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On the Lipschitz Modulus of the Argmin Mapping in Linear Semi-Infinite Optimization

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Abstract This paper is devoted to quantify the Lipschitzian behavior of the optimal solutions set in linear optimization under perturbations of the objective function and the right hand side of the constraints (inequalities). In our model, the set indexing the constraints is assumed to be a compact metric space and all coefficients depend continuously on the index. The paper provides a lower bound on the Lipschitz modulus of the optimal set mapping (also called argmin mapping), which, under our assumptions, is single-valued and Lipschitz continuous near the nominal parameter. This lower bound turns out to be the exact modulus in ordinary linear programming, as well as in the semi-infinite case under some additional hypothesis which always holds for dimensions $n \leqslant 3$. The expression for the lower bound (or exact modulus) only depends on the nominal problem's coefficients, providing an operative formula from the practical side, specially in the particular framework of ordinary linear programming, where it constitutes the sharp Lipschitz constant. In the semi-infinite case, the problem of whether or not the lower bound equals the exact modulus for $n > 3$ under weaker hypotheses (or none) remains as an open problem.

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1 Introduction

We consider the parametrized linear semi-infinite programming (LSIP, in brief) problem, in $\mathbb{R}^n$,

$$\begin{aligned} \pi(c,b): & \operatorname{Inf} c'x \\ & \text{s.t. } a_t'x \geqslant b_t, \quad t \in T, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the vector of decision variables, regarded as a column-vector, y' denotes the transpose of $y \in \mathbb{R}^n$, the index set T is a compact metric space, $a \in C(T, \mathbb{R}^n)$ is a given function, and $c \in \mathbb{R}^n$ and $b \in C(T, \mathbb{R})$ are regarded as the parameters. We denote by $\sigma(b)$ the constraint system associated to $\pi(c,b)$, i.e., $\sigma(b) := \left\{a_t'x \geqslant b_t, \quad t \in T\right\}$. The parameter space $\mathbb{R}^n \times C(T, \mathbb{R})$ is endowed with the norm

$$\|(c,b)\| := \max\{\|c\|, \|b\|_\infty\}, \tag{2}$$

where $\mathbb{R}^n$ is equipped with any given norm, $\|\cdot\|$, and $\|b\|_\infty := \max_{t\in T} |b_t|$. Sometimes along the paper $\|\cdot\|_\infty$ is also used for representing the supremum norm with respect to certain subsets of T.

This paper is focused on measuring the Lispchitzian behavior of the *optimal set mapping* (also called *argmin mapping*), $\mathcal{F}^* : \mathbb{R}^n \times C(T, \mathbb{R}) \rightrightarrows \mathbb{R}^n$, which assigns to each parameter (c,b) the *optimal set* of $\pi(c,b)$; i.e.,

$$\mathcal{F}^*(c,b) := \arg\min\{c'x \mid x \in \mathcal{F}(b)\},$$

where

$$\mathcal{F}(b) := \{x \in \mathbb{R}^n \mid a_t'x \geqslant b_t, \quad \text{for all } t \in T\}$$

is the set of feasible solutions of $\sigma(b)$.

Our analysis is concerned with the *Aubin property* (also called *pseudo-Lipschitz*) of $\mathcal{F}^*$, which is satisfied at $((\overline{c}, \overline{b}), \overline{x}) \in gph\mathcal{F}^*$ (the graph of $\mathcal{F}^*$) if there exist some neighborhoods U of $\overline{x}$ and V of $(\overline{c}, \overline{b})$ and a constant $\kappa > 0$ such that

$$d\big(x^2, \mathcal{F}^*(c^1, b^1)\big) \leqslant \kappa d\big((c^1, b^1), (c^2, b^2)\big), \tag{3}$$

for all $(c^1, b^1), (c^2, b^2) \in V$, and all $x^2 \in U \cap \mathcal{F}^*(c^2, b^2)$, where, as usual, $d(x, \emptyset) = +\infty$.

Aubin property of $\mathcal{F}^*$ is known to be equivalent to the *metric regularity* of the inverse mapping

$$\mathcal{G}^* := \left(\mathcal{F}^*\right)^{-1},$$

where $(c, b) \in \mathcal{G}^*(x) \Leftrightarrow x \in \mathcal{F}^*(c, b)$. At this point, we recall that $\mathcal{G}^*$ is metrically regular at $\overline{x} \in \mathbb{R}^n$ for $(\overline{c}, \overline{b}) \in \mathcal{G}^*(\overline{x})$ if there exist neighborhoods U of $\overline{x}$ and V of $(\overline{c}, \overline{b})$ and a constant $\kappa > 0$ such that

$$d\big(x, \mathcal{F}^*(c, b)\big) \leqslant \kappa d\big((c, b), \mathcal{G}^*(x)\big), \tag{4}$$

for all $x \in U$ and all $(c, b) \in V$. Observe that in this case, provided that U and V are open sets, $\mathcal{G}^*$ is also metrically regular at any $(x, (c, b)) \in (U \times V) \cap gph\, \mathcal{G}^*$.

The infimum of κ for which Eq. 4 holds (for some associated neighborhoods U and V) coincides with the infimum of κ verifying Eq. 3 (for possibly smaller neighborhoods; see, e.g., [17]). This infimum is referred to as the *modulus of metric regularity* (or *regularity modulus*) of $\mathcal{G}^*$ at $\overline{x}$ for $(\overline{c}, \overline{b}) \in \mathcal{G}^*(\overline{x})$, and is denoted by $reg\mathcal{G}^*(\overline{x} \mid (\overline{c}, \overline{b}))$. We define $reg\mathcal{G}^*(\overline{x} \mid (\overline{c}, \overline{b})) = +\infty$ when $\mathcal{G}^*$ is not metrically regular at $\overline{x}$ for $(\overline{c}, \overline{b})$.

Corollary 4.7 in [17] shows that the Aubin property of $\mathcal{F}^*$ at $((\overline{c}, \overline{b}), \overline{x}) \in gph\mathcal{F}^*$ is equivalent to the *strong Lipschitz stability* of $\mathcal{F}^*$ at $(\overline{c}, \overline{b})$ (i.e., single-valuedness and Lipschitz continuity of $\mathcal{F}^*$ in a neighborhood of $(\overline{c}, \overline{b})$, since $\mathcal{F}^*$ is convex-valued).

Attending to the previous paragraphs, the following conditions are equivalent:

(a) $\mathcal{G}^*$ is metrically regular at $\overline{x}$ for $(\overline{c}, \overline{b}) \in \mathcal{G}^*(\overline{x})$;
(b) $\mathcal{F}^*$ satisfies the Aubin property at $((\overline{c}, \overline{b}), \overline{x}) \in gph\mathcal{F}^*$;
(c) $\mathcal{F}^*$ is strongly Lipschitz stable (s.L.s.) at $(\overline{c}, \overline{b})$ with $\mathcal{F}^*(\overline{c}, \overline{b}) = \{\overline{x}\}$.

When confined to the finite case (T finite) this equivalence can be alternatively approached from [20, Thm. 1.2]: for a set-valued mapping between finite-dimensional spaces, the single-valuedness and Lipschitz continuity is equivalent to *premonotonicity* together with Aubin property. (See Remark 1 in Section 2 for additional details and references.)

From now on, we adopt (c) as the usual formulation in our statements. Assuming that $\mathcal{F}^*$ is s.L.s. at $(\overline{c}, \overline{b})$, and writing

$$\mathcal{F}^*(c, b) = \{x(c, b)\}$$

for (c, b) near $(\overline{c}, \overline{b})$, the regularity modulus of $\mathcal{G}^*$ can trivially be expressed as the *Lipschitz modulus* of $\mathcal{F}^*$ at the nominal parameter; i.e.,

$$reg\mathcal{G}^*\big(\overline{x} \mid \big(\overline{c}, \overline{b}\big)\big) = lip\mathcal{F}^*\big(\overline{c}, \overline{b}\big) := \limsup_{\substack{(c,b),(\widetilde{c},\widetilde{b}) \to (\overline{c},\overline{b}) \\ (c,b) \neq (\widetilde{c},\widetilde{b})}} \frac{\big\| x(c, b) - x(\widetilde{c}, \widetilde{b}) \big\|}{\big\| (c, b) - (\widetilde{c}, \widetilde{b}) \big\|}. \tag{5}$$

Here 'lim sup' is understood, as usual, as the supremum of all possible 'sequential $\limsup_{r \to +\infty}$' in which (c, b) and $(\widetilde{c}, \widetilde{b})$ are respectively replaced by (c^r, b^r) and $(\widetilde{c}^r, \widetilde{b}^r)$, with $r = 1, 2, ...$

The main goal of this paper is to compute (or estimate) this modulus. Specifically, our aim is to provide an operative exact formula (or bound) for $lip\mathcal{F}^*(\overline{c}, \overline{b})$, only involving the nominal elements $\overline{x}$ and $(\overline{c}, \overline{b})$. To the authors knowledge, the results given here provide new stuff even in the case of ordinary linear programming (see Section 5).

The metric regularity is a quantitative stability property widely used in both theoretical and computational studies in the field of variational analysis. Different authors have recently provided deep studies on this topic from different perspectives (see, for instance, Dontchev et al. [9], in relation to the modulus and radius of metric regularity, and Ioffe [15] for a survey about metric regularity and its connection with subdifferential calculus in Banach spaces). The reader is addressed to Mordukhovich [24] and Rockafellar and Wets [27] for a comprehensive overview on variational analysis, and Klatte and Kummer [17] for the study of this subject in connection with optimization theory. Studies on the metric regularity of systems of convex inequalities and its relation to different constraint qualifications can be traced out from Li [23] and Zheng and Ng [29]. See also Henrion and Klatte [14] for details about metric regularity of certain parametrized semi-infinite systems. A different approach to quantitative stability (through the concept of nondegenerancy) of parametrized optimization problems having nonisolated optima is provided in Bonnans and Shapiro [2]. In relation *sharp Lipschitz constants* for the feasible and the optimal set in ordinary linear programming, the reader is addressed to [21] and [22]. More references concerning this (finite) case are given in Section 5.

Immediate antecedents to the present work can be found in [4] and [5]. The first of these papers deals with the metric regularity of constraint systems (no objective function is considered). Specifically, it provides a formula for the regularity modulus of

$$\mathcal{G} := \mathcal{F}^{-1},$$

in terms of the system's data (see Proposition 1). Paper [5] is concerned with the metric regularity property of convex optimization problems under canonical perturbations. When confined to the linear semi-infinite context (Eq. 1), the referred paper shows that the metric regularity of $\mathcal{G}^*$ turns out to be equivalent to different stability criteria spread out in the literature, as the strong Lipschitz stability of $\mathcal{F}^*$, referred above, and the *strong Kojima stability* of $\mathcal{F}^*$ [19], under *Mangasarian-Fromovitz constraint qualification* (see also [16, 18, 28]). In particular, [5] provides an algebraic characterization of the metric regularity of $\mathcal{G}^*$ (KKT conditions with some additional requirement), gathered in Proposition 2(b), which constitutes the starting point of the present paper. In fact, with the aim of computing $lip\mathcal{F}^*(\overline{c}, \overline{b})$, this condition suggests the strategy of considering suitable finite subproblems of $\pi(c, b)$ with exactly n constraints.

There also exist in the literature different contributions to the stability theory of linear semi-infinite optimization problems focussed on the continuity of the associated set-valued mappings. Chapters 6 and 10 in [12] present a general overview about the stability of the feasible and the optimal set, respectively, of LSIP problems in a more general context with no continuity assumption on the data $t \mapsto \binom{a_t}{b_t}$ and allowing perturbations of all coefficients. See also [13] for details about the (Berge) lower semicontinuity of $\mathcal{F}$, and [6] for lower and upper semicontinuity of $\mathcal{F}^*$ in the referred general context. The reader is also addressed to [3] and [11] for the analysis of continuity properties of $\mathcal{F}$ and $\mathcal{F}^*$ in our framework of continuous data. In relation to the sensitivity analysis, some results about Lipschitz constants for the optimal value in connection with the so-called *distance to ill-posedness* can be traced out from [7] and [26] (the first one in the linear semi-infinite context without continuity assumptions, and the second in the conic context).

At this moment, we summarize the structure and main contributions of the paper. Section 2 contains the necessary definitions and preliminary results. Section 3 introduces a key reformulation of the right hand side of Eq. 5, showing that both c

and $\widetilde{c}$ can be replaced by the nominal parameter $\overline{c}$. In other words, under the strong Lipschitz stability assumption,

$$lip\mathcal{F}^*\left(\overline{c}, \overline{b}\right) = lip\mathcal{F}^*(\overline{c}, \cdot)\left(\overline{b}\right),$$

where $\mathcal{F}^*(\overline{c}, \cdot) : C(T, \mathbb{R}) \rightrightarrows \mathbb{R}^n$ is given by $\mathcal{F}^*(\overline{c}, \cdot)(b) = \mathcal{F}^*(\overline{c}, b)$. This fact clarifies the role played by the objective function in the value of $lip\mathcal{F}^*(\overline{c}, \overline{b})$. Roughly speaking, it determines which ingredients (x, b) should be used (those such that x is an optimal point for $\pi(\overline{c}, b)$).

Proposition 2(b) yields the idea of relating $lip\mathcal{F}^*(\overline{c}, \overline{b})$ to the Lipschitz moduli associated to appropriate finite subproblems: those formed by considering all blocks of n constraints involved in the Karush–Kuhn–Tucker conditions. Taking this idea into account, Section 4 starts by calculating the Lipschitz moduli associated to the referred finite subproblems. After that, a lower bound on $lip\mathcal{F}^*(\overline{c}, \overline{b})$ is provided (Theorem 1). Section 5 shows that this lower bound equals the exact modulus in the finite case. In fact, in this case, $lip\mathcal{F}^*(\overline{c}, \overline{b})$ turns out to be the sharp Lipschitz constant in a neighborhood of $(\overline{c}, \overline{b})$. Theorem 2 in Section 6 establishes that the lower bound equals the modulus under a certain additional assumption, called condition (H). This condition (H) always holds in $\mathbb{R}^n$ with $n \leqslant 3$ (Proposition 7) while it can hold or not in $\mathbb{R}^4$ (Examples 4 and 3, respectively).

