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Positive Solutions for Nonlinear Periodic Problems with the Scalar p-Laplacian

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Abstract We study the existence of positive solutions for a nonlinear periodic problem driven by the scalar p-Laplacian and having a nonsmooth potential. We impose a nonuniform nonresonance condition at $+\infty$ and a uniform nonresonance condition at 0^+. Using degree theoretic argument based on a fixed point index for multifunctions, we prove the existence of a strict positive solution.

Keywords Nonsmooth potential · Scalar p-Laplacian · Generalized subdifferential · Weighted eigenvalue problem · Nonuniform nonresonance · Fixed point index

Mathematics Subject Classifications (2000) 34B15 · 34B18

1 Introduction

In this paper we study the existence of positive solutions for the following nonlinear periodic problem with nonsmooth potential (hemivariational inequality):

$$\begin{cases} -\big(|x'(t)|^{p-2}x'(t)\big)' \in \partial j\big(t, x(t)\big) & \text{for a.a. } t \in T = [0, b], \\ x(0) = x(b), \ x'(0) = x'(b), \end{cases} \tag{1.1}$$

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with $1 < p < +\infty$. In this problem the potential function $j(t, \zeta)$ is a measurable function which is only locally Lipschitz and not necessarily C^1 in the ζ-variable. By $\partial j(z, \zeta)$ we denote the generalized subdifferential of $\zeta \longmapsto j(t, \zeta)$ (see Section 2).

Recently there have been some interesting works on the question of existence of positive solutions for scalar differential equations driven by the p-Laplacian differential operator. We mention the works of Agarwal-Lü-O'Regan [1], Ben Naoum-De Coster [4], De Coster [5], Manásevich-Njoku-Zanolin [7] and Wang [10]. In all these works the problem under consideration has Dirichlet or more general Sturm-Liouville boundary condition (which do not cover the periodic case), the potential function j is C^1 in the ζ-variable and the method of proof either uses the time-map function (see Ben Naoum-De Coster [4], De Coster [5] and Manásevich-Njoku-Zanolin [7]) or fixed point theorems of the expansion-comparison type (see Agarwal-Lü-O'Regan [1] and Wang [10]). None of the aforementioned work cover the case of periodic problems with a nonsmooth potential.

Our approach is degree theoretic and based on the fixed point index for certain nonconvex valued multifunctions, which was introduced recently by Bader [3]. Moreover, asymptotically at $+\infty$ we permit partial interaction (nonuniform nonresonance) of the "slope" $\left\{\frac{u}{\zeta^{p-1}}\right\}_{u\in\partial j(t,\zeta)}$ with first two eigenvalues of the negative scalar p-Laplacian with periodic boundary condition. The first to use the tool of fixed point index (the Leray–Schauder fixed point index) in the study of positive solutions for elliptic boundary value problems, was Amann [2].

Finally we should mention that boundary value problems with a nonsmooth potential function (known as hemivariational inequalities), arise in many engineering applications. The book of Naniewicz–Panagiotopoulos [8] contains many such applications.

2 Mathematical Background

In this section, for the convenience of the reader, we review the main mathematical tools which we will use in the analysis of problem (1.1).

First we recall the spectrum of the negative scalar p-Laplacian, with periodic boundary conditions. So we consider the following nonlinear eigenvalue problem:

$$\begin{cases} -\left(|x'(t)|^{p-2}x'(t)\right)' = \lambda|x(t)|^{p-2}x(t) & \text{for a.a. } t \in T = [0, b], \\ x(0) = x(b),\ x'(0) = x'(b), \end{cases} \tag{2.1}$$

with $1 < p < +\infty$, $\lambda \in \mathbb{R}$. A number $\lambda \in \mathbb{R}$ is an eigenvalue if for this particular λ problem (2.1) above has a nontrivial solution $x \in C^1_{\text{per}}(T)$, with

$$C^1_{\text{per}}(T) = \left\{x \in C^1(T) : x(0) = x(b),\ x'(0) = x'(b)\right\},$$

known as an eigenfunction corresponding to the eigenvalue λ. Multiplying the differential equation in Eq. 2.1 with $x(t)$ and integrating over $T = [0, b]$, after an integrating by parts we obtain

$$\lambda = \frac{\|x'\|_p^p}{\|x\|_p^p} \geqslant 0.$$

So a necessary condition for Eq. 2.1 to have a nontrivial solution is that $\lambda \geqslant 0$. Note that $\lambda_0 = 0$ is an eigenvalue with corresponding eigenspace $\mathbb{R}$ (the constant functions). Every nonconstant eigenfunction must change sign. It can be shown that the eigenvalues of Eq. 2.1 are given by the sequence

$$\left\{\mu_{2n} = \left(\frac{2n\pi_p}{b}\right)^p\right\}_{n\geqslant 0},$$

where $\pi_p = \frac{2\pi(p-1)^{\frac{1}{p}}}{p\sin\frac{\pi}{p}}$. If $p=2$, then $\pi_2 = \pi$ and we recover the well known spectrum of the negative Laplacian with periodic boundary conditions, namely

$$\left\{\mu_{2n} = \left(\frac{2n\pi}{b}\right)^2\right\}_{n\geqslant 0}.$$

Note that the eigenfunctions $u \in C^1_{\mathrm{per}}(T)$ of the negative p-Laplacian satisfy

$$u(t) \neq 0 \quad \text{for a.a. } t \in T,$$

more precisely they have a finite number of zero points.

Our method of proof will also use the spectrum of the following weighted version of problem Eq. 2.1:

$$\begin{cases} -\big(|x'(t)|^{p-2}x'(t)\big)' = \big(\lambda + g(t)\big)|x(t)|^{p-2}x(t) & \text{for a.a. } t \in T = [0,b], \\ x(0) = x(b),\ x'(0) = x'(b), \end{cases} \tag{2.2}$$

with $1 < p < +\infty$, $g \in L^1(T)$, $\lambda \in \mathbb{R}$. This problem was studied by Zhang [11], who proved that Eq. 2.2 has a double sequence of eigenvalues $\{\underline{\lambda}_{2n}\}_{n\geqslant 0}$ and $\{\overline{\lambda}_{2n}\}_{n\geqslant 0}$, such that

$$-\infty < \overline{\lambda}_0(g) < \underline{\lambda}_2(g) \leqslant \overline{\lambda}_2(g) < \ldots < \underline{\lambda}_{2n}(g) \leqslant \overline{\lambda}_{2n}(g) < \ldots,$$

with

$$\underline{\lambda}_{2n}(g) \longrightarrow +\infty \quad \text{as } n \to +\infty.$$

If $p=2$ (linear eigenvalue problem), then these two sequences are all the eigenvalues of Eq. 2.2. If $p \neq 2$ (nonlinear eigenvalue problem), we do not know if this is the case.

Let Y, Z be two Hausdorff topological spaces and let $G\colon Y \longrightarrow 2^Z \setminus \{\emptyset\}$ be a multifunction. We say that G is upper semicontinuous, if for every nonempty, closed set $C \subseteq Z$, the set

$$G^-(C) = \{y \in Y : G(y) \cap C \neq \emptyset\}$$

is closed. If the space Y is regular and the multifunction G has closed values, then upper semicontinuity of G implies that

$$\operatorname{Gr} G = \{(y,z) \in Y \times Z : z \in G(y)\}$$

is closed. The converse is true, if G is locally compact, namely if for every $y \in Y$ we can find a neighbourhood U of y such that $\overline{G(U)}$ is compact in Z. For details we refer to Gasiński–Papageorgiou [6].

Now let X and V be two Banach spaces and $K\colon X \supseteq D \longrightarrow V$. We say that K is completely continuous, if for every sequence $\{x_n\}_{n\geqslant 1} \subseteq D$ converging weakly to $x \in D$ (i.e., $x_n \xrightarrow{w} x \in D$), we have that

$$K(x_n) \longrightarrow K(x) \qquad \text{in } V,$$

i.e., K is sequentially continuous from D with relative weak topology of X into V with the norm topology. If X is reflexive and $D \subseteq X$ is a nonempty, closed and convex set, then the complete continuity of K implies that K is compact, i.e., K is continuous and maps bounded sets in D onto relatively compact sets in V.

