tag{2.4}$$

for all $\zeta \in N_S(x)$ and $\zeta' \in N_S(x')$ with $\|\zeta\| \leqslant \rho$ and $\|\zeta'\| \leqslant \rho$. In fact, in [27] it is proved that Eq. 2.4 characterizes the ρ-prox-regularity. So, in the particular case where $\rho = +\infty$, we obtain the monotonicity of the normal cone to S and hence by [13] one recovers the fact that the ρ-prox-regularity of the closed set S with $\rho = \infty$ corresponds to its convexity. It is also shown in [27] that the ρ-prox-regularity of S is equivalent to

$$\begin{cases} \text{the continuous differentiability over } U_\rho(S) \\ \text{of the square distance function } d_S^2(\cdot) \text{ to } S. \end{cases} \tag{2.5}$$

Further (see, [10, 12, 27])

$$\nabla\left((1/2)d_S^2\right)(x) = x - \operatorname{proj}_S(x) \quad \text{for any } x \in U_\rho(S) \tag{2.6}$$

and for any positive number $\delta < \rho$ and $x, x' \in U_\delta(S)$ one has

$$\|\operatorname{proj}_S(x) - \operatorname{proj}_S(x')\| \leqslant \frac{\rho}{\rho - \delta}\|x - x'\|. \tag{2.7}$$

Another characterization of the ρ-prox-regularity that we will need in the third section is the following one due to M. Bounkhel and L. Thibault.

Proposition 2.1 [4, Theorem 3.1] *The closed set S is ρ-prox-regular if and only if for all $x \in U_\rho(S)$ and all $\zeta \in \partial d_S(x)$ one has*

$$\langle \zeta, x' - x \rangle \leqslant \frac{8}{\rho - d_S(x)} \|x' - x\|^2 + d_S(x') - d_S(x)$$

for all $x' \in H$ with $d_S(x') \leqslant \rho$.

For other properties of ρ-prox-regular sets, we refer to [6, 10, 12, 17, 27]. We emphasize, for example, that for prox-regular sets all the *normal cones coincide*: Proximal, Fréchet, Mordukhovich, and Clarke normal cones.

Let now T_0, T be two nonnegative real numbers with $T_0 < T$ and, for each $t \in [T_0, T]$, let $C(t)$ be a closed ρ-prox-regular set in H. One says that the closed $C(t)$ moves in an absolutely continuous way with $t \in [T_0, T]$ when there exists an absolutely continuous function $v(\cdot)$ from $[T_0, T]$ into $\mathbb{R}$ such that for all $x \in H$

$$|d(x, C(t)) - d(x, C(s))| \leqslant |v(t) - v(s)| \tag{2.8}$$

for all $s, t \in [T_0, T]$.

We can now state the main result of the paper.

Theorem 2.1 *Assume that the closed sets $C(t)$ are ρ-prox-regular and move in an absolutely continuous way, that is, Eq. 2.8 holds with an absolutely continuous function $v(\cdot)$ over $[T_0, T]$.*

(a) *If the function $v(\cdot)$ is Lipschitzian with a Lipschitz modulus γ, then there exists a positive constant θ, depending only on γ and ρ, (in fact any $\theta < \min\{\frac{\rho}{3\gamma}, T - T_0\}$ is appropriate) such that for any $\lambda > 0$ the regularized differential equation*

$$\begin{cases} \dot{u}_\lambda(t) = -(1/2\lambda)\nabla d^2_{C(t)}(u_\lambda(t)) \\ u_\lambda(T_0) = a \end{cases}$$

is well defined on $[T_0, T_0 + \theta]$ and has a unique solution $u_\lambda(\cdot)$ on $[T_0, T_0 + \theta]$ with $\|\dot{u}_\lambda(t)\| \leqslant \gamma$. Further the family $(u_\lambda(\cdot))_\lambda$ converges uniformly on $[T_0, T_0 + \theta]$, when $\lambda \downarrow 0$, to a solution $u(\cdot)$ of (E) over the interval $[T_0, T_0 + \theta]$.
Dividing $[T_0, T]$ in a finite number of intervals with length less or equal than θ yields the existence of a unique solution $u(\cdot)$ of (E) over $[T_0, T]$. Further $\|\dot{u}(t)\| \leqslant \gamma$ for almost all $t \in [T_0, T]$.

(b) *If $v(\cdot)$ is merely absolutely continuous, then a change of time variable reduces the problem to (a) and gives a unique solution $u(\cdot)$ of (E); the mapping $u(\cdot)$ further verifies $\|\dot{u}(t)\| \leqslant |\dot{v}(t)|$ for almost all $t \in [T_0, T]$.*

The proof of the theorem will be established in the next section through a series of lemmas. The support function of the set $C(t)$ which plays a crucial role in the development of the regularization process of [23] is useless in our framework since the set $C(t)$ is not assumed to be convex.

3 Proof of the Theorem

We start the proof of Theorem 2.1 with the following lemma.

