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Relaxation Results for Hybrid Inclusions

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Abstract The Filippov–Ważewski relaxation theorem describes when the set of solutions to a differential inclusion is dense in the set of solutions to the relaxed (convexified) differential inclusion. This paper establishes relaxation results for a broad range of hybrid systems which combine differential inclusions, difference inclusions, and constraints on the continuous and discrete motions induced by these inclusions. The relaxation results are used to deduce continuous dependence on initial conditions of the sets of solutions to hybrid systems.

Keywords Relaxation · Hybrid systems · Differential inclusions · Difference inclusions · Constraints · Continuous dependence on initial conditions

Mathematics Subject Classifications (2000) 37N35 · 34A60 · 34A37 · 39A99 · 54C60

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1 Introduction

The Filippov–Ważewski relaxation theorem describes when solutions to a differential inclusion

$$\dot{z}(t) \in F(z(t)) \text{ for almost all } t \in [0, T], \tag{1}$$

form a dense subset, in the uniform metric, of the set of solutions to a relaxed differential inclusion

$$\dot{z}(t) \in \text{con}\, F(z(t)) \text{ for almost all } t \in [0, T]. \tag{2}$$

In Eq. 2, con $F(z)$ stands for the closed convex hull of $F(z)$, and the key assumption of the relaxation result is that F be Lipschitz continuous (in the sense of set-valued mappings). The goal of this paper is to extend the relaxation result to the setting of hybrid inclusions.

Hybrid inclusions are a convenient framework for modeling and analysis of hybrid dynamical systems. Such systems combine continuous evolution (flow) of their states with discontinuous evolution (jumps), and are abundant in applications. In particular, they result from applying hybrid feedback to (nonhybrid) control systems, as required by the pursuit of robustness of feedback to measurement error; see, for example, [11, 21, 27, 28, 32].

Following [15, 16], we work with hybrid inclusions that can be symbolically written as

$$\mathcal{H}: \qquad \begin{cases} \dot{x} \in F(x) & x \in C \\ x^+ \in G(x) & x \in D \end{cases}. \tag{3}$$

That is, the state x can flow according to the *flow map* F when it is in the *flow set* C, and it can also jump according to the *jump map* G from the *jump set* D. A precise definition of a solution to $\mathcal{H}$ is stated in Section 2. We note that closely related models of hybrid inclusions or "hybrid automata" can be found, for example, in [4, 6, 7, 12, 25]. See also Section 7.

By a relaxed hybrid inclusion we understand

$$\mathcal{H}^{\text{con}}: \qquad \begin{cases} \dot{x} \in \text{con}\, F(x) & x \in C \\ x^+ \in G(x) & x \in D \end{cases}. \tag{4}$$

That is, the relaxed hybrid inclusion $\mathcal{H}^{\text{con}}$ is obtained from the original $\mathcal{H}$ by convexification of the flow map F.[1] We are interested in describing when solutions to $\mathcal{H}$ are dense in the set of solutions to $\mathcal{H}^{\text{con}}$, with respect to graphical distance. Such

[1] Convexification of F represents the consideration of generalized solutions to the differential inclusion a la Krasovskii; see [24]. Such generalized solutions reflect the effect of vanishing perturbations on the inclusion, and as such, agree with generalized solutions a la Hermes; see [18] and [17]. Corresponding notions of generalized solutions to difference equations or inclusions do not lead to convexification of the right-hand side, and thus, convexification of G is not considered; see [23] (the same comments apply to generalized solutions for hybrid systems; see [30]).

a distance notion measures the distance between solutions of hybrid systems as the distance between their graphs, appears as a natural and appropriate notion to use for hybrid systems (see [15, 16, 25, 30] for some of its applications), and is closely related to Skorokhod topology applied to solutions to hybrid systems (as used, for example, by [7, 12]). In particular, solutions to two hybrid systems can be close to one another, in the graphical distance, even if their domains are not the same; this is particularly important in the hybrid setting (see Example 4.9).

One of the motivations for the pursuit of hybrid relaxation results is the analysis of stability properties for hybrid systems, including hybrid systems resulting from application of hybrid feedback algorithms to (classical) nonlinear control systems. For example, the classical Filippov–Ważewski relaxation theorem was the key component in deriving Lyapunov equivalent characterizations of input-to-state stability (see [33, Theorem 1]), a concept that has proven valuable in studying robustness of asymptotic stability in nonlinear control systems (see [31] and references therein). Studying this concept in the hybrid setting yields a natural motivation for hybrid relaxation results.

Obviously, a hybrid relaxation theorem will require the flow map F to be Lipschitz continuous. However, this will be far from sufficient: the geometry of the flow and jump sets, viability properties of the flow set under the flow map, and (semi)continuity of the jump map G will all be relevant. In particular, conditions involving the flow map and tangent cones to the flow set and the jump set will be required (similar conditions have been used in the analysis of capture basins and periodic solutions to hybrid systems in [4, 5]).

A special case of the hybrid system (3) and its relaxed version (4) is the constrained differential inclusion $\dot{x} \in F(x)$, $x \in C$ and its relaxed version $\dot{x} \in \operatorname{con} F(x)$, $x \in C$. Sufficient conditions for relaxation in such a case do exist in the literature; see [14] and the references therein. The conditions in [14] exclude the existence of points on the boundary of C at which $F(x)$ is outward pointing. As such points often exist in hybrid systems, we propose conditions for hybrid relaxation that in the special case of constrained inclusions yield an alternative to the conditions of [14].

The relaxation theorem will also lead to results on continuous dependence of the solution sets to hybrid inclusions on initial conditions (these are new even for the case of the flow and the jump maps being functions). Mild assumptions on the regularity of the data of $\mathcal{H}$ ensure that, for the relaxed hybrid inclusion $\mathcal{H}^{\text{con}}$, this dependence is outer (or upper) semicontinuous: the limit of a graphically convergent sequence of solutions to $\mathcal{H}^{\text{con}}$ is a solution to $\mathcal{H}^{\text{con}}$; see [16] (the said mild assumptions are in part motivated by accounting for the effects of measurement error in a general hybrid control system; see [30]). The relaxation results, thanks to the fact that solutions to $\mathcal{H}$ are solutions to $\mathcal{H}^{\text{con}}$, will imply under stronger assumptions that each solution to $\mathcal{H}^{\text{con}}$ with initial point ξ can be approximated by a solution to $\mathcal{H}^{\text{con}}$ from any initial point close enough to ξ. This is exactly inner (or lower) semicontinuous dependence of solution sets to $\mathcal{H}^{\text{con}}$ on initial conditions, which, combined with the generically present outer semicontinuous dependence, leads to continuous dependence. Related results on continuous dependence appeared, under quite restrictive conditions, in [1, 25, 34], and, in a setting more related to the current one, in [7] (still, [7] excluded multiple jumps at an instant and the flows were unconstrained).

A preliminary report on the results of this paper appeared as a conference note [8].

2 Preliminaries

2.1 Hybrid Systems

Following [16], we now make the concept of a solution to the hybrid inclusion $\mathcal{H}$ in Eq. 3 precise (similar concepts of a solution, and the use of generalized time domains, can be also found, for example, in [4, 6, 12]). A subset $S \subset \mathbb{R}_{\geqslant 0} \times \mathbb{N}$ is a *compact hybrid time domain* if $S = \bigcup_{j=0}^{J-1} \left([t_j, t_{j+1}], j\right)$ for some finite sequence of times $0 = t_0 \leqslant t_1 \leqslant t_2 \ldots \leqslant t_J$ (here, $\mathbb{R}_{\geqslant 0} = [0, \infty)$ and $\mathbb{N} = \{0, 1, 2, \ldots\}$). The set S is a *hybrid time domain* if for all $(T, J) \in S$, $S \cap ([0, T] \times \{0, 1, \ldots J\})$ is a compact hybrid time domain; equivalently, if S is a union of a finite or infinite sequence of intervals $[t_j, t_{j+1}] \times \{j\}$, with the last interval, if it exists, possibly of the form $[t_j, T)$ with T finite or $T = +\infty$. For any hybrid time domain S, $\sup_t S = \sup\{t \mid \exists j \text{ with } (t, j) \in S\}$, $\sup_j S = \sup\{j \mid \exists t \text{ with } (t, j) \in S\}$, $\sup S = (\sup_t S, \sup_j S)$, and length $S = \sup_t S + \sup_j S$. For a compact hybrid time domain, the suprema are in fact maxima.

A function $x : S \to \mathbb{R}^n$ is a *hybrid arc* if S is a hybrid time domain and $t \mapsto x(t, j)$ is locally absolutely continuous for each $j \in \mathbb{N}$. Given a hybrid arc x, we will write $\operatorname{dom} x$ for the domain of x.

As suggested by Eq. 3, the data of the hybrid inclusion includes the flow map F, the flow set C, the jump map G, the jump set D, and will also include the state space $\mathcal{O}$. Technical assumptions on the data will be given when needed; the general assumptions, in force throughout the paper, are as follows.

Standing Assumption

- The set $\mathcal{O} \subset \mathbb{R}^n$ is open;
- The sets C and D are subsets of $\mathcal{O}$;
- $F : \mathcal{O} \rightrightarrows \mathbb{R}^n$ is a set-valued mapping with nonempty and compact values;
- $G : \mathcal{O} \rightrightarrows \mathbb{R}^n$ is a set-valued mapping.

A hybrid arc $x : \operatorname{dom} x \to \mathcal{O}$ is a *solution to the hybrid system* $\mathcal{H}$ if $x(0, 0) \in C \cup D$ and:

(S1) For all $j \in \mathbb{N}$ and almost all t such that $(t, j) \in \operatorname{dom} x$,

$$x(t, j) \in C, \qquad \dot{x}(t, j) \in F(x(t, j)); \tag{5}$$

(S2) For all $(t, j) \in \operatorname{dom} x$ such that $(t, j+1) \in \operatorname{dom} x$,

$$x(t, j) \in D, \qquad x(t, j+1) \in G(x(t, j)). \tag{6}$$

A solution is called *compact* if its graph is compact (equivalently, if its domain is compact) and *complete* if its domain is unbounded.

2.2 Graph Distance

Given a closed unit ball $\mathbb{B} \subset \mathbb{R}^k$ in some norm and two sets $S_1, S_2 \subset \mathbb{R}^k$, the *Pompeiu–Hausdorff distance* between S_1 and S_2 is

$$d(S_1, S_2) = \inf\{\eta \geqslant 0 \mid S_1 \subset S_2 + \eta\mathbb{B}, S_2 \subset S_1 + \eta\mathbb{B}\}.$$

If, given a sequence of sets S_i, we have $d(S_i, S) \to 0$ as $i \to \infty$ for some set S, then the sets S_i converge to S (for sequential definitions of set convergence, see

[29, Chapter 4]). The converse is not true, unless all S_i's and S are contained in some bounded set; see [29, Example 4.13]. Given two (set-valued or not) mappings $M_1, M_2 : \mathbb{R}^l \rightrightarrows \mathbb{R}^m$, the *graphical distance* between M_1 and M_2 is

$$\mathrm{d_{gph}}(M_1, M_2) = d(\mathrm{gph}\, M_1, \mathrm{gph}\, M_2),$$

where $d(\cdot, \cdot)$ is the Pompeiu–Hausdorff distance in $\mathbb{R}^l \times \mathbb{R}^m$.

We will be particularly interested in set-valued mappings from $\mathbb{R}^2$ to $\mathbb{R}^n$ (hybrid arcs can be understood in that way). For convenience, we will consider $\mathbb{B} \subset \mathbb{R}^{n+2}$ given by $I\!B^1 \times I\!B^1 \times I\!B^n$, where $I\!B^1$ and $I\!B^n$ are Euclidean balls in $\mathbb{R}$ and $\mathbb{R}^n$ (we will usually skip the superscript, and write $I\!B$ for the Euclidean ball in the appropriate, from the context, space). In other words, $\mathbb{B}$ is the unit ball in the norm $\max\{|\alpha|, |\beta|, \|\gamma\|\}$ for $\alpha, \beta \in \mathbb{R}$, $\gamma \in \mathbb{R}^n$. Here and in what follows, $\|\cdot\|$ is the Euclidean norm in the appropriate space (usually $\mathbb{R}^n$).

