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On Attractors for Multivalued Semigroups Defined by Generalized Semiflows

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Abstract In this paper we extend results from Semigroup Theory on existence and characterization of attractors in order to include multivalued semigroups $T(t)$ defined by generalized semiflows $\mathscr{G}$. In particular we show that, if $\mathscr{G}$ is continuous, possesses a Lyapunov function, and $\mathscr{G}$ has a global attractor $\mathscr{A}$ which is maximal compact invariant, then $\mathscr{A} = W^u(Z(\mathscr{G}))$, where $Z(\mathscr{G})$ is the stationary solutions set and $W^u(Z(\mathscr{G}))$ is the unstable set of $Z(\mathscr{G})$. We introduce the φ-attractor concept which does not enjoy any uniformity on time of attraction and we prove, under suitable conditions, that the global φ-attractor $\widehat{N}$ is the set of asymptotic states described by $Z(\mathscr{G})$.

Keywords Generalized semiflows · Multivalued semigroups · Stationary solutions · Attractors · φ-attraction

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1 Introduction

The lack of uniqueness does not constitute an obstruction to obtaining the existence of global attractors for differential problems. In fact, a wide class of Cauchy problems

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usually termed as ill-posed enjoy sufficient compactness and dissipativity properties to prove the existence of a maximal compact invariant set which attracts all the bounded sets. There are several authors dealing with this issue, among which we name Ball, [1], Melnik and Valero, [8], and Carvalho and Gentile, [4]. We also cite [2] where it can be found a good description of the two first works and a comparison between them. In a very short way, we could say that Melnik and Valero consider multivalued operators mapping initial data u_0 from the phase space X onto the set of all possible states at a time t, $T(t)u_0 \subset X$ while Ball looks at solutions as the fundamental block in his considerations, as we briefly describe below. The ideas in the third work are deeply connected with both of them, since it deals with multivalued semigroups constructed by considering all possible solutions starting from given initial data. The main distinction on the study of the asymptotic behavior in [4] is that the multivalued semigroup considered there enjoy a strong regularity property in such way that it eventually becomes a semigroup. This condition seems to be hard to deal with, but actually it can be easily obtained, by comparison methods, for a class of problems involving maximal monotone operators with increasing resolvent and for which the uniqueness is not a trivial question, [3].

In this work we are concerned with multivalued semigroups $T(t)$ defined by generalized semiflows $\mathcal{G}$, that means, every point $\zeta \in T(t)x$, $x \in X$ (X a complete metric space, $t \in [0, \infty)$) is such that $\zeta = \varphi(t)$, where φ is a solution in $\mathcal{G}$ and $\varphi(0) = x$. Most definitions from Semigroup Theory can be extended in a very natural way to the multivalued framework, but some of them accept more than one extension. This is the case when we consider the existence of a certain time τ after which some condition must be satisfied, as occurs in the concepts of attraction, absorption and dissipativity or eventual properties. In Semigroup Theory, this time can be uniformly chosen on bounded sets or not, and these are the only two possibilities. Here, in this context, we have to consider the possibility of a time, after which something happens, being uniform on bounded sets, uniform on points or not uniform in any way. By uniformity on points we mean that given an arbitrary point $x \in X$, there is a time $\tau = \tau(x)$ after which all solutions $\varphi \in \mathcal{G}$ starting at x enjoy a certain property. This distinction generates a new group of definitions which does not make sense in single-valued theory and some of them seem to play a special role in multivalued semigroup context. In this text we call φ-concepts those which are defined without supposing any uniformity on time, that means, we add the prefix φ to the words attraction, dissipativity, boundedness, etc, in order to indicate that we are not supposing that the times in such definitions are uniformly chosen in any sense. For example, we say that a set A $\varphi-$attracts some subset $M \subset X$ if given $\varepsilon > 0$, each solution in $\mathcal{G}$ starting at some point in M, eventually goes into the ε neighbourhood of A, $O_\varepsilon(A)$, and remains inside it. One of our main results states that, if a generalized semiflow $\mathcal{G}$ is φ-asymptotically compact and possesses a Lyapunov function, then there exists a minimal closed global φ-attractor $\widehat{N}$ for $\mathcal{G}$ and $\widehat{N}$ coincides with the set of stationary solutions in $\mathcal{G}$.

We organize this paper as follows. Section 2 contains the main results in the literature on the existence of attractors for differential problems without uniqueness and provides the background information for further discussions. We describe all relevant definitions and theorems in [1, 2, 8] and [4], and we also add some others easily extended from [5] and [6]. In Section 3 we obtain some characterizations of the attractor, as is done in [6] for semigroups, and we prove that, even in the multivalued case, the maximal compact invariant global attractor can be described in terms of the unstable set of the equilibrium states, if there is a Lyapunov function for the system. In Section 4 we discuss the φ-concepts.

2 Multivalued Semigroups - A Theory for Differential Problems without Uniqueness

In this section we introduce the general framework for studying attractors for differential problems without uniqueness. Most elements in this preliminary text is already published in [1, 2, 8] or [4], as we indicate at the beginning of each known result. We introduce some additional definitions and results which are easily extend from Classical Semigroup Theory, and we use strongly [7] to support our discussion.

2.1 Definitions, Notations and Preliminary Facts

Let (X, d) be a complete metric space and let $P(X)$, $B(X)$, $C(X)$, $K(X)$, and $BC(X)$ denote respectively non-empty, non-empty and bounded, non-empty and closed, non-empty and compact and non-empty bounded and closed subsets of X. For $x \in X$ and $A, B \in P(X)$, and $\varepsilon > 0$ we set

$$\begin{array}{ll} d(x, A) & \doteq \inf_{a\in A}\{d(x, a)\}; \\ \operatorname{dist}(A, B) & \doteq \sup_{a\in A}\{d(a, B)\} = \sup_{a\in A} \inf_{b\in B}\{d(a, b)\}; \\ d_H(A, B) & \doteq \max\{\operatorname{dist}(A, B), \operatorname{dist}(B, A)\} : \\ O_\varepsilon(A) & \doteq \{z \in X; d(z, A) < \varepsilon\}. \end{array}$$

Definition 1 [1] A ***generalized semiflow*** $\mathcal{G}$ on X is a family of maps $\varphi\colon [0, \infty) \to X$ satisfying the conditions:

(H1) For each $z \in X$ there exists at least one $\varphi \in \mathcal{G}$ with $\varphi(0) = z$.
(H2) If $\varphi \in \mathcal{G}$ and $\tau \geqslant 0$, then $\varphi^\tau \in \mathcal{G}$, where $\varphi^\tau(t) := \varphi(t + \tau)$, $\forall\, t \in [0, \infty)$.
(H3) If $\varphi, \psi \in \mathcal{G}$, and $\psi(0) = \varphi(t)$ for some $t \geqslant 0$, then $\theta \in \mathcal{G}$, where

$$\theta(\tau) \doteq \begin{cases} \varphi(\tau) & \text{for } \tau \in [0, t] \\ \psi(\tau - t) & \text{for } \tau \in (t, \infty) \end{cases}$$

(H4) If $\{\varphi_j\}_{j=1}^{\infty} \subset \mathcal{G}$ and $\varphi_j(0) \to z$, then there exists a subsequence $\{\varphi_\mu\}$ of $\{\varphi_j\}$ and $\varphi \in \mathcal{G}$ with $\varphi(0) = z$ such that $\varphi_\mu(t) \to \varphi(t)$ for each $t \geqslant 0$.

Definition 2 We say that $\mathcal{G}$ is a ***continuous generalized semiflow*** if each $\varphi \in \mathcal{G}$ is a continuous map from $[0, \infty)$ on X.

Definition 3 A ***multivalued semigroup*** $\{T(t)\}_{t\geqslant 0}$ ***defined by*** $\mathcal{G}$ is a family of multivalued operators $T(t)\colon P(X) \to P(X)$ such that, for each $t \geqslant 0$,

$$T(t)E \doteq \{\varphi(t);\ \varphi \in \mathcal{G} \text{ with } \varphi(0) \in E\}.$$

It is well known that the family $\{T(t)\}_{t\geqslant 0}$ satisfies

Proposition 1 [2]

(a) *$\{T(t)\}_{t\geqslant 0}$ is a semigroup on $P(X)$, i.e., $T(0) = Id_{P(X)}$ and $T(t+s) = T(t)T(s)$ for all $t, s \geqslant 0$;*
(b) *$T(t)$ is monotone with respect to the partial order of set inclusion, i.e., $E \subset F$ implies $T(t)E \subset T(t)F$ for all $t \geqslant 0$;*
(c) *$T(t)x$ is compact for each $x \in X$;*
(d) *If $\{K_n\}_{n\geqslant 1}$ is a sequence of compact subsets of X such that $dist(K_n, K) \to 0$ as $n \to \infty$ then $dist(T(t)K_n, T(t)K) \to 0$ for each $t \geqslant 0$; and*
(e) *$T(t)\colon X \to K(X)$ is an upper semicontinuous map and it has closed graph for each $t \geqslant 0$.*

Definition 4 The ***positive orbit*** of $\varphi \in \mathscr{G}$ and $E \subset X$ are given by $\gamma^+(\varphi) \doteq \{\varphi(t);\ t \geqslant 0\}$ and $\gamma^+(E) \doteq \bigcup_{t\geqslant 0} T(t)E$. If $\tau > 0$, $\gamma_\tau^+(\varphi) \doteq \{\varphi(t); t \geqslant \tau\}$ and $\gamma_\tau^+(E) \doteq \bigcup_{t\geqslant\tau} T(t)E$.