We point out that the announced lower bound (or exact modulus) only depends on the coefficients of the nominal problem $\pi(\overline{c}, \overline{b})$, not involving problems $\pi(c, b)$ in a neighborhood of it. Moreover, it is given in terms of the norm of the inverse matrices associated to the referred blocks of n constraints, which makes it have practical advantages, mainly in the particular case of linear programming problems (T finite). Finally, in Section 7 we underline some conclusions about the main contributions of the paper, specifying the case in which the calculus of the Lipschitz modulus remains as open problem.

2 Preliminaries

In this section we provide further notation and some preliminary results. Given $\emptyset \neq X \subset \mathbb{R}^k$, $k \in \mathbb{N}$, we denote by $conv(X)$ and $cone(X)$ the *convex hull* and the *conical convex hull* of X, respectively. If now X is a subset of any topological space, $int(X)$ and $cl(X)$ will represent the interior and the closure of X, respectively. It is assumed that $cone(X)$ always contains the zero-vector, 0_k, and so $cone(\emptyset) = \{0_k\}$.

The *dual norm* of $\|\cdot\|$ is denoted by $\|\cdot\|_*$; i.e., for $u \in \mathbb{R}^k$,

$$\|u\|_* := \max\{u'z \mid \|z\| \leqslant 1\}.$$

The characteristic cone associated with problem in Eq. 1 is given by

$$K(b) := cone\left(\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix},\ t \in T;\ \begin{pmatrix} 0_n \\ -1 \end{pmatrix}\right\}\right).$$

When $\sigma(b)$ is consistent, the non-homogeneous Farkas Lemma characterizes those linear inequalities $u'x \geqslant v$ which are consequences of $\sigma(b)$ [i.e., are satisfied by all the points of $\mathcal{F}(b)$] as those verifying

$$\begin{pmatrix} u \\ v \end{pmatrix} \in cl(K(b)),$$

i.e., there exist sequences $\{\lambda^r\} \subset \mathbb{R}_+^{(T)}$ and $\{\mu^r\} \subset \mathbb{R}_+$ such that

$$\begin{pmatrix} u \\ v \end{pmatrix} = \lim_r \left\{ \sum_{t\in T} \lambda_t^r \begin{pmatrix} a_t \\ b_t \end{pmatrix} + \mu^r \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \right\},$$

where $\mathbb{R}_+^{(T)}$ denotes the set of all the functions $\lambda\colon T \to \mathbb{R}_+$ taking positive values at finitely many points of T, and $\lim_r$ should be interpreted as $\lim_{r\to+\infty}$.

When $K(b)$ is closed, $\sigma(b)$ is said to be a Farkas-Minkowski system (see [12]). If $x \in \mathcal{F}(b)$, we introduce the set of active indices and its associated cone of active constraints,

$$T_b(x) = \{t \in T \mid a_t' x = b_t\} \text{ and } A_b(x) = cone(\{a_t \mid t \in T_b(x)\}), \tag{6}$$

assuming $A_b(x) = \{0_n\}$ if $T_b(x) = \emptyset$. A point $x^0 \in \mathcal{F}(b)$ such that $T_b(x^0) = \emptyset$ is said to be a Slater point of $\sigma(b)$. In this case, a standard continuity and compactness argument yields the existence of a positive scalar ρ such that $a_t' x^0 \geqslant b_t + \rho$ for all $t \in T$. We say that $\sigma(b)$ satisfies the Slater condition if it has at least a Slater point.

Sometimes it will be convenient to extend definition in Eq. 6 not only to (continuous) functions b defined on the whole T, but also to (continuous) functions defined on some subset $D \subset T$. Specifically, when $\beta \in C(D, \mathbb{R})$, with $D \subset T$, we consider

$$T_\beta(x) = \{t \in D \mid a_t' x = \beta_t\}. \tag{7}$$

Lemma 1 (see [12]) *Let $(c, b) \in \mathbb{R}^n \times C(T, \mathbb{R})$ and $x \in \mathbb{R}^n$. One has:*

(a) *If $\sigma(b)$ satisfies the Slater condition, then $K(b)$ is closed;*
(b) *Karush–Kuhn–Tucker (KKT) conditions: If $x \in \mathcal{F}(b)$ and $c \in A_b(x)$ then $x \in \mathcal{F}^*(c, b)$. The converse holds when $K(b)$ is closed.*

The following proposition provides a characterization of the metric regularity of $\mathcal{G}$ as well as an expression for its regularity modulus at $(x, b) \in gph\mathcal{G}$, which can be derived from [4, Cor. 3.2]

Proposition 1 *$\mathcal{G}$ is metrically regular at x for $b \in \mathcal{G}(x)$ if and only if $\sigma(b)$ satisfies the Slater condition, and in this case we have*

$$reg\,\mathcal{G}(x \mid b) = (\inf\{\|u\|_* \mid u \in conv(\{a_t,\ t \in T_b(x)\})\})^{-1}.$$

Proof The characterization of the metric regularity is given in [4, Thm. 2.1]. With respect to the modulus, [4, Cor. 3.2] yields

$$reg\,\mathcal{G}(x \mid b) = \left(\inf\left\{\|u\|_* \mid \begin{pmatrix} u \\ u'x \end{pmatrix} \in conv\left(\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix},\ t \in T\right\}\right)\right\}\right)^{-1}.$$

Let us see that $\binom{u}{u'x} \in conv\left(\left\{\binom{a_t}{b_t},\ t \in T\right\}\right)$ if and only if $u \in conv(\{a_t, t \in T_b(x)\})$. The 'if' condition is immediate. To see the converse, write

$$\begin{pmatrix} u \\ u'x \end{pmatrix} = \sum_{t\in T} \lambda_t \begin{pmatrix} a_t \\ b_t \end{pmatrix} \tag{8}$$

for some $\lambda \in \mathbb{R}_+^{(T)}$ such that $\sum_{t\in T} \lambda_t = 1$. Then, multiplying both sides by $\binom{x}{-1}$ we obtain

$$0 = \sum_{t\in T} \lambda_t (a_t' x - b_t).$$

Since x is feasible for $\sigma(b)$, each $\lambda_t(a_t'x - b_t)$ must be zero, and thus $\lambda_t = 0$ if $t \notin T_b(x)$. Then, the first block in Eq. 8 provides $u \in conv(\{a_t,\ t \in T_b(x)\})$. □

Theorem 16 in [5] provides different characterizations of the strong Lipschitz stability of $\mathcal{F}^*$ at a given $(\overline{c}, \overline{b})$. One of them is condition (b) in the following proposition, which is formulated exclusively in terms of the nominal data (i.e., $\overline{c}$, $\overline{b}$, and $\overline{x}$) and plays a key role in our analysis. Next, Proposition 3 provides additional information, taken out from [5, Proposition 9]. In the sequel, $|D|$ will denote the cardinality of D.

Proposition 2 *Let* $(\overline{c}, \overline{b}) \in \mathbb{R}^n \times C(T, \mathbb{R})$. *The following conditions are equivalent*:

(a) $\mathcal{F}^*$ *is s.L.s. at* $(\overline{c}, \overline{b})$ *(i.e., single-valued and Lipschitz continuous in a neighborhood of* $(\overline{c}, \overline{b})$*) with* $\mathcal{F}^*(\overline{c}, \overline{b}) = \{\overline{x}\}$;

(b) $\sigma(\overline{b})$ *satisfies the Slater condition and there is no* $D \subset T_{\overline{b}}(\overline{x})$ *with* $|D| < n$ *such that* $\overline{c} \in cone(\{a_t,\ t \in D\})$.

Proposition 3 *Assume that* $\mathcal{F}^*$ *is s.L.s. at* $(\overline{c}, \overline{b})$ *with* $\mathcal{F}^*(\overline{c}, \overline{b}) = \{\overline{x}\}$. *Then*

(a) *There exist* $t_1, ..., t_n \in T_{\overline{b}}(\overline{x})$ *such that* $\{a_{t_1}, ..., a_{t_n}\}$ *is a basis of* $\mathbb{R}^n$, *and positive scalars* $\lambda_1, ..., \lambda_n$ *such that* $\overline{c} = \sum_{i=1}^n \lambda_i a_{t_i}$. *In other words,* $\overline{c} \in int(cone(\{a_{t_1}, ..., a_{t_n}\}))$.

(b) *A neighborhood* V *of* $(\overline{c}, \overline{b})$ *exists in such a way that* $\mathcal{F}^*(c, b)$ *consists of a unique point,* $x(c, b)$, *for* $(c, b) \in V$, *and Proposition* 2(b) *fulfills when replacing* $(\overline{c}, \overline{b})$ *by any* $(c, b) \in V$ *and* $\overline{x}$ *is replaced by* $x(c, b)$.

Remark 1 Lipschitz single-valuedness of $\mathcal{F}^*$ does not imply uniqueness of KKT multipliers (see Example 1 below). From a different view, we can find in the literature different contributions about the Lipschitz single-valuedness of KKT pairs (x, λ). The papers [10] and [20] tackle the Lipschitzian behavior of the KKT mapping. In our linear context, and assuming that T is finite, this mapping assigns to each (c, b) the set of all (x, λ) such that $x \in \mathcal{F}^*(c, b)$ and λ is a vector of associated KKT multipliers. Specifically, (c, b) and (x, λ) are related through the *variational condition*

$$\begin{pmatrix} c \\ b \end{pmatrix} \in \begin{pmatrix} A'\lambda \\ Ax \end{pmatrix} + \begin{pmatrix} 0_n \\ \partial\delta_{\mathbb{R}_+^T}(\lambda) \end{pmatrix},$$

where A is the matrix whose rows are a_t', A' is its transpose, and $\partial\delta_{\mathbb{R}_+^T}$ is the subdifferential of the (convex) indicator function of $\mathbb{R}_+^T$. Then, one can apply the results of [10] and [20] to characterize the Lipschitz single-valuedness of the KKT mapping near a nominal $((\overline{c}, \overline{b}), (\overline{x}, \overline{\lambda}))$. Actually, the first of these papers deals with a more general class of mappings consisting of the sum of a linear mapping plus the normal cone mapping to a polyhedral set. The second paper provides a general characterization of the Lipschitz single-valuedness of a multifunction, in finite dimensions, which is applied to what the authors call *solution multifunctions*, which also include KKT mappings.

Since this paper is devoted to determine $lip\mathcal{F}^*(\overline{c}, \overline{b})$, from now on we shall assume that $\mathcal{F}^*$ is s.L.s. at $(\overline{c}, \overline{b})$.

3 First Results on the Lipschitz Modulus

In this section, we provide an expression for the Lipschitz modulus of $\mathcal{F}^*$ at $(\overline{c}, \overline{b})$, which clarifies the role played by vector $\overline{c}$ in relation to this modulus. Roughly speaking, $\overline{c}$ decides which ingredients must be considered in the calculus of $lip\mathcal{F}^*(\overline{c}, \overline{b})$: those pairs (x, b) such that x is an optimal point of $\pi(\overline{c}, b)$.

First, we need the following lemma, which ensures that it is not restrictive to consider a fixed vector $\overline{c}$ in the objective function when (x, b) is close enough to $(\overline{x}, \overline{b})$.

Lemma 2 *Assume that* $\mathcal{F}^*$ *is s.L.s. at* $(\overline{c}, \overline{b})$ *with* $\mathcal{F}^*(\overline{c}, \overline{b}) = \{\overline{x}\}$. *If* $\{((c^r, b^r), x^r)\} \subset gph\mathcal{F}^*$ *converges to* $((\overline{c}, \overline{b}), \overline{x})$, *then* $((\overline{c}, b^r), x^r) \in gph\mathcal{F}^*$ *for r large enough.*

Proof Let $\{((c^r, b^r), x^r)\} \subset gph\mathcal{F}^*$ converge to $((\overline{c}, \overline{b}), \overline{x})$. For each r large enough (w.l.o.g., for all r), $\mathcal{F}^*$ is s.L.s. at (c^r, b^r). Then, applying Proposition 3(a), there exists $D_r := \left\{t_1^r, ..., t_n^r\right\} \subset T_{b^r}(x^r)$ such that $\{a_{t_1^r}, ..., a_{t_n^r}\}$ is a basis of $\mathbb{R}^n$ and

$$c^r = \sum_{i=1}^{n} \lambda_i^r a_{t_i^r}, \tag{9}$$

for some $\lambda_i^r > 0$ for each $i = 1, ..., n$.

Next we prove that $\left\{\sum_{i=1}^n \lambda_i^r\right\}_r$ must be bounded.

Otherwise, we may assume that $\gamma_r := \sum_{i=1}^n \lambda_i^r \to +\infty$ and, considering suitable subsequences if necessary, each $\left\{\frac{\lambda_i^r}{\gamma_r}\right\}$ converges to certain $\mu_i \geqslant 0$, for $i = 1, ..., n$, with $\sum_{i=1}^n \mu_i = 1$. The compactness of T together with the continuity of $t \mapsto a_t$ allows us to assume (taking subsequences if necessary) $\lim_r t_i^r = t_i \in T_{\overline{b}}(\overline{x})$ and $\lim_r a_{t_i^r} = a_{t_i}$, for $i = 1, ..., n$. Then, letting $r \to +\infty$ after dividing by γ_r both sides of Eq. 9, we have $0_n = \sum_{i=1}^n \mu_i a_{t_i}$, and hence $0_{n+1} = \sum_{i=1}^n \mu_i \binom{a_{t_i}}{b_{t_i}}$. This contradicts the Slater condition at $\sigma(\overline{b})$ (just multiply both sides by $\binom{x^0}{-1}$), where x^0 is any Slater point of $\sigma(\overline{b})$).