Lemma 3.1 *Let $z : [T_0, T] \to H$ be an absolutely continuous mapping and let $g(t) := d(z(t), C(t))$ for all $t \in [T_0, T]$. Assume that $d(z(t), C(t)) < \rho$ for all $t \in [T_0, T]$. Then for a.e. $t \in [T_0, T]$*

$$\dot{g}(t)g(t) \leqslant \langle \dot{z}(t), z(t) - \operatorname{proj}_{C(t)}(z(t))\rangle + g(t)|\dot{v}(t)|.$$

Proof Put $\varphi(t, x) := (1/2)d^2(x, C(t))$ for all $t \in [T_0, T]$ and all $x \in H$. The function g is absolutely continuous because, by Eq. 2.8,

$$|g(t) - g(s)| \leqslant \|z(t) - z(s)\| + |v(t) - v(s)|$$

for all $s, t \in [T_0, T]$. Fix any $t \in]T_0, T[$ such that g, v, and z are derivable at t. Observe that, according to Eqs. 2.5 and 2.6, the function $\varphi(t, \cdot)$ is continuously differentiable around $z(t)$ and that

$$\nabla_2\varphi(t, z(t)) = z(t) - \operatorname{proj}_{C(t)}(z(t)). \tag{3.1}$$

Write for $s \in]0, T - t[$ small enough

$$\begin{aligned}
&(1/2s)\left[g(t+s)^2 - g(t)^2\right] \\
&\quad = (1/s)[\varphi(t+s, z(t+s)) - \varphi(t, z(t+s))] + (1/s)[\varphi(t, z(t+s)) - \varphi(t, z(t))] \\
&\quad \leqslant (1/2)[d(z(t+s), C(t+s)) + d(z(t+s), C(t))] \cdot (1/s)|v(t+s) - v(t)| + \\
&\qquad + (1/s)[\varphi(t, z(t+s)) - \varphi(t, z(t))].
\end{aligned} \tag{3.2}$$

As z is derivable at t, there exists $\varepsilon(s) \underset{s\downarrow 0}{\longrightarrow} 0$ such that

$$z(t+s) = z(t) + s\dot{z}(t) + s\varepsilon(s)$$

and this yields

$$\begin{aligned}
&(1/s)[\varphi(t, z(t+s)) - \varphi(t, z(t))] \\
&\quad \leqslant (1/s)[\varphi(t, z(t) + s\dot{z}(t)) - \varphi(t, z(t))] + \\
&\qquad + (1/2)\|\varepsilon(s)\|\,[d(z(t) + s\dot{z}(t) + s\varepsilon(s)), C(t)) + d(z(t) + s\dot{z}(t), C(t))].
\end{aligned}$$

Putting this inequality and Eq. 3.2 together we obtain

$$\begin{aligned}
&(1/2s)[g(t+s)^2 - g(t)^2] \\
&\quad \leqslant (1/2)[d(z(t+s), C(t+s)) + d(z(t+s), C(t)] \cdot \\
&\qquad \cdot (1/s)|v(t+s) - v(t)| + (1/s)[\varphi(t, z(t) + s\dot{z}(t)) - \varphi(t, z(t))] + \eta(s)
\end{aligned}$$

for some $\eta(s) \underset{s\downarrow 0}{\longrightarrow} 0$. Taking $s \downarrow 0$, it follows that

$$\dot{g}(t)g(t) \leqslant g(t)|\dot{v}(t)| + \langle \nabla_2\varphi(t, z(t)), \dot{z}(t)\rangle$$

and the proof is complete according to Eq. 3.1. □

Fix any positive real number $\theta < T - T_0$ such that for every measurable subset $A \subset [T_0, T]$ with $\text{meas}(A) \leqslant \theta$ one has $\int_A |\dot{v}(s)|\, ds < \rho/3$. Observe that for any $x \in B(a, \rho/3)$ and any $t \in [T_0, T_0 + \theta]$ we have

$$d(x, C(t)) \leqslant d(a, C(T_0)) + \|x - a\| + \int_{T_0}^{t} |\dot{v}(s)|\, ds < 2\rho/3. \tag{3.3}$$

Then, according to Eq. 2.5, for any $t \in [T_0, T + \theta]$ the mapping $x \mapsto f(t, x) := \nabla(1/2)d^2_{C(t)}(x)$ is well defined on $B(a, \rho/3)$ and by Eq. 2.6

$$f(t, x) = x - \text{proj}_{\,C(t)}(x).$$

Further, for any $x_1, x_2 \in B(a, \rho/3)$ we have, by Eq. 3.3, the inequalities

$$d(x_i, C(t)) < 2\rho/3 < \rho$$

for $i = 1, 2$, and hence taking $\delta = 2\rho/3$, the inequality (2.7) yields

$$\|\text{proj}_{\,C(t)}(x_1) - \text{proj}_{\,C(t)}(x_2)\| \leqslant (\rho/\,(\rho - \delta))\|x_1 - x_2\| = 3\|x_1 - x_2\|.$$

Therefore, the mapping $f(t, \cdot)$ is Lipschitzian over the ball $B(a, \rho/3)$.

On the other hand, for any $x \in B(a, \rho/3)$, we claim that the mapping $f(\cdot, x)$ is Bochner integrable on $[T_0, T_0 + \theta]$. If H is separable, the argument is straightforward because the continuity of the function $t \mapsto d(x, C(t))$ (see Eq. 2.8) easily yields the weak measurability of the bounded mapping $f(\cdot, x)$ and hence, under the separability of H, the Bochner measurability and integrability of $f(x, \cdot)$. In fact, even if H is separable or not, one can (see [3]) prove more, say $f(x, \cdot)$ is continuous on $[T_0, T_0 + \lambda]$.