Definition 5 We say that there exists a ***complete orbit*** through $x \in X$ if there is a map $\psi : \mathbb{R} \to X$ such that, for any $s \in \mathbb{R}$, $\psi^s|_{\mathbb{R}^+} \in \mathscr{G}$ and $\psi(0) = x$. In this case, *the complete orbit of ψ* is given by

$$\gamma(\psi) = \text{Im}\ \psi = \{\psi(t), t \in \mathbb{R}\}.$$

We also say that ψ ***is a complete orbit through*** x.

Definition 6 We say that a complete orbit $\psi : \mathbb{R} \to X$ is ***stationary*** if $\psi(t) = z$, for all $t \in \mathbb{R}$ for some $z \in X$. We set

$$Z(\mathscr{G}) \doteq \{z \in X :\ \text{there exists a complete orbit } \psi \text{ such that } \psi(t) = z\ \ \forall\, t \in \mathbb{R}\}.$$

Remark 1 We observe that $z \in Z(\mathscr{G})$ can be called a ***stationary solution*** in $\mathscr{G}$ once we have that $z \in Z(\mathscr{G})$ if and only if $\exists\, \phi \in \mathscr{G}$ such that $\phi(t) = z$ for all $t \geqslant 0$.

When talking about differential equations or inclusions, we also have that $z \in Z(\mathscr{G})$ if and only if $z \in T(t)z$ for all $t \geqslant 0$, so we can refer to z as an ***equilibrium*** of $T(t)$.

Remark 2 It follows from (H_4) that $Z(\mathscr{G})$ is a closed set.

Definition 7 We say that a subset $A \subset X$ is ***positively invariant*** if $T(t)A \subset A, \forall\, t \geqslant 0$, A is ***negatively invariant*** if $A \subset T(t)A, \forall\, t \geqslant 0$, A is ***invariant*** if $T(t)A = A, \forall\, t \geqslant 0$, and A is ***quasi-invariant*** if for each $z \in A$ there exists a complete orbit ψ through z and $\psi(t) \in A$ for all $t \in \mathbb{R}$.

Remark 3 Invariant ⇒ quasi-invariant ⇒ negatively invariant.

Definition 8 If $\varphi \in \mathcal{G}$, ψ is a complete orbit and $E \subset X$,

- $\omega(\varphi) \doteq \{z \in X; \varphi(t_j) \to z, t_j \to +\infty\}$.
- $\alpha(\psi) \doteq \{z \in X; \psi(t_j) \to z, t_j \to -\infty\}$.
- $\omega_B(E) \doteq \{z \in X; \exists\, \varphi_j \in \mathcal{G}, \{\varphi_j(0)\} \subset E, \{\varphi_j(0)\} \in B(X)$, and there is $\{t_j\} \subset \mathbb{R}^+,\ t_j \to +\infty$ with $\varphi_j(t_j) \to z\}$.
- $\omega(E) \doteq \bigcap_{t\geqslant 0} \overline{\gamma_t^+(E)}$.

Remark 4 $\omega(\varphi) = \bigcap_{t\geqslant 0} \overline{\gamma_t^+(\varphi)}$.

Remark 5 [8] The set $\omega(E)$ consists of the limits of all converging sequences $\{\xi_n\}$, where $\xi_n \in T(t_n)E,\ t_n \to +\infty$. $\omega_B(E) \subset \omega(E)$ and $\omega_B(E) = \omega(E)$ if E is bounded. So we denote $\omega_B(B) = \omega(B)$ if $B \in B(X)$.

Definition 9 We say that $A \subset X$ ***attracts*** a set $E \subset X$ if for any $\varepsilon > 0$ there exists $\tau = \tau(\varepsilon, E) \geqslant 0$ such that $T(t)E \subset O_\varepsilon(A)$ for all $t \geqslant \tau$, or equivalently, $\mathrm{dist}(T(t)E, A) \to 0$ as $t \to +\infty$.

Definition 10

(a) The subset A is a ***global B-attractor*** if it attracts all bounded subsets of X.
(b) The subset A is a ***global point attractor*** if it attracts all points of X.

Definition 11 The generalized semiflow $\mathcal{G}$ is ***eventually bounded*** if for any $B \in B(X)$ there exists $\tau = \tau(B) \geqslant 0$ such that $\gamma_\tau^+(B) \in B(X)$.

Definition 12

(a) $\mathcal{G}$ is ***bounded dissipative or B-dissipative*** if there is a bounded global B-attractor for $\mathcal{G}$.
(b) $\mathcal{G}$ is ***point dissipative*** if there is a bounded global point attractor for $\mathcal{G}$.
(c) We say that $\mathcal{G}$ is ***φ-dissipative*** if there is a bounded set B_0 such that, for any $\varphi \in \mathcal{G},\ \varphi(t) \in B_0$ for all sufficiently large t.

Remark 6 Bounded dissipative $\Rightarrow$ point dissipative $\Rightarrow$ φ-dissipative. (**φ-dissipativity** is called **point dissipativity** in [1]).

Definition 13 $\mathcal{G}$ is ***compact*** if, for any sequence $\{\varphi_j\} \subset \mathcal{G}$ with $\{\varphi_j(0)\} \in B(X)$, there exists a subsequence $\{\varphi_{j_k}\} \subset \{\varphi_j\}$ such that $\{\varphi_{j_k}(t)\}$ is convergent for each $t > 0$.

Definition 14 $\mathcal{G}$ is ***asymptotically compact*** if, for any sequence $\{\varphi_j\} \subset \mathcal{G}$ with $\{\varphi_j(0)\} \in B(X)$, and for any sequence $\{t_j\}$, $t_j \to +\infty$, the sequence $\{\varphi_j(t_j)\}$ has a convergent subsequence; (Or equivalently, for any $B \in B(X)$, each sequence $\{\xi_n\}$, with $\xi_n \in T(t_n)B$, $\forall\, n \in \mathbb{N}$, and $t_n \to +\infty$, contains a convergent subsequence).

Definition 15 $\mathcal{G}$ is ***conditionally asymptotically compact*** if, for any $B \in B(X)$ such that $\gamma_{\tau(B)}^+(B) \in B(X)$ for some $\tau(B) \geqslant 0$, each sequence $\{\xi_n\}$, with $\xi_n \in T(t_n)B$, $\forall\, n \in \mathbb{N}$, and $t_n \to +\infty$, contains a convergent subsequence. (In [8], this condition is referred as ***asymptotically upper semicompact***).

Lemma 1 *Let $\mathcal{G}$ be asymptotically compact. If $B \in BC(X)$ is negatively invariant, then B is compact.*

Proposition 2 [1]

(a) *If $\mathcal{G}$ is asymptotically compact, then $\mathcal{G}$ is eventually bounded.*
(b) *If $\mathcal{G}$ is eventually bounded and compact, then $\mathcal{G}$ is asymptotically compact.*

Proposition 3 [2] *$\mathcal{G}$ is asymptotically compact, if and only if $\mathcal{G}$ is eventually bounded and conditionally asymptotically compact.*

2.2 Attractors for $\mathcal{G}$

In this section we give conditions on a generalized semiflow $\mathcal{G}$ so that $\omega(B)$ attracts B for all $B \in B(X)$, and we prove that these conditions imply that $\overline{\bigcup_{x\in X}\omega(x)}$ is the only minimum closed global point attractor and $\overline{\bigcup_{B\in B(X)}\omega(B)}$ is the only minimum closed global B-attractor. In order to do this, the following lemma will be essential.

Lemma 2 *Let $\mathcal{G}$ be a generalized semiflow and let $F \in C(X)$. If F attracts a subset $A \in P(X)$, then $\omega_B(A) \subset \omega(A) \subset F$. If $\omega(A)$ attracts A, then it must be the unique minimal closed set which attracts A. Also, for any $A \in P(X)$ and for any $\tau \geqslant 0$, we have $\omega(\gamma_\tau^+(A)) = \omega(A)$.*

Theorem 1

(i) *If $F \subset X$ is a closed global point attractor, then $\overline{\bigcup_{x\in X}\omega(x)} \subset F$. Particularly, if $\overline{\bigcup_{x\in X}\omega(x)}$ is a global point attractor, then it must be the unique minimal closed global point attractor $\widehat{M}$.*
(ii) *If for each $x \in X$, $\omega(x)$ atracts x, then $\mathcal{G}$ has the unique minimal closed global point attractor $\widehat{M}$ and $\widehat{M} = \overline{\bigcup_{x\in X}\omega(x)}$.*

Theorem 2

(i) *If $F \subset X$ is a closed global B-attractor, then $\overline{\bigcup_{B\in B(X)}\omega(B)} \subset F$, in particular, if $\overline{\bigcup_{B\in B(X)}\omega(B)}$ is a global B-attractor, then it must be the unique minimal closed global B-attractor M.*
(ii) *If for each $B \in B(X)$, $\omega(B)$ atracts B, then $\mathcal{G}$ has the unique minimal closed global B-attractor M and*

$$M = \overline{\bigcup_{B\in B(X)}\omega(B)}.$$

Lemma 3 *Let $\mathcal{G}$ be a generalized semiflow and $K \in K(X)$. If K attracts $A \in P(X)$, then each sequence $\{\xi_n\}$, with $\xi_n \in T(t_n)A$ and $t_n \to +\infty$, contains a subsequence that converges to some point of K.*

The above statements can be proved using the arguments as the analogous results in univalued case (see [7]). We only observe that in the multivalued context ω-limit sets are not necessarily positively invariant. The next proposition follows Lemma 3.1.1 [5], and can be easily proved by using Ball's arguments as is done in Lemma 3.4 (i), [1].