Consider the condition

$$|t - s| \leqslant \varepsilon, \quad \|x(t, j) - y(s, j)\| \leqslant \varepsilon. \tag{7}$$

Given two hybrid arcs $x : \mathrm{dom}\, x \to \mathbb{R}^n$, $y : \mathrm{dom}\, y \to \mathbb{R}^n$ and $\varepsilon < 1$, $\mathrm{d_{gph}}(x, y) \leqslant \varepsilon$ if and only if

- For each $(t, j) \in \mathrm{dom}\, x$ there exists $(s, j) \in \mathrm{dom}\, y$ such that Eq. 7 holds, and
- For each $(s, j) \in \mathrm{dom}\, y$ there exists $(t, j) \in \mathrm{dom}\, x$ such that Eq. 7 holds.

(If $\varepsilon \geqslant 1$ is considered, $\mathrm{d_{gph}}(x, y) \leqslant \varepsilon$ translates to inequalities involving $x(t, j)$ and $y(s, j')$ with $j \neq j'$; in contrast to the inequality in Eq. 7.)

2.3 Filippov–Ważewski Theorem

The version of the Filippov–Ważewski relaxation theorem given below is a direct combination of [13, Theorem 1] and [3, Chapter 2, Section 4, Theorem 2] (cf. [13, Theorem 3]). A set-valued mapping $\Gamma : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is *locally Lipschitz* continuous if for each $x \in \mathbb{R}^n$ there exists a neighborhood U of x and a constant $L > 0$ such that for each $x', x'' \in U$, $\Gamma(x') \subset \Gamma(x'') + L\|x' - x''\| I\!B$.

Theorem 2.1 *Let $\Gamma : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be locally Lipschitz continuous and have nonempty and compact values. For any absolutely continuous $z : [0, T] \to \mathbb{R}^n$ such that $\dot{z}(t) \in \mathrm{con}\, \Gamma(z(t))$ for almost all $t \in [0, T]$ and any $\varepsilon > 0$ there exists $\delta > 0$ such that, for any $y_0 \in z(0) + \delta I\!B$, there exists an absolutely continuous $y : [0, T] \to \mathbb{R}^n$ such that $y(0) = y_0$, $\dot{y}(t) \in \Gamma(y(t))$ and $\|z(t) - y(t)\| \leqslant \varepsilon$ for almost all $t \in [0, T]$.*

The conclusion of the theorem could in fact be termed "strong relaxation", to underline that appropriate solutions to the inclusion $\dot{y}(t) \in \Gamma(y(t))$ exist from all initial points sufficiently close to $z(0)$, rather than just from some initial point. Such distinction will become particularly important in the hybrid setting.

To see how the conclusions of Theorem 2.1 fail in absence of local Lipschitz continuity of Γ, it is enough to consider the case of $\Gamma : \mathbb{R} \to \mathbb{R}$ being a function given by $\Gamma(z) = \sqrt{|z|}$. Local Lipschitz continuity is, in general, necessary even if the conclusions of the theorem above are weakened to say that for any ε there exists

$y\colon [0, T] \to \mathbb{R}^n$ such that $\dot{y}(t) \in \Gamma(y(t))$ and $\|z(t) - y(t)\| \leqslant \varepsilon$ for almost all $t \in [0, T]$, without restricting the initial value $y(0)$. An example to this effect was given by [26]; see also the example in [3, Chapter 2, Section 4].

3 General Hybrid Relaxation Results

We begin by describing some possible failures of relaxation for the case of hybrid systems. These are due to the relationship between the flow map F, the flow set C, and the jump set D rather than the lack of regularity in F or in the jump map G. Of course, continuity of G has bearing on relaxation; this is addressed in some detail in Remarks 3.5 and 6.5.

(a) The presence of the constraint $z \in C$ during flows may prohibit the very existence of any solutions to $\dot{z} \in F(z)$ even if solutions to the relaxed inclusion $\dot{z} \in \operatorname{con} F(z)$ satisfying the constraint do exist. For example, in $\mathbb{R}$ let $C = \{0\}$, $F(\xi) = \{-1, 1\}$ for all $\xi \in \mathbb{R}$. The relaxed inclusion $\dot{z} \in \operatorname{con} F(z)$ has a unique constant solution that remains in C but this solution does not satisfy $\dot{z} \in F(z)$. It can also happen that solutions to the inclusion $\dot{z} \in F(z)$ exist and remain in C but only on time intervals far shorter than the solutions to the relaxed inclusion.

(b) Solutions to the relaxed inclusion $\dot{x} \in \operatorname{con} F(x)$ can graze D, making a jump possible, even when all nearby solutions to $\dot{x} \in F(x)$ flow by D. For example, in $\mathbb{R}^3$, let

$$C = \mathbb{R}^3, \quad D = \left\{\xi \in \mathbb{R}^3 \mid \xi_1 \geqslant 1, \xi_2 \leqslant 0\right\}, \quad F(\xi) = \left(1, \xi_3^2, \{-1, 1\}\right),$$

with $(0, 0, 0) \in G(1, 0, 0)$ and otherwise arbitrary G. For the initial condition $\xi^0 = (0, 0, 0)$, the relaxed hybrid system in Eq. 4 has a "periodic" solution that flows from ξ^0 to $(1, 0, 0)$, jumps back to ξ^0, flows again to $(1, 0, 0)$, etc. Any solution to the non-relaxed system (3) is strictly increasing in the second coordinate during flow. Thus, only solutions x with $x_2(0, 0) < 0$ can satisfy $x_2(t, 0) \in D$ for some $t > 0$. If such a solution then jumps, $x_2(t, 1) = 0$, consequently $x_2(t', 1) > 0$ for all $t' > t$, and the solution never intersects D again. In short, solutions to Eq. 4 jump at most once, and in particular, no solutions to Eq. 3 are close (in the graphical distance) to the mentioned periodic solution to the relaxed system (4).

(c) Solutions to the relaxed inclusion $\dot{x} \in \operatorname{con} F(x)$ hit the boundary of C, cannot flow further while remaining in C, but also hit D and lead to a solution to the relaxed hybrid system in Eq. 4 that then jumps, even when nearby solutions to $\dot{x} \in F(x)$ also flow out of C but do not hit D. For example, alter the system in (b) above to have $C = \{\xi \in \mathbb{R}^3 \mid \xi_1 \leqslant 1\}$. The "periodic" solution to Eq. 4 mentioned in (b) still exists, while all solutions to Eq. 3 cease to exist in time about 1 without ever jumping, or in time about 2 after jumping once.

Related examples can be found in [8] (see also Examples 5.3 and 5.4 for failures of "infinite time horizon" relaxation).

We will say that a hybrid arc x *initially flows* if for some $\varepsilon > 0$, $[0, \varepsilon] \times \{0\} \subset \operatorname{dom} x$, and *initially jumps* if $(1, 0) \in \operatorname{dom} x$.

Definition 3.1 Given $x_0 \in C \cup D$, strong relaxation for initially flowing (respectively, initially jumping) solutions from x_0 relative to C (respectively, relative to D) is possible if

> for any compact solution $x\colon \operatorname{dom} x \to \mathbb{R}^n$ to $\mathcal{H}^{\mathrm{con}}$ with $x(0,0) = x_0$ that initially flows (respectively, that initially jumps) and for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $y_0 \in (x(0,0) + \delta\mathbb{B}) \cap C$ (respectively, for any $y_0 \in (x(0,0) + \delta\mathbb{B}) \cap D$) there exist a hybrid arc $y\colon \operatorname{dom} y \to \mathbb{R}^n$ with compact $\operatorname{dom} y$ and $y(0,0) = y_0$ that is a solution to $\mathcal{H}$ and $d_{\mathrm{gph}}(x, y) \leqslant \varepsilon$, and moreover, if $x(T, J) \in D$, where $(T, J) = \max \operatorname{dom} x$, then $y(\tau, J) \in D$, where $(\tau, J) = \max \operatorname{dom} y$.

A natural example of systems where the strong relaxation for initially flowing or initially jumping solutions is possible provided by systems in which the flow dynamics governed by a nonconvex inclusion are in a sense separated from the dynamics and jump maps governing the jumps (for such systems, one could abandon graphical distance and rely on uniform distance, as the hybrid domains of a solution to $\mathcal{H}^{\mathrm{con}}$ and of a "nearby" solution to $\mathcal{H}$ turn out the same, and rely on the Filippov–Ważewski result, Theorem 2.1 to deduce relaxation. An illustration of when the domains are necessarily different and the uniform distance is not adequate will be given in Example 4.9). Such hybrid systems arise in hybrid modeling of so-called switching systems. We give some details in the following example.

Example 3.2 Let $\mathcal{O}_1 \in \mathbb{R}^{n_1}$, $\mathcal{O}_2 \in \mathbb{R}^{n_2}$ be open sets; $\hat{q}$ be a positive integer; $Q = \{1, 2, \ldots, \hat{q}\}$; let $F_q\colon \mathcal{O}_1 \rightrightarrows \mathbb{R}^{n_1}$, $q \in Q$, and $F_0\colon \mathcal{O}_2 \rightrightarrows \mathbb{R}^{n_2}$ be set-valued mappings with nonempty and compact values, with the values of F_0 also being convex; C^y, D^y be subsets of $\mathcal{O}_2$; and finally, $G_0\colon \mathcal{O}_2 \rightrightarrows \mathbb{R}^{n_2}$ be a set-valued mapping. Consider a hybrid system

$$\begin{cases} \dot{x} \in F_q(x),\ \ \dot{y} \in F_0(y), & y \in C^y \\ q^+ \in Q,\ \ y^+ = G_0(y), & y \in D^y \end{cases} \tag{8}$$

on the state space $\mathcal{O}_1 \times \mathcal{O}_2 \times \mathbb{R}$ and with the variable (x, y, q). The "discrete" variable q remains constant during flows, the variable x remains constant during jumps, and the variable determining the times of jumps, y, may change during both flows and jumps.

Such a system can be cast in the form Eq. 3. Indeed, one considers the flow map $(F_q(x), F_0(y), 0)$, the flow set $\mathcal{O} \times C^y \times Q$, the jump map $(x, G_0(y), Q)$, and the jump set $\mathcal{O} \times D^y \times Q$.

Suppose that, for each $q \in Q$, continuous time relaxation is possible for $\dot{x} \in F_q(x)$ (in the sense that the conclusions of Theorem 2.1 hold), as is guaranteed if F_q is locally Lipschitz. It is straightforward to verify that strong relaxation for initially flowing (respectively, initially jumping) solutions relative to C (respectively, relative to D) is possible, for any initial point.

A special case of Eq. 8 is provided by the system

$$\begin{cases} \dot{x} \in F_q(x),\ \ \dot{\tau} = 1, & \tau \in C^\tau \\ q^+ \in Q,\ \ \tau^+ = 0, & \tau \in D^\tau \end{cases} \tag{9}$$

where the variable is the triple (x, τ, q) and τ is a real-valued "timer" variable. With $C^\tau = [0, T]$ and $D^\tau = \{T\}$ for some $T > 0$, the solutions to this hybrid system with $\tau(0, 0) = 0$ correspond to switching between vector fields F_q every T units of time. When $C^\tau = [0, \infty)$ and $D^\tau = [T, \infty)$, the times in between switches are at least T. Such a case is sometimes referred to as "dwell-time switching with dwell-time T"; see, for example, [19] for details on these, and other kinds of "switching signals". Strong relaxation for initially flowing (respectively, initially jumping) solutions relative to C (respectively, relative to D) is then possible for Eq. 9 as long as, for each $q \in Q$, continuous time relaxation is possible for $\dot{x} \in F_q(x)$, in the sense of Theorem 2.1 (further special cases of Eq. 8, with a slightly more involved "timer" dynamics, also model average dwell-time and reverse average dwell-time switching signals; see [20]. The conclusion about relaxation still applies to those cases).

The same conclusion can be made for Eq. 8 if additional state constraints are present, that is, the following system is considered:

$$\begin{cases} \dot{x} \in F_q(x), \;\; \dot{y} \in F_0(y), \;\;\; x \in C_q^x, \;\; y \in C^y \\ q^+ \in Q, \;\; y^+ = G_0(y), \;\;\; y \in D^y \end{cases}$$

and for each $q \in Q$, the map F_q and the set C_q meet the sufficient conditions for continuous-time relaxation with constraints as given in [14], a simple version of which is stated in [14, Lemma 4.1].