Suppose that $C \subseteq V$ and $D \subseteq X$ are nonempty, closed and convex sets and $G\colon C \longrightarrow 2^D \setminus \{\emptyset\}$ is a multifunction with weakly compact and convex values, which is upper semicontinuous from C with the relative norm topology of V into D with the relative weak topology of X. Also suppose that $K\colon D \longrightarrow C$ is completely continuous and set

$$S = K \circ G : C \longrightarrow 2^C \setminus \{\emptyset\}.$$

Assume that S is compact, i.e., S maps bounded sets onto relatively compact ones (this is the case if for example G maps bounded sets to bounded sets and X is reflexive; also note that in the present setting the compactness of S implies that S is also upper semicontinuous). The important feature that we want to emphasize, is that S need not have convex values. This corresponds to the case of nonlinear boundary value problems with multivalued nonlinearities. Finally let U be a bounded (relatively) open subset of C such that

$$\mathrm{Fix}(S) \cap \partial U = \emptyset,$$

where

$$\mathrm{Fix}(S) = \big\{x \in C : \ x \in S(x)\big\}$$

(the set of fixed points of S). For such triples (S, U, C), Bader [3] defined a fixed point index, denoted by $i_C(S, U)$, which exhibits the usual properties. If $S_0 = K_0 \circ G_0$ and $S_1 = K_1 \circ G_1$, then we say that S_0 and S_1 are homotopic, if there exist an upper semicontinuous multifunction $\vartheta\colon [0,1] \times C \longrightarrow 2^D \setminus \{\emptyset\}$ (where D is equipped with the relative weak topology) with weakly compact and convex values such that

$$\vartheta(0,\cdot) = G_0 \quad \text{and} \quad \vartheta(1,\cdot) = G_1$$

and a sequentially continuous map $\xi\colon [0,1] \times D \longrightarrow C$ (where D again is equipped with the relative weak topology) such that

$$\xi(0,\cdot) = K_0 \quad \text{and} \quad \xi(1,\cdot) = K_1.$$

We set

$$H(t,x) = \xi\big(\vartheta(t,x)\big)$$

and in the homotopy invariance property of the index, we require

$$x \notin H(t,x) \quad \forall t \in T, \quad x \in \partial U$$

and that H is compact (i.e., upper semicontinuous and maps bounded sets to relatively compact sets).

Let X be a Banach space and let $\varphi\colon X \longrightarrow \mathbb{R}$ be a locally Lipschitz function. The generalized derivative of φ at $x \in X$ in the direction $h \in X$ is defined by

$$\varphi^0(x; h) = \limsup_{\substack{x' \to x \\ \lambda \searrow 0}} \frac{\varphi(x' + \lambda h) - \varphi(x')}{\lambda}.$$

The function $h \longmapsto \varphi^0(x; h)$ is sublinear, continuous and so it is the support function of a nonempty, w^*-compact and convex set $\partial\varphi(x)$, defined by

$$\partial\varphi(x) = \big\{x^* \in X^* : \langle x^*, h\rangle_X \leqslant \varphi^0(x; h) \text{ for all } h \in X\big\}.$$

The multifunction $x \longmapsto \partial\varphi(x)$ is known as the generalized subdifferential of φ. If $\varphi \in C^1(X)$, then $\partial\varphi(x) = \{\varphi'(x)\}$. The generalized subdifferential has a rich calculus, which generalizes the subdifferential calculus of continuous, convex functions. For details we refer to Gasiński–Papageorgiou [6, pp. 48–62].

3 Auxiliary Results

The hypotheses on the nonsmooth potential are the following:

$\underline{H(j)}$ $j\colon T \times \mathbb{R} \longrightarrow \mathbb{R}$ is a function, such that

(i) $j(\cdot, \zeta)$ is measurable for all $\zeta \in \mathbb{R}$;

(ii) $j(t, \cdot)$ is locally Lipschitz for almost all $t \in T$ and $\partial j(t, \zeta) \subseteq \mathbb{R}_+$ for almost all $t \in T$ and all $\zeta \geqslant 0$;

(iii) for every $r > 0$, there exists $a_r \in L^1(T)_+$, such that

$$|u| \leqslant a_r(t),$$

for almost all $t \in T$, all $\zeta \in \mathbb{R}$ with $|\zeta| \leqslant r$ and all $u \in \partial j(t, \zeta)$;

(iv) there exist functions $\vartheta_1, \vartheta_2 \in L^1(T)_+$, such that

$$0 = \mu_0 \leqslant \vartheta_1(t) \leqslant \vartheta_2(t) \leqslant \mu_2 \quad \text{for almost all } t \in T,$$

where the first and third inequalities are strict on sets (not necessarily the same) of positive measure and

$$\vartheta_1(t) \leqslant \liminf_{\zeta \to +\infty} \frac{u}{\zeta^{p-1}} \leqslant \limsup_{\zeta \to +\infty} \frac{u}{\zeta^{p-1}} \leqslant \vartheta_2(t),$$

uniformly for almost all $t \in T$ and all $u \in \partial j(t, \zeta)$;

(v) there exists $\eta > 0$ and $\widehat{\eta} \in L^1(T)_+$, such that

$$\eta \leqslant \liminf_{\zeta \to 0^+} \frac{u}{\zeta^{p-1}} \leqslant \limsup_{\zeta \to 0^+} \frac{u}{\zeta^{p-1}} \leqslant \widehat{\eta}(t),$$

uniformly for almost all $t \in T$ and all $u \in \partial j(t, \zeta)$.

Remark 3.1 Hypothesis $H(j)$(iv) is a nonuniform nonresonance condition at $+\infty$ in the spectral interval $[\mu_0, \mu_2]$. Hypothesis $H(j)$(v) is a uniform nonresonance condition at 0^+ to the right of the first eigenvalue $\mu_0 = 0$. The following nonsmooth locally Lipschitz function $j\colon \mathbb{R} \longrightarrow \mathbb{R}$ (for simplicity we drop the dependence in t), satisfies hypotheses $H(j)$:

$$j(\zeta) = \begin{cases} \ln|\zeta| & \text{if } \zeta < -1, \\ -\cos\frac{\pi}{2}|\zeta|^{\frac{p}{2}} & \text{if } \zeta \in [-1,1], \\ \frac{\xi}{p}|\zeta|^p - \frac{\xi}{p} & \text{if } \zeta > 1, \end{cases}$$

with $\xi \in (0, \mu_2)$.

In the sequel by $\sigma(p)$ we denote the eigenvalues of the weighted eigenvalue problem (2.2). We denote

$$W^{1,p}_{per}(0,b) = \{x \in W^{1,p}(0,b) : x(0) = x(b)\}$$

and recall that the embedding $W^{1,p}_{per}(0,b) \subseteq C(T)$ is compact (this also justifies the evaluation at $t=0$ and $t=b$ in the definition of $W^{1,p}_{per}(0,b)$). In what follows by $\langle \cdot, \cdot \rangle$ we denote the duality bracket for the pair $\left(W^{1,p}_{per}(0,b), W^{1,p}_{per}(0,b)^*\right)$. Consider the nonlinear operator $A\colon W^{1,p}_{per}(0,b) \longrightarrow W^{1,p}_{per}(0,b)^*$, defined by

$$\langle A(x), y \rangle = \int_0^b \left|x'(t)\right|^{p-2} x'(t) y'(t)\, dt \qquad \forall x, y \in W^{1,p}_{per}(0,b).$$

It is easy to check that A is strictly monotone, demicontinuous, hence it is maximal monotone. Also let $K\colon W^{1,p}_{per}(0,b) \longrightarrow L^{p'}(T) \subseteq W^{1,p}_{per}(0,b)^*$ (with $\frac{1}{p} + \frac{1}{p'} = 1$) be the bounded, continuous, strictly monotone (hence maximal monotone) map, defined by

$$K(x)(\cdot) = \left|x(\cdot)\right|^{p-2} x(\cdot).$$

The first result is an easy observation concerning the spectrum of the weighted eigenvalue problem (2.2).