Once we know that $\left\{\sum_{i=1}^n \lambda_i^r\right\}_r$ is bounded, we may assume w.l.o.g. that each $\left\{\lambda_i^r\right\}_r$ converges to certain $\lambda_i \geqslant 0$, $i = 1, ..., n$, and Eq. 9 leads us to $\overline{c} = \sum_{i=1}^n \lambda_i a_{t_i} \in cone(\{a_{t_i},\ i = 1, ..., n\})$. Thus $D := \{t_1, ..., t_n\} \subset T_{\overline{b}}(\overline{x})$, and the strong Lipschitz stability hypothesis [see Proposition 2 and Proposition 3(a)]) entails $\overline{c} \in int(cone\{a_{t_i},\ t_i \in D\})$; therefore, the continuity of $t \mapsto a_t$ yields $\overline{c} \in int(cone\{a_{t_i^r},\ i = 1, ..., n\})$ for r large enough and, so, appealing again to Lemma 1 and recalling $D_r = \left\{t_1^r, ..., t_n^r\right\} \subset T_{b^r}(x^r)$, we conclude $x^r \in \mathcal{F}^*(\overline{c}, b^r)$. □

The following proposition provides the announced reformulation of Eq. 5 when confined to our context of linear problems. Observe that $\overline{c}$ remains unperturbed in the statement of this proposition. Recall that $x(c,b)$ represents the unique optimal solution of $\pi(c,b)$ for (c,b) close enough to $(\overline{c},\overline{b})$, provided that $\mathcal{F}^*$ is s.L.s. at $(\overline{c},\overline{b})$.

Proposition 4 *Assume that $\mathcal{F}^*$ is s.L.s. at $(\overline{c},\overline{b})$. Then*

$$lip\mathcal{F}^*\big(\overline{c},\overline{b}\big)=\limsup_{\substack{b,\widetilde{b}\to\overline{b}\\ b\neq\widetilde{b}}}\frac{\big\|x(\overline{c},b)-x\big(\overline{c},\widetilde{b}\big)\big\|}{\big\|b-\widetilde{b}\big\|_\infty}\ \big(=:lip\mathcal{F}^*\big(\overline{c},\cdot\big)\big(\overline{b}\big)\big). \tag{10}$$

Proof According to the definitions [see Eq. 5], $lip\mathcal{F}^*(\overline{c},\overline{b})=\sup L_0$, where L_0 is the set of all the limits in the form

$$\limsup_r\frac{\big\|x\big(c^r,b^r\big)-x\big(\widetilde{c}^r,\widetilde{b}^r\big)\big\|}{\big\|\big(c^r,b^r\big)-\big(\widetilde{c}^r,\widetilde{b}^r\big)\big\|}$$

such that $(c^r,b^r),(\widetilde{c}^r,\widetilde{b}^r)\to(\overline{c},\overline{b})$ and $(c^r,b^r)\neq(\widetilde{c}^r,\widetilde{b}^r)$, for all r. Here all (c^r,b^r) and $(\widetilde{c}^r,\widetilde{b}^r)$ are assumed to be close enough $(\overline{c},\overline{b})$ to ensure the uniqueness of optimal solution. Moreover, $lip\mathcal{F}^*(\overline{c},\cdot)(\overline{b})$ is the supremum of L, where

$$L:=\left\{\limsup_r\frac{\big\|x\big(\overline{c},b^r\big)-x\big(\overline{c},\widetilde{b}^r\big)\big\|}{\big\|b^r-\widetilde{b}^r\big\|_\infty}\ \middle|\ b^r,\widetilde{b}^r\to\overline{b};\ b^r\neq\widetilde{b}^r\text{ for all }r\right\}.$$

Let us see that $\sup L=\sup L_0$. It is immediate that $L\subset L_0$ and then $\sup L\leqslant\sup L_0$.

Now, if $\alpha_0=\limsup_r\frac{\|x(c^r,b^r)-x(\widetilde{c}^r,\widetilde{b}^r)\|}{\|(c^r,b^r)-(\widetilde{c}^r,\widetilde{b}^r)\|}\in L_0$, let us see that $\alpha_0\leqslant\sup L$, in the non-trivial case $\alpha_0>0$. We can write $\alpha_0=\lim_k\frac{\|x(c^{r_k},b^{r_k})-x(\widetilde{c}^{r_k},\widetilde{b}^{r_k})\|}{\|(c^{r_k},b^{r_k})-(\widetilde{c}^{r_k},\widetilde{b}^{r_k})\|}$, where $\{(c^{r_k},b^{r_k})\}$ and $\{(\widetilde{c}^{r_k},\widetilde{b}^{r_k})\}$ are suitable subsequences.

Let us see that $b^{r_k}\neq\widetilde{b}^{r_k}$ for k large enough. Otherwise, we would have $b^{r_{k_p}}=\widetilde{b}^{r_{k_p}}$, $p=1,2,...$ for some subsequences, and the previous lemma, together with the uniqueness of optimal solutions around $(\overline{c},\overline{b})$, would yield, for a certain $p_0\in\mathbb{N}$, that $x(c^{r_{k_p}},b^{r_{k_p}})=x(\overline{c},b^{r_{k_p}})=x(\overline{c},\widetilde{b}^{r_{k_p}})=x(\widetilde{c}^{r_{k_p}},\widetilde{b}^{r_{k_p}})$ for $p\geqslant p_0$, yielding the contradiction $\alpha_0=0$.

So, $\alpha_0\leqslant\limsup_k\frac{\|x(c^{r_k},b^{r_k})-x(\widetilde{c}^{r_k},\widetilde{b}^{r_k})\|}{\|b^{r_k}-\widetilde{b}^{r_k}\|}=\limsup_k\frac{\|x(\overline{c},b^{r_k})-x(\overline{c},\widetilde{b}^{r_k})\|}{\|b^{r_k}-\widetilde{b}^{r_k}\|}\in L$, where the equality comes again from Lemma 2 and the uniqueness of optimal solutions around $(\overline{c},\overline{b})$. Hence $\sup L\geqslant\sup L_0$. □

As commented above, the previous proposition yields a better understanding of the Lipschitz modulus in our context. The expression of $lip\mathcal{F}^*(\overline{c},\overline{b})$ given in Eq. 10 still involves parameters and points in a neighborhood of the nominal one. Nevertheless it will help in searching a formula for $lip\mathcal{F}^*(\overline{c},\overline{b})$ in terms of the nominal problem's data, which constitutes the main goal in the rest of the paper.

4 A Lower Bound on the Lipschitz Modulus

In order to determine the Lipschitz modulus of $\mathcal{F}^*$ at $(\overline{c},\overline{b})$, condition (b) in Proposition 2 (together with Carathéodory's Theorem) suggests the strategy of considering subsets of n active constraints at $\overline{x}$ for $\overline{b}$ (where $\{\overline{x}\}=\mathcal{F}^*(\overline{c},\overline{b})$), in such way that $\overline{c}$ belongs to the associated active cone (see also Proposition 3(a)). In other words, we investigate the possibility of expressing $lip\mathcal{F}^*(\overline{c}, \overline{b})$ in terms of $\mathcal{T}_{\overline{b}}(\overline{x})$, where in general we define

$$\mathcal{T}_b(x) := \{D \subset T_b(x) \mid |D| = n \text{ and } \overline{c} \in cone(\{a_t,\ t \in D\})\}. \tag{11}$$

Note that $\overline{c}$ remains fixed in the definition of $\mathcal{T}_b(x)$, according to Lemma 2.

Now we intend to relate the Lipschitz modulus of 'the whole' $\mathcal{F}^*$ to the Lipschitz moduli associated with subproblems constrained only by inequalities with indices in D, for each $D \in \mathcal{T}_{\overline{b}}(\overline{x})$. Proposition 5 below determines the latter Lipschitz moduli. First, we introduce some notation used hereafter.

Given $\emptyset \neq D \subset T$, $|D| < \infty$, we consider $\mathcal{F}_D^* : \mathbb{R}^n \times \mathbb{R}^D \rightrightarrows \mathbb{R}^n$ defined by

$$\mathcal{F}_D^*(c, \beta) := \arg\min \left\{c'x \mid a_t'x \geqslant \beta_t \text{ for all } t \in D\right\}.$$

Elements of $\mathbb{R}^D$ are functions defined on a discrete domain and may be viewed as restrictions to D of continuous functions of $C(T, \mathbb{R})$, due to the fact that T is a Hausdorff space. With this notation our aim is, given $(\overline{c}, \overline{b}) \in C(T, \mathbb{R})$ at which $\mathcal{F}^*$ is s.L.s., to relate $lip\mathcal{F}^*(\overline{c}, \overline{b})$ with $\left\{lip\mathcal{F}_D^*(\overline{c}, \overline{b}_D) : D \in \mathcal{T}_{\overline{b}}(\overline{x})\right\}$, where $\overline{b}_D := (\overline{b}_t)_{t\in D}$.

The next example is intended to illustrate these ingredients and to provide motivation for some forthcoming results.

Example 1 Consider the problem in $\mathbb{R}^2$

$$\pi\left(\overline{c}, \overline{b}\right) := \text{Inf}\,\{2x_1 + x_2 \mid x_1 \geqslant 0,\ x_2 \geqslant 0,\ x_1 + x_2 \geqslant 0\},$$

where the constraints are associated with indices 1, 2 and 3 in the same order in which they are written. First note that $\mathcal{F}^*$ is s.L.s. at $(\overline{c}, \overline{b})$ with $\mathcal{F}^*(\overline{c}, \overline{b}) = \{\overline{x}\}$, where $\overline{c} = (2, 1)'$, $\overline{b} = 0_3$, and $\overline{x} = 0_2$, according to Proposition 2(b). In this case

$$\mathcal{T}_{\overline{b}}(\overline{x}) = \{\{1, 2\}, \{1, 3\}\}.$$

Attending to KKT conditions, one can easily check in this example that, for (c, b) close enough to $(\overline{c}, \overline{b})$ and $D \in \mathcal{T}_{\overline{b}}(\overline{x})$, $\mathcal{F}_D^*(c, b_D)$ coincides with the solution set of the linear system $\left\{a_t'x = b_t,\ t \in D\right\}$. This situation holds in general, as we observe below in Remark 2, and it underlies the fact that the Lipschitz modulus of $\mathcal{F}_D^*$ only involves the referred system of equations (see Proposition 5).

From now on, given $D \subset T$, with $|D| = n$, let A_D denote the matrix whose rows are a_t' for each $t \in D$ (given in some prefixed order). Sometimes we identify A_D with the endomorphism $\mathbb{R}^n \ni x \mapsto A_D x \in \mathbb{R}^D$. It is well-known that A_D is metrically regular, at any point of its graph, if and only if A_D is non-singular, and in such a case $regA_D$ does not depend on the point of $gphA_D$ at which is evaluated, and

$$regA_D = \left\|A_D^{-1}\right\| \tag{12}$$

(see, for instance, [9, Example 1.1]). For our choice of norms, provided that A_D is non-singular, we have

$$\left\|A_D^{-1}\right\| := \max_{\|y\|_\infty \leqslant 1} \left\|A_D^{-1}y\right\| = \max_{y\in\{-1,1\}^n} \left\|A_D^{-1}y\right\| = \left(\min_{\|\lambda\|_1=1} \left\|A_D'\lambda\right\|_*\right)^{-1}.$$

The second equality comes from the use of $\|\cdot\|_\infty$ in $\mathbb{R}^D$, together with the fact that $\{-1, 1\}^n$ is the set of extreme points of the associated closed unit ball and the function to be maximized is convex. The last equality is a straightforward consequence of Proposition 1.

As a consequence of Eq. 12, and applying Proposition 4, we obtain the following expected formula for the Lipschitz modulus of $\mathcal{F}_D^*$:

Proposition 5 *Let $D \subset T$, with $|D| = n$. Assume that $\mathcal{F}_D^*$ is s.L.s. at $(\overline{c}, \overline{\beta}) \in \mathbb{R}^n \times \mathbb{R}^D$. Then, one has*

$$lip\mathcal{F}_D^*(\overline{c}, \overline{\beta}) = \left\| A_D^{-1} \right\| .$$

Proof Under the strong Lipschitz stability assumption we have $\overline{c} = \sum_{t\in D} \lambda_t a_t$, where $\lambda_t > 0$ for all $t \in D$ and $\{a_t,\ t \in D\}$ is a basis of $\mathbb{R}^n$ (it comes straightforwardly from the specification of Proposition 3(a) to $\mathcal{F}_D^*$). So, for $x \in \mathbb{R}^n$ and $\beta \in \mathbb{R}^D$, the KKT conditions yield $((\overline{c}, \beta), x) \in gph\mathcal{F}_D^*$ if and only if $x = A_D^{-1}\beta$. Applying now Proposition 4 one concludes

$$lip\mathcal{F}_D^*(\overline{c}, \overline{\beta}) = \limsup_{\substack{\beta,\widetilde{\beta}\to\overline{\beta} \\ \beta\neq\widetilde{\beta}}} \frac{\left\| A_D^{-1}(\beta - \widetilde{\beta}) \right\|}{\left\| \beta - \widetilde{\beta} \right\|_\infty} = \left\| A_D^{-1} \right\| ,$$

which completes the proof. □

Remark 2 Under the assumptions of the previous proposition, in fact we have

$$\mathcal{F}_D^*(c, \beta) = \left\{ A_D^{-1}\beta \right\}$$

for each $(c, \beta) \in \mathbb{R}^n \times \mathbb{R}^D$ such that $\|c - \overline{c}\| < \varepsilon$, for a certain $\varepsilon > 0$. In fact, the strong Lipschitz stability assumption ensures the existence of $\varepsilon > 0$ such that $c \in int(cone(\{a_t,\ t \in D\}))$ whenever $\|c - \overline{c}\| < \varepsilon$ (*see* [12, Thm. A.7]). Then, the only point satisfying KKT conditions for the linear problem

$$Min\ \left\{ c'x \mid a_t'x \geqslant \beta_t,\ t \in D \right\}$$

is $A_D^{-1}\beta$ (recall that $\{a_t,\ t \in D\}$ is a basis of $\mathbb{R}^n$).

The following theorem provides the aimed lower bound for the Lipschitz modulus of 'the whole' $\mathcal{F}^*$. Observe that subset D in Proposition 5 actually belongs to $\mathcal{T}_{\overline{\beta}}(\overline{x})$.