Now, we state a hybrid relaxation result that involves assumptions on the continuity of the jump map G and on some general properties of solutions of the differential inclusion given by the flow map F, constrained by the flow set C, and having a "target set" D. Later, sufficient conditions for the said general properties of F, C, and D will be given in terms of F and the tangent cones to C and D.

The following two differential inclusions will play a role in the analysis: the relaxed constrained inclusion

$$\dot{z}(t) \in \operatorname{con} F(z(t)), \quad z(t) \in C \quad \text{ for almost all } t \in [0, T], \tag{10}$$

and the constrained inclusion

$$\dot{y}(t) \in F(y(t)), \quad y(t) \in C \quad \text{ for almost all } t \in [0, \tau]. \tag{11}$$

Also, the following property of certain mappings will be used: a set-valued mapping $\Delta\colon \mathcal{O} \rightrightarrows \mathbb{R}^n$ is *inner semicontinuous* relative to D if for any $x \in D$, any sequence $x_i \to x$ with $x_i \in D$, and any $y \in \Delta(x)$ there exist $y_i \in \Delta(x_i)$ such that $y = \lim y_i$.

Assumption 3.3 The hybrid system $\mathcal{H}$ satisfies the *hybrid relaxation conditions*, i.e.,

(a) The mappings $G^{\cap C}, G^{\cap D}, G^{\setminus}\colon \mathcal{O} \rightrightarrows \mathbb{R}^n$ defined at each $\xi \in \mathcal{O}$ by

$$G^{\cap C}(\xi) = G(\xi) \cap C, \quad G^{\cap D}(\xi) = G(\xi) \cap D, \quad G^{\setminus}(\xi) = G(\xi) \cap \big[\mathcal{O} \setminus (C \cup D)\big] \tag{12}$$

are inner semicontinuous relative to D;

(b) *Continuous-time strong relaxation with constraints and a target* is possible, that is, for any absolutely continuous $z\colon [0, T] \to \mathbb{R}^n$ that is a solution to the relaxed constrained inclusion (10) and any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $y_0 \in (z(0) + \delta\mathbb{B}) \cap C$ there exists an absolutely continuous $y\colon [0, \tau] \to \mathbb{R}^n$

with $y(0) = y_0$ that is a solution to the constrained inclusion (11) such that $d_{\text{gph}}(z, y) \leqslant \varepsilon$, and if additionally $z(T) \in D$ then also $y(\tau) \in D$.

The term "strong relaxation" in Assumption 3.3 (b) is used to differentiate that assumption from what could be termed "weak relaxation", which, for any $z\colon [0, T] \to \mathbb{R}^n$ that is a solution to Eq. 10 and any $\varepsilon > 0$, would call for the existence of $y\colon [0, \tau] \to \mathbb{R}^n$ that is a solution to Eq. 11 such that $d_{\text{gph}}(z, y) \leqslant \varepsilon$, and if additionally $z(T) \in D$ then also $y(\tau) \in D$.

Theorem 3.4 *Suppose that the hybrid system $\mathcal{H}$ satisfies the hybrid relaxation conditions. Then, for any $x_0 \in C \cup D$, strong relaxation for initially flowing (respectively, initially jumping) solutions from x_0 relative to C (respectively, relative to D) is possible (in the sense of Definition* 3.1).

Proof The proof will be an induction on the number of jumps for x. Let $J = \max_j \operatorname{dom} x$. If $J = 0$ (so no jumps) and x initially flows then the conclusion of the theorem reduces to assumption (b) [in the special case of $J = 0$ and x not initially flowing, x is a trivial hybrid arc (i.e., $\operatorname{dom} x = (0, 0)$) and the conclusions of the theorem are satisfied with $\varepsilon = \delta$ and by considering trivial arcs y with $\operatorname{dom} y = (0, 0)$ and given by $y(0, 0) = y_0$]. Now take any $J \geqslant 1$ and suppose that the conclusion of the theorem is valid for all hybrid arcs $x'(\cdot, \cdot)$ with compact domains and such that $\max_j \operatorname{dom} x' = J - 1$. Pick $\varepsilon > 0$ small enough so that $\operatorname{rge} x + \varepsilon I\!B \subset \mathcal{O}$, and let M be a bound on F on $\left(\operatorname{rge} x + \varepsilon I\!B\right) \cap C$. Let

$$\operatorname{dom} x = \bigcup_{j=0}^{J} \left[t_j, t_{j+1}\right] \times \{j\}.$$

Suppose first that $t_J = t_{J+1}$. If $x(t_J, J) \in C$ (respectively, $x(t_J, J) \in D$ or $x(t_J, J) \notin C \cup D$), then by inner semicontinuity of $G^{\cap C}$ (respectively, of $G^{\cap D}$ or $G^{\setminus}$), we have the existence of $\delta_1 \in (0, \varepsilon)$ such that for all $z \in x(t_J, J - 1) + \delta_1 I\!B$ there exists $z' \in G^{\cap C}(z)$ (respectively, $z' \in G^{\cap D}(z)$ or $z' \in G^{\setminus}(z)$) such that $z' \in x(t_J, J) + \varepsilon I\!B$. Now let x' be a truncation of x to $\operatorname{dom} x' = \bigcup_{j=0}^{J-1} [t_j, t_{j+1}] \times \{j\}$. Of course, x' is a solution to $\mathcal{H}^{\text{con}}$, and $x'(t_J, J - 1) \in D$. Pick $\delta_2 \in (0, \delta_1/(1 + 2M))$. By the inductive assumption, there exists $\delta > 0$, and for any $y_0 \in (x(0, 0) + \delta I\!B) \cap (C \cup D)$ a hybrid arc y', with compact $\operatorname{dom} y'$ and with $y'(0, 0) = y_0$, that is a solution to $\mathcal{H}$, such that

$$d_{\text{gph}}(x', y') \leqslant \delta_2,$$

and such that $y'(t'_J, J - 1) \in D$, where $(t'_J, J - 1) = \max \operatorname{dom} y'$. We need to argue that $y'(t'_J, J - 1) \in x(t_J, J - 1) + \delta_1 I\!B$. As $d_{\text{gph}}(x', y') \leqslant \delta_2$, there exists $(t, J - 1) \in \operatorname{dom} x'$ with $|t'_J - t| \leqslant \delta_2$, $\|y'(t'_J, J - 1) - x'(t, J - 1)\| \leqslant \delta_2$. Then

$$\begin{aligned}
&\|y'(t'_J, J - 1) - x'(t_J, J - 1)\| \\
&\quad \leqslant \|y'(t'_J, J - 1) - x'(t, J - 1)\| + \|x'(t, J - 1) - x'(T_J, J - 1)\| \\
&\quad \leqslant \delta_2 + M|t - t_J| \leqslant \delta_2 + M\left(|t - t'_J| + |t'_J - t_J|\right) \\
&\quad \leqslant \delta_2(1 + 2M),
\end{aligned}$$

where we have used the fact that $|t'_J - t_J| \leqslant \delta_2$ (this holds as $(t'_J, J-1)$, $(t_J, J-1)$ are the "endpoints" of domains of y' and x'). So, $y'(t'_J, J-1) \in x(t_J, J-1) + \delta_1 \mathbb{B}$, and since $y'(t'_J, J-1) \in D$, we have the existence of some $z' \in G^{\cap C}(y'(t'_J, J-1))$ (respectively, $z' \in G^{\cap D}(y'(t'_J, J-1)))$ such that $z' \in x(t_J, J) + \varepsilon \mathbb{B}$. Now, consider a hybrid arc $y\colon \operatorname{dom} y \to \mathbb{R}^n$ with $\operatorname{dom} y = \operatorname{dom} y' \cup (t'_J, J)$ given by

$$y(t, j) = \begin{cases} y'(t, j) & \text{for } (t, j) \in \operatorname{dom} y', \\ \quad z' & \text{for } (t, j) = (t'_J, J). \end{cases}$$

As $\mathrm{d}_{\mathrm{gph}}(x', y') \leqslant \delta_2 \leqslant \varepsilon$, $|t'_J - t_J| \leqslant \delta_2 \leqslant \varepsilon$, $\|z' - x(t_J, J)\| \leqslant \varepsilon$, we get $\mathrm{d}_{\mathrm{gph}}(x, y) \leqslant \varepsilon$.

Now suppose that $t_J < t_{J+1}$. In particular, $x(t_J, J) \in C$. By assumption (b) we have the existence of $\delta_3 \in (0, \varepsilon/2)$ such that, for all $z_0 \in (x(t_J, J) + \delta_3 \mathbb{B}) \cap C$, there exists an absolutely continuous $z\colon [0, \tau] \to \mathbb{R}^n$ solving Eq. 11 and such that $\mathrm{d}_{\mathrm{gph}}(z, x^J) \leqslant \varepsilon/2$, where $x^J\colon [0, t_{J+1} - t_J] \to \mathbb{R}^n$ is given by $x^J(t) = x(t_J + t, J)$. Moreover, if $x(t_{J+1}, J) \in D$, we can have $z(\tau) \in D$. By inner semicontinuity of $G^{\cap C}$, we have the existence of $\delta_1 \in (0, \delta_3)$ such that for all $z \in x(t_J, J-1) + \delta_1 \mathbb{B}$ there exists $z_0 \in G^{\cap C}(z)$ such that $z_0 \in (x(t_J, J) + \delta_3 \mathbb{B}) \cap C$. Now let x' be as in the paragraph above, with the current δ_1 pick $\delta_2 \in (0, \delta_1/(1 + 2M))$, and using the inductive assumption, pick $\delta > 0$ and y' also according to the description in the paragraph above. Consider a hybrid arc $y\colon \operatorname{dom} y \to \mathbb{R}^n$ with $\operatorname{dom} y = \operatorname{dom} y' \cup ([t'_J, t'_J + \tau], J)$ given by

$$y(t, j) = \begin{cases} \ y'(t, j) & \text{for } \quad\ \ (t, j) \in \operatorname{dom} y', \\ z(t - t'_J) & \text{for } (t, j) \in [t'_J, t'_J + \tau] \times \{J\}. \end{cases}$$

As $\mathrm{d}_{\mathrm{gph}}(x', y') \leqslant \delta_2 \leqslant \varepsilon/2$, $|t'_J - t_J| \leqslant \delta_2 \leqslant \varepsilon/2$, and $\mathrm{d}_{\mathrm{gph}}(z, x^J) \leqslant \varepsilon/2$, we have $\mathrm{d}_{\mathrm{gph}}(x, y) \leqslant \varepsilon$. □

Remark 3.5 While it is not true that (a) and (b) are necessary for the conclusions of Theorem 3.4 to hold, they are close to being necessary. More precisely, $G^{\cap D}$ and $G^{\setminus}$ must be inner semicontinuous at each $\xi \in D$, while $G^{\cap C}$ need not be inner semicontinuous at $\xi \in D$ if $G^{\cap C}(\xi) \subset C \cup D$ and no solutions to Eq. 10 exist from any point $\xi' \in G^{\cap C}(\xi)$.

Definition 3.6 Given $x_0 \in C \cup D$, strong relaxation for all solutions from x_0 is possible if

(SR) For any compact $x\colon \operatorname{dom} x \to \mathbb{R}^n$ with $x(0, 0) = x_0$ that is a solution to $\mathcal{H}^{\mathrm{con}}$ and for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $y_0 \in (x_0 + \delta \mathbb{B}) \cap (C \cup D)$ there exist a hybrid arc $y\colon \operatorname{dom} y \to \mathbb{R}^n$ with compact $\operatorname{dom} y$ and $y(0, 0) = y_0$ that is a solution to $\mathcal{H}$ and $\mathrm{d}_{\mathrm{gph}}(x, y) \leqslant \varepsilon$, and moreover, if $x(T, J) \in D$, where $(T, J) = \max \operatorname{dom} x$, then $y(\tau, J) \in D$, where $(\tau, J) = \max \operatorname{dom} y$.

In contrast to Definition 3.1, the definition above calls for the existence of y from any initial point y_0, independently of whether such y_0 is in C or in D, and independently of whether x flows first or jumps first.