Proposition 3.2 *If $\vartheta_1, \vartheta_2 \in L^1(T)_+$ are such that for some $n \geqslant 1$, we have*

$$\mu_{2n} \leqslant \vartheta_1(t) \leqslant \vartheta_2(t) \leqslant \mu_{2n+2} \quad \textit{for almost all } t \in T,$$

with the first and third inequalities strict on sets (in general different) of positive measure and $g \in L^1(T)_+$ satisfies

$$\vartheta_1(t) \leqslant g(t) \leqslant \vartheta_2(t) \quad \textit{for almost all } t \in T,$$

then problem (2.2) has nonzero eigenvalues and zero is not a limit point of them.

Proof Due to the monotonicity of the eigenvalues

$$\{\underline{\lambda}_{2n}(g)\}_{n\geqslant 1} \quad \text{and} \quad \{\overline{\lambda}_{2n}(g)\}_{n\geqslant 0}$$

on the weight function g (see Zhang [11]), we have

$$\overline{\lambda}_{2n}(g) \leqslant \overline{\lambda}_{2n}(\vartheta_1) < \overline{\lambda}_{2n}(\mu_{2n}) = 0 \tag{3.1}$$

and

$$0 = \underline{\lambda}_{2n+2}(\mu_{2n+2}) < \underline{\lambda}_{2n+2}(\vartheta_2) \leqslant \underline{\lambda}_{2n+2}(g). \tag{3.2}$$

Moreover, from Zhang [11] we know that, if $\lambda \in \mathbb{R}$ is an eigenvalue of Eq. 2.2, then

$$\lambda \in \bigcup_{k\geqslant 1} \left[\underline{\lambda}_{2k}(g), \overline{\lambda}_{2k}(g)\right] \cup \left(-\infty, \overline{\lambda}_0(g)\right]. \tag{3.3}$$

From Eqs. 3.1, 3.2 and 3.3 it follows that $\lambda \neq 0$.

Suppose that we can find a sequence $\{\lambda_n\}_{n\geqslant 1} \subseteq \sigma(p)$, such that $\lambda_n \longrightarrow 0$. To every eigenvalue λ_n corresponds an eigenfunction $u_n \in C^1_{per}(T)$, $u_n \neq 0$, such that

$$\begin{cases} -\left(|u_n'(t)|^{p-2}u_n'(t)\right)' = \left(\lambda_n + g(t)\right)\left|u_n(t)\right|^{p-2}u_n(t) & \text{for a.a. } t \in T = [0, b], \\ u_n(0) = u_n(b), \;\; u_n'(0) = u_n'(b). \end{cases} \tag{3.4}$$

Note that Eq. 3.4 is $(p-1)$-homogeneous. So we may assume that

$$\|u_n\| \;=\; 1 \qquad \forall n \geqslant 1.$$

So passing to a suitable subsequence if necessary, we may assume that

$$u_n \overset{w}{\longrightarrow} u \quad \text{in } W^{1,p}_{per}(0, b)$$

and

$$u_n \longrightarrow u \quad \text{in } C(T).$$

We rewrite Eq. 3.4 as the following equivalent equation in $W^{1,p}_{per}(0, b)^*$:

$$A(u_n) \;=\; (\lambda_n + g)K(u_n) \qquad \forall n \geqslant 1, \tag{3.5}$$

so

$$\langle A(u_n), u_n - u\rangle \;=\; \int_0^b (\lambda_n + g)|u_n|^{p-2}u_n(u_n - u)\, dt \;\longrightarrow\; 0. \tag{3.6}$$

But A being maximal monotone it is generalized pseudomonotone (see Gasiński–Papageorgiou [6, p. 84]). So from Eq. 3.6 it follows that

$$\langle A(u_n), u_n\rangle \;\longrightarrow\; \langle A(u), u\rangle,$$

thus

$$\|u_n'\|_p \;\longrightarrow\; \|u'\|_p.$$

Recall that $u_n' \overset{w}{\to} u'$ in $L^p(T)$. Since the space $L^p(T)$ is uniformly convex, it has the Kadec-Klee property and so

$$u_n \longrightarrow u \quad \text{in } W^{1,p}_{per}(0, b).$$

Passing to the limit as $n \to +\infty$ in Eq. 3.5, we obtain

$$A(u) \;=\; gK(u), \qquad \|u\| = 1. \tag{3.7}$$

Let $\varphi \in C^1_c(0, b)$. Since

$$\left(|u'|^{p-2}u'\right)' \;\in\; W^{-1,p'}(0, b) \;=\; W^{1,p}_0(0, b)^*$$

(see e.g., Gasiński–Papageorgiou [6, p. 9]) and if by $\langle \cdot, \cdot \rangle_0$ we denote the duality brackets for the pair $\big(W_0^{1,p}(0,b), W^{-1,p'}(0,b)\big)$, we have

$$\big\langle -\big(|u'|^{p-2}u'\big)', \varphi\big\rangle_0 \;=\; \int_0^b g|u|^{p-2}u\varphi\, dt. \tag{3.8}$$

Because the embedding $C_c^1(0,b) \subseteq W^{1,p}_{per}(0,b)$ is dense, from Eq. 3.8, we infer that

$$\begin{cases} -\big(|u'(t)|^{p-2}u'(t)\big)' = g(t)\big|u(t)\big|^{p-2}u(t) & \text{for a.a. } t \in T = [0,b], \\ u(0) = u(b). \end{cases} \tag{3.9}$$

From Eq. 3.9 it follows that

$$|u'|^{p-2}u' \;\in\; W^{1,1}(0,b)$$

and so $u' \in C(T)$, which means that $u \in C^1(T)$. If in Eq. 3.7 we act with a test function $\psi \in W^{1,p}_{per}(0,b)$, we perform an integration by parts and we use Eq. 3.9, to obtain

$$\big|u'(0)\big|^{p-2}u'(0)\psi(0) \;=\; \big|u'(b)\big|^{p-2}u'(b)\psi(b) \qquad \forall \psi \in W^{1,p}_{per}(0,b),$$

so

$$u'(0) \;=\; u'(b).$$

So $u \in C^1_{per}(T)$ is a nontrivial solution of

$$\begin{cases} -\big(|u'(t)|^{p-2}u'(t)\big)' = g(t)\big|u(t)\big|^{p-2}u(t) & \text{for a.a. } t \in T = [0,b], \\ u(0) = u(b),\ u'(0) = u'(b). \end{cases}$$

But this means that $\lambda = 0 \in \sigma(p)$, which contradicts the first part of the proof. □

So because of Proposition 3.2, we can find $\varepsilon_0 \in (0,1)$, such that

$$(-\varepsilon_0, \varepsilon_0) \cap \sigma(p) \;=\; \emptyset.$$

Let $\varepsilon \in (0, \varepsilon_0)$ and consider the following auxiliary periodic problem

$$\begin{cases} -\big(|x'(t)|^{p-2}x'(t)\big)' + \varepsilon\big|x(t)\big|^{p-2}x(t) = h(t) & \text{for a.a. } t \in T = [0,b], \\ x(0) = x(b),\ x'(0) = x'(b). \end{cases} \tag{3.10}$$

Proposition 3.3 *For every $h \in L^1(T)$ problem (3.10) has a unique solution $V_\varepsilon(h) \in C^1(T)$ and the solution map*

$$L^1(T) \ni h \longmapsto V_\varepsilon(h) \in W^{1,p}_{per}(0,b)$$

is completely continuous, i.e., if $h_n \xrightarrow{w} h$ in $L^1(T)$, then $V_\varepsilon(h_n) \longrightarrow V(h)$ in $W^{1,p}_{per}(0,b)$.

Proof We consider the operator $S_\varepsilon\colon W^{1,p}_{per}(0,b) \longrightarrow W^{1,p}_{per}(0,b)^*$, defined by

$$S_\varepsilon(x) \;=\; A(x) + \varepsilon K(x).$$

Then we have

$$\left\langle S_\varepsilon(x), x\right\rangle = \|x'\|_p^p + \varepsilon\|x\|_p^p \geqslant \varepsilon\|x\|^p,$$

so S_ε is coercive.