Proposition 4 *Let $\mathcal{G}$ be a generalized semiflow and $A \in P(X)$ such that $\omega(A)$ is a non-empty compact set which attracts A. Then $\omega(A)$ is quasi-invariant and so negatively invariant. The same is valid for $\omega_B(A)$ if $\omega_B(A)$ is a non-empty compact set which attracts A.*

The next lemma states the essential compactness we need in order to obtain good conditions on ω-limit sets. Its proof is evident.

Lemma 4 *Let $\mathcal{G}$ be a generalized semiflow and $A \in P(X)$. If each sequence $\{\xi_n\}$, with $\xi_n \in T(t_n)A$ and $t_n \to +\infty$, contains a convergent subsequence in X, then $\omega(A)$ is the minimal closed non-empty set which attracts A and also, $\omega(A)$ is compact and quasi-invariant.*

By Lemmas 3 and 4, we obtain immediately the following two results:

Theorem 3 *$\omega(A)$ is a non-empty quasi-invariant and compact set which attracts A, if and only if, each sequence $\{\xi_n\}$, with $\xi_n \in T(t_n)A$ and $t_n \to +\infty$, contains a convergent subsequence in X.*

Theorem 4 *Let $\mathcal{G}$ be a generalized semiflow and $K \in K(X)$. If K attracts $A \in P(X)$, then $\omega(A)$ is a non-empty minimal closed set which attracts A, and $\omega(A)$ is compact and quasi-invariant.*

Proposition 5 *Let $\mathcal{G}$ be a generalized semiflow and let $A \in P(X)$ be such that $\overline{\gamma^+(A)} \in K(X)$. Then $\omega(A)$ is a non-empty compact set which attracts A and $\omega(A)$ is quasi-invariant.*

Proof Let $\{x_n\} \subset A$. By (H_1) there is $\{\varphi_n\} \subset \mathcal{G}$ such that $\varphi_n(0) = x_n$. Consider a sequence $t_n \to +\infty$ and the sequence $\{\varphi_n(t_n)\}$. Then $\{\varphi_n(t_n)\}$ has a convergent subsequence $\varphi_{n_k}(t_{n_k}) \to y \in \omega(A)$. Therefore $\omega(A) \neq \emptyset$. Since $\omega(A) \subset \overline{\gamma^+(A)}$ it follows that $\omega(A)$ is compact. Also, we have that $\omega(A)$ attracts A. If not, there is $\varepsilon_0 > 0$ and sequences $t_j \to \infty$, $\{\varphi_j\} \subset \mathcal{G}$ with $\varphi_j(0) \in A$ such that $\varphi_j(t_j) \notin O_{\varepsilon_0}(\omega(A))$, what is a contradiction. So, it follows from Proposition 4 that $\omega(A)$ is quasi-invariant. □

Lemma 5 [1] *Let $\mathcal{G}$ be asymptotically compact.*

(i) *Let $B \in B(X)$. Then $\omega(B)$ is non-empty, compact, quasi-invariant, and is the minimal closed set which attracts B. If $T(t_0)\omega(B) \subset B$ for some $t_0 \geqslant 0$, then $\omega(B)$ is invariant.*
(ii) *If $\varphi \in \mathcal{G}$ then $\omega(\varphi)$ is non-empty, compact, quasi-invariant. Also, we have* $\lim_{t\to+\infty} d(\varphi(t), \omega(\varphi)) = 0$.
(iii) *If ψ is a bounded complete orbit then $\alpha(\psi)$ is non-empty, compact, quasi-invariant, and* $\lim_{t\to-\infty} d(\psi(t), \alpha(\psi)) = 0$.

Theorem 5 *Let $\mathscr{G}$ be a conditionally asymptotically compact generalized semiflow and $A \in P(X)$. If there exists $\tau \geqslant 0$ such that $\gamma_\tau^+(A) \in B(X)$, then $\omega(A)$ is a non-empty compact quasi-invariant set and $\omega(A)$ is the minimal closed set which attracts A.*

Proof Consider an arbitrary sequence $\{\xi_n\}$ with $\xi_n \in T(t_n)A$ and $t_n \to +\infty$. We may choose a subsequence $t_{n_k} \to +\infty$ satisfying $t_{n_k} \geqslant \tau$, $\forall\, k \in \mathbb{N}$. We have that $\gamma_\tau^+(\gamma_\tau^+(A)) \in B(X)$ and $T(t_{n_k})A = T(t_{n_k} - \tau)T(\tau)A$, $\forall\, k \in \mathbb{N}$. Since $\mathscr{G}$ is conditionally asymptotically compact and $\xi_{n_k} \in T(t_{n_k} - \tau)\gamma_\tau^+(A)$, we obtain that $\{\xi_{n_k}\}$ has a convergent subsequence in X. Then, it follows from Lemma 4 that $\omega(A)$ is a non-empty minimal closed set which attracts A, and $\omega(A)$ is compact and quasi-invariant. □

The following lemma motivates the definition of another compactness property.

Lemma 6 [7] *Let $\{L_n\}_{n=1}^\infty$ be a decreasing sequence of sets in a complete metric space $X: L_1 \supset L_2 \supset \ldots \supset L_n \supset \ldots$, satisfying the following condition. For each $n \in \mathbb{N}$ there exists a compact set $K_n \subset X$ and a number $\varepsilon_n > 0$ such that*

$$L_n \subset O_{\varepsilon_n}(K_n) \quad \text{and} \quad \varepsilon_n \to 0 \ \text{ as } n \to +\infty.$$

Then for every given $y_n \in L_n$, $\forall\, n \in \mathbb{N}$, the sequence $\{y_n\}_{n=1}^\infty$ contains a convergent subsequence.

Definition 16 We say that a generalized semiflow $\mathscr{G}$ possesses ***B-asymptotically compact property (B-ACP)*** if for any $B \in B(X)$ such that $\gamma_{t_1(B)}^+(B) \in B(X)$ for a certain $t_1(B) \geqslant 0$, there is a time $t_2(B) \geqslant t_1(B)$ such that for any $t \geqslant t_2(B)$, there exists a compact set $K(B,t) \subset X$ and $\varepsilon(B,t) > 0$ satisfying

$$T(t)B \subset O_{\varepsilon(B,t)}(K(B,t)) \quad \text{and} \quad \varepsilon(B,t) \to 0 \ \text{ as } t \to +\infty.$$

By using (H_3), Lemmas 6, 4, 2, and with the same arguments, we can extend Theorem 4.12 in [7] to the multivalued case.

Theorem 6 *Let $\mathscr{G}$ be a generalized semiflow with B-ACP and $A \in P(X)$. If exists $\tau \geqslant 0$ such that $\gamma_\tau^+(A) \in B(X)$, then $\omega(A)$ is a non-empty compact quasi-invariant set, and it is the minimal closed set which attracts A.*

So we have as a consequence of Theorem 1 (ii), Theorem 2 (ii) and Theorem 6 the following theorem:

Theorem 7 *Let $\mathscr{G}$ be an eventually bounded generalized semiflow with B-ACP. Then $\mathscr{G}$ has the unique minimal closed global point attractor $\widehat{M}$ and $\mathscr{G}$ has the unique minimal closed global B-attractor M. Also, we have*

$$\widehat{M} = \overline{\bigcup_{x \in X} \omega(x)}, \qquad M = \overline{\bigcup_{B \in B(X)} \omega(B)}.$$

Lemma 7 [8] *Let $B \in B(X)$ a negatively invariant set. If Z is a global B-attractor, then $B \subset \overline{Z}$. (So, if $\mathscr{A}$ is a global B-attractor, closed, bounded and negatively invariant, then $\mathscr{A}$ is minimal among all closed global B-attractors).*

Some other classes of generalized semiflow inspired in Semigroup Theory are now introduced and we make a comparison between them.

Definition 17 Let $\mathcal{G}$ be a generalized semiflow.

(a) We say that $\mathcal{G}$ is of ***class*** $\mathcal{K}^*$ if $T(t): X \to P(X)$ is compact for some $t_1 > 0$, i.e., for each $B \in B(X)$, its image $T(t_1)B$ is relatively compact in X.
(b) We say that $\mathcal{G}$ is of ***class B-***$\mathcal{K}$ if for each $B \in B(X)$, there exists $t_1 = t_1(B) \geqslant 0$ such that $T(t)B$ is relatively compact for each $t \geqslant t_1$.
(c) $\mathcal{G}$ is called ***asymptotically smooth*** if, for any non-empty closed bounded positively invariant set $B \subset X$, there is a compact set $J \subset B$ such that J attracts B.
(d) We say that $\mathcal{G}$ is ***uniformly compact for t large*** if for each $B \in B(X)$ there exists a $t(B) \geqslant 0$ such that $\gamma^+_{t(B)}(B)$ is relatively compact in X.