Corollary 3.7 *Under the assumptions of Theorem* 3.4*, for any*

$$x_0 \in (C \setminus D) \cup (D \setminus C) \cup (\operatorname{int} C \cap \operatorname{int} D),$$

strong relaxation for (all) solutions from x_0 is possible.

4 "Viability Conditions" for Hybrid Relaxation

Now, we give assumptions on the data of the hybrid system $\mathcal{H}$ that will imply the hybrid relaxation conditions of Assumption 3.3. Throughout this section, we pose the following conditions, which strengthen the standing assumption:

Assumption 4.1

(A1) $C, D \subset \mathcal{O}$ are relatively closed in $\mathcal{O}$;
(A2) $F\colon \mathcal{O} \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and locally bounded relative to C and has nonempty values on C.

The mapping F is *outer semicontinuous* relative to C if for any $x \in C$, any sequence $x_i \to x$ with $x_i \in C$, and any convergent $y_i \in F(x_i)$ with $\lim y_i = y$, we have $y \in F(x)$. It is *locally bounded* relative to C if for any $x \in C$ there exists a relative neighborhood $X \subset C$ of x such that $F(X)$ is bounded.

Some further background material is needed. Given a set $S \subset \mathbb{R}^n$ and $\xi \in S$, $T_S(\xi)$ will denote the *tangent cone* to S at ξ, that is the set of all vectors $w \in \mathbb{R}^n$ such that

$$w = \lim_{i\to\infty} \frac{\xi_i - \xi}{\tau_i}$$

where $\xi_i \in S$, $\xi_i \to \xi$, and $\tau_i \searrow 0$ (this cone is sometimes called the contingent cone, or the Bouligand tangent cone). Also, $M_S(\xi)$ will denote the *Dubovitskii–Miliutin tangent cone* to S at ξ, that is, the set

$$M_S(\xi) = \mathbb{R}^n \setminus T_{\mathbb{R}^n\setminus S}(\xi).$$

For an alternate definition, see [2, Definition 4.3.1].

The result below summarizes some basic viability and invariance results. More specifically, (a) is [2, Proposition 3.4.1], (b) is [2, Proposition 3.4.2], (c) immediately follows from the definition [2, Definition 4.3.1] of $M_{\operatorname{int} D}(z_0)$ and [2, Corollary 5.3.2].

Theorem 4.2 *Let $\Gamma\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be an outer semicontinuous, locally bounded mapping with nonempty and convex values. Let S be a closed set. Consider the differential inclusion*

$$\dot z(t) \in \Gamma(z(t)) \tag{13}$$

(a) *If $z\colon [0, T] \to \mathbb{R}^n$ is a solution to* Eq. 13 *and $z(t) \in S$ for all $t \in [0, T]$, then*

$$\Gamma(z(0)) \cap T_S(z(0)) \neq \emptyset.$$

(b) *Given $z_0 \in S$, if, for all ξ in some neighborhood of z_0,*

$$\Gamma(\xi) \cap T_S(\xi) \neq \emptyset,$$

then there exists $T > 0$ and a solution $z\colon [0, T] \to \mathbb{R}^n$ to Eq. 13 *with $z(0) = z_0$.*

(c) *If* $\operatorname{int} S \neq \emptyset$*, $z_0 \in \partial S$, Γ is locally Lipschitz continuous, and*

$$\Gamma(z_0) \cap M_{\operatorname{int} S}(z_0) \neq \emptyset,$$

then there exists $T > 0$ and a solution $z\colon [0, T] \to \mathbb{R}^n$ to Eq. 13 *such that $z(t) \in$* $\operatorname{int} S$ *for all $t \in (0, T]$.*

Theorem 4.5 will relate the assumptions below to Assumption 3.3, part (b) about continuous-time strong relaxation with constraints and a target.

Assumption 4.3

(1) F is locally Lipschitz continuous on a neighborhood of C;
(2) For all $\xi \in \partial C \cap \mathcal{O}$ with $\operatorname{con} F(\xi) \cap T_C(\xi) \neq \emptyset$, we have

$$-\operatorname{con} F(\xi) \subset M_{\mathcal{O}\setminus C}(\xi)$$

and there exists $r_1 > 0$ such that

$$\operatorname{con} F(z) \subset M_{\operatorname{int} C}(z) \quad \forall z \in \partial C \cap (\xi + r_1 I\!B);$$

(3) For all $\xi \in \operatorname{int} C \cap \partial D$ we have

$$\operatorname{con} F(\xi) \cap M_{\operatorname{int} D}(\xi) \neq \emptyset;$$

(4) For all $\xi \in \partial C \cap \partial D \cap \mathcal{O}$ with $\operatorname{con} F(\xi) \cap T_C(\xi) = \emptyset$, there exists $r_2 > 0$ such that

$$\partial C \cap (\xi + r_2 I\!B) \subset D.$$

The three examples of failure of relaxation in hybrid systems, given at the beginning of the current section, correspond precisely to conditions (2), (3), and (4) above. That is, the example (a) violates condition (2) (while (3) and (4) can be satisfied by adding $D = \mathbb{R}$); (b) violates condition (3) (while (2) and (4) were satisfied); and similarly, (c) violates (4). This does not mean though that Assumption 4.3 is necessary for Theorem 4.5 below.

Lemma 4.4 *Let $z\colon [0, T] \to \mathbb{R}^n$ be a solution to* Eq. 10. *If* (2) *of Assumption* 4.3 *holds, then $z(t) \in \operatorname{int} C$ for all $t \in (0, T)$, and if $z(T) \in \partial C$, then* $\operatorname{con} F(z(T)) \cap T_C(z(T)) = \emptyset$.

Proof Suppose that $z(\tau) \in \partial C$ for some $\tau \in (0, T)$. Then $\operatorname{con} F(z(\tau)) \cap T_C(z(\tau)) \neq \emptyset$, by (a) of Theorem 4.2. Let $y\colon [0, \tau] \to \mathbb{R}$ be given by $y(s) = z(\tau - s)$. Then $y(0) = z(\tau)$ while, by (2), $\dot{y}(0) \in -\operatorname{con} F(y(0)) \subset M_{\mathcal{O}\setminus C}(y(0))$. By (c) of Theorem 4.2, for some $s > 0$, $y(t) \in \mathcal{O} \setminus C$ for all $t \in (0, s)$. This contradicts $z(t) \in C$ for all $t \in [0, T]$. □

Theorem 4.5 *If* (1) *and* (2) *of Assumption* 4.3 *hold, then for each solution $x\colon [0, T] \to \mathbb{R}^n$ of* Eq. 10 *and each $\varepsilon > 0$ there exists $\delta > 0$ such that for all $y_0 \in (x(0) + \delta I\!B) \cap C$ there exists a solution $y\colon [0, \tau] \to \mathbb{R}^n$ to* Eq. 11 *with $y(0) = y_0$ and $\mathrm{d}_{\mathrm{gph}}(x, y) \leqslant \varepsilon$. If additionally* (3) *and* (4) *hold, and $x(T) \in D$, then y can be chosen so that also $y(\tau) \in D$. In particular, Assumption* 4.3 *implies Assumption* 3.3 (*b*).

Proof Pick any absolutely continuous $x\colon [0, T] \to \mathbb{R}^n$ solving Eq. 10 and any $\varepsilon > 0$. If $T = 0$, then one can just take $x = y$ (and these are just "trivial" arcs on the interval $[0, 0]$). From now on, suppose $T > 0$.

Let $\varepsilon_1 \in (0, \varepsilon)$ be such that F is defined on $\operatorname{rge} x + \varepsilon_1 I\!B$, and on that set, F is Lipschitz continuous with constant K and bounded by M.

Pick $\varepsilon_2, \varepsilon_3 \in (0, \varepsilon_1)$ so that

$$\begin{aligned}
&\varepsilon_2 + M\varepsilon_3 \leqslant \varepsilon_1,\ 2\varepsilon_3 < T, \text{ and} \\
&\quad \text{if } x(0) \in \partial C \text{ then } \varepsilon_2 + M\varepsilon_3 \leqslant r_1 \text{ while } x([\varepsilon_3, T-\varepsilon_3]) + \varepsilon_2 I\!B \subset \operatorname{int} C; \\
&\quad \text{if } x(0) \in \operatorname{int} C \text{ then } x([0, T-\varepsilon_3]) + \varepsilon_2 I\!B \subset \operatorname{int} C.
\end{aligned} \tag{14}$$

Above, r_1 is associated with $x(0) \in \partial C$ as in assumption (2). The Filippov–Ważewski theorem now says that there exists $\delta \in (0, \varepsilon_2)$ such that, for all $y_0 \in x(0) + \delta I\!B$, there exists $y\colon [0, T-\varepsilon_3] \to \mathbb{R}^n$ satisfying $\dot{y}(t) \in F(y(t))$ for almost all $t \in [0, T-\varepsilon_3]$ and $\|x(t) - y(t)\| \leqslant \varepsilon_2$ for all $t \in [0, T-\varepsilon_3]$. If $x(0) \in \operatorname{int} C$, then Eq. 14 implies that $y(t) \in C$ for all $t \in [0, T-\varepsilon_3]$. This is also true if $x(0) \in \partial C$, but for all $t \in [\varepsilon_3, T-\varepsilon_3]$. In the latter case, for $t \in [0, \varepsilon_3]$ we have $y(t) \in x_0 + \varepsilon_2 I\!B + \varepsilon_3 M I\!B \subset x_0 + r_1 I\!B$. This is sufficient to guarantee that if $y(0) \in C$, then $y(t) \in C$ for $t \in [0, \varepsilon_3]$; c.f. [2, Theorem 4.3.6]. Consequently, for any $y_0 \in (x_0 + \delta I\!B) \cap C$ there exists $y\colon [0, T-\varepsilon_3] \to \mathbb{R}^n$ satisfying $\dot{y}(t) \in F(y(t))$ for almost all $t \in [0, T-\varepsilon_3]$, $y(t) \in C$ and $\|x(t) - y(t)\| \leqslant \varepsilon_2 \leqslant \varepsilon$ for all $t \in [0, T-\varepsilon_3]$. Thus, for any such y, and for any $t \in [T-\varepsilon_3, T]$, we have

$$\begin{aligned}
\|x(t) - y(T-\varepsilon_3)\| &\leqslant \|x(t) - x(T-\varepsilon_3)\| + \|x(T-\varepsilon_3) - y(T-\varepsilon_3)\| \\
&\leqslant M\varepsilon_3 + \varepsilon_2 \leqslant \varepsilon_1 \leqslant \varepsilon,
\end{aligned}$$

and $t - (T-\varepsilon_3) \leqslant \varepsilon_3 \leqslant \varepsilon$. So, for any such y we have $d_{\mathrm{gph}}(x, y) \leqslant \varepsilon$. This completes the proof of the case of $x(T) \notin D$.

Now suppose that x, as assumed above, also has $x(T) \in D$. If $x(T) \in \operatorname{int} D$, then we can pick $\varepsilon_2, \varepsilon_3 \in (0, \varepsilon_1)$ as in Eq. 14 so that additionally, $x(T-\varepsilon_3) + \varepsilon_2 I\!B \subset D$. Then y, obtained as above, satisfy $y(T-\varepsilon_3) \in D$.