Clearly S_ε is also maximal monotone. But a maximal monotone, coercive operator is surjective (see Gasiński–Papageorgiou [6, p. 80]). So we can find $x \in W^{1,p}_{per}(0,b)$, such that

$$S_\varepsilon(x) = h \tag{3.11}$$

(recall that $L^1(T) \subseteq W^{1,p}_{per}(0,b)^*$). The operator S_ε is strictly monotone (since both operators A and K are strictly monotone). So the solution $x \in W^{1,p}_{per}(0,b)$ is unique. As in the proof of Proposition 1.1, from Eq. 1.1 we deduce that $x = V_\varepsilon(h) \in C^1_{per}(T)$ and it solves problem (3.10).

Next we show that the solution map $V_\varepsilon\colon L^1(T) \longrightarrow W^{1,p}_{per}(0,b)$ is completely continuous. To this end suppose that

$$h_n \overset{w}{\longrightarrow} h \quad \text{in } L^1(T).$$

Let us set

$$x_n = V_\varepsilon(h_n) \in C^1_{per}(T) \quad \forall n \geqslant 1.$$

We have

$$\begin{cases} -\left(|x_n'(t)|^{p-2}x_n'(t)\right)' + \varepsilon\left|x_n(t)\right|^{p-2}x_n(t) = h_n(t) & \text{for a.a. } t \in T = [0,b], \\ x_n(0) = x_n(b),\ x_n'(0) = x_n'(b). \end{cases}$$

Multiplying this equation with $x_n(t)$, integrating over $T = [0,b]$ and performing an integration by parts, we have

$$\|x_n'\|_p^p + \varepsilon\|x_n\|_p^p = \int_0^b h_n x_n\, dt \leqslant \|h_n\|_1\|x_n\|_\infty,$$

so

$$\varepsilon\|x_n\|^p \leqslant c_1\|x_n\| \qquad \forall n \geqslant 1,$$

for some $c_1 > 0$. Thus the sequence $\{x_n\}_{n\geqslant 1} \subseteq W^{1,p}_{per}(0,b)$ is bounded.

So we may assume that

$$x_n \overset{w}{\longrightarrow} x \quad \text{in } W^{1,p}_{per}(0,b)$$

and

$$x_n \longrightarrow x \quad \text{in } C(T).$$

We have

$$A(x_n) + \varepsilon K(x_n) = h_n \qquad \forall n \geqslant 1. \tag{3.12}$$

Using as a test function $x_n - x \in W^{1,p}_{per}(0,b)$, we obtain

$$\langle A(x_n), x_n - x\rangle + \varepsilon\int_0^b |x_n|^{p-2}x_n(x_n - x)\, dt = \int_0^b h_n(x_n - x)\, dt \qquad \forall n \geqslant 1.$$

Evidently

$$\int_0^b |x_n|^{p-2} x_n (x_n - x)\, dt \;\longrightarrow\; 0$$

and

$$\int_0^b h_n (x_n - x)\, dt \;\longrightarrow\; 0.$$

Therefore, we have

$$\lim_{n\to+\infty} \big\langle A(x_n), x_n - x \big\rangle \;=\; 0,$$

from which as before (see the proof of Proposition 3.2), we deduce that

$$x_n \;\longrightarrow\; x \quad \text{in } W^{1,p}_{per}(0,b).$$

Passing to the limit as $n \to +\infty$ in Eq. 3.12, we obtain

$$A(x) + \varepsilon K(x) \;=\; h,$$

so

$$\begin{cases} -\big(|x'(t)|^{p-2}x'(t)\big)' + \varepsilon \big|x(t)\big|^{p-2} x(t) = h(t) & \text{for a.a. } t \in T = [0,b], \\ x(0) = x(b),\; x'(0) = x'(b), \end{cases}$$

thus $x = V_\varepsilon(h)$. Therefore by Urysohn's criterion for convergence sequences, we conclude that for the original sequence, we have

$$x_n \;=\; V_\varepsilon(h_n) \;\longrightarrow\; x \;=\; V_\varepsilon(h) \quad \text{in } W^{1,p}_{per}(0,b),$$

so V_ε is completely continuous. □

We consider the Lipschitz continuous truncation map $\tau \colon \mathbb{R} \longrightarrow \mathbb{R}_+$, defined by

$$\tau(\zeta) \;=\; \begin{cases} 0 & \text{if } \zeta \leqslant 0, \\ \zeta & \text{if } \zeta > 0. \end{cases}$$

We set $j_+(t,\zeta) = j\big(t, \tau(\zeta)\big)$. Evidently for all $\zeta \in \mathbb{R}$, the function $t \longmapsto j_+(t,\zeta)$ is measurable and for almost all $t \in T$, the function $\zeta \longmapsto j_+(t,\zeta)$ is locally Lipschitz. Also from the nonsmooth chain rule (see Gasiński–Papageorgiou [6, pp. 54–55]), we have

$$\partial j_+(t,\zeta) \;=\; \begin{cases} \{0\} & \text{if } \zeta < 0, \\ \operatorname{conv}\{r \partial j(t,0) : \; r \in [0,1]\} & \text{if } \zeta = 0, \\ \partial j(t,\zeta) & \text{if } \zeta > 0. \end{cases} \tag{3.13}$$

We consider the multifunction $G \colon W^{1,p}_{per}(0,b) \longrightarrow 2^{L^1(T)}$, defined by

$$G(x) \;=\; \big\{ u \in L^1(T) : \; u(t) \in \partial j_+\big(t, x(t)\big) \text{ for a.a. } t \in T \big\}.$$

Proposition 3.4 *If hypotheses* $H(j)$(i), (ii) *and* (iii) *hold, then* G *has nonempty, weakly compact and convex values in* $L^1(T)$ *and it is upper semicontinuous from* $W^{1,p}_{per}(0,b)$

into $L^1(T)_w$ (by $L^1(T)_w$ we denote the Lebesgue space $L^1(T)$ furnished with the weak topology).

Proof By definition (see Section 2), for every $\xi \in \mathbb{R}$, we have

$$j_+^0\big(t, x(t); \xi\big) = \limsup_{\substack{v \to x(t) \\ r \searrow 0}} \frac{j_+(t, v + r\xi) - j_+(t, v)}{r} = \inf_{\varepsilon,\delta>0} \sup_{\substack{|v - x(t)| < \delta \\ 0 < r < \varepsilon}} \frac{j_+(t, v + r\xi) - j_+(t, v)}{r} = \inf_{n,m\geqslant 1} \sup_{\substack{|v - x(t)| < \frac{1}{n} \\ 0 < r < \frac{1}{m},\, v, r \in \mathbb{Q}}} \frac{j_+(t, v + r\xi) - j_+(t, v)}{r}. \tag{3.14}$$

From hypotheses $H(j)$(i) and (ii), we have that the function $(t, \zeta) \longmapsto j_+(t, \zeta)$ is $\mathcal{L}(T) \times \mathcal{B}(\mathbb{R})$-measurable with $\mathcal{L}$ being the Lebesgue σ-field of T and with $\mathcal{B}(\mathbb{R})$ being the Borel σ-field. Hence Eq. 3.14 implies that the function $t \longmapsto j_+^0\big(t, x(t); \xi\big)$ is Lebesgue measurable. Also we know that the function $\xi \longmapsto j_+^0\big(t, x(t); \xi\big)$ is continuous. From the definition of the generalized subdifferential, we have

$$\operatorname{Gr} \partial j_+\big(\cdot, x(\cdot)\big) = \big\{(t, u) \in T \times \mathbb{R} : u\xi \leqslant j_+^0\big(t, x(t); \xi\big) \text{ for all } \xi \in \mathbb{R}\big\}.$$

Let $\{\xi_n\}_{n\geqslant 1}$ be an enumeration of the rationals in $\mathbb{R}$. Exploiting the continuity of the function $\xi \longmapsto j_+^0\big(t, x(t); \xi\big)$, we have

$$\operatorname{Gr} \partial j_+\big(\cdot, x(\cdot)\big) = \big\{(t, u) \in T \times \mathbb{R} : u\xi_n \leqslant j_+^0(t, x(t); \xi_n),\ n \geqslant 1\big\} \in \mathcal{L}(T) \times \mathcal{B}(\mathbb{R}).$$

Invoking the Yankov-von Neumann-Aumann selection theorem (see Gasiński–Papageorgiou [6, p. 23]), we obtain a Lebesgue measurable function $u\colon T \longrightarrow \mathbb{R}$, such that

$$u(t) \in \partial j_+\big(t, x(t)\big) \quad \text{for a.a. } t \in T.$$

If $r = \|x\|_\infty$, because of hypothesis $H(j)$(iii), we have

$$\big|u(t)\big| \leqslant a_r(t) \quad \text{for a.a. } t \in T,$$

i.e., $u \in L^1(T)$.