Fix now any real number $\lambda > 0$ and consider the differential equation over $[T_0, T_0 + \theta[\times B(a, \rho/3)$

$$\left(E^*_\lambda\right) \begin{cases} \dot{u}(t) = -(1/2\lambda)\nabla d^2_{C(t)}(u(t)) \\ u(T_0) = a. \end{cases}$$

The results just obtained above ensure us that this differential equation has a (unique) solution $u_\lambda(\cdot)$ defined on its *maximal* interval of existence $[T_0, T_\lambda[\subset [T_0, T_0 + \theta[$.

The following lemma provides an upper bound for the derivative in t of the distance function from $u_\lambda(t)$ to $C(t)$.

Lemma 3.2 *Put $g_\lambda(t) := d(u_\lambda(t), C(t))$ for any $t \in [T_0, T_\lambda[$. Then g_λ is locally absolutely continuous on $[T_0, T_\lambda[$ and*

$$\dot{g}_\lambda(t) \leqslant |\dot{v}(t)| - (1/\lambda)g_\lambda(t) \text{ a.e. } t \in [T_0, T_\lambda[.$$

Further, for all $t \in [T_0, T_\lambda[$

$$g_\lambda(t) \leqslant e^{-t/\lambda} \int_{T_0}^{t} |\dot{v}(s)| e^{s/\lambda}\, ds.$$

Proof As $d(u_\lambda(t), C(t)) < 2\rho/3$ for any $t \in [T_0, T_\lambda[$ by Eq. 3.3, we may take $z(t) = u_\lambda(t)$ in Lemma 3.1. So using (E_λ^*), we obtain (with $\varphi(t,x) = (1/2)d^2(x, C(t))$) for almost all $t \in [T_0, T_\lambda[$

$$\begin{aligned} \dot{g}_\lambda(t) g_\lambda(t) &\leqslant g_\lambda(t)|\dot{v}(t)| + \langle \nabla_2 \varphi(t, u_\lambda(t)), \dot{u}_\lambda(t) \rangle \\ &= g_\lambda(t)|\dot{v}(t)| - (1/\lambda)\|\nabla_2 \varphi(t, u_\lambda(t))\|^2 \\ &= g_\lambda(t)|\dot{v}(t)| - (1/\lambda)(g_\lambda(t))^2 \end{aligned}$$

because

$$\|\nabla_2 \varphi(t, u_\lambda(t))\| = \|u_\lambda(t) - \operatorname{proj}_{C(t)}(u_\lambda(t))\| = g_\lambda(t).$$

Fix any $t \in]T_0, T_\lambda[$ where $\dot{g}_\lambda(t)$, $\dot{v}(t)$, and $\dot{u}_\lambda(t)$ exist. The latter inequality entails

$$\dot{g}_\lambda(t) \leqslant |\dot{v}(t)| - (1/\lambda) g_\lambda(t) \quad \text{if } g_\lambda(t) > 0$$

and the inequality still holds if $g_\lambda(t) = 0$ because in this case the derivability of g_λ at t ensures that $\dot{g}_\lambda(t) = 0$. Indeed since $g_\lambda \geqslant 0$ and $\lim_{s\to 0}(1/s)g_\lambda(t+s)$ exists, we have simultaneously $\lim_{s\downarrow 0}(1/s)g_\lambda(t+s) \geqslant 0$ and $\lim_{s\uparrow 0}(1/s)g_\lambda(t+s) \leqslant 0$ and hence $\dot{g}_\lambda(t) = 0$. So the first inequality of the lemma is proved. The second one is a consequence of the first one and of the Gronwall inequality. □

The third lemma establishes an upper bound of the norm of the derivative of the solution $u_\lambda(\cdot)$ of (E_λ^*).

Lemma 3.3 *For almost all* $t \in [T_0, T_\lambda[$ *one has*

$$\|\dot{u}_\lambda(t)\| = (1/\lambda) g_\lambda(t) \leqslant (1/\lambda) e^{-t/\lambda} \int_{T_0}^{t} |\dot{v}(s)| e^{s/\lambda}\, ds$$

and hence

$$\|\dot{u}_\lambda(t)\| \leqslant (1/\lambda) \int_{T_0}^{t} |\dot{v}(s)|\, ds. \tag{3.4}$$

If in addition the function $\dot{v}(\cdot)$ *is in* $L^p([T_0, T])$, *with* $p \in]1, \infty]$, *then for almost all* $t \in [T_0, T_\lambda[$

$$\|\dot{u}_\lambda(t)\| \leqslant (1/\lambda)\big(\lambda/q\big)^{1/q} \|\dot{v}\|_p, \tag{3.5}$$

where $1/p + 1/q = 1$.

Proof Writing

$$g_\lambda(t) = d(u_\lambda(t), C(t)) = \|u_\lambda(t) - \operatorname{proj}_{C(t)}(u_\lambda(t))\|$$

and observing, by (E_λ^*) and Eq. 2.6, that for almost all $t \in [T_0, T_\lambda[$

$$\lambda \dot{u}_\lambda(t) = -(1/2)\nabla d^2_{C(t)}(u_\lambda(t)) = -(u_\lambda(t) - \text{proj}_{\,C(t)}(u_\lambda(t))),$$

we obtain

$$\|\dot{u}_\lambda(t)\| = (1/\lambda)\|u_\lambda(t) - \text{proj}_{\,C(t)}(u_\lambda(t))\| = (1/\lambda)g_\lambda(t).$$