Now suppose that $x(T) \in \partial D$ and $x(T) \in \operatorname{int} C$. Let $x_e\colon [0, T_e]$, $T_e > T$, be any absolutely continuous arc such that

$$\begin{aligned}
&\operatorname{rge} x_e \subset \operatorname{rge} x + \varepsilon_1 I\!B, \quad \dot{x}_e(t) \in F(x_e(t)) \text{ for } t \in [0, T_e], \\
&x(t) = x_e(t) \text{ for } t \in [0, T],
\end{aligned} \tag{15}$$

and furthermore, such that $x_e(t) \in \operatorname{int} D$ for all $t \in (T, T_e]$ (this is possible thanks to assumption (3) and by recalling Theorem 4.2 (c)) and $x_e(t) \in \operatorname{int} C$ for all $t \in [T, T_e]$. Then also $x_e(t) \in \operatorname{int} C$ for all $t \in (0, T_e]$. Thus, we can pick $T_1 \in (T, T_e]$, $\varepsilon_2, \varepsilon_3 \in (0, \varepsilon_1)$ as in Eq. 14, and $\varepsilon_4 \in (0, \varepsilon_2)$ so that $T_1 - \varepsilon_3 \leqslant T$, $x_e([T-\varepsilon_3, T_1]) + \varepsilon_2 I\!B \subset \operatorname{int} C$, and $x(T_1) + \varepsilon_4 I\!B \subset \operatorname{int} D$. The Filippov–Ważewski theorem implies that there exists $\delta \in (0, \varepsilon_4)$ such that, for all $y_0 \in x(0) + \delta I\!B$, there exists $y\colon [0, T_1] \to \mathbb{R}^n$ satisfying $\dot{y}(t) \in F(y(t))$ for almost all $t \in [0, T_1]$ and $\|x(t) - y(t)\| \leqslant \varepsilon_4$ for all $t \in [0, T_1]$. As in the discussion of the case of $x(T) \notin D$ we can now conclude that if $y_0 \in (x(0) + \delta I\!B) \cap C$, then the corresponding y satisfies $y(t) \in C$ for all $t \in [0, T_1]$. Obviously, $y(T_1) \in D$. It remains to argue that $d_{\mathrm{gph}}(x, y) \leqslant \varepsilon$. For each $t \in [0, T]$, we have $\|x(t) - y(t)\| \leqslant \varepsilon_4 \leqslant \varepsilon$. For $t \in (T, T_1]$, we have

$$\begin{aligned}
\|y(t) - x(T)\| &\leqslant \|y(t) - y(T)\| + \|y(T) - x(T)\| \\
&\leqslant M|T_1 - T| + \varepsilon_4 \leqslant M\varepsilon_3 + \varepsilon_2 \leqslant \varepsilon.
\end{aligned}$$

This completes the argument.

Finally, suppose $x(T) \in \partial D \cap \partial C \cap \mathcal{O}$. Again, let $x_e\colon [0, T_e]$, $T_e > T$, be any absolutely continuous arc such that Eq. 15 holds. For some (arbitrarily small) $t > 0$, $x_e(T+t) \notin C$ (thanks to Lemma 4.4 and Theorem 4.2 (a)). Thus, we can pick $T_1 \in (T, T_e]$, $\varepsilon_2, \varepsilon_3 \in (0, \varepsilon_1)$ as in Eq. 14, and $\varepsilon_4 \in (0, \varepsilon_2)$ so that $T_1 - T \leqslant \varepsilon_3$, $\varepsilon_2 + M\varepsilon_3 \leqslant r_2$ and $(x_e(T_1) + \varepsilon_4 \mathbb{B}) \cap C = \emptyset$. Here, r_2 is associated with $x(T) \in \partial C \cap \partial D$ through assumption (4). The Filippov–Ważewski theorem, and the discussion above, implies that there exists $\delta \in (0, \varepsilon_4)$ such that, for all $y_0 \in (x(0) + \delta \mathbb{B}) \cap C$, there exists $y\colon [0, T_1] \to \mathbb{R}^n$ satisfying $\dot{y}(t) \in F(y(t))$ for almost all $t \in [0, T_1]$, $\|x(t) - y(t)\| \leqslant \varepsilon_4$ for all $t \in [0, T_1]$, and $y(t) \in C$ for all $t \in [0, T - \varepsilon_3]$, in fact $y(t) \in \operatorname{int} C$ for all $t \in (0, T - \varepsilon_3]$. On the other hand, $y(T_1) \in x_e(T_1) + \varepsilon_4 \mathbb{B}$, so $y(T_1) \notin C$. For each y under discussion, let T_y be the minimum of times $t \geqslant T - \varepsilon_3$ such that $y(t) \in \partial C$. It must be that $T_y \in (T - \varepsilon_3, T_1)$. Then

$$\begin{aligned}\|y(T_y) - x(T)\| &\leqslant \|y(T_y) - y(T)\| + \|y(T) - x(T)\| \\ &\leqslant M\varepsilon_3 + \varepsilon_4 \leqslant M\varepsilon_3 + \varepsilon_2 \leqslant r_2.\end{aligned}$$

So $y(T_y) \in \partial C \cap (x(T) + r_1 \mathbb{B}) \subset D$. It remains to argue that, if we let y' be the truncation of y from $[0, T_1]$ to $[0, T_y]$, then $\mathrm{d}_{\mathrm{gph}}(x, y') \leqslant \varepsilon$. For each $t \in [0, T - \varepsilon_3]$, we have $t \in [0, T_y]$ and $\|x(t) - y'(t)\| \leqslant \varepsilon_4 \leqslant \varepsilon$. For $t \in [T - \varepsilon_3, T]$, we have $t - (T - \varepsilon_3) \leqslant \varepsilon$ and

$$\begin{aligned}\|x(t) - y'(T - \varepsilon_3)\| &\leqslant \|x(t) - x(T - \varepsilon_3)\| + \|x(T - \varepsilon_3) - y'(T - \varepsilon_3)\| \\ &\leqslant M\varepsilon_3 + \varepsilon_2 \leqslant \varepsilon_1 \leqslant \varepsilon.\end{aligned}$$

So, for any $t \in [0, T]$ we can find $s \in [0, T_y]$ with $|t - s| \leqslant \varepsilon$ and $\|x(t) - y'(s)\| \leqslant \varepsilon$. Similarly, for any $t \in (T - \varepsilon_3, T_y]$, we have

$$\begin{aligned}\|y'(t) - x(T)\| &\leqslant \|y'(t) - y(T)\| + \|y(T) - x(T)\| \\ &\leqslant M\varepsilon_3 + \varepsilon_4 \leqslant M\varepsilon_3 + \varepsilon_2 \leqslant \varepsilon,\end{aligned}$$

and $|t - T| \leqslant \varepsilon_3 \leqslant \varepsilon$. So, for any $t \in [0, T_y]$ there is $s \in [0, T]$ with $|t - s| \leqslant \varepsilon$ and $\|x(t) - y'(s)\| \leqslant \varepsilon$. This completes the proof. □

Corollary 4.6 *Assume that the mappings $G^{\cap C}, G^{\cap D}, G^{\setminus}\colon \mathcal{O} \to \mathbb{R}^n$ are inner semicontinuous relative to D and that Assumption 4.3 holds. Then for any*

$$x_0 \in (C \setminus D) \cup (D \setminus C) \cup (\operatorname{int} C \cap \operatorname{int} D),$$

strong relaxation is possible for solutions from x_0 (in the sense of Definition 3.6). Strong relaxation is also possible for solutions from any x_0 if additionally

(a) $x_0 \in \partial C \cap \operatorname{int} D$ *and additionally* $T_C(x_0) \cap \operatorname{con} F(x_0) = \emptyset$;
(b) $x_0 \in \operatorname{int} C \cap \partial D$, *or more generally, if* $x_0 \in C \cap \partial D$ *and* $\operatorname{con} F(x_0) \cap M_{\operatorname{int} D}(x_0) \neq \emptyset$ *and there exists* $r_3 > 0$ *such that* $(x_0 + r_3 \mathbb{B}) \cap D \subset C$;
(c) $x_0 \in \partial C \cap \partial D$, $\operatorname{int} C \cap \operatorname{int} D = \emptyset$, *and* $\operatorname{con} F(x_0) \cap T_C(x_0) = \emptyset$.

Proof The first conclusion just summarizes Theorem 3.4, Theorem 4.5, and Corollary 3.7.

For (a), the tangent cone solution implies that solutions x to $\mathcal{H}^{\mathrm{con}}$ with $x(0, 0) = x_0$ that initially flow do not exist. For the solutions that initially jump, we know that local relaxation at x_0 is possible, as $x_0 \in \operatorname{int} D$ and thanks to Theorem 3.4 and Theorem 4.5.

For (b), given a solution that initially flows, the conclusion comes from $(x_0 + r_3\mathbb{B}) \cap (C \cup D) = (x_0 + r_3\mathbb{B}) \cap C$, Theorem 3.4 and Theorem 4.5. Let x be a compact solution to $\mathcal{H}^{\text{con}}$ with $x(0, 0) = x_0$ that initially jumps and pick $\varepsilon > 0$. Let $\delta_1 > 0$ be such that for all $y_0 \in (x_0 + \delta_1\mathbb{B}) \cap D$ there exists a solution x_{y_0} to $\mathcal{H}$ with $x_{y_0}(0, 0) = y_0$ and such that $\mathrm{d}_{\text{gph}}(x, x_{y_0}) < \varepsilon/2$; the existence of such δ_1 follows from Theorem 3.4 and Theorem 4.5. Since $(x_0 + r_3\mathbb{B}) \cap D \subset C$, one has $M_{\text{int}\,D}(x_0) \subset T_C(x_0)$, which combined with $\text{con}\, F(x_0) \cap M_{\text{int}\,D}(x_0) \neq \emptyset$ yields $\text{con}\, F(x_0) \cap T_C(x_0) \neq \emptyset$. Let $r_1 > 0$ come from (2) of Assumption (4.3), and let $\varepsilon_1 \in (0, \min\{\varepsilon/4, r_1/2\}$. Let $T \in (0, \varepsilon/2)$ and $z\colon [0, T] \to \mathbb{R}^n$ be a solution to $\dot{z}(t) \in \text{con}\, F(z(t))$ with $z(0) = x_0$ and such that $z(T) + \varepsilon_2 \in \text{int}\, D$ for some $\varepsilon_2 \in (0, \varepsilon_1)$ while $\|x_0 - z(t)\| \leqslant \varepsilon_1$ for all $t \in [0, T]$ [such solution exists by Theorem 4.2 (c)]. By the Filippov–Ważewski theorem, there exists $\delta_2 > 0$ such that for all $y_0 \in (x_0 + \delta_2\mathbb{B}) \cap C$ there exists a solution $z_{y_0}\colon [0, T] \to \mathbb{R}^n$ to $\dot{z}_{y_0}(t) \in F(z_{y_0}(t))$ such that $\|z(t) - z_{y_0}(t)\| \leqslant \varepsilon_2$ for $t \in [0, T]$. In particular, $\|x_0 - z_{y_0}(t)\| \leqslant \varepsilon/2$ for all $t \in [0, T]$, which implies, by (2) of Assumption (4.3), that $z_{y_0}(t) \in C$ for all $t \in [0, T]$. Now, given any $y_0 \in (x_0 + \delta_2\mathbb{B}) \cap C$, we can consider a solution to $\mathcal{H}$ given by $y(t, 0) = z_{y_0}(t)$ for $t \in [0, T]$, $y(T + t, j) = x_{y_0}(t, j)$ for all $(t, j) \in \text{dom}\, x_{y_0}$. For such y we have $\mathrm{d}_{\text{gph}}(x, y) \leqslant \varepsilon$. Letting $\delta = \min\{\delta_1, \delta_2\}$ finishes the argument.

In case (c), there are no initially flowing solutions from x_0, and also by (4) of Assumption (4.3), $\partial C \cap (x + r_2\mathbb{B}) \subset D$. It now suffices to show that from each point in $C \setminus D$ sufficiently close to x_0, there exists a solution to $\dot{z} \in F(z)$ that then enters $\mathcal{O} \setminus C$ in small amount of time, and so, crosses ∂C (at a point that is also in D). This comes from applying the Filippov–Ważewski relaxation theorem to a solution to $\dot{z} \in \text{con}\, F(z)$ from x_0, which, by $\text{con}\, F(x_0) \subset M_{\mathcal{O}\setminus C}(x_0)$, enters $\mathcal{O} \setminus C$ instantly. Details of the argument are similar to what has been done in (b). □

Example 4.7 For the system in Eq. 9, as discussed in Example 3.2, and when $C^\tau = [0, T]$ and $D^\tau = \{T\}$ or $C^\tau = [0, \infty)$ and $D^\tau = [T, \infty)$, strong relaxation, for all solutions, is possible from any x_0. Indeed, the assumptions of Corollary 4.6, including conditions (a), (b), and (c) are satisfied.

Remark 4.8 In several special cases, some of the tangent cone conditions (2), (3), or (4) of Assumption 4.3, are automatically satisfied.