So we have proved that for every $x \in W^{1,p}_{per}(0, b)$, $G(x) \neq \emptyset$ and it is easily seen to be closed and convex. Moreover, for every $x \in W^{1,p}_{per}(0, b)$, we have

$$G(x) \subseteq W_r = \big\{u \in L^1(T) : \big|u(t)\big| \leqslant a_r \text{ for a.a. } t \in T, \big\}$$

where $r = \|x\|_\infty$. So by the Dunford–Pettis theorem (see Gasiński–Papageorgiou [6, p. 723]), it follows that $G(x)$ is also weakly compact in $L^1(T)$.

The fact that $G(x) \subseteq W_r$, implies that the multifunction G is locally compact into $L^1(T)_w$ (i.e., into $L^1(T)$ equipped with the weak topology). Also since $L^1(T)$ is separable, the weakly compact subsets of $L^1(T)$ furnished with the relative weak topology are metrizable. So in order to prove the claimed upper semicontinuity of

G, it suffices to show that $\mathrm{Gr}\, G$ is sequentially closed in $W^{1,p}_{per}(0,b) \times L^1(T)_w$. To this end let $\{(x_n, h_n)\}_{n \geqslant 1} \subseteq \mathrm{Gr}\, G$ and assume that

$$x_n \longrightarrow x \quad \text{in } W^{1,p}_{per}(0,b)$$

and

$$h_n \xrightarrow{w} h \quad \text{in } L^1(T).$$

Since the generalized subdifferential $\zeta \longmapsto \partial j_+(t,\zeta)$ has closed graph, from Gasiński–Papageorgiou [6, p. 31], we have that

$$h(t) \in \operatorname{conv} \limsup_{n \to +\infty} \partial j_+\big(t, x_n(t)\big) \subseteq \partial j_+\big(t, x(t)\big) \quad \text{for a.a. } t \in T,$$

so

$$(x,h) \in \mathrm{Gr}\, G,$$

i.e., G is upper semicontinuous from $W^{1,p}_{per}(0,b)$ into $L^1(T)_w$. □

Now let

$$C_+ = \big\{x \in W^{1,p}_{per}(0,b) : x(t) \geqslant 0 \text{ for all } t \in T\big\}.$$

This is the positive cone of the ordered Banach space $W^{1,p}_{per}(0,b)$. It is a closed, convex cone, hence a retract of $W^{1,p}_{per}(0,b)$.

Lemma 3.5 *If hypotheses $H(j)$(i), (ii) and (iii) hold, then for every $\varepsilon > 0$, we have*

$$V_\varepsilon \circ (\varepsilon K + G) \subseteq C_+.$$

Proof Let $x \in C_+$ and consider $y \in \big[V_\varepsilon \circ (\varepsilon K + G)\big](x)$. We have

$$A(y) + \varepsilon K(y) = \varepsilon K(x) + u,$$

with $u \in G(x)$. Acting with the test function $-y^- \in W^{1,p}_{per}(0,b)$, we obtain

$$\big\|(y^-)'\big\|_p^p + \varepsilon \|y^-\|_p^p = -\varepsilon \int_0^b |x|^{p-2} x y^- \, dt - \int_0^b u y^- \, dt,$$

so

$$\varepsilon \|y^-\|^p \leqslant -\varepsilon \int_0^b |x|^{p-2} x y^- \, dt - \int_0^b u y^- \, dt.$$

Since $x \in C_+$, we have

$$-\varepsilon \int_0^b |x|^{p-2} x y^- \, dt \leqslant 0.$$

Also since

$$u(t) \in \partial j_+\big(t, x(t)\big) \quad \text{for a.a. } t \in T,$$

by hypothesis $H(j)$(ii) and Eq. 3.13, we have that

$$u(t) \geqslant 0 \quad \text{for a.a. } t \in T$$

and so

$$-\int_0^b uy^- \, dt \leqslant 0.$$

So finally $\varepsilon \|y^-\|^p \leqslant 0$, which implies that $y^- \equiv 0$, hence $y \in C_+$. □

This lemma implies that we can consider the fixed point index

$$i_{C_+}\big(V_\varepsilon \circ (\varepsilon K + G),\, B_r^+\big),$$

with $r > 0$, where $B_r^+ = B_r \cap C_+$.

Proposition 3.6 *If hypotheses $H(j)$ hold, then we can find $R_0 > 0$, such that*

$$i_{C_+}\big(V_\varepsilon \circ (\varepsilon K + G),\, B_R^+\big) = 1 \qquad \forall R \geqslant R_0.$$

Proof For $g \in L^1(T)$ we consider the admissible homotopy $H_1(\beta, x)$, defined by

$$H_1(\beta, x) = \big[V_\varepsilon \circ \big(\varepsilon K + \beta G + (1-\beta) g K\big)\big](x).$$

Claim 1 There exist a function $g_0 \in L^1(T)$ and $R_0 > 0$, such that

$$x \notin H_1(\beta, x) \qquad \forall\, \beta \in [0,1], \qquad x \in \partial B_R^+, \qquad R \geqslant R_0. \tag{3.15}$$

Suppose that Eq. 3.15 is not true. This means that we can find sequences $\{\beta\}_{n\geqslant 1} \subseteq [0,1]$ and $\{x_n\}_{n\geqslant 1} \subseteq C_+$, such that

$$\beta_n \longrightarrow \beta \in [0,1],$$
$$\|x_n\| \longrightarrow +\infty$$

and

$$A(x_n) = \beta_n u_n + (1-\beta_n) g K(x_n), \tag{3.16}$$

with

$$u_n \in G(x_n) \qquad \forall n \geqslant 1.$$

We set

$$y_n = \frac{x_n}{\|x_n\|} \qquad \forall n \geqslant 1.$$

By passing to a subsequence if necessary, we may assume that

$$y_n \stackrel{w}{\longrightarrow} y \quad \text{in } W^{1,p}_{per}(0,b),$$
$$y_n \longrightarrow y \quad \text{in } C(T).$$

For every $\delta > 0$ and $n \geqslant 1$, we define the set

$$D_{\delta,n} = \left\{ t \in T : \; x_n(t) > 0, \; \vartheta_1(t) - \delta \leqslant \frac{u(t)}{x_n(t)^{p-1}} \leqslant \vartheta_2(t) + \delta \right.$$
$$\left. \text{for all } u(t) \in \partial j_+\big(t, x_n(t)\big) = \partial j\big(t, x_n(t)\big) \right\}.$$

Note that for every $t \in \{y > 0\}$, we have that $x_n(t) \longrightarrow +\infty$ and so from hypothesis $H(j)$(iv), we have that

$$\chi_{D_{\varepsilon,n}}(t) \longrightarrow 1 \quad \text{for a.a. } t \in \{y > 0\}.$$

By virtue of hypotheses $H(j)$(iii), (iv) and Eq. 3.15, we see that for almost all $t \in T$, all $\zeta \in \mathbb{R}$ and all $u \in \partial j_+(t, \zeta)$, we have

$$|u| \leqslant \widehat{a}(t) + \widehat{c}(t)|x|^{p-1},$$

with $\widehat{a}, \widehat{c} \in L^1(T)_+$, so

$$\frac{|u_n(t)|}{\|x_n\|^{p-1}} \leqslant \frac{\widehat{a}(t)}{\|x_n\|^{p-1}} + \widehat{c}(t) y_n(t)^{p-1} \tag{3.17}$$

(since $y_n \geqslant 0$). Thus the sequence $\left\{\frac{u_n}{\|x_n\|^{p-1}}\right\}_{n\geqslant 1} \subseteq L^1(T)$ is uniformly integrable.