Taking the inequality of Lemma 3.2 into account, we get the first inequality of the lemma. To see the second inequality, it suffices to write, for a.e $t \in [T_0, T_\lambda[$, according to the first one

$$\begin{aligned}\|\dot{u}_\lambda(t)\| &\leqslant (1/\lambda)e^{-t/\lambda}\int_{T_0}^{t} |\dot{v}(s)|e^{s/\lambda}\,ds \\ &\leqslant (1/\lambda)e^{-t/\lambda}e^{t/\lambda}\int_{T_0}^{t} |\dot{v}(s)|\,ds = (1/\lambda)\int_{T_0}^{t} |\dot{v}(s)|\,ds.\end{aligned}$$

Assume now that $\dot{v} \in L^p([0, T])$. Like above, write for a.e. $t \in [T_0, T_\lambda[$

$$\begin{aligned}\|\dot{u}_\lambda(t)\| &\leqslant (1/\lambda)e^{-t/\lambda}\|\dot{v}\|_p \left[\int_{T_0}^{t} e^{qs/\lambda}\,ds\right]^{1/q} \\ &= (1/\lambda)e^{-t/\lambda}\|\dot{v}\|_p \left[(\lambda/q)(e^{qt/\lambda} - e^{qT_0/\lambda})\right]^{1/q} \\ &\leqslant (1/\lambda)(\lambda/q)^{1/q}\|\dot{v}\|_p.\end{aligned}$$

□

The inequality (3.4) tells us that the mapping $u_\lambda(\cdot)$ is Lipschitzian on $[T_0, T_\lambda[$ with a same Lipschitz modulus $(1/\lambda \int_{T_0}^{T} |\dot{v}(s)|\,ds)$ on all $[T_0, T_\lambda[$. So (T_λ being finite), the limit $u_\lambda(T_\lambda) := \lim_{t\uparrow T_\lambda} u_\lambda(t)$ exists in H and the extended mapping $u_\lambda(\cdot)$ is Lipschitzian on $[T_0, T_\lambda]$ and $\|u_\lambda(T_\lambda) - a\| \leqslant \rho/3$.

To have a Lipschitz modulus of $u_\lambda(\cdot)$ independent of λ, the inequality (3.5) leads us to assume that $v(\cdot)$ is Lipschitzian with a constant modulus γ on all $[T_0, T]$, i.e., $\|\dot{v}(t)\| \leqslant \gamma$ for a.e. $t \in [T_0, T]$, to get that $u_\lambda(\cdot)$ is also Lipschitzian on $[T_0, T_\lambda]$ with γ as a Lipschitz modulus. Under this assumption, we may suppose that the above positive number θ also verifies $\gamma\theta < \rho/3$. Then for $u_\lambda(T_\lambda) := \lim_{t\uparrow T_\lambda} u_\lambda(t)$ obtained above we have

$$\|u_\lambda(T_\lambda) - a\| = \|u_\lambda(T_\lambda) - u_\lambda(T_0)\| \leqslant \gamma(T_\lambda - T_0) < \rho/3 \tag{3.6}$$

and hence $u_\lambda(.)$ (extended at T_λ) is a Lipschitzian solution over the closed interval $[T_0, T_\lambda]$. Further $T_\lambda = T_0 + \theta$ since otherwise Eq. 3.6 would allow us to extend $u_\lambda(\cdot)$ on the right of T_λ in a solution to the differential equation (E_λ^*) with the range of the extension of $u_\lambda(\cdot)$ included in $B(a, \rho/3)$, which would be in contradiction with the maximality of the interval $[T_0, T_\lambda[$.

Our analysis establishes that, for any real number $\lambda > 0$, the differential equation relative now to the closed interval $[T_0, T_0 + \theta]$ and denoted by

$$(E_\lambda)\begin{cases}\dot{u}(t) = -(1/2\lambda)\nabla d^2_{C(t)}(u(t)) \\ u(T_0) = a\end{cases}$$

has a unique Lipschitzian solution $u_\lambda(\cdot)$ on all the closed interval $[T_0, T_0+\theta]$ with $u_\lambda([T_0, T_0+\theta]) \subset B(a, \rho/3)$.

Lemma 3.4 *Assume that $v(\cdot)$ is Lipschitzian with modulus γ on $[T_0, T]$ and put $\alpha := 4\gamma^2$ and $\beta := 6\gamma/\rho$. Then for all positive numbers $\lambda, \mu < \rho/(2\gamma)$ one has*

$$\|u_\lambda(t) - u_\mu(t)\| \leqslant \alpha(\lambda+\mu)\int_{T_0}^{t} \exp(\beta(t-s))\, ds$$

for all $t \in [T_0, T_0+\theta]$.