Theorem 8 *Let $\mathcal{G}$ be a generalized semiflow, then*

(i) *$\mathcal{G}$ is of class $\mathcal{K}^* \Rightarrow \mathcal{G}$ is of class B-$\mathcal{K} \Rightarrow \mathcal{G}$ is conditionally asymptotically compact.*
(ii) *$\mathcal{G}$ possesses B-ACP $\Leftrightarrow \mathcal{G}$ is conditionally asymptotically compact.*
(iii) *$\mathcal{G}$ possesses B-ACP $\Rightarrow \mathcal{G}$ is asymptotically smooth.*
(iv) *$\mathcal{G}$ uniformly compact for t large $\Leftrightarrow \mathcal{G}$ is eventually bounded and it is of class B-$\mathcal{K}$.*
(v) *$\mathcal{G}$ is asymptotically compact $\Rightarrow \mathcal{G}$ eventually bounded.*
(vi) *$\mathcal{G}$ compact and eventually bounded $\Rightarrow \mathcal{G}$ is asymptotically compact.*
(vii) *$\mathcal{G}$ is asymptotically compact $\Leftrightarrow \mathcal{G}$ is conditionally asymptotically compact and eventually bounded.*

Lemma 8 [1] *Let $\mathcal{G}$ be φ-dissipative and eventually bounded. Then there exists a bounded set B_1 such that given any compact $K \subset X$ there exists $\varepsilon = \varepsilon(K) > 0$, $t_1 = t_1(K) > 0$, such that $T(t)O_\varepsilon(K) \subset B_1$ for all $t \geqslant t_1$.*

Remark 7 It follows as a corollary of the proof of Lemma 3.5 in [1]: Let $\mathcal{G}$ be an eventually bounded generalized semiflow. If there is a bounded set B_0 such that, for any $\varphi \in \mathcal{G}$ and $\varepsilon > 0$, $\varphi(t) \in O_\varepsilon(B_0)$ for all sufficiently large t then from eventual boundedness we know that for each $\delta > 0$ there exists $\tau(B_0, \delta) \geqslant 0$ such that $B_1 \doteq \gamma^+_{\tau(B_0,\delta)}(O_\delta(B_0)) \in B(X)$, and moreover for each compact set $K \subset X$, there exists $\varepsilon = \varepsilon(K, \delta) > 0$ and $t_1 = t_1(K, \delta) \geqslant 0$ such that $T(t)O_\varepsilon(K) \subset B_1$ for all $t \geqslant t_1$.

This result appears in Semigroup Theory [6, 7], and its proof involves the continuity of each operator $T(t)$. Once we admit $T(t)$ to be a multivalued operator it loses the property of being continuous, but the multivalued semigroup maintains the essential atraction property.

Lemma 8 says that if $\mathcal{G}$ is φ-dissipative and eventually bounded, then there exists a bounded set B_1 that absorbs neighbourhoods of compact sets. If moreover $\mathcal{G}$ possesses B-ACP, then there exists a bounded set which absorbs bounded sets. See next proposition.

Proposition 6 *Let $\mathscr{G}$ be an eventually bounded generalized semiflow and suppose that $\mathscr{G}$ is φ-dissipative. If $\mathscr{G}$ possesses B-ACP, or equivalently, if $\mathscr{G}$ is conditionally asymptotically compact, then $\mathscr{G}$ possesses a bounded set which absorbs bounded sets. Therefore, in particular $\mathscr{G}$ is B-dissipative.*

Proof Let B_1 be as Remark 7. Statement: B_1 absorbs bounded sets. In fact, as $\mathscr{G}$ is eventually bounded, for each $B \in B(X)$ there exists a $t(B) \geqslant 0$ such that $\gamma^+_{t(B)}(B) \in B(X)$. So, by Theorem 6, we obtain that $\omega(B)$ is a non-empty compact set which attracts B. So there exists $\varepsilon > 0$ and $t_0 \geqslant 0$ such that $T(t)O_\varepsilon(\omega(B)) \subset B_1$ and $T(t)B \subset O_\varepsilon(\omega(B))$ for all $t \geqslant t_0$. □

Theorem 9 [1] *Let $\mathscr{G}$ be a generalized semiflow. If $\mathscr{G}$ has a compact invariant global B-attractor, then $\mathscr{G}$ is φ-dissipative and asymptotically compact. Reciprocally, if $\mathscr{G}$ is φ-dissipative and asymptotically compact, then $\mathscr{G}$ has a compact invariant global B-attractor $\mathscr{A}$. The global B-attractor $\mathscr{A}$ is unique and given by*

$$\mathscr{A} = \bigcup_{B \in B(X)} \omega(B) = \omega_B(B_1) = \omega_B(X),$$

where $B_1 \in B(X)$ is as in Lemma 8. Furthermore $\mathscr{A}$ is the maximal compact invariant subset of X, and $\mathscr{A}$ is minimal among all closed global B-attractors.

Theorem 10 [8] *Let $\mathscr{G}$ be a generalized semiflow. If there exists a compact set $K \subset X$ such that it is a global B-attractor, then $\mathscr{G}$ has a global B-attractor compact invariant which is minimal among all closed global B-attractors.*

Now we give more equivalent conditions to existence of the maximal compact invariant global B-attractor.

Theorem 11 *Let $\mathscr{G}$ be a generalized semiflow. Then the following statements are equivalent:*

(I) *$\mathscr{G}$ is conditionally asymptotically compact and B-dissipative;*
(II) *$\mathscr{G}$ possesses B-ACP and is B-dissipative;*
(III) *$\mathscr{G}$ is conditionally asymptotically compact, eventually bounded and point dissipative;*
(IV) *$\mathscr{G}$ possesses B-ACP, is eventually bounded and point dissipative;*
(V) *$\mathscr{G}$ possesses B-ACP, is eventually bounded and φ-dissipative;*
(VI) *$\mathscr{G}$ is asymptotically compact and φ-dissipative;*
(VII) *$\mathscr{G}$ possesses a minimal non-empty closed global B-attractor which is the maximal compact invariant subset of X;*
(VIII) *$\mathscr{G}$ possesses a non-empty compact global B-attractor.*

Proof It is easy to see that $\mathscr{G}$ B-dissipative implies $\mathscr{G}$ eventually bounded and point dissipative; and point dissipative implies $\varphi-$dissipative. By Theorem 8 (ii) $\mathscr{G}$ possesses B-ACP $\Leftrightarrow$ $\mathscr{G}$ is conditionally asymptotically compact. Again by Theorem 8 (vii) and (ii) $\mathscr{G}$ is asymptotically compact $\Leftrightarrow$ $\mathscr{G}$ is conditionally asymptotically compact and eventually bounded $\Leftrightarrow$ $\mathscr{G}$ possesses B-ACP and $\mathscr{G}$ is eventually bounded. So we have $(I) \Leftrightarrow (II) \Rightarrow (III) \Leftrightarrow (IV) \Rightarrow (V) \Leftrightarrow (VI)$. By Theorem 9 $(VI) \Leftrightarrow (VII)$. Trivially $(VII) \Rightarrow (VIII)$. Finally, Lemma 3 guarantees that $(VIII) \Rightarrow (I)$. □

Remark 8 Let $\mathscr{G}$ be asymptotically compact, or equivalently, $\mathscr{G}$ possesses B-ACP and is eventually bounded, then:

(a) $\mathscr{G}$ is point dissipative $\Leftrightarrow \bigcup_{x \in X} \omega(x) \in B(X)$.
(b) $\mathscr{G}$ is B-dissipative $\Leftrightarrow \bigcup_{B \in B(X)} \omega(B) \in B(X)$.

Remark 9 [7] If for each $B \in B(X)$, $\omega(B)$ attracts B and $\omega(B) = \bigcup_{x \in B} \omega(x)$, then $\widehat{M} = M$.

Theorem 12 *Let $\mathscr{G}$ be asymptotically compact, or equivalently, $\mathscr{G}$ possesses B-ACP and is eventually bounded. Let $\widehat{M}$ and M be as in Theorem 7. If $\widehat{M} \in B(X)$, then for each $\delta > 0$, $M = \omega(O_\delta(\widehat{M}))$. Furthermore, M is the maximal compact invariant subset of X.*

Proof By Remark 7 and Theorem 9, $B_1 \doteq \gamma^+_{\tau(\widehat{M},\delta)}(O_\delta(\widehat{M})) \in B(X)$, and $\mathscr{A} \doteq \omega_B(B_1) = \bigcup_{B \in B(X)} \omega(B)$ is a compact invariant global B-attractor. Furthermore $\mathscr{A}$ is the maximal compact invariant subset of X, and $\mathscr{A}$ is minimal among all closed global B-attractors. Therefore $M = \mathscr{A}$ and we have

$$M = \overline{\bigcup_{B \in B(X)} \omega(B)} = \bigcup_{B \in B(X)} \omega(B) = \omega(B_1) = \omega\left(\gamma^+_{\tau(\widehat{M},\delta)}(O_\delta(\widehat{M}))\right) = \omega(O_\delta(\widehat{M})).$$

□

2.3 Eventual Semigroups

There is a class of multivalued semigroups which behaves, for large values of t, like a single-valued semigroup. This occurs when dealing with problems without uniqueness of solutions but enjoying strong regularizing properties. This class was introduced in [4], where the issue is about p-laplacian problems, $p > 2$. When considering parabolic problems perturbed by non-globally Lipschitz operators, uniqueness is a non trivial question but, under reasonable conditions, these problems enjoy enough regularity and absorption properties to allow uniqueness after some time has elapsed. Below we briefly describe those ideas putting them in the context of generalized semiflows.

Definition 18 We say that a generalized semiflow $\mathscr{G}$ defines an ***eventual semigroup*** if there exists a semigroup $\{S(t)\}_{t \geqslant 0}$ such that for any $B \in B(X)$, there exists $\tau_0 = \tau_0(B) > 0$ such that, if $\tau \geqslant \tau_0$ and $x_\tau \in T(\tau)B$ then for each $t \geqslant 0$, $T(t)x_\tau = S(t)x_\tau$, where $T(t)$ is the multivalued semigroup defined by $\mathscr{G}$.