Because of the Dunford–Pettis theorem (see Gasiński–Papageorgiou [6, p. 723]), at least for a subsequence, we can say that

$$\frac{u_n}{\|x_n\|^{p-1}} \xrightarrow{w} w_0 \quad \text{in } L^1(T).$$

Note that

$$\left\| \big(1 - \chi_{D_{\delta,n}}\big) \frac{u_n}{\|x_n\|^{p-1}} \right\|_{L^1(\{y>0\})} \longrightarrow 0,$$

so

$$\chi_{D_{\delta,n}} \frac{u_n}{\|x_n\|^{p-1}} \xrightarrow{w} w_0 \quad \text{in } L^1(\{y > 0\}).$$

From the definition of the set $D_{\delta,n}$, we have

$$\chi_{D_{\delta,n}}(t) \frac{u_n(t)}{\|x_n\|^{p-1}} = \chi_{D_{\delta,n}}(t) \frac{u_n(t)}{x_n^{p-1}(t)} y_n(t)^{p-1}$$
$$\leqslant \chi_{D_{\delta,n}}(t) \big(\vartheta_2(t) + \delta\big) y_n(t)^{p-1}.$$

If we pass to the limit as $n \to +\infty$ and we use Mazur's lemma, we obtain

$$w_0(t) \leqslant \big(\vartheta_2(t) + \delta\big) y(t)^{p-1} \quad \text{for a.a. } t \in \{y > 0\}.$$

Since $\delta > 0$ was arbitrary, we let $\delta \searrow 0$, to obtain

$$w_0(t) \leqslant \vartheta_2(t) y(t)^{p-1} \quad \text{for a.a. } t \in \{y > 0\}.$$

Moreover, from Eq. 3.17, it is clear that

$$w_0(t) = 0 \quad \text{for a.a. } t \in \{y = 0\}.$$

Since $y \in C_+$, we have that

$$T = \{y > 0\} \cup \{y = 0\}$$

and so from the above, we infer that

$$w_0(t) \leqslant \vartheta_2(t) y(t)^{p-1} \quad \text{for a.a. } t \in T. \tag{3.18}$$

In a similar fashion we also show that

$$\vartheta_1(t) y(t)^{p-1} \leqslant w_0(t) \quad \text{for a.a. } t \in T. \tag{3.19}$$

From Eqs. 3.18 and 3.19, it follows that

$$\vartheta_1(t) y(t)^{p-1} \leqslant w_0(t) \leqslant \vartheta_2(t) y(t)^{p-1} \quad \text{for a.a. } t \in T,$$

so

$$w_0(t) = g_0(t) y(t)^{p-1} \quad \text{and} \quad \vartheta_1(t) \leqslant g_0(t) \leqslant \vartheta_2(t) \quad \text{for a.a. } t \in T, \tag{3.20}$$

with $g_0 \in L^1(T)_+$. We return to Eq. 3.16 and divide with $\|x_n\|^{p-1}$. Because of the $(p-1)$-homogeneity of the operators A and K, we obtain

$$A(y_n) = \beta_n \frac{u_n}{\|x_n\|^{p-1}} + (1-\beta_n) g K(y_n). \tag{3.21}$$

In this operator equation, we act with the test function $y_n - y \in W^{1,p}_{per}(0,b)$. So we have

$$\langle A(y_n), y_n - y \rangle = \int_0^b \beta_n \frac{u_n}{\|x_n\|^{p-1}} (y_n - y)\, dt + \int_0^b (1-\beta_n) g |y_n|^{p-2} y_n (y_n - y)\, dt.$$

Note that

$$\beta_n \frac{u_n}{\|x_n\|^{p-1}} (y_n - y)\, dt \longrightarrow 0$$

and

$$\int_0^b (1-\beta_n) g |y_n|^{p-2} y_n (y_n - y)\, dt \longrightarrow 0.$$

Therefore, it follows that

$$\lim_{n \to +\infty} \langle A(y_n), y_n - y \rangle = 0.$$

As before from this convergence we infer that

$$y_n \longrightarrow y \quad \text{in } W^{1,p}_{per}(0,b). \tag{3.22}$$

So if in Eq. 3.21 we pass to the limit as $n \to +\infty$ and we use Eq. 3.22, we obtain

$$A(y) = \beta g_0 K(y) + (1-\beta) g K(y)$$

(see Eq. 3.20). We put $g = g_0 \in L^1(T)_+$ and we have

$$A(y) = g_0 K(y).$$

From this operator equation, as in the proof of Proposition 3.2, we obtain

$$\begin{cases} -\big(|y'(t)|^{p-2}y'(t)\big)' = g_0(t)\big|y(t)\big|^{p-2}y(t) & \text{for a.a. } t \in T = [0, b], \\ y(0) = y(b),\ y'(0) = y'(b). \end{cases} \tag{3.23}$$

Note that because of Eq. 3.22, we have $\|y\| = 1$, i.e., $y \neq 0$. So Eq. 3.23 implies that $0 \in \sigma(p)$, which contradicts Proposition 3.2 (see Eq. 3.20). This proves that Eq. 3.15 is indeed true and so Claim 1 is proved.

Claim 2 We have

$$i_{C_+}\big(V_\varepsilon \circ (\varepsilon + g_0)K,\, B_R^+\big) \;=\; 1 \qquad \forall R \geqslant R_0.$$

From the definition of the Leray–Schauder fixed point index (see Amann [2]), we have

$$i_{C_+}\big(V_\varepsilon \circ (\varepsilon + g_0)K,\, B_R^+\big) \;=\; d_{LS}\big(I - V_\varepsilon \circ (\varepsilon + g_0)K \circ r,\, B_R \cap r^{-1}(B_R^+), 0\big),$$

where r: $W_0^{1,p}(0, b) \longrightarrow C_+$ is an retraction. We consider the retraction

$$r(x) \;=\; x^+ \;=\; \max\{x, 0\}.$$

Then for this retraction consider the compact homotopy $\widehat{h}(\beta, x)$, defined by

$$\widehat{h}(\beta, x) \;=\; \beta V_\varepsilon \circ (\varepsilon + g_0)K \circ r(x).$$

First note that if $x = \widehat{h}(\beta, x)$, then

$$x = \beta V_\varepsilon\big((\varepsilon + g_0)K(x^+)\big),$$

so

$$\frac{1}{\beta^{p-1}} A(x) + \frac{1}{\beta^{p-1}} \varepsilon K(x) \;=\; (\varepsilon + g_0)K(x^+).$$

Using as a test function $-x^- \in W_{per}^{1,p}(0, b)$, we obtain

$$\|(x^-)'\|_p^p + \varepsilon \|x^-\|_p^p \;=\; 0,$$

so

$$\varepsilon \|x^-\|^p \;\leqslant\; 0,$$

i.e., $x^- = 0$ and so $x = x^+ \in C_+$.