(a) Suppose that $C = \mathcal{O}$. Then (2) and (4) are satisfied, as $\partial C \cap \mathcal{O} = \emptyset$.
(b) Suppose that $D = \partial C \cap \mathcal{O}$ (the case of $C = \mathcal{O}$ is not interesting in such situation). Then (3) is satisfied as $\text{int}\, C \cap \partial D = \emptyset$ and (4) is satisfied as $\partial C \cap \mathcal{O} = D$.
(c) Suppose $C \cup D = \mathcal{O}$ and $\text{int}\, C \cap \text{int}\, D = \emptyset$ (in light of the former, the latter means that $C \cap D = \partial C \cap \mathcal{O} = \partial D \cap \mathcal{O}$). Then (3) and (4) are satisfied, as $\text{int}\, C \cap \partial D = \emptyset$.
(d) Suppose that $\partial C \cap \partial D = \emptyset$. Then (4) is satisfied.

Also, a simple sufficient condition for (2) is that

($\star$) For all $x \in \partial C \cap \mathcal{O}$, $\text{con}\, F(x) \cap T_C(x) = \emptyset$,

which, in light of Theorem 4.2 (a) and (b), is equivalent to no solutions to $\dot{z} \in \text{con}\, F(z)$ from $\partial C \cap \mathcal{O}$ remaining in C for (any small) positive time.

Combining (b) of Corollary 4.6 and the case (a) of Remark 4.8 implies that when $C = \mathcal{O}$, if the conditions on G, as in (a) of Assumption 3.3, hold, and if $\operatorname{con} F(x) \cap M_{\operatorname{int} D}(x) \neq \emptyset$ for all $x \in \partial D \cap \mathcal{O}$, then strong relaxation is possible for solutions from all initial points. Also, for the case of either (b) or (c) of Remark 4.8, the said conditions on G and ($\star$) of Remark 4.8 are enough to guarantee strong relaxation from all initial points. Example 4.9 below illustrates this. We note though that if $D = \partial D \cap \mathcal{O} \subset \operatorname{int} C$, then Assumption 4.3 is never satisfied. Indeed, (3) is never true, as $\operatorname{int} D = \emptyset$.[2]

Example 4.9 On $\mathcal{O} = \mathbb{R}^2 \times (-1, 1)$, consider a hybrid system with

$$C = \{\xi \in \mathcal{O} \mid \xi_2 \geqslant \xi_1 - 1\},\ \ F(\xi) = \left(1, \xi_3^2, \{-1, 1\}\right),\ \ D = \partial C \cap \mathcal{O},\ \ G(\xi) = (0, 0, 0).$$

This falls under case (b) of Remark 4.8. Moreover, the condition ($\star$) of Remark 4.8 is satisfied: for each $\xi \in \partial C \cap \mathcal{O}$,

$$\operatorname{con} F(\xi) = \left(1, \xi_3^2, [-1, 1]\right), \qquad T_C(\xi) = \left\{v \in \mathbb{R}^3 \mid v_2 \geqslant v_1\right\},$$

and so $\operatorname{con} F(\xi) \cap T_C(\xi) = \emptyset$ as long as $\xi_2^2 < 1$. In light of Corollary 4.6 and Remark 4.8, strong relaxation is possible from any initial point in C.

For illustration purposes, consider a compact solution to Eq. 4 from $x_0 = (0, 0, 0)$ given by

$$x(t, j) = (t - j, 0, 0) \quad \text{for} \quad (t, j) \in \operatorname{dom} x = \bigcup_{j=0}^{T-1} [j, j+1] \times \{j\}$$

for some integer $T \geqslant 1$. Pick $\varepsilon \in (0, 1)$ and take $y_0 = x_0$. Let $\varepsilon' = \varepsilon/\sqrt{3}$ and find an arc $z_3\colon [0, 1 + \varepsilon'/T] \to \mathbb{R}$ so that $z_3(0) = 0$, $\dot{z}_3(t) \in \{-1, 1\}$ and $z_3^2(t) \leqslant \varepsilon'/(T + \varepsilon')$ for all $t \in [0, 1 + \varepsilon'/T]$. Set $z_2(t) = \int_0^t z_3^2(s)\, ds$, note that $z_2(t) \leqslant \varepsilon'/T$ for all $t \in [0, 1 + \varepsilon'/T]$, and let $\tau \in (1, 1 + \varepsilon'/T]$ be the smallest number such that $z_2(\tau) = \tau - 1$. Finally, set $z_1(t) = t$ for all $t \in [0, \tau]$ and note that $z_2(\tau) = z_1(\tau) - 1$, and so $(z_1, z_2, z_3)(\tau) \in D$. Consider

$$y(t, j) = (z_1, z_2, z_3)(t - j\tau) \quad \text{for} \quad (t, j) \in \operatorname{dom} y = \bigcup_{j=0}^{T-1} [j\tau, (j+1)\tau] \times \{j\}.$$

Then y is a solution to Eq. 3, $y(0, 0) = y_0$, and $\mathrm{d}_{\mathrm{gph}}(z, y) \leqslant \varepsilon$. To see the latter fact, take any $(t, j) \in \operatorname{dom} x$, note that $(t\tau, j) \in \operatorname{dom} y$ and $|t - t\tau| \leqslant T|1 - \tau| \leqslant \varepsilon' < \varepsilon$, and that furthermore

$$\begin{aligned}
\|x(t, j) - y(t\tau, j)\| &\leqslant \|(t - j, 0, 0) - (t - j\tau, z_2(t - j\tau), z_3(t - j\tau))\| \\
&\leqslant \sqrt{(j(\tau - 1))^2 + (\varepsilon'/T)^2 + (\varepsilon'/(T + \varepsilon'))^2} \\
&\leqslant \sqrt{\varepsilon'^2 + (\varepsilon'/T)^2 + (\varepsilon'/(T + \varepsilon'))^2} \\
&\leqslant \varepsilon'\sqrt{3} = \varepsilon.
\end{aligned}$$

[2] $D = \partial D \cap \mathcal{O} \subset \operatorname{int} C$ subsumes the case of the jump set being an $n - 1$ dimensional manifold and $C = \mathcal{O} = \mathbb{R}^n$, as studied in [7]. In such a case, the transversality condition used in [7] combined with the Filippov–Ważewski theorem show that (b) of Assumption 3.3 is in fact satisfied.

Note that, given x as above and $T > 1$, it is impossible to find y with $\operatorname{dom} y = \operatorname{dom} x$, even if the choice of $y(0,0)$ is free (it is possible for the case of $T = 1$). In particular, it is then impossible to find y with $\operatorname{dom} y = \operatorname{dom} x$ and $d_{\mathrm{gph}}(x, y) \leqslant \varepsilon$. This illustrates that is impossible, in general, to find a relaxed solution with the same time domain as the original one.

5 Application to Stability Analysis

As an example of an application of the relaxation results to stability theory, we show that uniform global asymptotic stability of a compact set for Eq. 3 implies that property for Eq. 4, in fact with the same convergence rate. Like in the case of continuous-time systems [33], this result can be used to generate converse Lyapunov theorems for input-to-state stability (ISS) in hybrid systems [9]. This is because a simple small-gain argument converts the ISS property for a hybrid system with inputs into asymptotic stability for an autonomous hybrid system with a set-valued (not necessarily convex-valued) flow map. When this asymptotic stability also implies asymptotic stability for the corresponding relaxed hybrid system, recent converse Lyapunov theorems [10] generate smooth Lyapunov functions that verify ISS for the original hybrid system with inputs.

Let $\mathcal{A} \subset \mathcal{O}$ be a compact set. A function $\omega\colon \mathcal{O} \to \mathbb{R}_{\geqslant 0}$ is a proper indicator of $\mathcal{A}$ with respect to $\mathcal{O}$ if it is continuous, $\omega(\xi) = 0$ if and only if $\xi \in \mathcal{A}$ and, given a sequence of points $\xi_i \in \mathcal{O}$, $\omega(\xi_i) \to \infty$ if $|\xi_i| \to \infty$ or if ξ_i converge to a point on the boundary of $\mathcal{O}$. An example of a proper indicator, for the case of $\mathcal{O} = \mathbb{R}^n$, is the distance from $\mathcal{A}$. A function $\beta\colon \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is a $\mathcal{KL}$ function if it is continuous, $r \mapsto \beta(r, s)$ is 0 at 0 and nondecreasing, $s \mapsto \beta(r, s)$ is nonincreasing and converges to 0 as $s \to \infty$. The set $\mathcal{A}$ is *uniformly globally asymptotically stable* for Eq. 3 if there exist a proper indicator ω of $\mathcal{A}$ with respect to $\mathcal{O}$ and a $\mathcal{KL}$ function β such that

$$\omega(x(t, j)) \leqslant \beta\left(\omega(x(0,0)), t + j\right) \quad \text{for all } (t, j) \in \operatorname{dom} x \tag{16}$$

for every solution x to Eq. 3.

Proposition 5.1 *Let ω be a proper indicator of a compact set $\mathcal{A} \subset \mathcal{O}$ with respect to $\mathcal{O}$ and β be a $\mathcal{KL}$ function. Suppose that Eq. 16 holds for every solution x to Eq. 3. If strong relaxation for initially flowing (respectively, initially jumping) solutions from each $x_0 \in C' \cup D'$ is possible relative to C' (respectively, relative to D') for the hybrid system*

$$\mathcal{H}' : \qquad \begin{cases} \dot{x} \in F'(x) & x \in C' \\ x^{+} \in G'(x) & x \in D' \end{cases} \tag{17}$$

with the data $\mathcal{O}' = \mathcal{O} \setminus \mathcal{A}$, $C' = C \setminus \mathcal{A}$, $D' = D \setminus \mathcal{A}$, $F' = F\big|_{\mathcal{O}\setminus\mathcal{A}}$, $G' = G\big|_{\mathcal{O}\setminus\mathcal{A}}$, then Eq. 16 holds for every solution x to Eq. 4.

Proof Suppose that, to the contrary, there exists a solution x to Eq. 4 and $(t, j) \in \operatorname{dom} x$ such that $\omega(x(t, j)) > \beta\left(\omega(x(0,0)), t + j\right)$. If $x(s, k) \notin \mathcal{A}$ for all $(s, k) \in \operatorname{dom} x$ with $s + k \leqslant t + j$, let $x' = x$ and $(t', j') = (t, j)$. In the opposite case, let $(T, J) \in$

dom x be the "last" of all those $(s,k) \in \operatorname{dom} x$, $s+k \leqslant t+j$, such that $x(s,k) \in \mathcal{A}$. Note that since $\omega(x(t,j)) > 0$, we have $T+J < t+j$. As $x(T,J) \in A$ and Eq. 16, $G(x(T,J)) \cap \mathcal{O} \subset \mathcal{A}$. By the definition of (T,J), it must be then the case that for all small enough $\delta > 0$, $(T+\delta, J) \in \operatorname{dom} x$. Fix $\gamma > 0$ so that $t+j-T-J-\gamma > 0$ and pick such $\delta \in (0,\gamma)$ small enough so that

$$\omega(x(t,j)) > \beta\left(\omega(x(T+\delta,J)),\, t+j-T-J-\gamma\right),$$

which is possible as $\omega(x(T,J)) = 0$, $\delta \mapsto \omega(x(T+\delta,J))$ is continuous, and by the properties of $\mathcal{KL}$ functions. Then, also by the property of β as a $\mathcal{KL}$ function,

$$\omega(x(t,j)) > \beta\left(\omega(x(T+\delta,J)),\, t+j-T-J-\delta\right).$$

Now, let $x'(s,k) = x(s-T-\delta, k-J)$ and $(t',j') = (t-T-\delta, j-J)$.