Theorem 1 *Assume that $\mathcal{F}^*$ is s.L.s. at $(\overline{c}, \overline{b})$ with $\mathcal{F}^*(\overline{c}, \overline{b}) = \{\overline{x}\}$. Then*

$$lip\mathcal{F}^*\left(\overline{c}, \overline{b}\right) \geqslant \sup_{D\in\mathcal{T}_{\overline{b}}(\overline{x})} lip\mathcal{F}_D^*\left(\overline{c}, \overline{b}_D\right) = \sup_{D\in\mathcal{T}_{\overline{b}}(\overline{x})} \left\| A_D^{-1} \right\| . \tag{13}$$

Proof The last equality comes trivially from Proposition 5. Suppose, reasoning by contradiction, that there exist $\alpha_1, \alpha_2 > 0$ and $D \in \mathcal{T}_{\overline{b}}(\overline{x})$ such that

$$lip\mathcal{F}^*\left(\overline{c}, \overline{b}\right) < \alpha_1 < \alpha_2 < lip\mathcal{F}_D^*\left(\overline{c}, \overline{b}_D\right). \tag{14}$$

The last inequality, together with the fact that $lip\mathcal{F}_D^*(\overline{c}, \overline{b}_D) = reg A_D$, yields the existence of two sequences, $\{x^r\} \subset \mathbb{R}^n$ converging to $\overline{x}$, and $\{\beta^r\} \subset \mathbb{R}^D$ converging to $\overline{b}_D$, such that, for each $r \in \mathbb{N}$,

$$\left\| x^r - A_D^{-1}\beta^r \right\| > \alpha_2 \left\| \beta^r - A_D x^r \right\|_\infty . \tag{15}$$

Write $\tilde{x}^r := A_D^{-1}\beta^r$, for each r, and observe that $\{\tilde{x}^r\}_{r\in\mathbb{N}}$ converges to $A_D^{-1}\overline{b}_D = \overline{x}$ (recall that $D \in \mathcal{T}_{\overline{b}}(\overline{x})$).

The proof is organized in four steps.

Step 1 We construct a sequence $\{\widetilde{b}^r\} \subset C(T, \mathbb{R})$ converging to $\overline{b}$ and verifying, for each r,

$$\mathcal{F}^*(\overline{c}, \widetilde{b}^r) = \{\tilde{x}^r\}. \tag{16}$$

Define, for each $t \in T$,

$$\widetilde{b}_t^r := (1 - \varphi_r(t))a_t'\tilde{x}^r + \varphi_r(t) \min\left\{a_t'x^r, a_t'\tilde{x}^r, \overline{b}_t\right\}, \tag{17}$$

where $\varphi_r : T \to [0, 1]$ is a continuous function satisfying

$$\varphi_r(t) = \begin{cases} 0 & \text{if } t \in D, \\ 1 & \text{if } \max\left\{\left|a_t'\tilde{x}^r - a_t'x^r\right|, \ a_t'\overline{x} - \overline{b}_t\right\} \geqslant \left(1 + \frac{1}{r}\right) \|\beta^r - A_D x^r\|_\infty . \end{cases} \tag{18}$$

The existence of φ_r is guaranteed by Urysohn's Lemma. To see this observe that, from Eq. 15, $\|\beta^r - A_D x^r\|_\infty > 0$ (otherwise, $\left\|x^r - A_D^{-1}\beta^r\right\| = 0$) and, if $t \in D$, then $a_t'\overline{x} - \overline{b}_t = 0$ and $\left|a_t'\tilde{x}^r - a_t'x^r\right| = \left|\beta_t^r - a_t'x^r\right| \leqslant \|\beta^r - A_D x^r\|_\infty$. Consequently, D and

$$\left\{t \in T \mid \max\left\{\left|a_t'\tilde{x}^r - a_t'x^r\right|, \ a_t'\overline{x} - \overline{b}_t\right\} \geqslant \left(1 + \frac{1}{r}\right) \|\beta^r - A_D x^r\|_\infty\right\}$$

are disjoint closed sets. If the second of these sets is empty, then we take $\varphi_r \equiv 0$. We establish the convergence of $\{\widetilde{b}^r\}$ to $\overline{b}$ in Step 4.
Note that the definition of $\widetilde{b}^r$ ensures $D \subset T_{\widetilde{b}^r}(\tilde{x}^r)$. Then $D \in \mathcal{T}_{\widetilde{b}^r}(\tilde{x}^r)$, recalling $\overline{c} \in cone(\{a_t, t \in D\})$. Moreover, $\tilde{x}^r \in \mathcal{F}(\widetilde{b}^r)$, $r = 1, 2, \ldots$, because of Eq. 17. Then, appealing to Lemma 1(b), we have $\tilde{x}^r \in \mathcal{F}^*(\overline{c}, \widetilde{b}^r)$ for each r. Taking $\lim_r \widetilde{b}^r = \overline{b}$ into account we conclude that, for r large enough, $\mathcal{F}^*(\overline{c}, \widetilde{b}^r)$ is a singleton, and therefore Eq. 16 holds.

Step 2 We construct a sequence $\{b^r\} \subset C(T, \mathbb{R})$ converging to $\overline{b}$, verifying, for each r,

$$x^r \in \mathcal{F}^*(\overline{c}, b^r) \tag{19}$$

and

$$\left\|\widetilde{b}^r - b^r\right\|_\infty \leqslant \left(1 + \frac{1}{r}\right) \|\beta^r - A_D x^r\|_\infty . \tag{20}$$

Define, for each $t \in T$,

$$b_t^r := (1 - \varphi_r(t))a_t'x^r + \varphi_r(t) \min\left\{a_t'x^r, a_t'\tilde{x}^r, \overline{b}_t\right\},$$

where $\varphi_r(t)$ is the same as in Eq. 18. We will see that $\{b^r\}$ converges to $\overline{b}$ in Step 4.
In order to prove Eq. 20, for any given r, let us consider the non-trivial case $\varphi_r(t) < 1$ (otherwise, $\widetilde{b}_t^r = b_t^r = \min\{a_t'x^r, a_t'\tilde{x}^r, \overline{b}_t\}$). In such a case we have

$$\left|\widetilde{b}_t^r - b_t^r\right| = (1 - \varphi_r(t)) \left|a_t'\tilde{x}^r - a_t'x^r\right| < \left(1 + \frac{1}{r}\right) \|\beta^r - A_D x^r\|_\infty .$$

Next we show that, for each r, $x^r \in \mathcal{F}^*(\overline{c}, b^r)$. Note that the definition of b^r entails $x^r \in \mathcal{F}(b^r)$ and $D \in \mathcal{T}_{b^r}(x^r)$ for all r. Then, appealing again to Lemma 1 (b), we obtain Eq. 19.

Step 3 Completion of the proof.
According to Eqs. 15, 16, and 20, we obtain

$$\begin{aligned} d\big(x^r, \mathcal{F}^*\big(\overline{c}, \widetilde{b}^r\big)\big) &= \left\|x^r - \tilde{x}^r\right\| > \alpha_2 \left\|\beta^r - A_D x^r\right\|_\infty \\ &\geqslant \alpha_2 \left(1 + \frac{1}{r}\right)^{-1} \left\|\widetilde{b}^r - b^r\right\|_\infty \\ &\geqslant \alpha_2 \left(1 + \frac{1}{r}\right)^{-1} d\big(\big(\overline{c}, \widetilde{b}^r\big), \mathcal{G}^*(x^r)\big), \end{aligned}$$

where the last inequality comes from Step 2, which entails $(\overline{c}, b^r) \in \mathcal{G}^*(x^r)$. Moreover, if r is large enough to obtain $\alpha_2(1 + \frac{1}{r})^{-1} > \alpha_1$, we finally conclude

$$d\big(x^r, \mathcal{F}^*\big(\overline{c}, \widetilde{b}^r\big)\big) > \alpha_1 d\big(\big(\overline{c}, \widetilde{b}^r\big), \mathcal{G}^*(x^r)\big),$$

which contradicts Eq. 14.

Step 4 Both sequences $\{\widetilde{b}^r\}$ and $\{b^r\}$ converge to $\overline{b}$.
Due to the complete analogy between both sequences, it suffices to show the result for one of them, say $\{\widetilde{b}^r\}$. For each $r \in \mathbb{N}$ and each $t \in T$ we can write

$$\begin{aligned} \left|\widetilde{b}_t^r - \overline{b}_t\right| &\leqslant (1 - \varphi_r(t))\left|a_t'\tilde{x}^r - \overline{b}_t\right| + \varphi_r(t)\left|\min\left\{a_t'x^r, a_t'\tilde{x}^r, \overline{b}_t\right\} - \overline{b}_t\right| \\ &= (1 - \varphi_r(t))\left|a_t'\tilde{x}^r - \overline{b}_t\right| + \varphi_r(t) \max\left\{\left[\overline{b}_t - a_t'x^r\right]_+, \left[\overline{b}_t - a_t'\tilde{x}^r\right]_+\right\}, \end{aligned} \tag{21}$$

where $[\alpha]_+ := \max\{\alpha, 0\}$ represents the positive part of $\alpha \in \mathbb{R}$. Moreover,

$$\left[\overline{b}_t - a_t'x^r\right]_+ \leqslant \left[\overline{b}_t - a_t'\overline{x}\right]_+ + \left[a_t'\overline{x} - a_t'x^r\right]_+ \leqslant \left|a_t'\overline{x} - a_t'x^r\right| \leqslant \left\|a_t\right\|_* \left\|\overline{x} - x^r\right\|,$$

where we make use of the feasibility of $\overline{x}$ for $\sigma(\overline{b})$; i.e., $\overline{b}_t - a_t'\overline{x} \leqslant 0$ for all $t \in T$. Through a completely analogous expression for $\left[\overline{b}_t - a_t'\tilde{x}^r\right]_+$, we conclude

$$\max\left\{\left[\overline{b}_t - a_t'x^r\right]_+, \left[\overline{b}_t - a_t'\tilde{x}^r\right]_+\right\} \leqslant \left\|a_t\right\|_* \max\left\{\left\|\overline{x} - x^r\right\|, \left\|\overline{x} - \tilde{x}^r\right\|\right\}. \tag{22}$$

Next we focus on providing upper bounds for $(1 - \varphi_r(t))\left|a_t'\tilde{x}^r - \overline{b}_t\right|$, in the right hand side of Eq. 21. In the non-trivial case $\varphi_r(t) < 1$ we have, according to Eq. 18,

$$\max\left\{\left|a_t'\tilde{x}^r - a_t'x^r\right|, \; a_t'\overline{x} - \overline{b}_t\right\} < \left(1 + \frac{1}{r}\right)\left\|\beta^r - A_D x^r\right\|_\infty,$$

and then

$$\begin{aligned} (1 - \varphi_r(t))\left|a_t'\tilde{x}^r - \overline{b}_t\right| &\leqslant \left|a_t'\tilde{x}^r - a_t'x^r\right| + \left|a_t'x^r - a_t'\overline{x}\right| + a_t'\overline{x} - \overline{b}_t \\ &\leqslant 2\left(1 + \frac{1}{r}\right)\left\|\beta^r - A_D x^r\right\|_\infty + \|a_t\|_* \left\|x^r - \overline{x}\right\|. \end{aligned} \tag{23}$$

From Eqs. 21, 22, and 23, we obtain, for each $r \in \mathbb{N}$,

$$\left\| \widetilde{b}^r - \overline{b} \right\|_\infty \leqslant 2 \left(1 + \frac{1}{r}\right) \left\| \beta^r - A_D x^r \right\|_\infty + \\ +(\max_{t \in T} \|a_t\|_*) \left(\left\| x^r - \overline{x} \right\| + \max \left\{ \left\| \overline{x} - x^r \right\|, \left\| \overline{x} - \tilde{x}^r \right\| \right\} \right) \underset{r \to \infty}{\longrightarrow} 0.$$

□

5 The Exact Modulus in the Finite Case

In this section we analyze the case when T is finite, i.e., the case of ordinary linear programming. There are different contributions and approaches to the Lipschitzian behavior of the feasible and the optimal set in this case. For example, Dontchev et al. [9] entails that a lower bound on the Lipschitz modulus of the optimal set mapping is given by the reciprocal to the *distance to strong metric irregularity*. See also Cheung et al. [8] in relation to different concepts of distance to ill-posedness. Another approach may be developed from the *analysis of local stability regions* in Nožička et al. [25] (recovered, e.g., in Sections 5.4 and 5.5 of Bank et al. [1]). Li [21] provides Lipschitz constants for feasible and optimal solutions of ordinary linear programs (with inequality and equality constraints) under perturbations of the right hand side of the constraint system (the objective function is not perturbed). The Lipschitz constant provided in [21] for the feasible set mapping is shown to be sharp, while the sharpness of the Lipschitz constant for the optimal set mapping is pointed out in that paper as an open problem (end of p. 38 in [21]). In this section we show that the last Lipschitz constant is not sharp in general (even for inequality constraints only) and provide the sharp Lipschitz constant by using one of the tools developed in Li [22].

Specifically, the Lipschitz constant given in [21], adapted to our setting 1, and assuming that $\mathcal{F}^*$ is s.L.s. at $(\overline{c}, \overline{b})$, is

$$\gamma := \max_{\substack{D \subset T,\ |D|=n \\ A_D \text{ non-singular}}} \left\| A_D^{-1} \right\|.$$

In the following paragraphs we show that

$$\alpha := \max_{D \in \mathcal{T}_{\overline{b}}(\overline{x})} \left\| A_D^{-1} \right\|$$

becomes the sharp Lipschitz constant for $\mathcal{F}^*$ in a neighborhood of $(\overline{c}, \overline{b})$. In fact, we prove that α is a Lipschitz constant for $\mathcal{F}^*$ in a neighborhood of $(\overline{c}, \overline{b})$. Then, Theorem 1 obviously implies that α is sharp and equals the Lipschitz modulus (i.e., the Lipschitz modulus works as a Lipschitz constant in this case). Observe that γ does not depend on vector $\overline{c}$ (indeed, Theorem 3.5 in [21] already shows that γ is sharp for a certain $\overline{c}$). In Example 4 we have $\gamma = \sqrt{65} \approx 8.0623$ and $\alpha = \sqrt{1837/32} \approx 7.5767$ (see Remark 4 in Section 6 for details).