Next, if for $x \in C_+$, we have

$$x \;=\; \beta V_\varepsilon\big((\varepsilon + g_0)K(x)\big),$$

so

$$A(x) \;=\; \big((\beta^{p-1} - 1)\varepsilon + \beta^{p-1} g_0\big)K(x)$$

and thus

$$\begin{cases} -\big(|x'(t)|^{p-2}x'(t)\big)' = \big((\beta^{p-1} - 1)\varepsilon + \beta^{p-1} g_0\big)\big|x(t)\big|^{p-2}x(t) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for a.a. } t \in T = [0, b], \\ x(0) = x(b),\ x'(0) = x'(b). \end{cases} \tag{3.24}$$

If $\beta \in (0, 1]$, then because of hypothesis $H(\mathrm{i})(\mathrm{iv})$, we have

$$0 = \mu_0 \leqslant \beta^{p-1}\vartheta_1(t) \leqslant \beta^{p-1}g_0(t) \leqslant \beta^{p-1}\vartheta_2(t) \leqslant \mu_2$$

and the first and last inequalities are strict on sets (in general different) of positive measure. Also

$$0 \geqslant (\beta^{p-1} - 1)\varepsilon \geqslant -\varepsilon > -\varepsilon_0$$

and so from Eq. 3.24 and Proposition 3.2, we infer that $x = 0$.

If $\beta = 0$, then clearly $x = 0$.

Therefore, from the homotopy invariance of the Leray–Schauder degree, we have that

$$d_{LS}\big(I - V_\varepsilon \circ (\varepsilon + g_0)K \circ r, B_R \cap r^{-1}(B_R^+), 0\big) = 1,$$

so

$$i_{C_+}\big(V_\varepsilon \circ (\varepsilon + g_0)K, B_R^+\big) = 1,$$

which proves Claim 2.

Using Claim 1, from the homotopy invariance of the fixed point index, we have

$$i_{C_+}\big(V_\varepsilon \circ (\varepsilon K + G), B_R^+\big) = i_{C_+}\big(V_\varepsilon \circ (\varepsilon + g_0), B_R^+\big). \tag{3.25}$$

Thus from Claim 2, we have

$$i_{C_+}\big(V_\varepsilon \circ (\varepsilon + G)K, B_R^+\big) = 1 \qquad \forall R \geqslant R_0.$$

□

We will prove something analogous for small balls.

Proposition 3.7 *If hypotheses $H(j)$ hold, then we can find $\varrho_0 > 0$, such that*

$$i_{C_+}\big(V_\varepsilon \circ (\varepsilon K + G)K, B_\varrho^+\big) = 0 \qquad \forall \varrho \in (0, \varrho_0].$$

Proof We consider the admissible homotopy $H_2(\beta, x)$, defined by

$$H_2(\beta, x) = \big[V_\varepsilon \circ \big(\varepsilon K + \beta\eta K + (1-\beta)G\big)\big](x) \qquad \forall \beta \in [0, 1], \qquad x \in W^{1,p}_{per}(0, b),$$

where $\eta > 0$ is as in hypothesis $H(j)(\mathrm{v})$.

Claim 1 There exists $\varrho_0 > 0$ small enough, such that

$$x \notin H_2(\beta, x) \qquad \forall \beta \in [0, 1], \qquad x \in \partial B_\varrho, \qquad \varrho \in (0, \varrho_0]. \tag{3.26}$$

Suppose that Eq. 3.26 is not true. Then we can find sequences $\{\beta_n\}_{n\geqslant 1} \subseteq [0, 1]$ and $\{x_n\}_{n\geqslant 1} \subseteq C_+$, such that

$$\beta_n \longrightarrow \beta \in [0, 1], \qquad \|x_n\| \longrightarrow 0$$

and

$$x_n \in H_2(\beta_n, x_n) \qquad \forall n \geqslant 1.$$

From the last inclusion, it follows that

$$A(x_n) \;=\; \beta_n \eta K(x_n) + (1-\beta_n)u_n \qquad \forall n \geqslant 1, \tag{3.27}$$

with $u_n \in G(x_n)$ for all $n \geqslant 1$. Let

$$y_n \;=\; \frac{x_n}{\|x_n\|} \qquad \forall n \geqslant 1.$$

Then we may assume (at least for a subsequence), that

$$y_n \xrightarrow{w} y \quad \text{in } W^{1,p}_{per}(0,b)$$

and

$$y_n \longrightarrow y \quad \text{in } C(T).$$

First suppose that by passing to a subsequence if necessary, we have $\beta_n < 1$ for all $n \geqslant 1$. From Eq. 3.27, we obtain

$$\begin{cases} -\big(|x_n'(t)|^{p-2}x_n'(t)\big)' = \beta_n \eta \big|x_n(t)\big|^{p-2}x_n(t) + (1-\beta_n)u_n(t) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for a.a. } t \in T = [0,b], \\ x_n(0) = x_n(b),\; x_n'(0) = x_n'(b). \end{cases} \tag{3.28}$$

From Stampacchia's theorem, we know that

$$x_n'(t) \;=\; 0 \quad \text{for a.a. } t \in \{x_n = 0\}.$$

So from Eq. 3.28, we have

$$(1-\beta_n)u_n(t) \;=\; 0 \quad \text{for a.a. } t \in \{x_n = 0\},$$

and thus

$$u_n(t) \;=\; 0 \quad \text{for a.a. } t \in \{x_n = 0\} \tag{3.29}$$

(since $\beta_n < 1$ for all $n \geqslant 1$). Because of hypotheses $H(j)$(ii), (iii), (iv), (v) and Eq. 3.29, we can find $\gamma \in L^1(T)_+$, such that

$$0 \;\leqslant\; u_n(t) \;\leqslant\; \gamma(t)x_n(t)^{p-1} \quad \text{for a.a. } t \in T,$$

so

$$\frac{|u_n(t)|}{\|x_n\|^{p-1}} \;\leqslant\; \gamma(t)y_n(t)^{p-1} \quad \text{for a.a. } t \in T$$

and thus the sequence $\left\{\frac{u_n}{\|x_n\|^{p-1}}\right\}_{m\geqslant 1} \subseteq L^1(T)$ is uniformly integrable.

By the Dunford–Pettis theorem (see Gasiński–Papageorgiou [6, p. 723]), we can say that

$$\frac{u_n}{\|x_n\|^{p-1}} \xrightarrow{w} f \quad \text{in } L^1(T).$$

Since $\{x_n\}_{n\geqslant 1} \subseteq C_+$ and $\|x_n\| \longrightarrow 0$, we have that

$$x_n(t) \longrightarrow 0^+ \quad \text{uniformly for all } t \in T.$$

For $\varepsilon > 0$ and $n \geqslant 1$, we consider the set

$$E_{\varepsilon,n} = \left\{t \in T : x_n(t) > 0,\ \eta - \varepsilon \leqslant \frac{u_n(t)}{x_n(t)^{p-1}} \leqslant \widehat{\eta} + \varepsilon\right\}$$

(see hypothesis $H(j)(\mathrm{v})$). Arguing as in the proof of Proposition 3.6, we can show that

$$\eta y(t)^{p-1} \leqslant f(t) \leqslant \widehat{\eta}(t) y(t)^{p-1} \quad \text{for a.a. } t \in T,$$

so

$$f(t) = g(t) y(t)^{p-1} \quad \text{for a.a. } t \in T,$$

with

$$g \in L^1(T) \quad \text{and} \quad \eta \leqslant g(t) \leqslant \widehat{\eta}(t) \quad \text{for a.a. } t \in T. \tag{3.30}$$

If we divide Eq. 3.27 with $\|x_n\|^{p-1}$, we obtain

$$A(y_n) = \beta_n \eta K(y_n) + (1-\beta_n) \frac{u_n}{\|x_n\|^{p-1}}, \tag{3.31}$$

so

$$\langle A(y_n), y_n - y\rangle = \int_0^b \beta_n \eta |y_n|^{p-2} y_n (y_n - y)\, dt + \int_0^b (1-\beta_n) \frac{u_n}{\|x_n\|^{p-1}}\, dt$$

and thus

$$\lim_{n \to +\infty} \langle A(y_n), y_n - y\rangle = 0.$$

From this convergence as before, we infer that

$$y_n \longrightarrow y \quad \text{in } W^{1,p}_{per}(0,b). \tag{3.32}$$

Therefore, if we pass to the limit as $n \to +\infty$ in Eq. 3.31, we obtain

$$A(y) = \beta \eta K(y) + (1-\beta) g K(y)$$

(see Eq. 3.30) and so

$$\begin{cases} -\left(|y'(t)|^{p-2} y'(t)\right)' = \left(\beta\eta + (1-\beta) g(t)\right) \left|y(t)\right|^{p-2} y(t) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for a.a. } t \in T = [0,b], \\ y(0) = y(b),\ y'(0) = y'(b). \end{cases} \tag{3.33}$$

Note that

$$h(t) = \beta\eta + (1-\beta) g(t) \geqslant \eta > 0 \quad \text{for a.a. } t \in T.$$

Integrating Eq. 3.33 over $T = [0,b]$, we obtain

$$0 = \int_0^b h(t) y(t)^{p-1}\, dt.$$

But because of Eq. 3.32, we have $\|y\| = 1$ and so $y \neq 0$. Therefore

$$\int_0^b h(t)y(t)^{p-1}\,dt \ > \ 0,$$

a contradiction. So if $\beta_n < 1$ for all $n \geqslant 1$, Eq. 3.26 holds.