Theorem 13 *Let $\mathscr{G}$ be a generalized semiflow which defines an eventual semigroup associated with the semigroup $\{S(t)\}_{t \geqslant 0}$ and let $B_0 \in B(X)$ be such that $\gamma^+_{\tau_1}(B_0) =$*

$\bigcup_{t\geqslant\tau_1} T(t)B_0 \in B(X)$ for some $\tau_1 = \tau_1(B_0) \geqslant 0$. If $\{S(t)\}_{t\geqslant 0}$ is of class $\mathscr{K}$ or class $\mathscr{A}\mathscr{K}$ or class B-$\mathscr{A}\mathscr{K}$ or if it possesses B-ACP (for these definitions see [7]), then

(i) *$\omega(B_0)$ is non-empty, compact and invariant by $\{S(t)\}_{t\geqslant 0}$;*
(ii) *$\omega(B_0)$ attracts B_0;*
(iii) *$\omega(B_0)$ is the minimal closed set which attracts B_0;*

Proof The case where $\{S(t)\}_{t\geqslant 0}$ is of class $\mathscr{K}$ was done in Theorem 2.1 in [4] where is used Theorem 2.1, [6]. For the other cases it is enough to repeat the same arguments in [4], and the result follows from Theorem 4.13, [7], and Proposition 3.4, [6] for $\{S(t)\}_{t\geqslant 0}$ of class $\mathscr{A}\mathscr{K}$ and class B-$\mathscr{A}\mathscr{K}$. Theorem 4.12 in [7] is used if $\{S(t)\}_{t\geqslant 0}$ possesses B-ACP. □

As a consequence we have

Theorem 14 *Let $\mathscr{G}$ be a B-dissipative generalized semiflow which defines an eventual semigroup $\{T(t)\}_{t\geqslant 0}$ associated with the semigroup $\{S(t)\}_{t\geqslant 0}$. Suppose that for any $B \in B(X)$, $\omega(B)$ is an invariant set under $\{T(t)\}_{t\geqslant 0}$. If the semigroup $\{S(t)\}_{t\geqslant 0}$ is of class $\mathscr{K}$ or class $\mathscr{A}\mathscr{K}$ or class B-$\mathscr{A}\mathscr{K}$ or if it possesses B-ACP, then $\mathscr{G}$ has a maximal compact invariant global B-attractor.*

Remark 10 As a complement of Theorem 8 (iii) we observe that if $\mathscr{G}$ defines an eventual semigroup $\{T(t)\}_{t\geqslant 0}$ associated with an asymptotically smooth semigroup, then $\mathscr{G}$ possesses B-ACP.

3 Characterizations for the Maximal Compact Invariant Global B-attractor

In this section we obtain some characterizations for the maximal compact invariant global B-atractor. In particular we prove that, even in the multivalued case, this set can be described in terms of the unstable set of the equilibrium states, if there is a Lyapunov function for the system. We also describe this atractor as union of bounded complete orbits (or precompact complete orbits) as it is done in single-valued case.

Theorem 15 *Let $\mathscr{G}$ be asymptotically compact and φ- dissipative. Then the maximal compact invariant global B-attractor $\mathscr{A}$ can be characterized by:*

(i) $\mathscr{A} = \bigcup_{B\in B(X)} \omega(B)$;
(ii) $\mathscr{A} = \omega_B(X)$;
(iii) $\mathscr{A} = \omega_B(B_1)$, *where $B_1 \in B(X)$ is the set in Lemma 8;*
(iv) $\mathscr{A} = \bigcup_{K\in K(X)} \omega(K)$;
(v) *$\mathscr{A}$ is the union of all complete bounded orbits in X;*
(vi) *$\mathscr{A}$ is the union of all complete precompact orbits in X;*
(vii) *$\mathscr{A}$ is the maximal invariant bounded set in X.*

Proof For parts (i), (ii) and (iii) see Theorem 9.

(iv): Since $\mathscr{A}$ is compact and invariant, $\omega(\mathscr{A}) = \mathscr{A}$. So, we have

$$\bigcup_{K\in K(X)} \omega(K) \subset \bigcup_{B\in B(X)} \omega(B) = \mathscr{A} = \omega(\mathscr{A}) \subset \bigcup_{K\in K(X)} \omega(K).$$

Therefore $\mathscr{A} = \bigcup_{K \in K(X)} \omega(K)$.

(v) and (vi): Since $\mathscr{A}$ is invariant, then it is quasi-invariant. Thus, given $x \in \mathscr{A}$, there exists a complete orbit ψ through x (i. e., $\psi(0) = x$) such that $\psi(t) \in \mathscr{A}$, $\forall\, t \in \mathbb{R}$. Consider $\gamma(\psi) = \operatorname{Im}\psi = \{\psi(t), t \in \mathbb{R}\} \subset \mathscr{A} \in B(X)$. Then $\gamma(\psi)$ is bounded. Moreover note that, $\gamma(\psi)$ is precompact, once $\overline{\gamma(\psi)} \subset \mathscr{A}$. Therefore we can conclude that $\mathscr{A}$ is included in the union of all complete bounded orbits in X and $\mathscr{A}$ is included also in the union of all complete precompact orbits in X.

On the other hand, if $x \in X$ and ψ_x is a complete bounded (or precompact) orbit in X through x, consider $\gamma(\psi_x) = \operatorname{Im}\psi_x = \{\psi_x(t), t \in \mathbb{R}\}$. Since $\gamma(\psi_x)$ is negatively invariant, we have,

$$\gamma(\psi_x) \subset \overline{\gamma(\psi_x)} \subset \bigcap_{\tau \geqslant 0} \overline{\gamma(\psi_x)} \subset \bigcap_{\tau \geqslant 0} \overline{\gamma_\tau^+(\gamma(\psi_x))} = \omega(\gamma(\psi_x)) \subset \bigcup_{B \in B(X)} \omega(B) = \mathscr{A}.$$

Therefore, $\bigcup_{x \in X} \gamma(\psi_x) \subset \mathscr{A}$.

(vii) If $D \subset X$ is a bounded and invariant subset of X, then

$$D \subset \bigcap_{t \geqslant 0} \overline{\gamma_t^+(D)} = \omega(D) \subset \bigcup_{B \in B(X)} \omega(B) = \mathscr{A}.$$

□

Definition 19 [1] Let $\mathscr{G}$ be a generalized semiflow. A map $V\colon X \to \mathbb{R}$ is a **Lyapunov function for** $\mathscr{G}$ if

(i) V is continuous;
(ii) $V(\varphi(t)) \leqslant V(\varphi(s))$ whenever $\varphi \in \mathscr{G}$ and $t \geqslant s \geqslant 0$;
(iii) If $V(\psi(t)) =$ constant for some complete orbit ψ and all $t \in \mathbb{R}$, then ψ is stationary.

Lemma 9 *Let $\mathscr{G}$ be a generalized semiflow, and $x \in X$ such that there exists a complete orbit ψ_x through x and a compact set K such that $\{\psi_x(t); t \leqslant 0\} \subset K$, then $\alpha(\psi_x)$ is quasi-invariant.*

Proof Let $z \in \alpha(\psi_x)$. Then there is a sequence $t_j \to -\infty$ (we may suppose without lost of generality that $\ldots \leqslant t_2 \leqslant t_1 \leqslant 0$) such that $\psi_x(t_j) \to z$. By definition of complete orbit we have $\psi_x^{t_j} \in \mathscr{G}$ and $\psi_x^{t_j}(0) \to z$. So by (H_4), there is a subsequence, which we do not relabel, and a solution $g_0 \in \mathscr{G}$ with $g_0(0) = z$, such that $\psi_x^{t_j}(t) \to g_0(t)$, $\forall\, t \geqslant 0$. Clearly $g_0(t) \in \alpha(\psi_x)$, $\forall\, t \geqslant 0$. Now consider $\tau_j \doteq t_j - 1$ $(\tau_j \leqslant 0)$. Since $\psi_x^{t_j - 1}(0) = \psi_x(t_j - 1) = \psi_x(\tau_j) \in K \in K(X)$, we have (after extraction of a further subsequence) that $\psi_x^{t_j-1}(0) = \psi_x(\tau_j) \to y$. By (H_4), there is a subsequence, which we do not relabel, and a solution $g_1 \in \mathscr{G}$ with $g_1(0) = y$ such that $\psi_x^{t_j-1}(t) \to g_1(t)$, $\forall\, t \geqslant 0$. Clearly $g_1(t) \in \alpha(\psi_x)$, $\forall\, t \geqslant 0$. Note that $g_1^1 = g_0$, since $g_1^1(t) = g_1(t+1) = \lim_{j \to +\infty} \psi_x^{t_j-1}(t+1) = \lim_{j \to +\infty} \psi_x^{t_j}(t) = g_0(t)$, $\forall\, t \geqslant 0$. Proceeding inductively, we find for each $r = 1, 2, \ldots,$ a solution $g_r \in \mathscr{G}$ such that $g_r^1 = g_{r-1}$ and $g_r(t) \in \alpha(\psi_x)$, $\forall\, t \geqslant 0$. Given $t \in \mathbb{R}$, we define $g(t)$ as the common value of $g_r(t+r)$ for $r \geqslant -t$. Then g is a complete orbit with $g(0) = g_0(0) = z$. In fact, given $s \in \mathbb{R}$ and $t \geqslant 0$, we have $g^s(t) = g(t+s) = g_r(t+s+r) = g_r^{s+r}(t)$, where $r \doteq \lceil -s \rceil$ is the minor integer value which is larger or equal to $-s$. (note that $s + r \geqslant 0$ e