Next we present the definitions and tools needed in this section. Let V be an open and convex subset of $\mathbb{R}^m$, $m \in \mathbb{N}$. In the following lines, $\mathbb{R}^m$ and $\mathbb{R}^n$ are endowed with arbitrary norms, both denoted by $\|\cdot\|$. A multifunction $\mathcal{S} : V \rightrightarrows \mathbb{R}^n$ is said to be

Hausdorff lower semicontinuous at $y \in V$ if $\lim_{z \to y}(\sup_{x \in \mathcal{S}(y)} d(x, \mathcal{S}(z))) = 0$. $\mathcal{S}$ is said to be λ-Lipschitz continuous on V, denoted by $\mathcal{S} \in Lip(\lambda)$ if

$$H(\mathcal{S}(y), \mathcal{S}(z)) \leqslant \lambda \|y - z\| \text{ for all } y, z \in V,$$

where $H(\mathcal{S}(y), \mathcal{S}(z))$ is the Hausdorff distance between $\mathcal{S}(y)$ and $\mathcal{S}(z)$, given by

$$H(\mathcal{S}(y), \mathcal{S}(z)) = \max\left\{ \sup_{x \in \mathcal{S}(y)} d(x, \mathcal{S}(z)), \sup_{x \in \mathcal{S}(z)} d(x, \mathcal{S}(y)) \right\}.$$

$\mathcal{S}$ is said to be locally upper Lipschitz continuous with modulo λ on V, denoted by $\mathcal{S} \in UL(\lambda)$, if for any $y \in V$ there exists a neighborhood W of y such that

$$\sup_{x \in \mathcal{S}(y)} d(x, \mathcal{S}(z)) \leqslant \lambda \|y - z\|, \text{ for all } z \in W.$$

Lemma 3 [22, Theorem 2.1] *If $\mathcal{S}$ is Hausdorff lower semicontinuous and $\mathcal{S} \in UL(\lambda)$ then $\mathcal{S} \in Lip(\lambda)$.*

In fact, Theorem 2.1 in [22] is stated for a mapping defined on the whole $\mathbb{R}^m$ (with our current notation), but the proof also works for mappings defined on convex subsets of $\mathbb{R}^m$.

Next we choose an appropriate neighborhood V of $(\overline{c}, \overline{b})$. Our current assumptions are: T is finite, $\mathcal{F}^*$ is s.L.s. at $(\overline{c}, \overline{b})$, and $\mathcal{F}^*(\overline{c}, \overline{b}) = \{\overline{x}\}$. Let V be an open and convex neighborhood of $(\overline{c}, \overline{b})$ such that $\mathcal{F}^*$ is single-valued and Lipschitz continuous (i.e., s.L.s.) on V. Since $\overline{c}$ will remain fixed in our analysis, let $V_{\overline{b}}$ be an open and convex neighborhood of $\overline{b}$ such that $\{\overline{c}\} \times V_{\overline{b}} \subset V$ and write $\mathcal{F}^*(\overline{c}, b) = \{x(b)\}$ for $b \in V_{\overline{b}}$. Then it is not restrictive to assume that $V_{\overline{b}}$ is small enough to guarantee

$$T_b(x(b)) \subset T_{\overline{b}}(\overline{x}), \tag{24}$$

and hence $\mathcal{T}_b(x(b)) \subset \mathcal{T}_{\overline{b}}(\overline{x})$, for $b \in V_{\overline{b}}$.

Let $\mathcal{F}^*(\overline{c}, \cdot)|_{V_{\overline{b}}}$ denote the restriction of $\mathcal{F}^*(\overline{c}, \cdot)$ to $V_{\overline{b}}$; i.e., $\mathcal{F}^*(\overline{c}, \cdot)|_{V_{\overline{b}}} : V_{\overline{b}} \rightrightarrows \mathbb{R}^n$ is given by

$$\mathcal{F}^*(\overline{c}, \cdot)\big|_{V_{\overline{b}}}(b) = \mathcal{F}^*(\overline{c}, b) = \{x(b)\}, \text{ for } b \in V_{\overline{b}}.$$

Proposition 6 *Under the previous assumptions, $\mathcal{F}^*(\overline{c}, \cdot)|_{V_{\overline{b}}} \in UL(\alpha)$.*

Proof Take any $b^0 \in V_{\overline{b}}$ and consider a neighborhood W_{b^0} of b^0 such that $W_{b^0} \subset V_{\overline{b}}$ (recall that $V_{\overline{b}}$ is open) and such that $T_b(x(b)) \subset T_{b^0}(x(b^0))$ for all $b \in W_{b^0}$. Let us see that

$$\left\|x(b) - x(b^0)\right\| \leqslant \alpha \left\|b - b^0\right\|_\infty \text{ for all } b \in W_{b^0}.$$

Pick $b \in W_{b^0}$. The choice of V entails in particular that $\mathcal{F}^*$ is s.L.s. at $(\overline{c}, b)$, and, according to Propositions 2 and 3 we conclude the existence of $D \in \mathcal{T}_b(x(b))$ such that $x(b) = A_D^{-1} b_D$. Our choice of W_{b^0} ensures $D \in \mathcal{T}_{b^0}(x(b^0))$ $(\subset \mathcal{T}_{\overline{b}}(\overline{x}))$, and thus $x(b^0) = A_D^{-1} b_D^0$. In this way,

$$\left\| x(b) - x\left(b^0\right) \right\| = \left\| A_D^{-1} b_D - A_D^{-1} b_D^0 \right\| \leqslant \left\| A_D^{-1} \right\| \left\| b_D - b_D^0 \right\|_\infty \leqslant \alpha \left\| b - b^0 \right\|_\infty . \qquad \square$$

Corollary 1 $\mathcal{F}^*(\overline{c}, \cdot)|_{V_{\overline{b}}} \in Lip(\alpha)$.

Proof The result follows from Lemma 3 and the previous proposition, once we have established that $\mathcal{F}^*(\overline{c}, \cdot)|_{V_{\overline{b}}}$ is Hausdorff lower semicontinuous (on $V_{\overline{b}}$). Slater condition and Theorem 5.1(iii) in [6] ensure the lower semicontinuity of $\mathcal{F}^*(\overline{c}, \cdot)|_{V_{\overline{b}}}$ in the sense of Berge, and this is equivalent to Hausdoff's when the mapping under consideration is compact-valued (see Lemma 2.2.3 in [1]). □

Corollary 2 *If T is finite and $\mathcal{F}^*$ is s.L.s. at $(\overline{c}, \overline{b})$ with $\mathcal{F}^*(\overline{c}, \overline{b}) = \{\overline{x}\}$, then*

$$lip\mathcal{F}^*\left(\overline{c}, \overline{b}\right) = \max_{D \in \mathcal{T}_{\overline{b}}(\overline{x})} \left\| A_D^{-1} \right\| .$$

Moreover, this quantity is the sharp Lipschitz constant for $\mathcal{F}^*$ in a neighborhood of $(\overline{c}, \overline{b})$.

Proof Just combine the previous corollary with Theorem 1 and Proposition 4. □

This expression gives rise to an easily implementable method for calculating the exact modulus, as far as it reduces to calculate a finite number of matrix norms.

Observe that this analysis cannot be extended (at least straightforwardly) to the semi-infinite case, where Eq. 24 is no longer true in general. Moreover, we cannot ensure that $x(b) = A_D^{-1} b_D$ holds for all b in a neighborhood of $\overline{b}$ *with the same* D, and this is a key fact in the finite case.

6 A Sufficient Condition for the Exact Modulus in the Semi-Infinite Case

This section introduces a certain algebraic condition under which the lower bound Eq. 13 on the Lipschitz modulus of $\mathcal{F}^*$ becomes the exact value of the modulus (see Theorem 2) in the semi-infinite case. In such a way, we provide a sufficient condition in order to have an expression of the Lipschitz modulus exclusively in terms of the nominal problem's data. As it is shown later, the referred sufficient condition is always satisfied for dimension $n \leqslant 3$, whereas it is no longer true for $n = 4$.

The next paragraphs are intended to motivate and illustrate the announced Theorem 2. Assume that $\mathcal{F}^*$ is s.L.s. at $(\overline{c}, \overline{b})$ and, according to Proposition 4, consider two sequences $\{b^r\}, \{\widetilde{b}^r\} \subset C(T, \mathbb{R})$ converging to $\overline{b}$, with $b^r \neq \widetilde{b}^r$ for r large enough. In the sequel $x(\overline{c}, b)$ stands for the unique optimal point of $\mathcal{F}^*(\overline{c}, b)$, for b near $\overline{b}$. The following example shows how we can improve the ratio $\frac{\|x(\overline{c}, b^r) - x(\overline{c}, \widetilde{b}^r)\|}{\|b^r - \widetilde{b}^r\|_\infty}$ by means of keeping points $x(\overline{c}, b^r)$ and $x(\overline{c}, \widetilde{b}^r)$, and replacing b^r and $\widetilde{b}^r$ by some β^r and $\widetilde{\beta}^r$ defined in an appropriate subset of T and verifying $\left\| \beta^r - \widetilde{\beta}^r \right\|_\infty \leqslant \left\| b^r - \widetilde{b}^r \right\|_\infty$. The construction of these new β^r and $\widetilde{\beta}^r$ is a key step in the proof of Theorem 2.

Example 2 Let us consider the parametrized problem, in $\mathbb{R}^2$, endowed with the Euclidean norm,

$$\begin{aligned}\pi(c,b): Inf \quad & c_1x_1 + c_2x_2 \\ s.t. \quad & x_1 + x_2 \geqslant b_1,\ t = 1, \\ & 2x_1 + x_2 \geqslant b_2,\ t = 2, \\ & -2x_1 + x_2 \geqslant b_3,\ t = 3, \\ & -x_1 + x_2 \geqslant b_4,\ t = 4.\end{aligned}$$

Fix $\overline{c} = (0, 1)'$ and $\overline{b} = 0_4$ and observe that, according to Proposition 2, $\mathcal{F}^*$ is s.L.s. at $(\overline{c}, \overline{b})$ and $\mathcal{F}^*(\overline{c}, \overline{b})$ consists of $\overline{x} = 0_2$. Here $\mathcal{T}_{\overline{b}}(\overline{x}) = \{\{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}\}$. In order to give a first (lower) estimation of $lip\mathcal{F}^*(\overline{c}, \overline{b})$, we consider the sequences given by, for $r = 1, 2, ...,$

$$b^r = \left(\frac{-1}{r}, \frac{-3}{r}, \frac{-1}{r}, \frac{1}{r}\right)' \text{ and } \widetilde{b}^r = \left(\frac{1}{r}, \frac{-4}{r}, \frac{-2}{r}, \frac{-2}{r}\right)',$$

and so

$$x^r := x(\overline{c}, b^r) = \left(\frac{-1}{r}, 0\right)' \text{ and } \widetilde{x}^r := x(\overline{c}, \widetilde{b}^r) = \left(\frac{1}{r}, 0\right)'.$$

Hence we have

$$\limsup_r \frac{\|x^r - \widetilde{x}^r\|}{\|b^r - \widetilde{b}^r\|_\infty} = \lim_r \frac{2/r}{3/r} = 2/3 \leqslant lip\mathcal{F}^*(\overline{c}, \overline{b}).$$

A way of replacing b^r and $\widetilde{b}^r$ by some β^r and $\widetilde{\beta}^r$ as commented above, maintaining the optimality of x^r and $\widetilde{x}^r$, respectively, consists of taking, for each r, some subsets of indices $D_r \in \mathcal{T}_{b^r}(x^r)$ and $\widetilde{D}_r \in \mathcal{T}_{\widetilde{b}^r}(\widetilde{x}^r)$ and defining $\beta^r,\ \widetilde{\beta}^r \in \mathbb{R}^{D_r \cup \widetilde{D}_r}$ as follows:

(a) $\beta_t^r = b_t^r$ for $t \in D_r$ and $\widetilde{\beta}_t^r = \widetilde{b}_t^r$ for $t \in \widetilde{D}_r$, in order to maintain KKT conditions,
(b) β_t^r as close as possible to $\widetilde{\beta}_t^r$, but keeping the feasibility of x^r, for $t \in \widetilde{D}_r \backslash D_r$,
(c) $\widetilde{\beta}_t^r$ as close as possible to β_t^r, but keeping the feasibility of $\widetilde{x}^r$, for $t \in D_r \backslash \widetilde{D}_r$.

In this example the only possibilities are $D_r = \{1, 4\}$ and $\widetilde{D}_r = \{1, 3\}$.

The new β^r, $\widetilde{\beta}^r \in \mathbb{R}^{\{1,3,4\}}$ are, for $r = 1, 2, ...,$

$$\beta^r = \left(\frac{-1}{r}, \frac{-2}{r}, \frac{1}{r}\right)' \text{ and } \widetilde{\beta}^r = \left(\frac{1}{r}, \frac{-2}{r}, \frac{-1}{r}\right)'$$

and the new ratio is

$$\frac{\|x^r - \widetilde{x}^r\|}{\|\beta^r - \widetilde{\beta}^r\|_\infty} = \frac{2/r}{2/r} = 1.$$

Observe that in the previous example, appealing to Eq. 7 with $D = \{1, 3, 4\}$, we have $T_{\beta^r}(x^r) = \{1, 4\}$ and $T_{\widetilde{\beta}^r}(\widetilde{x}^r) = \{1, 3, 4\}$, which have the common subset

$\widehat{D} = \{1, 4\} \in \mathcal{T}_{\overline{b}}(\overline{x})$. The existence of such a common subset for suitably chosen points and parameters constitutes another key idea in the proof of Theorem 2.