Proof Fix two any positive numbers λ and μ. Observe first that for any $t \in [T_0, T_0 + \theta]$, according to Eq. 3.3, $d(u_\lambda(t), C(t)) < 2\rho/3$ since $u_\lambda(t) \in B(a, \rho/3)$. Further we know by Eq. 2.7 that for any x_1, x_2 with $d(x_i, C(t)) < 2\rho/3$ we have

$$\|\mathrm{proj}_{\,C(t)}(x_1) - \mathrm{proj}_{\,C(t)}(x_2)\| \leqslant 3\|x_1 - x_2\|. \tag{3.7}$$

Recalling (see Eq. 2.6) that

$$(1/2)\nabla d^2_{C(t)}(x) = x - \mathrm{proj}_{\,C(t)}(x) \quad \text{for } d(x, C(t)) < \rho \tag{3.8}$$

and observing, according to Eq. 3.5 with $p = \infty$ and $q = 1$, that $\|\dot{u}_\lambda(t)\| \leqslant \gamma$, we see by (E_λ) that for a.e. t $\in [T_0, T_0+\theta]$

$$-(\rho/\gamma)\dot{u}_\lambda(t) \in N_{C(t)}(\mathrm{proj}_{\,C(t)}(u_\lambda(t))) \text{ and } \|(\rho/\gamma)\dot{u}_\lambda(t)\| \leqslant \rho.$$

So according to Eq. 2.4, for a.e. $t \in [T_0, T_0+\theta]$

$$\begin{aligned}&\langle -\dot{u}_\lambda(t) + \dot{u}_\mu(t), \mathrm{proj}_{\,C(t)}(u_\lambda(t)) - \mathrm{proj}_{\,C(t)}(u_\mu(t))\rangle \\ &\quad \geqslant -(\gamma/\rho)\|\mathrm{proj}_{\,C(t)}(u_\lambda(t)) - \mathrm{proj}_{\,C(t)}(u_\mu(t))\|^2.\end{aligned}$$

As, by (E_λ) and Eq. 3.8,

$$\mathrm{proj}_{\,C(t)}(u_\lambda(t)) = u_\lambda(t) - \lambda\dot{u}_\lambda(t),$$

the last inequality above and Eq. 3.7 entail for a.e. $t \in [T_0, T_0+\theta]$

$$\langle -\dot{u}_\lambda(t) + \dot{u}_\mu(t), u_\lambda(t) - \lambda\dot{u}_\lambda(t) - u_\mu(t) + \mu\dot{u}_\mu(t)\rangle \geqslant -(3\gamma/\rho)\|u_\lambda(t) - u_\mu(t)\|^2$$

and hence for $\beta := 6\gamma/\rho$

$$\begin{aligned}&2\langle \dot{u}_\lambda(t) - \dot{u}_\mu(t), u_\lambda(t) - u_\mu(t)\rangle \\ &\quad \leqslant \beta\|u_\lambda(t) - u_\mu(t)\|^2 + 2\langle \dot{u}_\lambda(t) - \dot{u}_\mu(t), \lambda\dot{u}_\lambda(t) - \mu\dot{u}_\mu(t)\rangle.\end{aligned}$$

Putting $\alpha := 4\gamma^2$ and using the Lipschitz property with modulus γ of $u_\lambda(\cdot)$ we obtain for a.e. $t \in [T_0, T_0+\theta]$

$$d/dt(\|u_\lambda(t) - u_\mu(t)\|^2) \leqslant \alpha(\lambda+\mu) + \beta\|u_\lambda(t) - u_\mu(t)\|^2. \tag{3.9}$$

The Gronwall inequality allows us to conclude that for all $t \in [T_0, T_0 + \theta]$

$$\|u_\lambda(t) - u_\mu(t)\|^2 \leqslant \alpha(\lambda + \mu) \int_{T_0}^{t} \exp(\beta(t - s))\, ds$$

because $u_\lambda(T_0) - u_\mu(T_0) = 0$. □

Observe now that Proposition 2.1 yields that, for a closed ρ-prox-regular subset S of H, there exists some constant $\sigma > 0$ such that for any $x, x' \in U_S(2\rho/3)$ and any $\zeta \in \partial d_S(x)$

$$\langle \zeta, x' - x\rangle \leqslant d_S(x') - d_S(x) + \sigma \|x' - x\|^2. \tag{3.10}$$

Lemma 3.5 *The family $(u_\lambda(\cdot))_\lambda$ converges uniformly on $[T_0, T_0 + \theta]$ to a solution of the differential inclusion*

$$\begin{cases} -\dot{u}(t) \in N_{C(t)}(u(t)) \\ u(T_0) = a \end{cases}$$

over the interval $[T_0, T_0 + \theta]$.

Proof Lemma 3.4 ensures that the family $(u_\lambda(\cdot))_\lambda$ (for $\lambda \downarrow 0$) verifies the Cauchy property with respect to the norm of uniform convergence on the space $\mathcal{C}([T_0, T_0 + \theta], H)$ of continuous mappings from $[T_0, T_0 + \theta]$ into H. This family then converges uniformly on $[T_0, T_0 + \theta]$ to a continuous mapping $u(\cdot)$ when $\lambda \downarrow 0$.

Now observe that if S is a ρ-prox-regular subset of H then for $x \in U_S(2\rho/3)$ and $\xi = (1/2)\nabla d_S^2(x)$ with $\xi \neq 0$, we necessarily have $d_S(x) \neq 0$ (since for $x \in S$ we obviously have $0 \in \widehat{\partial}\,((1/2)d_S^2)(x))$ and hence

$$\nabla d_S(x) = \frac{1}{2\sqrt{d_S^2(x)}} \nabla(d_S^2)(x) = \frac{1}{d_S(x)}\xi.$$

For such ξ, the inequality (3.10) yields for all $x' \in U_S(2\rho/3)$

$$\langle \xi, x' - x\rangle \leqslant d_S(x) d_S(x') - d_S^2(x) + \sigma d_S(x)\|x' - x\|^2$$

and the latter inequality still holds if $\xi = 0$ because in such a case $d_S(x) = 0$.