$-(t+s) \leqslant -s \leqslant r)$. As $g_r \in \mathscr{G}$, by (H_2), $g_r^{s+r} \in \mathscr{G}$. Therefore $g^s = g_r^{s+r} \in \mathscr{G}$. Thus g is a complete orbit with $g(0) = z$ and $g(t) \in \alpha(\psi_x)$, $\forall\, t \geqslant 0$. Therefore $\alpha(\psi_x)$ is quasi-invariant. □

Lemma 10 *Let $\mathscr{G}$ be a generalized semiflow and suppose there exists a Lyapunov function $V : X \to \mathbb{R}$ for $\mathscr{G}$. Let $x \in X$ such that there exists a complete orbit ψ_x through x and a compact set K with $\{\psi_x(t); t \leqslant 0\} \subset K$. Then $\alpha(\psi_x) \subset Z(\mathscr{G})$.*

Proof Let $\{t_j\}$ be a sequence such that $t_j \to -\infty$ as $j \to +\infty$ and $\ldots t_j < \ldots < t_2 < t_1 \leqslant 0$. By hypothesis there is a convergent subsequence $\{\psi_x(t_{j_\ell})\} \subset \{\psi_x(t_j)\}$. By Definition 16 (ii) we obtain that $\{V(\psi_x(t_{j_\ell}))\}$ is a nondecreasing sequence. In fact, $t_{j_{\ell+1}} < t_{j_\ell}$ then $t_{j_\ell} = t_{j_{\ell+1}} + r$, $r > 0$. So $\psi_x(t_{j_\ell}) = \psi_x(t_{j_{\ell+1}} + r) = \psi_x^{t_{j_{\ell+1}}}(r)$ and $\psi_x^{t_{j_{\ell+1}}} \in \mathscr{G}_B$. Then $V(\psi_x(t_{j_\ell})) = V(\psi_x^{t_{j_{\ell+1}}}(r)) \leqslant V(\psi_x^{t_{j_{\ell+1}}}(0)) = V(\psi_x(t_{j_{\ell+1}}))$. So we have that

$$\lim_{\ell \to +\infty} V(\psi_x(t_{j_\ell})) = d \doteq \sup\{V(\psi_x(t_{j_\ell})),\ \ell \in \mathbb{N}\},$$

and this limit does not depend on the chosen sequence $\{t_j\}$.

Let $y \in \alpha(\psi_x)$. Then $y = \lim_{j \to +\infty} \psi_x(t_j)$, $t_j \to -\infty$. So $\lim_{\ell \to +\infty} V(\psi_x(t_{j_\ell})) = d$ and $\lim_{\ell \to +\infty} \psi_x(t_{j_\ell}) = y$, $t_{j_\ell} \to -\infty$. Also, we have $\lim_{\ell \to +\infty} V(\psi_x(t_{j_\ell})) = V(y)$. By uniqueness of the limit, $V(y) = d$. We know from Lemma 9 that $\alpha(\psi_x)$ is quasi-invariant. Then there is a complete orbit $\widetilde{\psi}$ through y such that $\widetilde{\psi}(t) \in \alpha(\psi_x)$, $\forall\, t \in \mathbb{R}$. Therefore $V(\widetilde{\psi}(t)) = d = V(y)$, $\forall\, t \in \mathbb{R}$, i.e., $\widetilde{\psi}$ is a stationary solution and $y \in Z(\mathscr{G})$. □

Theorem 16 *Let $\mathscr{G}$ be a continuous generalized semiflow and suppose that there exists a Lyapunov function $V : X \to \mathbb{R}$ for $\mathscr{G}$, and $\mathscr{G}$ has a maximal compact invariant global B-attractor $\mathcal{A}$. Then $\mathcal{A} = W^u(Z(\mathscr{G}))$ where $W^u(Z(\mathscr{G}))$ is the unstable set of $Z(\mathscr{G})$ given by $\{y \in X$; there is a complete orbit φ_y through y and $d(\varphi_y(-t), Z(\mathscr{G})) \overset{t \to +\infty}{\longrightarrow} 0\}$.*

Proof Let $x \in \mathscr{A}$. Since $\mathscr{A}$ is invariant there is a complete orbit ϕ_x through x such that $\phi_x(t) \in \mathscr{A}$, $\forall\, t \in \mathbb{R}$. Since $\mathscr{A}$ is compact $\alpha(\phi_x) \subset Z(\mathscr{G})$. Also, we have that $\phi_x(-t) \to \alpha(\phi_x)$. In fact, $\lim_{t \to +\infty} d(\phi_x(-t), \alpha(\phi_x)) = 0 \Leftrightarrow \forall\, \varepsilon > 0$ there is $\tau_\varepsilon \geqslant 0$ such that $\phi_x(-t) \in O_\varepsilon(\alpha(\phi_x)))$, $\forall\, t \geqslant \tau_\varepsilon$. Suppose, by contradiction, that there is $\varepsilon_0 > 0$ such that for any $k \in \mathbb{N}$, there exists $t_k \geqslant k$ with $\phi_x(-t_k) \notin O_{\varepsilon_0}(\alpha(\phi_x))$. Once $\{\phi_x(-t_k)\}_{k=1}^{\infty} \subset \mathscr{A}$ and $\mathscr{A}$ is compact, we have that $\{\phi_x(-t_k)\}_{k=1}^{\infty}$ has a convergent subsequence which converges to a point of $\alpha(\phi_x))$, what is impossible. Therefore $\lim_{t \to +\infty} d(\phi_x(-t), \alpha(\phi_x)) = 0$. So $x \in W^u(Z(\mathscr{G}))$ and $\mathscr{A} \subset W^u(Z(\mathscr{G}))$.

On the other hand, let $x \in W^u(Z(\mathscr{G}))$. Then there is a complete orbit ϕ_x through x such that $\phi_x(-t) \to Z(\mathscr{G})$. Once $Z(\mathscr{G}) \subset \mathscr{A}$ and $\mathscr{A}$ is a global B-attractor, given $\varepsilon_0 > 0$, there exists $\tau_0 \geqslant 0$ such that $T(t)x \subset O_{\varepsilon_0}(\mathscr{A}) \in B(X)$ and $\phi_x(-t) \in O_{\varepsilon_0}(\mathscr{A})$, $\forall\, t \geqslant \tau_0$. Since $\mathscr{G}$ is continuous $\gamma(\phi_x) \doteq \{\phi_x(t);\ t \in \mathbb{R}\} \in B(X)$, and so $\mathscr{A}$

attracts $\gamma(\phi_x)$. Once $\gamma(\phi_x)$ is negatively invariant, given $\varepsilon > 0, x \in O_\varepsilon(\mathscr{A})$. Therefore $x \in \mathscr{A}$ and $W^u(Z(\mathscr{G})) \subset \mathscr{A}$. □

4 Global $\varphi-$attractor

In this section we introduce a new group of definitions which we call "φ-concepts", that means, we add the prefix φ to the words attraction, dissipativity, boundedness, etc, in order to indicate that we are not supposing that the times in such definitions are uniformly chosen in any sense. One of our main results in this work verifies that, if a generalized semiflow $\mathscr{G}$ is φ-asymptotically compact and possesses a Lyapunov function, then there exists a minimal closed global φ-attractor $\widehat{N}$ for $\mathscr{G}$ and $\widehat{N}$ coincides with the set of stationary solutions in $\mathscr{G}$, see Theorem 21.

Definition 20 Let $\mathscr{G}$ be a generalized semiflow and $A \in P(X)$. We define

$$\omega_\varphi(A) \doteq \bigcup_{\varphi\in\mathscr{G},\varphi(0)\in A} \omega(\varphi).$$

Definition 21

(a) We say that A ***φ-attracts*** a set $M \in P(X)$ if for any $\varepsilon > 0$ and $\varphi \in \mathscr{G}$ with $\varphi(0) \in M$, there exists a $t_0 = t_0(\varphi, \varepsilon) \geqslant 0$ such that $\varphi(t) \in O_\varepsilon(A)$, $\forall\, t \geqslant t_0$. We say that A φ-attracts x if A φ-attracts $\{x\}$.
(b) We say that A is a ***global φ-attractor*** if A $\varphi-$attracts all points $x \in X$.
(c) We say that $\mathscr{G}$ is ***eventually φ-bounded*** if for any $\varphi \in \mathscr{G}$, there exists $t_0 = t_0(\varphi) \geqslant 0$ such that $\gamma^+_{t_0}(\varphi) \in B(X)$.
(d) We say that $\mathscr{G}$ is ***φ-asymptotically compact*** if, for any $\varphi \in \mathscr{G}$ and for any sequence $t_j \to +\infty$, the sequence $\{\varphi(t_j)\}$ has a convergent subsequence in X.
(e) We say that $\mathscr{G}$ is ***φ-conditionally asymptotically compact*** if for each $\varphi \in \mathscr{G}$ such that $\gamma^+_{\tau_0}(\varphi) \in B(X)$ for some $\tau_0 = \tau_0(\varphi) \geqslant 0$, each sequence $\{\varphi(t_n)\}$ with $t_n \to +\infty$, has a convergent subsequence in X.

Remark 11 We have

(i) $\mathscr{G}$ is φ-dissipative if and only if, there is a bounded global φ-attractor for $\mathscr{G}$.
(ii) B-attraction ⇒ Point attraction ⇒ φ-attraction.
(iii) $\mathscr{G}$ asymptotically compact ⇒ $\mathscr{G}$ is φ-asymptotically compact.
(iv) $\mathscr{G}$ eventually bounded ⇒ $\mathscr{G}$ is eventually φ-bounded.
(v) $\mathscr{G}$ is φ-asymptotically compact ⇔ $\mathscr{G}$ is φ-conditionally asymptotically compact and eventually φ-bounded.