In fact, there are two key ideas underlying the proof of Theorem 2: given $b^r, \widetilde{b}^r \to \overline{b}$, and x^r, $\widetilde{x}^r$, $\overline{x}$ being the unique points of $\mathcal{F}^*(\overline{c}, b^r)$, $\mathcal{F}^*(\overline{c}, \widetilde{b}^r)$, and $\mathcal{F}^*(\overline{c}, \overline{b})$, respectively, the first idea is, as pointed out above, to improve the ratio $\frac{\|x^r - \widetilde{x}^r\|}{\|b^r - \widetilde{b}^r\|_\infty}$ by maintaining points and changing parameters to get $\frac{\|x^r - \widetilde{x}^r\|}{\|\beta^r - \widetilde{\beta}^r\|_\infty}$. The second idea is to improve the latter ratio by maintaining parameters and changing one point, say $\widetilde{x}^r$, by an appropriate $\tilde{x}^r + \delta u^r$. For developing this second idea we need an additional hypothesis, called (H), used in Theorem 2 in order to guarantee the existence of the referred common subset of indices.

Note that condition (H) below is referred to the point $((\overline{c}, \overline{b}), \overline{x}) \in gph\mathcal{F}^*$.

(H): *If* $T_1, T_2, T_3 \subset T_{\overline{b}}(\overline{x})$ *verify*

$$\overline{c} \in cone(\{a_t,\ t \in T_1 \cup T_2\}) \cap cone(\{a_t,\ t \in T_1 \cup T_3\}) \tag{25}$$

and

$$\overline{c} \notin cone(\{a_t,\ t \in T_1\}), \tag{26}$$

then there exists a subset $\widetilde{T}_1 \subset T_1$ *such that* $|\widetilde{T}_1| \leqslant n - 1$ *and* Eq. 25 *is fulfilled when* T_1 *is replaced by* $\widetilde{T}_1$ (*obviously* Eq. 26 *is also preserved*).

Remark 3 According to Carathéodory's Theorem, it is not restrictive to assume that sets T_1, T_2, and T_3 involved in the statement of condition (H) verify $|T_i| \leqslant n$, for $i = 1, 2, 3$.

The following technical lemmas are also used in the next theorem.

Lemma 4 *If* $\mathcal{F}^*$ *is s.L.s. at* $(\overline{c}, \overline{b})$ *with* $\mathcal{F}^*(\overline{c}, \overline{b}) = \{\overline{x}\}$, *and* D *is a finite subset of* T *containing some subset* $\widehat{D} \in \mathcal{T}_{\overline{b}}(\overline{x})$, *then* $\mathcal{F}^*_D$ *is s.L.s. at* $(\overline{c}, \overline{b}_D)$.

Proof According to Proposition 2(b), we only have to note that $\overline{x} \in \mathcal{F}^*_D(\overline{c}, \overline{b}_D)$, which is a consequence of $\widehat{D} \subset D$ and $\widehat{D} \in \mathcal{T}_{\overline{b}}(\overline{x})$. □

Lemma 5 *Let* $x, u \in \mathbb{R}^n$ *be given. Then, either* $\|x + \alpha u\| \geqslant \|x\|$ *for all* $\alpha > 0$ *or* $\|x - \alpha u\| \geqslant \|x\|$ *for all* $\alpha > 0$.

Proof Assume, reasoning by contradiction, the existence of $\alpha_1 > 0$ and $\alpha_2 > 0$ such that $\|x + \alpha_1 u\| < \|x\|$ and $\|x - \alpha_2 u\| < \|x\|$. Then we attain the following contradiction:

$$\begin{aligned}\|x\| &= \left\| \frac{\alpha_2}{\alpha_1 + \alpha_2}(x + \alpha_1 u) + \frac{\alpha_1}{\alpha_1 + \alpha_2}(x - \alpha_2 u) \right\| \\ &\leqslant \frac{\alpha_2}{\alpha_1 + \alpha_2} \|x + \alpha_1 u\| + \frac{\alpha_1}{\alpha_1 + \alpha_2} \|x - \alpha_2 u\| < \|x\|.\end{aligned}$$

□

Theorem 2 *Assume that $\mathcal{F}^*$ is s.L.s. at $(\overline{c}, \overline{b})$, with $\mathcal{F}^*(\overline{c}, \overline{b}) = \{\overline{x}\}$, and that condition* (H) *is satisfied at $((\overline{c}, \overline{b}), \overline{x})$. Then*

$$lip\mathcal{F}^*\left(\overline{c}, \overline{b}\right) = \sup_{D \in \mathcal{T}_{\overline{b}}(\overline{x})} \left\| A_D^{-1} \right\|.$$

Proof The inequality '$\geqslant$' has been established in Theorem 1. Suppose, proceeding by contradiction, that there exists $\alpha > 0$ such that

$$lip\mathcal{F}^*(\overline{c}, \overline{b}) > \alpha > \sup_{D \in \mathcal{T}_{\overline{b}}(\overline{x})} \left\| A_D^{-1} \right\|. \tag{27}$$

According to Proposition 4, the first inequality in Eq. 27 yields the existence of two sequences, $\{b^r\}$ and $\left\{\widetilde{b}^r\right\}$ in $C(T, \mathbb{R})$, both converging to $\overline{b}$, such that

$$\frac{\left\| x^r - \tilde{x}^r \right\|}{\left\| b^r - \widetilde{b}^r \right\|_\infty} > \alpha, \tag{28}$$

with $\{x^r\} = \mathcal{F}^*(\overline{c}, b^r)$ and $\left\{\tilde{x}^r\right\} = \mathcal{F}^*(\overline{c}, \widetilde{b}^r)$, where we assume that the whole $\{b^r\}$ and $\left\{\widetilde{b}^r\right\}$ are close enough to $\overline{b}$ to guarantee the uniqueness of optimal solutions.

For r large enough (w.l.o.g., for all r) $\mathcal{F}^*$ is s.L.s. at $(\overline{c}, b^r)$ and $(\overline{c}, \widetilde{b}^r)$, respectively [see Proposition 3(b)]. Then, appealing to Proposition 3(a), for each r, there exist $D_r \subset T_{b^r}(x^r)$ and $\widetilde{D}_r \subset T_{\widetilde{b}^r}(\tilde{x}^r)$ such that $\overline{c} \in cone(\{a_t,\ t \in D_r\}) \cap cone\left(\left\{a_t,\ t \in \widetilde{D}_r\right\}\right)$, with $|D_r| = \left|\widetilde{D}_r\right| = n$.

Next, for each r large enough, we construct β^r and $\widetilde{\beta}^r \in \mathbb{R}^{D_r \cup \widetilde{D}_r}$ in the line of Example 2, that is, verifying

$$\mathcal{F}^*_{D_r \cup \widetilde{D}_r}(\overline{c}, \beta^r) = \left\{x^r\right\}, \qquad \mathcal{F}^*_{D_r \cup \widetilde{D}_r}(\overline{c}, \widetilde{\beta}^r) = \left\{\tilde{x}^r\right\}, \tag{29}$$

and

$$\left\| \beta^r - \widetilde{\beta}^r \right\|_\infty \leqslant \left\| b^r - \widetilde{b}^r \right\|_\infty. \tag{30}$$

We define, for each r,

$$\beta_t^r := \begin{cases} a_t' x^r & \text{if } t \in T_1^r \cup T_2^r, \\ a_t' \tilde{x}^r & \text{if } t \in T_3^r, \end{cases} \quad \widetilde{\beta}_t^r := \begin{cases} a_t' \tilde{x}^r & \text{if } t \in T_1^r \cup T_3^r, \\ a_t' x^r & \text{if } t \in T_2^r, \end{cases}$$

where $\left\{T_1^r, T_2^r, T_3^r\right\}$ is the partition of $D_r \cup \widetilde{D}_r$ given by

$$\begin{aligned} T_1^r &:= (D_r \cap \widetilde{D}_r) \cup \left\{t \in \widetilde{D}_r \setminus D_r \mid a_t'(\tilde{x}^r - x^r) \geqslant 0\right\} \\ &\quad \cup \left\{t \in D_r \setminus \widetilde{D}_r \mid a_t'(\tilde{x}^r - x^r) \leqslant 0\right\}, \\ T_2^r &:= \left\{t \in D_r \setminus \widetilde{D}_r \mid a_t'(\tilde{x}^r - x^r) > 0\right\}, \\ T_3^r &:= \left\{t \in \widetilde{D}_r \setminus D_r \mid a_t'(\tilde{x}^r - x^r) < 0\right\}. \end{aligned}$$

Obviously, these sets verify, for each r,

$$D_r \subset T_1^r \cup T_2^r; \widetilde{D}_r \subset T_1^r \cup T_3^r,$$

and then

$$\overline{c} \in cone\left(\left\{a_t,\ t \in T_1^r \cup T_2^r\right\}\right) \cap cone\left(\left\{a_t,\ t \in T_1^r \cup T_3^r\right\}\right). \tag{31}$$

This entails $x^r \in \mathcal{F}^*_{D_r \cup \widetilde{D}_r}(\overline{c}, \beta^r)$ and $\tilde{x}^r \in \mathcal{F}^*_{D_r \cup \widetilde{D}_r}(\overline{c}, \widetilde{\beta}^r)$ (Eq. 31 provides the KKT conditions, and feasibility can easily be checked). Therefore, from Lemma 4, for each r large enough $\mathcal{F}^*_{D_r \cup \widetilde{D}_r}$ is s.L.s. at $(\overline{c}, \beta^r)$ and $(\overline{c}, \widetilde{\beta}^r)$, respectively, entailing Eq. 29. The reader can check that Eq. 30 also holds.

Let us see that there exists an infinite subset $N \subset \mathbb{N}$ such that every $r \in N$ has an associated

$$\widehat{D}_r \subset D_r \cup \widetilde{D}_r, \text{ with } \left|\widehat{D}_r\right| = n, \tag{32}$$

such that

$$\overline{c} \in cone\left(\left\{a_t,\ t \in \widehat{D}_r\right\}\right),\ A_{\widehat{D}_r} \text{ is invertible and } \left\| A^{-1}_{\widehat{D}_r} \right\| > \alpha. \tag{33}$$

Assume for the moment that such a subsequence $\left\{\widehat{D}_r\right\}_{r \in N}$ has been constructed. Then, we can complete the proof as follows: Let us write $\widehat{D}_r = \left\{t_1^r, ..., t_n^r\right\}$, for each $r \in N$. Due to the compactness of T, the continuity of $t \mapsto a_t$, the facts that $\widehat{D}_r \subset T_{\beta^r}(x^r) \cup T_{\widetilde{\beta}^r}(\tilde{x}^r)$ (note that $T_1^r \cup T_2^r \subset T_{\beta^r}(x^r)$, $T_1^r \cup T_3^r \subset T_{\widetilde{\beta}^r}(\tilde{x}^r)$, and $T_1^r \cup T_2^r \cup T_3^r = D_r \cup \widetilde{D}_r$), $x^r \to \overline{x}$, $\tilde{x}^r \to \overline{x}$, and the definition of β^r and $\widetilde{\beta}^r$, we conclude, for some appropriate subsequence of r's, that $\left\{t_i^r\right\}_r$ converges to some $t_i \in T_{\overline{b}}(\overline{x})$. Write $\widehat{D} := \{t_1, ..., t_n\}$. Following a similar argument to the proof of Lemma 2, we conclude in a first step that the sequences of coefficients generating $\overline{c}$ as a linear combination of $\left\{a_t,\ t \in \widehat{D}_r\right\}$ are bounded and, in a second step, we conclude $\overline{c} \in cone\left(\left\{a_t,\ t \in \widehat{D}\right\}\right)$ and hence $\widehat{D} \in \mathcal{T}_{\overline{b}}(\overline{x})$. Due to this, the strong Lipschitz stability of $\mathcal{F}^*$ guarantees, appealing to Proposition 3(a), that $A_{\widehat{D}}$ is non-singular. Finally, letting $r \to +\infty$ in the last part of Eq. 33 we conclude $\alpha \leqslant \left\| A^{-1}_{\widehat{D}} \right\|$, which contradicts Eq. 27.

So, the rest of the proof is devoted to construct $\widehat{D}_r$, for infinitely many r's. We shall distinguish two (not disjoint) cases:

Case 1: $N_1 := \left\{r \in \mathbb{N} \mid \overline{c} \in cone\left(\left\{a_t,\ t \in T_1^r\right\}\right)\right\}$ is infinite.
Since $T_1^r \subset T_{\beta^r}(x^r) \cap T_{\widetilde{\beta}^r}(\tilde{x}^r)$ [recall Eq. 7] for all $r \in N_1$, appealing to Lemma 4, and assuming that r is large enough, we can use the specification of Proposition 3 (a) to $\mathcal{F}^*_{T_1^r}$ to find a subset $\widehat{D}_r \subset T_1^r$, with $\left|\widehat{D}_r\right| = n$, such that $\overline{c} \in cone\left(\left\{a_t,\ t \in \widehat{D}_r\right\}\right)$ and $\left\{a_t,\ t \in \widehat{D}_r\right\}$ is a basis of $\mathbb{R}^n$. Thus we have $\mathcal{F}^*_{\widehat{D}_r}\left(\overline{c}, \beta^r_{\widehat{D}_r}\right) = \{x^r\}$ and $\mathcal{F}^*_{\widehat{D}_r}\left(\overline{c}, \widetilde{\beta}^r_{\widehat{D}_r}\right) = \left\{\tilde{x}^r\right\}$, where $\beta^r_{\widehat{D}_r}$ and $\widetilde{\beta}^r_{\widehat{D}_r}$ represent, respectively, the restrictions of β^r and $\widetilde{\beta}^r$ to $\widehat{D}_r$. Hence, taking into account Eqs. 28 and 30 , we obtain, for each $r \in N_1$,

$$\alpha < \frac{\left\|x^r - \tilde{x}^r\right\|}{\left\|b^r - \widetilde{b}^r\right\|_\infty} \leqslant \frac{\left\|x^r - \tilde{x}^r\right\|}{\left\|\beta^r - \widetilde{\beta}^r\right\|_\infty} \leqslant \frac{\left\| A^{-1}_{\widehat{D}_r}\left(\beta^r_{\widehat{D}_r} - \widetilde{\beta}^r_{\widehat{D}_r}\right)\right\|}{\left\|\beta^r_{\widehat{D}_r} - \widetilde{\beta}^r_{\widehat{D}_r}\right\|_\infty} \leqslant \left\| A^{-1}_{\widehat{D}_r} \right\|. \tag{34}$$