We have constructed a solution x' to

$$\mathcal{H}'^{\,\mathrm{con}}: \qquad \begin{cases} \dot{x} \in \operatorname{con} F'(x) & x \in C' \\ x^+ \in \quad G'(x) & x \in D' \end{cases}$$

such that $\omega(x'(t',j')) > \beta\left(\omega(x'(0,0)), t'+j'\right)$ for some $(t',j') \in \operatorname{dom} x'$. By continuity of ω and β, there exists $\varepsilon > 0$ such that $\omega(\xi) > \beta\left(\omega(x'(0,0)), t+j'\right)$ whenever $\|\xi - x'(t',j')\| < \varepsilon$, $|t-t'| < \varepsilon$. The assumption about relaxation, applied with $y_0 = x'(0,0)$ and the ε just mentioned, immediately leads to a solution to Eq. 17, and hence to a solution to Eq. 3, that contradicts Eq. 16. $\square$

Sufficient conditions for strong relaxation for initially flowing (respectively, initially jumping) solutions to Eq. 17 from x_0 to be possible relative to C' (respectively, relative to D') can be given. For example, in light of Theorem 3.4 and Assumption 3.3, it is enough that:

(a') The mappings $G(\xi) \cap (C \setminus \mathcal{A})$, $G(\xi) \cap (D \setminus \mathcal{A})$, $G(\xi) \cap \left[\mathcal{O} \setminus (\mathcal{A} \cup C \cup D)\right]$ be inner semicontinuous relative to $D \setminus \mathcal{A}$;

(b') Continuous-time strong relaxation with constraints and a target is possible, with $C \setminus \mathcal{A}$, $D \setminus \mathcal{A}$ replacing C, D in (b) of Assumption 3.3.

In turn, viability conditions of Assumption 4.3 only need to be verified with $\mathcal{O}$ replaced by $\mathcal{O} \setminus \mathcal{A}$, in order to guarantee (b') above. More generally, we have the following "local" corollary of Theorem 4.5:

Corollary 5.2 *Let $x\colon [0,T] \to \mathbb{R}^n$ be a compact solution to Eq. 10 and $U \subset \mathcal{O}$ an open set such that* rge $x \subset U$. *If F is locally Lipschitz continuous on U and* (2) *of Assumption* 4.3 *holds with $\mathcal{O}$ replaced by U then for each $\varepsilon > 0$ there exists $\delta > 0$ such that for all $y_0 \in (x(0) + \delta\mathbb{B}) \cap C$ there exists a solution $y\colon [0,\tau] \to \mathbb{R}^n$ to* Eq. 11 *with $y(0) = y_0$ and $\mathrm{d}_{\mathrm{gph}}(x,y) \leqslant \varepsilon$. If additionally* (3) *and* (4) *hold with $\mathcal{O}$ replaced by U in* (4)*, and $x(T) \in D$, then y can be chosen so that also $y(\tau) \in D$.*

We now note that non-uniform global asymptotic stability of Eq. 3 fails to imply global asymptotic stability of Eq. 4, even if strong relaxation is possible.

Example 5.3 On $\mathcal{O} = \mathbb{R}^3$, let $\mathcal{A}$ be the singleton $(7,7,7)$ and consider

$$C = \{\xi \mid \xi_1 + \xi_2 \leqslant 1, \xi_2 \geqslant 0\},\ \ D = \overline{\mathcal{O} \setminus C},\ \ F(\xi) = (1-\xi_1, \xi_3^2, \{-1,1\}),\ \ G(\xi) = \mathcal{A}.$$

Then $\mathcal{A}$ is stable for both Eqs. 3 and 4: solutions from a (sufficiently small) neighborhood of $\mathcal{A}$ stay in that neighborhood, in fact they reach $\mathcal{A}$ in one jump. For Eq. 3, $\mathcal{A}$ is also attractive: all solutions converge to $\mathcal{A}$, in fact in finite time (the convergence is not uniform though; for example, solutions from $(0, 0, 0)$ can take an arbitrarily large amount of time to flow to D, and hence, an arbitrarily large amount of "hybrid time" to reach $\mathcal{A}$). For Eq. 4, $\mathcal{A}$ is not attractive: the constant hybrid arc $x(t, 0) = (1, 0, 0)$, $t \in \mathbb{R}_{\geqslant 0}$, is a solution to Eq. 4 and does not converge to $\mathcal{A}$. It can be verified directly that relaxation, in the sense of Definition 3.1, is possible, even though the viability conditions in Assumption 4.3 fail.

That global asymptotic stability of Eq. 3 fails to imply global asymptotic stability of Eq. 4 can be attributed to the failure of relaxation on the infinite (hybrid) time horizon. Note that in Example 5.3, the hybrid arc $x(t, 0) = (1, 0, 0)$, $t \in \mathbb{R}_{\geqslant 0}$, is a solution to Eq. 4 but there are no solutions to Eq. 3, from $(1, 0, 0)$ or otherwise, that remain close to x for all $t \in \mathbb{R}_{\geqslant 0}$ (such relaxation is possible for purely continuous-time unconstrained systems; see [22]). An even simpler example, one for which the viability conditions in Assumption 4.3 hold and where relaxation on infinite (hybrid) time fails can be given.

Example 5.4 On $\mathcal{O} = \mathbb{R}^2$ consider

$$C = D = \mathcal{O}, \quad F(\xi) = \left(\{-1, 1\}, \xi_1 + \xi_2^2\right), \quad G(\xi) = (0, 0).$$

The hybrid arc $x(0, 0) = (7, 7)$, $x(1, t) = (0, 0)$ for $t \in \mathbb{R}_{\geqslant 0}$ is a solution to Eq. 4 (it initially jumps and then only flows). Any solution to Eq. 3 that jumps has to jump to $(0, 0)$. If it only flows afterwards, its first coordinate grows (faster than exponentially) to ∞. Thus, the graph of such a solution is not close to the graph of x.

We note though that for a continuous-time system on $\mathbb{R}^2$ given by F above, [22, Theorem 1] guarantees that, given any $\varepsilon > 0$, there does exist a solution to $\dot{x} \in F(x)$ with $\|x(t)\| < \varepsilon$. However, that result does not guarantee that $x(0) = (0, 0)$; in fact, meeting this initial condition is impossible for the system under discussion (cf. the example in Section 4 of [22]).

6 Continuous Dependence

Throughout this section, we pose Assumption 4.1 and

Assumption 6.1

(A3) $G \colon \mathcal{O} \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and locally bounded relative to D, has nonempty values on D, and $G(x) \subset \mathcal{O}$ for all $x \in D$.

We will say that

- The *solution sets to $\mathcal{H}$ depend outer-semicontinuously on initial conditions* at x_0 if for each $\varepsilon > 0$, $M > 0$ there exists $\delta > 0$ such that, for any $y_0 \in x_0 + \delta \mathbb{B}$ and any solution y to $\mathcal{H}$ with $y(0,0) = y_0$ and length dom $y \leqslant M$ there exists a solution x to $\mathcal{H}$ with $x(0,0) = x_0$ and

$$d_{\mathrm{gph}}(x, y) \leqslant \varepsilon;$$

- The *solution sets to $\mathcal{H}$ depend inner-semicontinuously on initial conditions* at x_0 if for each $\varepsilon > 0$, $M > 0$ there exists $\delta > 0$ such that, for any $y_0 \in x_0 + \delta\mathbb{B}$ and any solution x to $\mathcal{H}$ with $x(0,0) = x_0$ and length dom $x \leqslant M$ there exists a solution y to $\mathcal{H}$ with $y(0,0) = y_0$ and

$$d_{\mathrm{gph}}(x, y) \leqslant \varepsilon;$$

- The *solution sets to $\mathcal{H}$ depend continuously on initial conditions* at x_0 if they depend both outer and inner-semicontinuously.

The solution sets to $\mathcal{H}$ depend outer-semicontinuously, inner-semicontinuously, and continuously *relative to $C \cup D$* if only $y_0 \in C \cup D$ are considered.

Note that the difference between what we called strong local relaxation and inner-semicontinuity is the uniformity in the latter: that δ works for all solutions from x_0 rather than for each solution we have a δ.

Proposition 6.2 *Suppose that strong relaxation for solutions from x_0 is possible and that all solutions to $\mathcal{H}^{\mathrm{con}}$ from x_0 that are not complete are bounded. Then solution sets to $\mathcal{H}^{\mathrm{con}}$ and the solution sets to $\mathcal{H}$ depend continuously on initial conditions at x_0, relative to $C \cup D$.*

Proof Outer semicontinuity for $\mathcal{H}^{\mathrm{con}}$ was shown in [16, Corollary 4.8]. The needed "local eventual boundedness assumption" is guaranteed here by the assumption of boundedness of the maximal and not complete solutions (that this is sufficient follows from the last two lines of the proof of [16, Theorem 4.6]).

We now claim that for each $\varepsilon > 0$, $M > 0$ there exists $\delta > 0$ such that, for any $y_0 \in x_0 + \delta\mathbb{B}$ and any solution x to $\mathcal{H}^{\mathrm{con}}$ with $x(0,0) = x_0$ and length dom $x \leqslant M$ there exists a solution y to $\mathcal{H}$ with $y(0,0) = y_0$ and $d_{\mathrm{gph}}(x, x) \leqslant \varepsilon$. This gives uniformity in relaxation, and entails the inner semicontinuity for both $\mathcal{H}$ and $\mathcal{H}^{\mathrm{con}}$. If the claim was false, then for some $\varepsilon > 0$, $M > 0$ there is a sequence $x_i \colon \mathrm{dom}\, x_i \to \mathbb{R}^n$ of solutions to $\mathcal{H}^{\mathrm{con}}$ with $x_i(0,0) = x_0$, $|\,\mathrm{dom}\, x_i| \leqslant M$, a sequence of points $y_{0,i} \to x_0$ for which all solutions $y \colon \mathrm{dom}\, y \to \mathbb{R}^n$ to $\mathcal{H}$ with $y(0,0) = y_{0,i}$ satisfy $d_{\mathrm{gph}}(x_i, y) > \varepsilon$. Again by the proof of [16, Theorem 4.6], the sequence of x_i's is bounded and a graphically convergent subsequence (which we do not relabel) can be picked (see [16, Theorem 4.4]), the limit x of which is a solution to $\mathcal{H}^{\mathrm{con}}$ with compact dom x satisfying length dom $x \leqslant M$. We have $d_{\mathrm{gph}}(x_i, x) \to 0$, and

$$\varepsilon < d_{\mathrm{gph}}(x_i, y) \leqslant d_{\mathrm{gph}}(x_i, x) + d_{\mathrm{gph}}(x, y),$$

which holds for all solutions $y \colon \mathrm{dom}\, y \to \mathbb{R}^n$ to $\mathcal{H}$ with $y(0,0) = y_{0,i}$. This contradicts strong relaxation at x_0.

Finally, the outer semicontinuity for $\mathcal{H}^{\mathrm{con}}$ and strong relaxation for solutions from x_0 combine to yield outer semicontinuity for $\mathcal{H}$. □

Recall that Corollaries 3.7 and 4.6 gave sufficient conditions for the strong relaxation for solutions from x_0 to be possible.

Example 6.3 A ball bouncing on the floor, with h denoting its height and v its velocity, can be modeled as the hybrid system, on $\mathbb{R}^2$, given by

$$\begin{cases} \dot{h} = v, \ \dot{v} = -g & h \geqslant 0, (h, v) \neq (0, 0) \\ \dot{h} = v, \ \dot{v} \in [-g, 0] & (h, v) = (0, 0) \\ h^+ = h, \ v^+ = -\gamma v & h = 0, \ v \leqslant 0 \end{cases} \quad . \tag{18}$$

Above, g is the acceleration due to gravity, and $\gamma \in (0, 1)$ represents dissipation of energy at each bounce.[3]

The dependence of solutions to Eq. 18 on initial conditions is not continuous at the point $h = 0$, $v = 0$. Indeed, the inner semicontinuity fails: there exists a flowing-only solution $h(t, 0) = 0$, $v(t, 0) = 0$ for $t \in \mathbb{R}_{\geqslant 0}$ from that initial point, while for initial conditions approaching the origin, the solutions experience infinitely many jumps in less and less time (they are Zeno solutions), and are not graphically close to the mentioned flowing-only solution (however, outer semicontinuity is present, as the Zeno solutions converge to the instantaneous Zeno solution from the origin: $h(0, j) = 0$, $v(0, j) = 0$ for $j \in \mathbb{N}$).