Lemma 11 *Let $F \in C(X)$. If F φ-attracts $A \in P(X)$, then $\omega(\varphi) \subset F$, for each $\varphi \in \mathscr{G}$ with $\varphi(0) \in A$, so $\overline{\omega_\varphi(A)} \subset F$. In particular, if $\omega_\varphi(A)$ φ-attracts A, then $\overline{\omega_\varphi(A)}$ will be the minimal closed set which φ-attracts A.*

Proof Let $\varphi \in \mathscr{G}$ with $\varphi(0) \in A$. Since F φ-attracts A, given $k > 0$, there exists $t_k = t_k(\varphi) \geqslant 0$ such that $\varphi(t) \in O_{1/2k}(F)$, $\forall\, t \geqslant t_k$. Let $z \in \omega(\varphi)$. Then $z = \lim_{j\to+\infty} \varphi(t_j)$

with $t_j \to +\infty$. We can extract a subsequence t_{j_k} such that $t_{j_k} \geqslant t_k$, $\forall\, k \in \mathbb{N}$, and $z = \lim_{k\to+\infty} \varphi(t_{j_k})$. Then, $z \in \bigcap_{k\in\mathbb{N}} O_{1/k}(F) = F$. Therefore $\omega(\varphi) \subset F$ and $\overline{\omega_\varphi(A)} \subset F$. □

Lemma 12 *Let $\mathcal{G}$ be a generalized semiflow, $A, M \in P(X)$, and $x \in X$. Then the following statements are equivalent:*

1. *A φ-attracts M;*
2. *For any $\varepsilon > 0$, and $\varphi \in \mathcal{G}$ with $\varphi(0) \in M$, there exists a $t(\varphi, \varepsilon) \geqslant 0$ such that $\gamma^+_{t(\varphi,\varepsilon)}(\varphi) \subset O_\varepsilon(A)$;*
3. *$\lim_{t\to+\infty} d(\varphi(t), A) = 0$, $\forall\, \varphi \in \mathcal{G}$ with $\varphi(0) \in M$.*
4. *For any $\tau \geqslant 0$, A φ-attracts $\gamma^+_\tau(M)$.*

Lemma 13 *Let $\mathcal{G}$ be a generalized semiflow. Then for any $A \neq \emptyset$ and for any $\tau \geqslant 0$, we have $\omega_\varphi(\gamma^+_\tau(A)) = \omega_\varphi(A)$.*

Remark 12 The two above Lemmas can be easily proved. We only observe that, for (4) $\Rightarrow$ (1) in Lemma 12 it is enough to suppose that A φ-attracts $\gamma^+_\tau(M)$ *for some* $\tau \geqslant 0$.

Theorem 17

(i) *If $F \subset X$ is a closed global φ-attractor, then $\overline{\bigcup_{x\in X} \omega_\varphi(x)} \subset F$. In particular, if $\overline{\bigcup_{x\in X} \omega_\varphi(x)}$ is a global φ-attractor, then it must be the unique minimal closed global φ-attractor $\widehat{N}$.*
(ii) *If for each $x \in X$, $\omega_\varphi(x)$ φ-attracts x, then $\mathcal{G}$ has the unique minimal closed global φ-attractor $\widehat{N}$ and $\widehat{N} = \overline{\bigcup_{x\in X} \omega_\varphi(x)} = \overline{\omega_\varphi(X)}$.*

Proof

(i) Since F is a closed global φ-attractor, it φ-attracts each $x \in X$. So, by Lemma 11, $\omega_\varphi(x) \subset F$, $\forall\, x \in X$. Therefore $\overline{\bigcup_{x\in X} \omega_\varphi(x)} \subset F$.
(ii) Let $\xi \in X$. Since $\omega_\varphi(\xi)$ φ-attracts ξ, we have that $\overline{\bigcup_{x\in X} \omega_\varphi(x)}$ φ-attracts ξ. Therefore $\overline{\bigcup_{x\in X} \omega_\varphi(x)}$ is a global φ-attractor. So, by (i), $\overline{\bigcup_{x\in X} \omega_\varphi(x)}$ is the unique minimal closed global φ-attractor $\widehat{N}$. □

Lemma 14 *Let $\mathcal{G}$ be a generalized semiflow and $K \in K(X)$. If $\varphi \in \mathcal{G}$ is such that $d(\varphi(t), K) \to 0$ as $t \to +\infty$, then each sequence $\{\varphi(t_n)\}$, with $t_n \to +\infty$, contains a convergent subsequence in X, and the limit belongs to K.*

Theorem 18 *Let $\mathcal{G}$ be a generalized semiflow and $A \in P(X)$. If for each $\varphi \in \mathcal{G}$ with $\varphi(0) \in A$, every sequence $\{\varphi(t_n)\}$ with $t_n \to +\infty$ contains a convergent subsequence in X, then $\omega_\varphi(A)$ is a non-empty set, φ-attracts A and $\omega_\varphi(A)$ is quasi-invariant. The set $\overline{\omega_\varphi(A)}$ is the minimal closed set which φ-attracts A. Moreover, each $\omega(\varphi)$ with $\varphi \in \mathcal{G}$ and $\varphi(0) \in A$, is a non-empty compact and quasi-invariant set, and $\lim_{t\to+\infty} d(\varphi(t), \omega(\varphi)) = 0$.*

(Compare this result with Lemma 4, and note that we don't have necessarily $\omega_\varphi(A)$ compact).

Proof Let $x \in A$, by (H_1), there exists $\varphi \in \mathcal{G}$ with $\varphi(0) = x \in A$. Choose a sequence $t_n \to +\infty$. By hypothesis $\{\varphi(t_n)\}$ contains a convergent subsequence $\varphi(t_{n_k}) \to \xi \in \omega(\varphi) \subset \omega_\varphi(A)$. Therefore $\omega(\varphi) \neq \emptyset$ and consequently $\omega_\varphi(A) \neq \emptyset$. Let us prove that $\omega_\varphi(A)$ φ-attracts A. Assume, on the contrary, that $\omega_\varphi(A)$ does not φ-attracts A, then there exist $\varepsilon_0 > 0$ and $\varphi \in \mathcal{G}$ with $\varphi(0) \in A$ such that for each $n \in \mathbb{N}$, $\varphi(t_n) \notin O_{\varepsilon_0}(\omega_\varphi(A))$ for some $t_n > n$. By hypothesis $\{\varphi(t_n)\}$ contains a convergent subsequence, $\varphi(t_{n_k}) \to \zeta \in \omega(\varphi) \subset \omega_\varphi(A)$, what is impossible. Therefore, $\omega_\varphi(A)$ φ-attracts A. Moreover, from Lemma 11 we have that $\overline{\omega_\varphi(A)}$ is the minimal closed set which φ-attracts A.

In order to prove that $\omega(\varphi)$, with $\varphi \in \mathcal{G}$ and $\varphi(0) \in A$, is quasi-invariant we proceed as it is done in Lemma 3.4, (i), [1], and construct a complete orbit ψ through z and $\psi(t) \in \omega(\varphi)$, $\forall\, t \in \mathbb{R}$. This also implies that $\omega_\varphi(A)$ is quasi-invariant.

Now let $\{y_n\}_{n=1}^{\infty} \subset \omega(\varphi)$, $\varphi(0) \in A$. Then, for each $n \in \mathbb{N}$, there exists $t_n > n$ such that $d(\varphi(t_n), y_n) < 1/n$. Since $\{\varphi(t_n)\}$ contains a subsequence converging to some point $\zeta \in \omega(\varphi)$, $\{y_n\}$ has a subsequence that converges to $\zeta \in \omega(\varphi)$. Therefore $\omega(\varphi)$ is compact. Now we prove that $\lim_{t\to+\infty} d(\varphi(t), \omega(\varphi)) = 0$, for $\varphi \in \mathcal{G}$ with $\varphi(0) \in A$. Assume, on the contrary, that it does not happen. Then there exists $\varepsilon_0 > 0$ such that for each $n \in \mathbb{N}$ $d(\varphi(t_n), \omega(\varphi)) > \varepsilon_0$ for some $t_n > n$. But this contradicts the hypothesis and the definition of $\omega(\varphi)$. □

As an immediate consequence of the two above results we have

Lemma 15 *Let $\mathcal{G}$ be a generalized semiflow and let $x \in X$ and $K \in K(X)$. If for each $\varphi \in \mathcal{G}$ with $\varphi(0) = x$, $d(\varphi(t), K) \to 0$ as $t \to +\infty$, then $\omega_\varphi(x)$ is a non-empty quasi-invariant set which φ-attracts x. The set $\overline{\omega_\varphi(x)}$ is the minimal closed set which φ-attracts x.*

Remark 13 Sentences (*ii*) and (*iii*) of Lemma 5 remain valid if we only suppose that $\mathcal{G}$ is φ-asymptotically compact.

Proposition 7 *Let $\mathcal{G}$ be a generalized semiflow. If $\mathcal{G}$ is φ-dissipative, then $\mathcal{G}$ is eventually φ-bounded and $\omega_\varphi(X) \in B(X)$. On another side, if $\mathcal{G}$ is φ- asymptotically compact and $\omega_\varphi(X) \in B(X)$, then $\mathcal{G}$ is φ-dissipative.*

Proof If $\mathcal{G}$ is φ-dissipative then there is $B_0 \in B(X)$ such that for any $\varphi \in \mathcal{G}$ there exists $t_0(\varphi) \geqslant 0$ such that $\varphi(t) \in B_0$, $\forall\, t \geqslant t_0(\varphi)$. So $\gamma^+_{t_0(\varphi)}(\varphi) \subset B_0 \in B(X)$ and therefore $\mathcal{G}$ is eventually φ-bounded. Note that $\omega(\varphi) = \bigcap_{t\geqslant 0} \overline{\gamma_t^+(\varphi)} \subset \overline{\gamma^+_{t_0(\varphi)}(\varphi)} \subset \overline{B_0}$. Then $\omega_\varphi(X) = \bigcup_{x\in X} \omega_\varphi(x) = \bigcup_{\varphi\in\mathcal{G},\, \varphi(0)\in X} \omega(\varphi) \subset \overline{B_0} \in B(X)$.