Case 2: $N_2 := \mathbb{N} \backslash N_1$ is infinite.
In this case hypothesis (H) will allow us to construct new points and parameters making the ratio in Eq. 28 increase. In a first stage, roughly speaking, we show the applicability of condition (H) for T_1^r, T_2^r and T_3^r when $r \in N_2$ is large enough. Since, for all r, $\left|T_1^r \cup T_2^r \cup T_3^r\right| = \left|D_r \cup \widetilde{D}_r\right| \leqslant 2n$, the set $\left\{\left|T_1^r\right|, \left|T_2^r\right|, \left|T_3^r\right|\right\}$ may present only a finite amount of cases, and then it is not restrictive to assume, by taking a suitable subsequence, still identified

with N_2 for simplicity, that those cardinalities do not depend on r. Write, for each r,

$$T_1^r = \left\{t_1^r, ..., t_k^r\right\}, \ T_2^r = \left\{t_{k+1}^r, ..., t_p^r\right\}, \ T_3^r = \left\{t_{p+1}^r, ..., t_q^r\right\}.$$

Due to the compactness of T, and taking again a suitable subsequence, we may assume that $\left\{t_i^r\right\}_r$ converges to some $t_i \in T_{\overline{b}}(\overline{x})$, for $i = 1, ..., q$ (the argument is similar to the one preceding Case 1). Write

$$T_1 = \{t_1, ..., t_k\}, \ T_2 = \{t_{k+1}, ..., t_p\}, \ T_3 = \{t_{p+1}, ..., t_q\}.$$

From Eq. 31, and applying again a sequential argument similar to the proof of Lemma 2 we obtain

$$\overline{c} \in cone(\{a_t, \ t \in T_1 \cup T_2\}) \cap cone(\{a_t, \ t \in T_1 \cup T_3\}).$$

Moreover, we must have $\overline{c} \notin cone(\{a_t, \ t \in T_1\})$, because otherwise the strong Lipschitz stability assumption would yield $\overline{c} \in int(cone(\{a_t, \ t \in T_1\}))$, and so $\overline{c} \in cone\left(\left\{a_t, \ t \in T_1^r\right\}\right)$ for $r \in N_2$ large enough, which contradicts the definition of N_2. Therefore, we may apply hypothesis (H) to conclude the existence of $\widetilde{T}_1 \subset T_1$, with $\left|\widetilde{T}_1\right| \leqslant n - 1$ and such that

$$\overline{c} \in cone\left(\left\{a_t, \ t \in \widetilde{T}_1 \cup T_2\right\}\right) \cap cone\left(\left\{a_t, \ t \in \widetilde{T}_1 \cup T_3\right\}\right).$$

For simplicity let us write $\widetilde{T}_1 := \{t_1, ..., t_s\}$, $s \leqslant k$. Again, by the strong Lipschitz stability assumption, in fact we have

$$\overline{c} \in int\left(cone\left(\left\{a_t, \ t \in \widetilde{T}_1 \cup T_2\right\}\right)\right) \cap int\left(cone\left(\left\{a_t, \ t \in \widetilde{T}_1 \cup T_3\right\}\right)\right),$$

and then

$$\overline{c} \in int\left(cone\left(\left\{a_t, \ t \in \widetilde{T}_1^r \cup T_2^r\right\}\right)\right) \cap int\left(cone\left(\left\{a_t, \ t \in \widetilde{T}_1^r \cup T_3^r\right\}\right)\right) \quad (35)$$

for $r \in N_2$ large enough, where $\widetilde{T}_1^r := \{t_1^r, ..., t_s^r\}$. For simplicity we shall assume w.l.o.g. that every $r \in N_2$ fulfills Eq. 35.

In the next paragraphs we consider a fixed $r \in N_2$. Appealing to Lemma 5, we may consider a certain

$$u^r \in \left\{a_t, \ t \in \widetilde{T}_1^r\right\}^{\perp} \text{ with } \left\|u^r\right\| = 1 \quad (36)$$

(recall $\left|\widetilde{T}_1^r\right| = \left|\widetilde{T}_1\right| \leqslant n - 1$) such that

$$\left\|x^r - \tilde{x}^r - \delta u^r\right\| \geqslant \left\|x^r - \tilde{x}^r\right\| \text{ for all } \delta > 0.$$

For each r we consider

$$\delta^r := \min\left\{\frac{a_t'(x^r - \tilde{x}^r)}{a_t' u^r} \mid t \in T_2^r \text{ with } a_t' u^r < 0 \text{ or } t \in T_3^r \text{ with } a_t' u^r > 0\right\}.$$

Observe that δ^r is well defined, because otherwise we would have $a_t' u^r \geqslant 0$ for all $t \in T_2^r$ and $a_t' u^r \leqslant 0$ for all $t \in T_3^r$. According to Eqs. 35 and 36, the first entails $\overline{c}' u^r \geqslant 0$

and the latter yields $\overline{c}'u^r \leqslant 0$. Then, we obtain $u^r \in \left\{a_t,\ t \in \widetilde{T}_1^r \cup T_2^r\right\}^{\perp}$, whereas Eq. 35 entails $span\left\{a_t,\ t \in \widetilde{T}_1^r \cup T_2^r\right\} = \mathbb{R}^n$ (see, e.g., [12, Theorem A.7]). Moreover, the definitions of T_2^r and T_3^r entail $\delta^r > 0$.

Next we show that $\{\delta^r\}$ has some subsequence (still denoted by $\{\delta^r\}$ for simplicity) converging to zero. In fact, assuming w.l.o.g. that $\{u^r\}$ converges to some $u \in \mathbb{R}^n$ with $\|u\| = 1$, Eq. 36 entails, letting $r \to \infty$, $u \in \left\{a_t,\ t \in \widetilde{T}_1\right\}^{\perp}$, and the same argument which shows that δ^r is well defined may be used to establish the existence of either $t \in T_2$ with $a_t'u < 0$ or $t \in T_3$ with $a_t'u > 0$. Let $t_j,\ j \in \{k+1, ..., q\}$, be such a t. Obviously $a_{t_j}'u^r \neq 0$ for r large enough, and

$$0 \leqslant \limsup_r \delta^r \leqslant \limsup_r \frac{a_{t_j}'(x^r - \tilde{x}^r)}{a_{t_j}'u^r} = \frac{a_{t_j}'(\overline{x} - \overline{x})}{a_{t_j}'u} = 0. \tag{37}$$

Consider, for each r, the (sub)problems associated with $(\overline{c}, \hat{\beta}^r)$ and $(\overline{c}, \check{\beta}^r)$ determined by the new parameters $\hat{\beta}^r, \check{\beta}^r \in \mathbb{R}^{\widetilde{T}_1^r \cup T_2^r \cup T_3^r}$ defined as

$$\hat{\beta}_t^r := \begin{cases} a_t'x^r & \text{if } t \in \widetilde{T}_1^r, \\ a_t'x^r & \text{if } t \in T_2^r, \\ a_t'(\tilde{x}^r + \delta^r u^r) & \text{if } t \in T_3^r, \end{cases}$$

and

$$\check{\beta}_t^r := \begin{cases} a_t'\tilde{x}^r = a_t'(\tilde{x}^r + \delta^r u^r) & \text{if } t \in \widetilde{T}_1^r, \\ a_t'x^r & \text{if } t \in T_2^r, \\ a_t'(\tilde{x}^r + \delta^r u^r) & \text{if } t \in T_3^r. \end{cases}$$

Observe that $\left\|\hat{\beta}^r - \check{\beta}^r\right\|_\infty \leqslant \left\|\beta^r - \widetilde{\beta}^r\right\|_\infty$ because $\widetilde{T}_1^r \subset T_1^r$. Next we show that

$$\mathcal{F}^*_{\widetilde{T}_1^r \cup T_2^r \cup T_3^r}\left(\overline{c}, \hat{\beta}^r\right) = \left\{x^r\right\} \text{ and } \mathcal{F}^*_{\widetilde{T}_1^r \cup T_2^r \cup T_3^r}\left(\overline{c}, \check{\beta}^r\right) = \left\{\tilde{x}^r + \delta^r u^r\right\}.$$

To prove the first equality, since $\overline{c} \in cone\left(\left\{a_t,\ t \in \widetilde{T}_1^r \cup T_2^r\right\}\right)$, and the indices in $\widetilde{T}_1^r \cup T_2^r$ are active at x^r for $\hat{\beta}^r$, it is enough to show the feasibility of x^r for $\hat{\beta}^r$, and this follows from the choice of δ^r. In fact, condition '$\delta^r \leqslant \frac{a_t'(x^r - \tilde{x}^r)}{a_t'u^r}$ for all $t \in T_3^r$ with $a_t'u^r > 0$' is equivalent to '$a_t'x^r \geqslant a_t'(\tilde{x}^r + \delta^r u^r)$ for all $t \in T_3^r$'. Therefore, $x^r \in \mathcal{F}^*_{\widetilde{T}_1^r \cup T_2^r \cup T_3^r}(\overline{c}, \hat{\beta}^r)$, and this set is a singleton for r large enough.

Analogously, from $\overline{c} \in cone\left(\left\{a_t,\ t \in \widetilde{T}_1^r \cup T_3^r\right\}\right)$, where indices in $\widetilde{T}_1^r \cup T_3^r$ are active at $\tilde{x}^r + \delta^r u^r$ for $\check{\beta}^r$, we conclude the feasibility of $\tilde{x}^r + \delta^r u^r$ for $\check{\beta}^r$, now due to the fact that '$\delta^r \leqslant \frac{a_t'(x^r - \tilde{x}^r)}{a_t'u^r}$ for all $t \in T_2^r$ with $a_t'u^r < 0$ ' is equivalent to '$a_t'(\tilde{x}^r + \delta^r u^r) \geqslant a_t'x^r$ for all $t \in T_2^r$'. Thus, $\mathcal{F}^*_{\widetilde{T}_1^r \cup T_2^r \cup T_3^r}(\overline{c}, \check{\beta}^r) = \left\{\tilde{x}^r + \delta^r u^r\right\}$. Moreover, Eqs. 36 and 37 ensure that $\tilde{x}^r + \delta^r u^r \to \overline{x}$ as $r \to \infty$.

We have, due to the choice of u^r, appealing to Eqs. 28, 30 , and the previous paragraphs,

$$\frac{\left\|x^r - (\tilde{x}^r + \delta^r u^r)\right\|}{\left\|\hat{\beta}^r - \check{\beta}^r\right\|_\infty} \geqslant \frac{\left\|x^r - \tilde{x}^r\right\|}{\left\|\beta^r - \widetilde{\beta}^r\right\|_\infty} \geqslant \frac{\left\|x^r - \tilde{x}^r\right\|}{\left\|b^r - \widetilde{b}^r\right\|_\infty} > \alpha. \tag{38}$$

Finally, let us consider some $t^r_{j_1} \in T^r_2 \cup T^r_3$, $j_1 \in \{k+1, ..., q\}$, at which the minimum defining δ^r is attained. Then $t^r_{j_1}$ is active at x^r for $\hat{\beta}^r$ and at $\tilde{x}^r + \delta^r u^r$ for $\check{\beta}^r$. In fact,

$$a'_{t^r_{j_1}} x^r = a'_{t^r_{j_1}} (\tilde{x}^r + \delta^r u^r) = \hat{\beta}^r_{t^r_{j_1}} = \check{\beta}^r_{t^r_{j_1}}.$$

With these new parameters, $(\bar{c}, \hat{\beta}^r)$ and $(\bar{c}, \check{\beta}^r)$, and these new optimal points, x^r and $\tilde{x}^r + \delta^r u^r$, we are again in the same situation as Eqs. 29 and 30, and then we redefine the T^r_i's in this 'second stage' as

$$T_1^{2,r} := \widetilde{T}^r_1 \cup \left\{t^r_{j_1}\right\},$$

$$T_2^{2,r} := \begin{cases} T^r_2 \backslash \left\{t^r_{j_1}\right\}, & \text{if } t^r_{j_1} \in T^r_2, \\ T^r_2, & \text{if } t^r_{j_1} \in T^r_3, \end{cases}$$

$$T_3^{2,r} := \begin{cases} T^r_3, & \text{if } t^r_{j_1} \in T^r_2, \\ T^r_3 \backslash \left\{t^r_{j_1}\right\}, & \text{if } t^r_{j_1} \in T^r_3. \end{cases} \tag{39}$$

Once the previous process is done for all $r \in N_2$, we consider the new sets

$$N_{21} := \left\{r \in N_2 | \bar{c} \in cone\left(\left\{a_t,\ t \in T_1^{2,r}\right\}\right)\right\},\ N_{22} := N_2 \backslash N_{21}.$$

Again we distinguish two (not disjoint) cases.

Subcase 2.1: If N_{21} is infinite, then we can reproduce Case 1 with this new ingredients, and in the same way as in Eq. 34, we find, for each $r \in N_{21}$, a certain $\widehat{D}_r \subset T_1^{2,r}$, with $\left|\widehat{D}_r\right| = n$, such that $\bar{c} \in cone\left(\left\{a_t,\ t \in \widehat{D}_r\right\}\right)$ and $\left\{a_t,\ t \in \widehat{D}_r\right\}$ is a basis of $\mathbb{R}^n$. Thus we have $\mathcal{F}^*_{\widehat{D}_r}\left(\bar{c}, \hat{\beta}^r_{\widehat{D}_r}\right) = \{x^r\}$ and $\mathcal{F}^*_{\widehat{D}_r}\left(\bar{c}, \check{\beta}^r_{\widehat{D}_r}\right) = \left\{\tilde{x}^r + \delta^r u^r\right\}$. Hence, taking into account Eq. 38, we obtain,

$$\alpha < \frac{\left\|x^r - (\tilde{x}^r + \delta^r u^r)\right\|}{\left\|\hat{\beta}^r - \check{\beta}^r\right\|_\infty} \leqslant \frac{\left\|A^{-1}_{\widehat{D}_r}(\hat{\beta}^r_{\widehat{D}_r} - \check{\beta}^r_{\widehat{D}_r})\right\|}{\left\|\hat{\beta}^r_{\widehat{D}_r} - \check{\beta}^r_{\widehat{D}_r}\right\|_\infty} \leqslant \left\|A^{-1}_{\widehat{D}_r}\right\|.$$

Subcase 2.2: If N_{22} is infinite, then we can reproduce Case 2 with N_{22} instead of N_2. In this way we ensure the existence, for r large enough, of a certain $\widetilde{T}_1^{2,r}$ given by condition (H). We can also transfer a new index $t^r_{j_2}$ from the $T_2^{2,r} \cup T_3^{2,r}$ towards $\widetilde{T}_1^{2,r}$. Then we define the corresponding N_{221} and N_{222}, and so on.