However, solutions to Eq. 18 depend continuously on initial conditions at any other initial point (with $h \geqslant 0$). To see this, one can first consider Eq. 18 on the state space $\mathcal{O} = \mathbb{R}^2 \setminus \{0\}$ and note that all solutions to the original system except those from the origin are still solutions to the new system (the state $h = 0$, $v = 0$ is not reachable from any initial condition besides $h = 0$, $v = 0$ itself). Then, conditions of Corollary 4.6, including Assumption 4.3 can be verified. For example, (2) of Assumption 4.3 needs to be checked at points $(h, v) = x$ such that $x \in \partial C \cap \mathcal{O}$ and $F(x) \cap T_C(x) \neq \emptyset$, i.e., at points where $h = 0$ and $v > 0$ ($T_C(x) = \mathbb{R}_{\geqslant 0} \times \mathbb{R}$ for all $x \in \partial C \cap \mathcal{O} = \{x \,|\, h = 0, v \neq 0\}$ while $F(x) = (v, -g)$). One then has $-F(x) \subset M_{\mathcal{O} \setminus C}(x) = (-\infty, 0) \times \mathbb{R}$ and for all nearby points in ∂C, $F(x) \subset M_{\operatorname{int} C}(x) = (0, \infty) \times \mathbb{R}$. Hence (2) of Assumption 4.3 is satisfied. Condition (3) of Assumption 4.3 is met vacuously, (4) is easy to check (at points where $h = 0$, $v < 0$), and as for such points, $F(x) \cap T_C(x) = \emptyset$, condition (c) of Corollary 4.6 is met. Consequently, by Corollary 4.6, strong relaxation is possible at each initial point, and Proposition 6.2 yields continuous dependence.

We note that essentially the same argument as in the proof of Proposition 6.2 shows a certain “uniformity in inner semicontinuity”, which is a counterpart to “uniformity in outer semicontinuity” as in [16, Corollary 4.8]. More specifically, one obtains:

Corollary 6.4 *Let $K \subset \mathcal{O}$ be a compact set such that for each $x_0 \in K$, strong relaxation is possible for solutions from x_0 and such that all not complete solutions to $\mathcal{H}^{\mathrm{con}}$ from x_0 are bounded. Then, for any $\varepsilon > 0$ and $M > 0$, there exists $\delta > 0$ such that, for any $x_0 \in K$, any $y_0 \in x_0 + \delta \mathbb{B}$ and any solution x to $\mathcal{H}^{\mathrm{con}}$ with $x(0, 0) = x_0$ and* length dom $x \leqslant M$ *there exists a solution y to $\mathcal{H}$ with $y(0, 0) = y_0$ and $\mathrm{d_{gph}}(x, x) \leqslant \varepsilon$.*

While the solution sets to $\mathcal{H}$ depend continuously on initial conditions at x_0 as in Proposition 6.2, these sets need not be closed under graphical convergence of

[3] The inclusion $\dot{v} \in [-g, 0]$ is used to allow for the natural constant flowing solution from $h = 0$, $v = 0$ while preserving outer semicontinuity and convex-valuedness of the flow map.

solutions. That is, the limit of a graphically convergent sequence of solutions x_i to $\mathcal{H}$, with $x_i(0,0) \to x_0$, need not be a solution to $\mathcal{H}$. It will be a solution to $\mathcal{H}^{\mathrm{con}}$, as more generally, the limit of a graphically convergent sequence of solutions x_i to $\mathcal{H}^{\mathrm{con}}$, with $x_i(0,0) \to x_0$, is a solution to $\mathcal{H}^{\mathrm{con}}$; see [16, Theorem 4.4].

We conclude the section by revisiting Assumption 3.3 (a) in presence of Assumption 4.1.

Remark 6.5 The outer semicontinuity of G implies outer semicontinuity of $G^{\cap C}$ and $G^{\cap D}$ defined in Eq. 12. Indeed, this follows from C, D being relatively closed in $\mathcal{O}$, and the fact that $G(\xi) \subset \mathcal{O}$ for all $\xi \in D$.

Inner semicontinuity of G does not imply inner semicontinuity of $G^{\cap C}$ or $G^{\cap D}$ (for example, inner semicontinuity of $G^{\cap C}$ fails at $\xi \in \operatorname{int} D$ if $G(\xi) \cap C \neq \emptyset$ but $G(\xi') \cap C = \emptyset$ for some ξ' arbitrarily close to ξ). The reverse implication, that inner semicontinuity of $G^{\cap C}$, $G^{\cap D}$ implies that of G, will be true if one has $G(\xi) \subset C \cup D$ for all $\xi \in D$, as then $G(\xi) = G^{\cap C}(\xi) \cup G^{\cap D}(\xi)$.

As we already noted, $G^{\cap C}$, $G^{\cap D}$ are outer semicontinuous, so (a) of Assumption 3.3 means that $G^{\cap C}$ and $G^{\cap D}$ are actually continuous (as set-valued mappings) relative to D. This continuity does not mean that $G^{\cap C}(\xi) \neq \emptyset$ for all $\xi \in D$, similarly, it may be that $G^{\cap D}(\xi) = \emptyset$ for some $\xi \in D$.

Given any connected subset S of D, it must be the case that either $G^{\cap C}(\xi) \neq \emptyset$ for all $\xi \in S$, or $G^{\cap C}(\xi) = \emptyset$ for all $\xi \in S$. Similarly for $G^{\cap D}$.

In the special case of $G(\xi) \subset C \setminus D$ for all $\xi \in D$, $G^{\cap D}$ is empty-valued, hence inner semicontinuous relative to D. Also then, $G^{\cap C} = G$ on D, so (a) of Assumption 3.3 is equivalent to continuity of G on D.

7 Special Case: Systems with Logic Variables

Let $Q \subset \mathbb{Z}^{n_d}$ be a set, and for each $q \in Q$ let $\mathcal{O}_q \subset \mathbb{R}^{n_c}$ be an open set, let C_q, $D_q \subset \mathcal{O}_q$, and let $F_q\colon \mathcal{O}_q \rightrightarrows \mathbb{R}^{n_c}$, $G_q\colon \mathcal{O}_q \rightrightarrows \mathbb{R}^{n_c} \times \mathbb{R}^{n_d}$ be (set-valued) mappings. Consider the hybrid system given by

$$\mathcal{H}_q: \qquad \begin{cases} \dot{\xi} & \in F_q(\xi) \quad \xi \in C_q \\ \begin{bmatrix} \xi^+ \\ q^+ \end{bmatrix} & \in G_q(\xi) \quad \xi \in D_q \end{cases} . \tag{19}$$

In particular, during flow, the so called "discrete variable" or "mode" q remains constant (i.e., $\dot{q} = 0$). Such a system can be thought of as a special case of Eq. 3, by considering the variable $x = (\xi, q)$, any open $\mathcal{O} \subset \mathbb{R}^{n_c} \times \mathbb{R}^{n_d}$ such that $\mathcal{O} \cap \mathbb{R}^{n_c} \times \mathbb{Z}^{n_d} = \bigcup_{q \in Q} \mathcal{O}_q \times \{q\}$, the sets

$$C = \bigcup_{q \in Q} C_q \times \{q\}, \qquad D = \bigcup_{q \in Q} D_q \times \{q\}, \tag{20}$$

and the mappings $F\colon \mathcal{O} \rightrightarrows \mathbb{R}^{n_c} \times \mathbb{R}^{n_d}$, $G\colon \mathcal{O} \rightrightarrows \mathbb{R}^{n_c} \times \mathbb{R}^{n_d}$ given by

$$F(\xi, q) = F_q(\xi) \times \{0\}, \qquad G(\xi, q) = G_q(\xi).$$

Such reformulation is what makes possible translating relaxation results, Theorems 3.4 and 4.5, to the system in Eq.19 and its relaxed version:

$$\mathcal{H}_q^{\text{con}}: \qquad \begin{cases} \dot{\xi} & \in \operatorname{con} F_q(\xi) \quad \xi \in C_q \\ \begin{bmatrix} \xi^+ \\ q^+ \end{bmatrix} & \in \quad G_q(\xi) \qquad \xi \in D_q \end{cases} \tag{21}$$

Given two hybrid arcs $x = (\xi, q) : \operatorname{dom} x \to \mathbb{R}^{n_c} \times \mathbb{R}^{n_d}$, $y = (\eta, r) : \operatorname{dom} y \to \mathbb{R}^{n_c} \times \mathbb{R}^{n_d}$ their graphical distance will be $\mathrm{d_{gph}}(x, y) = d(\operatorname{gph} x, \operatorname{gph} y)$, where d is the Pompeiu–Hausdorff distance in $\mathbb{R}^2 \times \mathbb{R}^{n_c} \times \mathbb{R}^{n_d} \times \mathbb{R}$ defined with the ball

$$\mathbb{B} = I\!B^1 \times I\!B^1 \times I\!B^{n_c} \times I\!B^{n_d}$$

that corresponds to the norm $\max\{|\alpha|, |\beta|, \|\gamma\|, \|\delta\|\}$ where $\alpha, \beta \in \mathbb{R}$, $\gamma \in \mathbb{R}^{n_c}$, $\delta \in \mathbb{R}^{n_d}$. Consider the condition

$$|t - s| \leqslant \varepsilon, \quad \|\xi(t, j) - \eta(s, j)\| \leqslant \varepsilon, \quad q(t, j) = r(s, j). \tag{22}$$

Given two hybrid arcs $x = (\xi, q)$, $y = (\eta, r)$ and $\varepsilon < 1$, $\mathrm{d_{gph}}(x, y) \leqslant \varepsilon$ if and only if

- For each $(t, j) \in \operatorname{dom} x$ there exists $(s, j) \in \operatorname{dom} y$ such that Eq. 22 holds, and
- For each $(s, j) \in \operatorname{dom} y$ there exists $(t, j) \in \operatorname{dom} x$ such that Eq. 22 holds.

In the current setting, Theorem 3.4 reduces to the following.

Corollary 7.1 *Suppose that for each $q \in Q$,*

(a) *The mappings $G_q^{\cap C}, G_q^{\cap D}, G_q^{\setminus} : \mathcal{O}_q \rightrightarrows \mathbb{R}^{n_c} \times \mathbb{R}^{n_d}$ defined by*

$$G_q^{\cap C}(x) = G_q(x) \cap C, \quad G_q^{\cap D}(x) = G_q(x) \cap D, \quad G_q^{\setminus}(x) = G_q(x) \cap \big[\mathcal{O} \setminus (C \cup D)\big]$$

with C and D as in Eq. 20 are inner semicontinuous relative to D_q;

(b) *Assumption* 3.3 (*b*) *holds with F_q, C_q, D_q replacing F, C, D, respectively.*

Then, for any $\xi_0 \in C_{q_0} \cup D_{q_0}$, strong relaxation for initially flowing (respectively, initially jumping) solutions from (ξ_0, q_0) relative to C_{q_0} (respectively, relative to D_{q_0}) is possible, that is: for any compact solution $x = (\xi, q) : \operatorname{dom} x \to \mathbb{R}^{n_c} \times \mathbb{R}^{n_d}$ to $\mathcal{H}_q^{\text{con}}$ with $\xi(0, 0) = \xi_0$, $q(0, 0) = q_0$ that initially flows (respectively, that initially jumps) and for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $\eta_0 \in (\xi(0, 0) + \delta I\!B) \cap C_{q_0}$ (respectively, for any $\eta_0 \in (x(0, 0) + \delta I\!B) \cap D_{q_0}$) there exist a hybrid arc $y = (\eta, r) : \operatorname{dom} y \to \mathbb{R}^{n_c} \times \mathbb{R}^{n_d}$ with compact dom y *and $\eta(0, 0) = \eta_0$, $r(0, 0) = q_0$ that is a solution to $\mathcal{H}_q$ and $\mathrm{d_{gph}}(x, y) \leqslant \varepsilon$, and moreover, if $\xi(T, J) \in D_{q(T,J)}$, where $(T, J) = \max \operatorname{dom} x$, then $\eta(\tau, J) \in D_{r(\tau,J)}$, where $(\tau, J) = \max \operatorname{dom} y$.*

Since during flows for hybrid systems in Eqs. 19, 21 the variable q remains constant, the conditions in Assumption 4.3 that are sufficient for (b) of Assumption 3.3 can be applied to Eqs. 19, 21 in each q separately. Thus, if, for each $q \in Q$, the sets C_q, D_q and the mapping F_q satisfy Assumption 4.3 (with C_q, D_q, F_q replacing C, D, F), (b) of Corollary 7.1 is satisfied.

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