On the other hand, if $\beta_n = 1$ for all $n \geqslant n_0$, then

$$A(x_n) \ = \ \eta K(x_n) \qquad \forall n \geqslant 1,$$

so

$$A(y_n) \ = \ \eta K(y_n) \qquad \forall n \geqslant 1.$$

As above acting with the test function $y_n - y \in W^{1,p}_{per}(0,b)$, we obtain

$$\lim_{n\to+\infty} \langle A(y_n), y_n - y\rangle \ = \ 0,$$

so

$$y_n \ \longrightarrow \ y \quad \text{in } W^{1,p}_{per}(0,b),$$

with $\|y\| = 1$, so $y \neq 0$.

In the limit as $n \to +\infty$, we obtain

$$A(y) \ = \ \eta K(y),$$

so

$$\begin{cases} -\big(|y'(t)|^{p-2}y'(t)\big)' = \eta\big|y(t)\big|^{p-2}y(t) & \text{for a.a. } t \in T = [0,b], \\ y(0) = y(b),\ \ y'(0) = y'(b). \end{cases} \tag{3.34}$$

Since $\eta > 0$ and $y \in C_+ \setminus \{0\}$, by integrating Eq. 3.34, as before we reach a contradiction.

So in all cases Eq. 3.26 is true and thus we have proved Claim 1.

Claim 2 We have

$$i_{C_+}\big(V_\varepsilon \circ (\varepsilon+\eta)K, B^+_\varrho\big) \ = \ 0 \qquad \forall \varrho \in (0,\varrho_0]. \tag{3.35}$$

For this purpose let $\xi \in C_{per}(T)$ be such that $\xi \neq 0$ and

$$\xi(t) \ \geqslant \ 0 \qquad \forall t \in T.$$

We consider the admissible homotopy $H_3(\beta, x)$, defined by

$$H_3(\beta,x) \ = \ \big(V_\varepsilon \circ (\varepsilon+\eta)K\big)(x) + \beta\xi.$$

We show that

$$x \neq H_3(\beta,x) \qquad \forall \beta \in [0,1], \qquad x \in \partial B^+_r, \qquad r > 0. \tag{3.36}$$

Indeed, if this is not the case, then we can find $\beta_0 \in [0,1]$, $r_0 > 0$ and $x \in \partial B^+_{r_0}$, such that

$$A(x) \ = \ \eta K(x) + \beta_0\xi,$$

so

$$\begin{cases} -\left(|x'(t)|^{p-2}x'(t)\right)' = \eta\left|x(t)\right|^{p-2}x(t) + \beta_0\xi(t) & \text{for a.a. } t \in T = [0, b], \\ x(0) = x(b), \ x'(0) = x'(b). \end{cases}$$

Since $x \in \partial B^+_{r_0}$, integrating over $T = [0, b]$, we obtain

$$0 = \int_0^b \eta x(t)^{p-1}\,dt + \beta_0 \int_0^b \xi(t)\,dt > 0,$$

a contradiction. So Eq. 3.36 is true and the homotopy invariance of the fixed index implies that

$$i_{C_+}\left(V_\varepsilon \circ (\varepsilon + \eta)K, B^+_\varrho\right) = i_{C_+}\left(V_\varepsilon \circ (\varepsilon + \eta)K + \xi, B^+_\varrho\right) \qquad \forall \varrho \in (0, \varrho_0]. \tag{3.37}$$

If $i_{C_+}\left(V_\varepsilon \circ (\varepsilon + \eta)K + \xi, B^+_\varrho\right) \neq 0$, then we can find $x \in B^+_\varrho$, such that

$$\begin{cases} -\left(|x'(t)|^{p-2}x'(t)\right)' = \eta\left|x(t)\right|^{p-2}x(t) + \xi(t) & \text{for a.a. } t \in T = [0, b], \\ x(0) = x(b), \ x'(0) = x'(b). \end{cases}$$

Since $\xi \neq 0$, we have that $x \neq 0$. Integrating the above equation, as above we reach a contradiction. So we must have that

$$i_{C_+}\left(V_\varepsilon \circ (\varepsilon + \eta)K + \xi, B^+_\varrho\right) = 0 \qquad \forall \varrho \in (0, \varrho_0] \tag{3.38}$$

and thus from Eq. 3.37 we conclude that Eq. 3.35 holds and so Claim 2 is proved.

Using Claim 1 and the homotopy invariance of the fixed point index, we have

$$i_{C_+}\left(V_\varepsilon \circ (\varepsilon K + G), B^+_\varrho\right) = i_{C_+}\left(V_\varepsilon \circ (\varepsilon + \eta)K, B^+_\varrho\right) \qquad \forall \varrho \in (0, \varrho_0]. \tag{3.39}$$

Then from Claim 2, Eqs. 3.37 and 3.38, we conclude that

$$i_{C_+}\left(V_\varepsilon \circ (\varepsilon K + G), B^+_\varrho\right) = 0 \qquad \forall \varrho \in (0, \varrho_0].$$

□

4 Positive Solutions

Using the auxiliary results of the previous section and the properties of the fixed point index, we obtain the following existence theorem.

Theorem 4.1 *If hypotheses $H(j)$ hold, then problem Eq. 1.1 has a solution $x \in C^1(T)$, such that*

$$x(t) > 0 \qquad \forall t \in (0, b)$$

and

$$x'(0) = x'(b) = 0 \quad \textit{if} \quad x(0) = x(b) = 0.$$

Proof From Propositions 3.6 and 3.7 and the additivity property of the fixed point index, we have

$$\begin{aligned} & i_{C_+}\big(V_\varepsilon \circ (\varepsilon K + G),\, B_R^+ \setminus \overline{B}_\varrho^+\big) \\ & \quad = i_{C_+}\big(V_\varepsilon \circ (\varepsilon K + G),\, B_R^+\big) - i_{C_+}\big(V_\varepsilon \circ (\varepsilon K + G),\, B_\varrho^+\big) \\ & \quad = 1 - 0 \;=\; 1, \end{aligned}$$

with $\varrho \in (0, \varrho_0]$ and $R \geqslant R_0$. Then from the solution property of the fixed point index, we can find $x \in B_R^+ \setminus \overline{B}_\varrho^+$, hence $x \neq 0$, such that

$$A(x) \;=\; u,$$

with $u \in G(x)$, so

$$\begin{cases} -\big(|x'(t)|^{p-2}x'(t)\big)' = u(t) \geqslant 0 & \text{for a.a. } t \in T = [0, b], \\ x(0) = x(b),\; x'(0) = x'(b). \end{cases}$$

Hence $x \in C^1_{per}(T)$. By virtue of the strong maximum principle of Vázquez [9] (see also Gasiński–Papageorgiou [6, p. 116]), we have that

$$x(t) \;>\; 0 \qquad \forall t \in (0, b)$$

and

$$x'(0) \;=\; x'(b) \;=\; 0 \quad \text{if} \quad x(0) \;=\; x(b) \;=\; 0.$$

□

Remark 4.2 An interesting open question is whether in hypothesis $H(j)$(iv), we can replace the spectral interval $[0, \mu_2]$ by any interval $[\mu_{2n}, \mu_{2n+2}]$, $n \geqslant 1$.

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