Then for a.e. $t \in [T_0, T_0 + \theta]$ we have for all $x' \in H$ with $d_{C(t)}(x') < 2\rho/3$

$$\langle -\dot{u}_\lambda(t), x' - u_\lambda(t)\rangle \leqslant (1/\lambda) d_{C(t)}(u_\lambda(t))[d_{C(t)}(x') - d_{C(t)}(u_\lambda(t)) + \sigma\|x' - u_\lambda(t)\|^2]$$

and hence

$$\langle -\dot{u}_\lambda(t), x' - u_\lambda(t)\rangle \leqslant \gamma[d_{C(t)}(x') - d_{C(t)}(u_\lambda(t)) + \sigma\|x' - u_\lambda(t)\|^2], \tag{3.11}$$

since $(1/\lambda) d_{C(t)}(u_\lambda(t)) = \|\dot{u}_\lambda(t)\| \leqslant \gamma$, according to Lemma 3.3.

By the inequality $\|\dot{u}_\lambda(t)\| \leqslant \gamma$ for a.e. $t \in [T_0, T_0 + \theta]$, choose a sequence $\lambda_n \downarrow 0$ such that the sequence $(\dot{u}_{\lambda_n}(\cdot))_n$ converges weakly in $L_H^2([T_0, T_0 + \theta])$ to some mapping $w(\cdot)$. For each $y \in H$, writing for $T_0 \leqslant s \leqslant t \leqslant T_0 + \theta$

$$u_{\lambda_n}(t) - u_{\lambda_n}(s) = \int_s^t \dot{u}_{\lambda_n}(\tau)\, d\tau,$$

we see that

$$\langle u_{\lambda_n}(t) - u_{\lambda_n}(s), y\rangle = \int_{T_0}^{T_0+\theta} \langle \dot{u}_{\lambda_n}(\tau), y\mathbb{1}_{[s,t]}(\tau)\rangle \, d\tau,$$

which easily yields

$$u(t) - u(s) = \int_s^t w(\tau)\, d\tau,$$

that is, $u(\cdot)$ is absolute continuous and $\dot{u}(\cdot) = w(\cdot)$ a.e..

Further, by Mazur Lemma, some sequence of convex combinations of the form $\left(\sum_{k=n}^{m(n)} r_{k,n} u_{\lambda_k}\right)_n$ (with $r_{k,n} \geqslant 0$ and $\sum_{k=n}^{m(n)} r_{k,n} = 1$) converges strongly in $L^2_H([T_0, T_0 + \theta])$ to $\dot{u}(\cdot)$. Extracting a subsequence if necessary, we may suppose that the above sequence of convex combinations converges a.e. to $\dot{u}(\cdot)$, that is, pointwise on some set $A \subset [T_0, T_0 + \theta]$ such that $[T_0, T_0 + \theta] \setminus A$ has Lebesgue measure 0. Fix any $t \in A$ and any $x' \in H$ with $d_{C(t)}(x') < 2\rho/3$. We observe that

$$\left| \sum_{k=n}^{m(n)} r_{k,n} \langle \dot{u}_{\lambda_k}(t), u(t) - u_{\lambda_k}(t)\rangle \right| \leqslant \gamma \sum_{k=n}^{m(n)} r_{k,n} \|u(t) - u_{\lambda_k}(t)\|$$

and hence

$$\sum_{k=n}^{m(n)} r_{k,n} \langle \dot{u}_{\lambda_k}(t), u(t) - u_{\lambda_k}(t)\rangle \underset{n\to\infty}{\longrightarrow} 0 \tag{3.12}$$

since it is easily seen that $\sum_{k=n}^{m(n)} r_{k,n} \|u(t) - u_{\lambda_k}(t)\| \underset{n\to\infty}{\longrightarrow} 0$ because $\|u(t) - u_{\lambda_n}(t)\| \underset{n\to\infty}{\longrightarrow} 0$ and $\sum_{k=n}^{m(n)} r_{k,n} = 1$. The convergence (3.12) and the equality

$$\begin{aligned}&\sum_{k=n}^{m(n)} r_{k,n} \langle \dot{u}_{\lambda_k}(t), x' - u_{\lambda_k}(t)\rangle \\ &\quad = \left\langle \sum_{k=n}^{m(n)} r_{k,n} \dot{u}_{\lambda_k}(t), x' - u(t) \right\rangle + \sum_{k=n}^{m(n)} r_{k,n} \langle \dot{u}_{\lambda_k}(t), u(t) - u_{\lambda_k}(t)\rangle\end{aligned}$$

give

$$\sum_{k=n}^{m(n)} r_{k,n} \langle \dot{u}_{\lambda_k}(t), x' - u_{\lambda_k}(t)\rangle \underset{n\to\infty}{\longrightarrow} \langle \dot{u}(t), x' - u(t)\rangle. \tag{3.13}$$

Writing by Eq. 3.11

$$\begin{aligned}&\sum_{k=n}^{m(n)} r_{k,n} \langle -\dot{u}_{\lambda_k}(t), x' - u_{\lambda_k}(t)\rangle \\ &\quad \leqslant \gamma \left[d_{C(t)}(x') - \sum_{k=n}^{m(n)} r_{k,n} d_{C(t)}(u_{\lambda_k}(t)) + \sigma \sum_{k=n}^{m(n)} r_{k,n} \|x' - u_{\lambda_k}(t)\|^2 \right],\end{aligned}$$

and passing to the limit with $n \to \infty$ we obtain through Eq. 3.13

$$\langle -\dot{u}(t), x' - u(t)\rangle \leqslant \gamma[d_{C(t)}(x') - d_{C(t)}(u(t)) + \sigma\|x' - u(t)\|^2]. \tag{3.14}$$