Since the original T^r_2 and T^r_3, for all $r \in \mathbb{N}$, have cardinalities no larger than n, this 'transferring process' is finite. Specifically, for some $m \leqslant 2n$ we attain our goal of ensuring the existence of infinitely many $\widehat{D}_r$'s verifying

$$\widehat{D}_r \subset T_1^{m,r} \text{ and } \bar{c} \in cone\left(\left\{a_t,\ t \in \widehat{D}_r\right\}\right),$$

in which case we proceed as in Subcase 2.1 (or Case 1). Note that this stage m happens at most when some of sets $T_2^{m,r}$ or $T_3^{m,r}$ is empty. This completes the proof. □

The following proposition establishes that hypothesis (H) is superfluous in the previous theorem for dimensions $n \leqslant 3$.

Proposition 7 *Assume that* $n \leqslant 3$ *and* $\mathcal{F}^*$ *is s.L.s. at* $(\overline{c}, \overline{b})$, *with* $\mathcal{F}^*(\overline{c}, \overline{b}) = \{\overline{x}\}$. *Then, condition* (H) *is satisfied at* $((\overline{c}, \overline{b}), \overline{x})$.

Proof Take $T_1, T_2, T_3 \subset T_{\overline{b}}(\overline{x})$ verifying Eqs. 25 and 26. According to Remark 3, it is not restrictive to assume that T_1, T_2, and T_3 are finite.

For $n \in \{1, 2\}$ it is easy to check that any closed and convex pointed cone (containing no lines) in the form $cone(S)$, where S is a finite subset of $\mathbb{R}^n$, is generated at most by n elements of S. This is no longer the situation for $n \geqslant 3$.

In the case $n = 1$, appealing to Carathéodory's Theorem, we may assume from Eqs. 25 and 26 that $|T_2| = |T_3| = 1$ and $T_1 = \emptyset$. So, this case is trivial and from now on we suppose $n \geqslant 2$.

Because of the Slater condition, there exists $x^0 \in \mathbb{R}^n$ such that,

$$a_t'(x^0 - \overline{x}) > 0 \text{ for all } t \in T_1 \cup T_2 \cup T_3. \tag{40}$$

Now, take an orthonormal basis $\tilde{B} := \left\{\tilde{e}_1, \tilde{e}_2, ..., \tilde{e}_n\right\}$ in $\mathbb{R}^n$ such that $\tilde{e}_1 := \frac{x^0 - \overline{x}}{\|x^0 - \overline{x}\|}$. With respect to this basis, we ensure that the first coordinate of a_t is non-negative, i.e. $a_{t,1} > 0$, for all $t \in T_1 \cup T_2 \cup T_3$. Then, attending to the statement we intend to prove, it is not restrictive to assume that $a_{t,1} = 1$, for each $t \in T_1 \cup T_2 \cup T_3$.

We may also assume w.l.o.g. that the first coordinate of $\overline{c}$ equals 1, because $\overline{c} \in cone(\{a_t,\ t \in T_1 \cup T_2\})$ yields the existence of $\lambda \in \mathbb{R}_+^{T_1 \cup T_2}$ with $\overline{c} = \sum_{t \in T_1 \cup T_2} \lambda_t a_t$, and therefore, appealing to Eq. 40,

$$\tilde{e}_1' \overline{c} = \sum_{t \in T_1 \cup T_2} \lambda_t \tilde{e}_1' a_t > 0.$$

(Note that the strong Lipschitz stability assumption yields $\overline{c} \neq 0_n$, and thus some λ_t must be positive.)

In this setting in which all vectors involved have 1 as their first coordinate, conical combinations (i.e., linear combinations with nonnegative scalars) are the same as convex combinations. Moreover, we can work in the space of the last $n-1$ coordinates. In this way our problem is posed in $\mathbb{R}^{n-1}$.

In the case $n = 2$ (now viewed in $\mathbb{R}$), attending to Carathéodory's Theorem (see also Remark 3), the non-trivial case is $|T_1| = 2$, $|T_2| = |T_3| = 1$, and in this case Eqs. 25 and 26 mean that the projections (on their second coordinates) of $\{a_t,\ t \in T_1\}$ and $\{a_t,\ t \in T_2 \cup T_3\}$ are, respectively, in the two opposite open half-lines having the projection of $\overline{c}$ as their common extreme point. Thus we can choose any $\widetilde{T}_1 \subset T_1$ with $\left|\widetilde{T}_1\right| = 1$.

In the case $n = 3$ let us write, for some nonnegative scalars λ_t^1, $t \in T_1$, and λ_t^2, $t \in T_2$,

$$\overline{c} = \sum_{t \in T_1} \lambda_t^1 a_t + \sum_{t \in T_2} \lambda_t^2 a_t = \sum_{t \in T_1} \lambda_t^1 a_t + \gamma v^2, \tag{41}$$

where $\gamma := \sum_{t\in T_2} \lambda_t^2$ and $v^2 := \gamma^{-1} \sum_{t\in T_2} \lambda_t^2 a_t$. Note that $\gamma > 0$ because of Eq. 26, and we have $\sum_{t\in T_1} \lambda_t^1 + \gamma = 1$. Thus, Eq. 41 leads us to

$$0_3 = \sum_{t\in T_1} \lambda_t^1 (a_t - \overline{c}) + \gamma (v^2 - \overline{c}),$$

and then

$$v^2 = \overline{c} - \gamma^{-1} \sum_{t\in T_1} \lambda_t^1 (a_t - \overline{c}) \in \overline{c} - cone\, \{a_t - \overline{c},\ t \in T_1\}. \tag{42}$$

An analogous calculation, starting from $T_1 \cup T_3$ instead of $T_1 \cup T_2$, yields $\overline{c} = \sum_{t\in T_1} \tilde{\lambda}_t^1 a_t + \delta v^3$, for some nonnegative scalars such that $\sum_{t\in T_1} \tilde{\lambda}_t^1 + \delta = 1$, and

$$v^3 \in \overline{c} - cone\, \{a_t - \overline{c},\ t \in T_1\}.$$

Moreover, taking into account that the first coordinate of all elements in $\{a_t,\ t \in T_1;\ \overline{c}\}$ is 1, Eq. 26 can be written as $\overline{c} \notin conv(\{a_t,\ t \in T_1\})$, and this is equivalent to $0_3 \notin conv\, \{a_t - \overline{c},\ t \in T_1\}$, which implies that $cone\, \{a_t - \overline{c},\ t \in T_1\}$ (which can be viewed in $\mathbb{R}^2$) is pointed. Then this cone can be generated by (at most) two vectors, say $a_{t_1} - \overline{c}$ and $a_{t_2} - \overline{c}$, with $\widetilde{T}_1 := \{t_1, t_2\} \subset T_1$.

Writing Eq. 42 as

$$v^2 = \overline{c} - \mu_1 (a_{t_1} - \overline{c}) - \mu_2 (a_{t_2} - \overline{c}),$$

for certain nonnegative scalars μ_1 and μ_2, and recalling the definition of v^2, we obtain

$$\overline{c} = \frac{\mu_1}{1+\mu_1+\mu_2} a_{t_1} + \frac{\mu_2}{1+\mu_1+\mu_2} a_{t_2} + \frac{\gamma^{-1}}{1+\mu_1+\mu_2} \sum_{t\in T_2} \lambda_t^2 a_t.$$

Analogous expression is obtained with T_3 instead of T_2. □

The following examples show that condition (H) can either be satisfied or not for dimension $n = 4$.

Example 3 (*Condition* (H) *may fail in* $\mathbb{R}^4$) Consider the problem $\pi(\overline{c}, \overline{b})$, associated to $(\overline{c}, \overline{b}) = ((0, 0, 2, 1)', 0_6)$, given by

$$\begin{aligned} &\text{Inf } 2x_3 + x_4 \\ &\text{s.t.} \begin{pmatrix} 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & 1 & 1 \\ \frac{1}{4} & 1 & 4 & 1 \\ -\frac{1}{4} & -1 & 4 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \geqslant 0_6. \end{aligned}$$

It can easily be checked that $\mathcal{G}^*$ is metrically regular at 0_4 for $(\overline{c}, \overline{b})$, i.e., $\mathcal{F}^*$ is s.L.s. at $(\overline{c}, \overline{b})$. In fact, one has

$$\mathcal{T}_{\overline{b}}(0_4) = \{\{1,2,3,6\}, \{1,2,4,5\}, \{1,2,5,6\}, \{1,3,4,6\}, \{2,3,4,5\}, \{3,4,5,6\}\}.$$

Then Eqs. 25 and 26 hold for

$$T_1 = \{1,2,3,4\},\ T_2 = \{6\}\ , T_3 = \{5\}.$$

However, there is no subset $\widetilde{T}_1 \subset T_1$ with $|\widetilde{T}_1| = 3$ such that $\widetilde{T}_1 \cup \{6\}$ and $\widetilde{T}_1 \cup \{5\}$ simultaneously belong to $\mathcal{T}_{\overline{b}}(0_4)$.

Example 4 (*Condition* (H) *may hold in* $\mathbb{R}^4$) Consider the problem $\pi(\overline{c}, \overline{b})$ coming from replacing $\overline{c}$ in the previous example by $\overline{c} = (0, 0.7, 2, 1)'$.

One can check that

$$\mathcal{T}_{\overline{b}}(0_4) = \{\{1, 2, 3, 5\}, \{2, 3, 4, 5\}, \{2, 3, 5, 6\}\}.$$

Consider T_1, T_2, and T_3 satisfying Eqs. 25 and 26. If $|T_1| \leqslant 3$, it is enough to take $\widetilde{T}_1 = T_1$.

Suppose now that $|T_1| \geqslant 4$. Equation 25 implies that both $T_1 \cup T_2$ and $T_1 \cup T_3$ do contain $\{2, 3, 5\}$. Observe that $\mathcal{T}_{\overline{b}}(0_4)$ consists of all possibilities of adding one element to $\{2, 3, 5\}$. So, $|T_1| \geqslant 4$, together with Eq. 26, entails $\{2, 3, 5\} \not\subset T_1$. In this way, $1 \leqslant |T_1 \cap \{2, 3, 5\}| \leqslant 2$. Moreover, $\{2, 3, 5\} \setminus T_1 \subset T_2 \cap T_3$, because all indices in $\{2, 3, 5\}$ are necessary for generating $\overline{c}$ as a conical combination. Thus, it is enough to consider as $\widetilde{T}_1$ any subset of T_1 which contains $T_1 \cap \{2, 3, 5\}$ plus any element $t_0 \in T_1 \setminus \{2, 3, 5\}$. Note that such a $\widetilde{T}_1$ verifies $|\widetilde{T}_1| \leqslant 3$ and both $\widetilde{T}_1 \cup T_2$ and $\widetilde{T}_1 \cup T_3$ do contain $\{t_0, 2, 3, 5\}$.

Remark 4 In Example 3 we have

$$\max_{D \in \mathcal{T}_{\overline{b}}(\overline{x})} \left\| A_D^{-1} \right\| = \max_{\substack{D \subset T,\ |D|=n \\ A_D \text{ non-singular}}} \left\| A_D^{-1} \right\| = \left\| A_{\{3,4,5,6\}}^{-1} \right\| = \sqrt{65} \approx 8.0623.$$

Example 4 (which was already referred to in Section 5) has the same constraint system as Example 3, but different objective function. In it we have

$$\max_{D \in \mathcal{T}_{\overline{b}}(\overline{x})} \left\| A_D^{-1} \right\| = \left\| A_{\{2,3,5,6\}}^{-1} \right\| = \sqrt{1837/32} \approx 7.5767.$$

7 Conclusions and Open Problem

At this moment we summarize the main contributions of the paper, devoted to calculate the Lipschitz modulus of $\mathcal{F}^*$ at a point $(\overline{c}, \overline{b})$ under the strong Lipschitz stability assumption at this point.

- In a first step, the Lipschitz modulus for problems in $\mathbb{R}^n$ with exactly n constraints is determined in Proposition 5:

$$lip\mathcal{F}_D^*(\overline{c}, \overline{b}) = \left\| A_D^{-1} \right\|.$$

- In general, one has (Theorem 1)

$$lip\mathcal{F}^*(\overline{c}, \overline{b}) \geqslant \sup_{D \in \mathcal{T}_{\overline{b}}(\overline{x})} \left\| A_D^{-1} \right\|. \tag{43}$$

- In ordinary linear programming (T finite) equality holds, i.e.,

$$lip\mathcal{F}^*(\overline{c}, \overline{b}) = \max_{D \in \mathcal{T}_{\overline{b}}(\overline{x})} \left\| A_D^{-1} \right\|,$$

and the right hand side of the previous expression is in fact the sharp Lipschitz constant of $\mathcal{F}^*$ in a neighborhood of $(\overline{c}, \overline{b})$. This provides a negative answer to an open problem pointed out in [21, p. 38].

- If additionally condition (H) is satisfied, then Eq. 43 becomes an equality also in the semi-infinite case (Theorem 2).
- Condition (H) is always satisfied for dimensions $n \leqslant 3$ (Proposition 7), and may be held or not for higher dimensions (Examples 3 and 4).

We point out the fact that the right hand side of Eq. 43 only involves the nominal problem's data and this expression is easily implementable, for instance in MATLAB.

The possibility of establishing the equality in Eq. 43 by weakening or removing condition (H), in the semi-infinite case, remains as an open problem. The fact that condition (H) is always satisfied when $n \leqslant 3$ introduces some difficulties (in relation to geometrical intuition) when dealing with examples or counterexamples.

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