Recalling (see Lemma 3.3) that $d_{C(t)}(u_\lambda(t)) = \lambda|\dot{u}_\lambda(t)| \leqslant \lambda\gamma$ we see that $d_{C(t)}(u_{\lambda_n}(t)) \to 0$ when $n \to \infty$ and hence $d_{C(t)}(u(t)) = 0$, that is, $u(t) \in C(t)$. Taking any $x' \in C(t)$ (i.e. $d_{C(t)}(x') = 0$) in Eq. 3.14 we see that

$$\langle -\dot{u}(t), x' - u(t)\rangle \leqslant \gamma\sigma\|x' - u(t)\|^2,$$

which gives

$$-\dot{u}(t) \in \widehat{N}_{C(t)}(u(t)) \subset N_{C(t)}(u(t)).$$

Finally the uniform convergence of $(u_\lambda)_\lambda$ to u implies $u(T_0) = a$. The proof of the lemma is then complete. □

The existence result over all the interval $[T_0, T]$ can be obtained through the existence of the truncated interval above under the Lipschitzian behavior of the variation function $v(\cdot)$.

Lemma 3.6 *Assume that the moving set $C(t)$ moves in a Lipschitzian way over the interval $[T_0, T]$, that is, the above function $v(\cdot)$ is Lipschitzian with some constant $\gamma \geqslant 0$ as a Lipschitz modulus over $[T_0, T]$. Then the differential inclusion (E) has a unique solution over all the interval $[T_0, T]$.*

Proof The uniqueness has been already discussed and proved in [15]. Fix an integer $k \in \mathbb{N}$ such that $(T - T_0)/k < \rho/(3\gamma)$. Without loss of generality we may then take for the positive real number θ that has been fixed above with $\theta < \rho/(3\gamma)$ the real number $(T - T_0)/k$, i.e., $\theta = (T - T_0)/k$. Put $T_i = T_0 + i\theta$ for $i = 0, 1, \cdots, k$. Lemma 3.5 provides a Lipschitzian solution u_1 (with γ as Lipschitz modulus) on the interval $[T_0, T_1]$ of the differential inclusion

$$\begin{cases} -\dot{u}(t) \in N_{C(t)}(u(t)) \\ u(T_0) = a. \end{cases}$$

As $u_1(T_1) \in C(T_1)$ we may apply again Lemma 3.5 with T_1 in place of T_0 and with $u_1(T_1)$ as initial condition to obtain a Lipschitzian solution (with γ as Lipschitz modulus) over the second closed interval $[T_1, T_2]$ for the differential inclusion

$$\begin{cases} -\dot{u}(t) \in N_{C(t)}(u(t)) \\ u(T_1) = u_1(T_1). \end{cases}$$

We may proceed in this way up to the last closed interval $[T_{k-1}, T_k]$. Defining the mapping $u(\cdot)$ on all the interval $[T_0, T]$ by putting $u(t) := u_i(t)$ for any $t \in [T_{i-1}, T_i]$, with $i = 1, \cdots, k$, it is easily seen that $u(\cdot)$ provides a Lipschitzian (with γ as Lipschitz modulus) solution over $[T_0, T]$ of the differential inclusion (E). □

The last step is the reduction of the absolutely continuous case to the Lipschitzian one. So, we suppose now that the function $v(\cdot)$ is absolutely continuous on $[T_0, T]$.

Put $w(t) := \int_{T_0}^{t} |\dot{v}(r)|\,dr$ for each $t \in [T_0, T]$. The function $w(\cdot)$ is non negative, $w(T_0) = 0$, and for all $x \in H$ and $t_1, t_2 \in [T_0, T]$ with $t_1 \leqslant t_2$

$$|d(x, C(t_2)) - d(x, C(t_1))| \leqslant \int_{t_1}^{t_2} |\dot{v}(r)|\,dr = w(t_2) - w(t_1). \tag{3.15}$$

We follow now the method of reduction of Moreau [23]. Observe that Eq. 3.15 entails that when $w(t_1) = w(t_2)$ we have $C(t_1) = C(t_2)$. This allows us to define a set-valued mapping $D : [0, w(T_0)] \rightrightarrows H$ by putting

$$D(s) = C(t) \quad \text{for } w(t) = s$$

and Eq. 3.15 gives for all $x \in H$ and $s_1, s_2 \in [0, w(T)]$ with $s_i = w(t_i)$

$$\begin{aligned} |d(x, D(s_2)) - d(x, D(s_1))| &= |d(x, C(t_2)) - d(x, C(t_1))| \\ &\leqslant |w(t_2) - w(t_1)| = |s_2 - s_1|, \end{aligned}$$

that is, the closed ρ-prox-regular set $D(s)$ moves in a Lipschitzian way with 1 as Lipschitz modulus. Lemma 3.6 says that the differential inclusion

$$(I_D) \begin{cases} \dot{z}(s) \in -N_{D(s)}(z(s)) \\ z(0) = a \end{cases}$$

has a Lipschitzian solution $z(\cdot)$ on $[0, w(T)]$ with 1 as Lipschitz modulus. Put $u(t) := z(w(t))$ for all $t \in [T_0, T]$. As

$$u(t_2) - u(t_1) = \int_{t_1}^{t_2} |\dot{v}(r)|\dot{z}(w(r))\,dr$$

for all $t_1, t_2 \in [T_0, T]$ with $t_1 < t_2$, it is easily seen through (I_D) and the cone property of $N_{D(s)}(\cdot)$ that $u(\cdot)$ is a solution of (E) on $[T_0, T]$ and that $\|\dot{u}(t)\| \leqslant |\dot{v}(t)|$ for a.e. $t \in [T_0, T]$.

This completes the proof of Theorem 2.1.

References

1. Aussel, D., Daniilidis, A., Thibault, L.: Subsmooth sets: functional characterizations and related concepts. Trans. Amer. Math. Soc. **4**, 1275–1301 (2005)
2. Benabdellah, H.: Existence of solutions to the nonconvex sweeping process. J. Differential Equations **164**, 286–295 (2000)
3. Bernard, F., Thibault, L., Zlateva, N.: Prox-regular sets and epigraphs in uniformly convex Banach spaces: normal regularity and other properties (to appear)
4. Bounkhel, M., Thibault, L.: Nonconvex sweeping process and prox-regularity in Hilbert space. J. Nonlinear Convex Anal. **6**, 359–374 (2001)
5. Brogliato, B., Daniliidis, A., Lemaréchal, C., Acary, V.: On the equivalence between complementarity systems, projected systems and differential inclusions. Syst. Control. Lett. **6**, 45–51 (2005)
6. Canino, A.: p-Convex sets and geodesics. J. Differential Equations **75**, 118–157 (1988)
7. Castaing, C.: Version aléatoire du problème de rafle par un convexe, Séminiare d'Analyse Convexe de Montpellier, Exposé 1 (1974)
8. Castaing, C., Duc Ha, T.X., Valadier, M.: Evolution equations governed by the sweeping process. Set-Valued Anal. **1**, 109–139 (1993)
9. Castaing, C., Valadier, M.: Convex analysis and measurable multifunctions. Lecture Notes in Mathematics, vol. 580. Springer Berlin (1977)

10. Clarke, F.H., Stern, R.J., Wolenski, P.R.: Proximal smoothness and the lower C^2 property. J. Convex Anal. **2**, 117–144 (1995)
11. Colombo, G., Goncharov, V.V.: The sweeping process without convexity. Set-Valued Anal. **7**, 357–374 (1999)
12. Colombo, G., Goncharov, V.V.: Variational inequalities and regularity properties of closed sets in Hilbert spaces. J. Convex Anal. **8**, 197–221 (2001)
13. Correa, R., Jofre, A., Thibault, L.: Subdifferential monotonicity as characterization of convexity. Numer. Funct. Anal. Optim. **15**, 531–536 (1994)
14. Cornet, B.: Existence of slow solutions for a class of differential inclusions. J. Math. Anal. Appl. **96**, 130–147 (1983)
15. Edmond, J.F., Thibault, L.: Relaxation of an optimal control problem involving a perturbed sweeping process. Math. Programming **104**, 347–373 (2005)
16. Edmond, J.F., Thibault, L.: BV solutions of nonconvex sweeping process differential inclusions with perturbation. J. Differential Equations **226**, 135–179 (2006)
17. Federer, H.: Curvature measures. Trans. Amer. Math. Soc. **93**, 418–491 (1959)
18. Henry, C.: An existence theorem for a class of differential equations with multivalued right-hand side. J. Math. Anal. Appl. **41**, 179–186 (1973)
19. Lozano, R., Brogliato, B., Egeland, O., Maschke, B.M.: Dissipative Systems Analysis and Control. Springer London, CCE Series (2000)
20. Marcellin, S., Thibault, L.: Evolution problems associated with primal lower nice functions. J. Convex Anal. **13**, 385–421 (2006)
21. Maury, B., Venel, J.: Un modèle de mouvement de foule. ESAIM Proc. **18**, 143–152 (2007)
22. Mordukhovich, B.S.: Variational analysis and generalized differentiation I and II. Comprehensive Studies in Mathematics, vol. 330 and 331. Springer (2005)
23. Moreau, J.J.: Rafle par un convexe variable I, Séminaire d'Analyse Convexe de Montpellier, Exposé 15 (1971)
24. Moreau, J.J.: Rafle par un convexe variable II, Séminaire d'Analyse Convexe de Montpellier, Exposé 3 (1972)
25. Moreau, J.J.: Evolution problem associated with a moving convex set in a Hilbert space. J. Differential Equations **26**, 347–374 (1977)
26. Monteiro Marques, M.D.P.: Differential Inclusions in Nonsmooth Mechanical Problems, Shocks and Dry Friction. Birkhauser (1993)
27. Poliquin, R.A., Rockafellar, R.T., Thibault, L.: Local differentiability of distance functions. Trans. Amer. Math. Soc. **352**, 5231–5249 (2000)
28. Rockafellar, R.T., Wets, R.J.B.: Variational Analysis. Comprehensive Studies in Mathematics, vol. 317. Springer (1998)
29. Thibault, L.: Sweeping process with regular and nonregular sets. J. Differential Equations **193**, 1–26 (2003)
30. Valadier, M.: Quelques problèmes d'entrainement unilatéral en dimension finie, Séminaire d'Analyse Convexe de Montpellier, Exposé 17 (1992)