On the other hand, if $\mathcal{G}$ is φ-asymptotically compact and $\omega_\varphi(X) \in B(X)$, then it follows from Theorem 18, that $\omega_\varphi(X) = \bigcup_{x\in X} \omega_\varphi(x)$ is a bounded global φ-attractor. Therefore $\mathcal{G}$ is φ-dissipative. □

Remark 14 Let $\mathcal{G}$ be a generalized semiflow, $A \in P(X)$, and $\varphi \in \mathcal{G}$ with $\varphi(0) \in A$. If $\overline{\gamma^+(\varphi)} \in K(X)$, or if $\mathcal{G}$ is a φ-conditionally asymptotically compact generalized semiflow and $\gamma^+(\varphi) \in B(X)$, then every sequence $\{\varphi(t_n)\}$ with $t_n \to +\infty$ contains a convergent subsequence. If it happens to each $\varphi \in \mathcal{G}$ with $\varphi(0) \in A$ we can apply Theorem 18.

Theorem 19 *Let $\mathcal{G}$ be a generalized semiflow with B-ACP and $A \in P(X)$. If there exists $\tau \geqslant 0$ such that $\gamma_\tau^+(A) \in B(X)$, then $\omega_\varphi(A)$ is a non-empty set, quasi-invariant and φ-attracts A.*

Proof Let $B \doteq \gamma_\tau^+(A)$. Then $\gamma_\tau^+(B) = \gamma_{2\tau}^+(A) \subset \gamma_\tau^+(A) \in B(X)$. Let $\varphi \in \mathcal{G}$ with $\varphi(0) \in B$ and consider an arbitrary sequence with $t_j \to +\infty$. From Lemma 6, we obtain that $\{\varphi(t_j)\}$ has a convergent subsequence. So, by Theorem 18, $\omega_\varphi(B)$ is a non-empty set, quasi-invariant and φ-attracts B. But, by Lemma 13, $\omega_\varphi(A) = \omega_\varphi(\gamma_\tau^+(A)) = \omega_\varphi(B)$. So $\omega_\varphi(A)$ is a non-empty set, quasi-invariant and φ-attracts $B = \gamma_\tau^+(A)$. Then by Lemma 12 (4) $\Rightarrow$ (1), $\omega_\varphi(A)$ φ-attracts A. □

Theorem 20 *If $\mathcal{G}$ is a φ-asymptotically compact generalized semiflow, then $\mathcal{G}$ has the unique non-empty minimal closed global φ-attractor $\widehat{N}$ and*

$$\widehat{N} = \overline{\bigcup_{x\in X} \omega_\varphi(x)} = \overline{\omega_\varphi(X)}.$$

Proof Let $\overline{x} \in X$ and let $\varphi \in \mathcal{G}$ with $\varphi(0) = \overline{x}$. Consider an arbitrary sequence $t_j \to +\infty$. Since $\mathcal{G}$ is φ-asymptotically compact, the sequence $\{\varphi(t_j)\}$ has a convergent subsequence. So, by Theorem 18, $\omega_\varphi(\overline{x})$ is a non-empty set, quasi-invariant and φ-attracts $\overline{x}$. Thus, by Theorem 17 (*ii*), $\mathcal{G}$ has the unique minimal closed global φ-attractor $\widehat{N}$ and $\widehat{N} = \overline{\bigcup_{x\in X} \omega_\varphi(x)} = \overline{\omega_\varphi(X)} \supset \omega_\varphi(\overline{x}) \neq \emptyset$. □

Proposition 8 *If $\mathcal{G}$ is an asymptotically compact generalized semiflow and φ- dissipative, then its minimal closed global φ-attractor $\widehat{N} = \overline{\omega_\varphi(X)}$ can be characterized by $\widehat{N} = \overline{\omega_\varphi(B_1)}$, where $B_1 \in B(X)$ is the set in the Lemma* 8.

Proof Let $B_1 \in B(X)$ as in the Lemma 8 and let $D \doteq \overline{\omega_\varphi(B_1)}$. Since $\mathcal{G}$ is asymptotically compact, Theorem 18 implies that $\omega_\varphi(B_1)$ is a non-empty set, quasi-invariant and φ-attracts B_1. We know, by Theorem 20, $\widehat{N} = \overline{\bigcup_{x\in X} \omega_\varphi(x)}$ is the unique non-empty minimal closed global φ-attractor. It remains to show that $\widehat{N} = D$. We have $D = \overline{\omega_\varphi(B_1)} \subset \overline{\bigcup_{x\in X} \omega_\varphi(x)} = \widehat{N}$.

On the other hand, if $y \in \bigcup_{x\in X} \omega_\varphi(x)$, then $y \in \omega_\varphi(x)$, for some $x \in X$. Then $y \in \omega(\varphi)$, for some $\varphi \in \mathcal{G}$ with $\varphi(0) = x$. By Lemma 5, $\omega(\varphi)$ is non-empty, compact, quasi-invariant, and $\lim_{t\to+\infty} d(\varphi(t), \omega(\varphi)) = 0$. Then if $K \doteq \omega(\varphi) \in K(X)$ we have from Lemma 8 that there exist $\varepsilon(K) > 0$, $t_1 = t_1(K) > 0$, such that $T(t)O_{\varepsilon(K)}(K) \subset B_1$ for all $t \geqslant t_1$. Let $0 < \varepsilon < \varepsilon(K)$. Then there is a $t(\varphi, \varepsilon) \geqslant 0$ such that $\varphi(t) \in O_\varepsilon(K)$, $\forall\, t \geqslant t(\varphi, \varepsilon)$. Then $\varphi^{t(\varphi,\varepsilon)}(t) \in T(t)O_\varepsilon(K) \subset T(t)O_{\varepsilon(K)}(K) \subset B_1$, $\forall\, t \geqslant t_1$. Since $\mathcal{G}$ is asymptotically compact, for any $\psi \in \mathcal{G}$ with $\psi(0) \in B_1$ and for any sequence $t_j \to +\infty$, we have that the sequence $\{\psi(t_j)\}$ has a convergent subsequence. Then, by Theorem 18, $\omega_\varphi(B_1)$ φ-attracts B_1. So, there is $\tau(\varphi, t_1, \varepsilon) \geqslant 0$ such that $\varphi^{t_1+t(\varphi,\varepsilon)}(t) \in O_\varepsilon(\omega_\varphi(B_1)) \subset O_\varepsilon(D)$, $\forall\, t \geqslant \tau(\varphi, t_1, \varepsilon)$. Then $\gamma^+_{t_1+t(\varphi,\varepsilon)+\tau(\varphi,t_1,\varepsilon)}(\varphi) \subset O_\varepsilon(D)$. Consider $t_0 \doteq t_1 + t(\varphi, \varepsilon) + \tau(\varphi, t_1, \varepsilon)$. Then $\omega(\varphi) = \bigcap_{t\geqslant 0} \overline{\gamma_t^+(\varphi)} \subset \overline{\gamma_{t_0}^+(\varphi)} \subset \overline{O_\varepsilon(D)}$. So $y \in \omega(\varphi) \subset \bigcap_{0<\varepsilon<\varepsilon(K)} \overline{O_\varepsilon(D)} = D$. Therefore $\bigcup_{x\in X} \omega_\varphi(x) \subset D$. Then $\widehat{N} = \overline{\bigcup_{x\in X} \omega_\varphi(x)} \subset \overline{D} = D$. Therefore $\widehat{N} = D \doteq \overline{\omega_\varphi(B_1)}$. □

The next two results are the principal results of this section.

Theorem 21 *If $\mathscr{G}$ is a φ-asymptotically compact generalized semiflow and possesses a Lyapunov function $V: X \to \mathbb{R}$, then its minimal closed global φ-attractor $\widehat{N}$ is non-empty and $\widehat{N} = Z(\mathscr{G})$.*

Proof Let $\varphi \in \mathscr{G}$. By Remark 13 we know that $\omega(\varphi)$ is non-empty, compact, quasi-invariant and $\lim_{t\to+\infty} d(\varphi(t), \omega(\varphi)) = 0$.

Let $\{t_j\}$ be such that $t_j \to +\infty$ as $j \to +\infty$ and $0 \leqslant t_1 < t_2 < \ldots < t_j < \ldots$. By Lemma 14, $\{\varphi(t_j)\}$ has a convergent subsequence $\{\varphi(t_{j_\ell})\}$ and so $\{V(\varphi(t_{j_\ell}))\}$ converges too. From Definition 16 (ii) we obtain that $\{V(\varphi(t_{j_\ell}))\}$ is a nonincreasing sequence, $V(\varphi(t_{j_{\ell+1}})) \leqslant V(\varphi(t_{j_\ell})) \leqslant \ldots \leqslant V(\varphi(0))$. So $\{V(\varphi(t_{j_\ell}))\}$ converges to its infimum,

$$\lim_{\ell\to+\infty} V(\varphi(t_{j_\ell})) = c \doteq inf\{V(\varphi(t_{j_\ell})),\ \ell \in \mathbb{N}\}.$$

and this limit does not depend on the chosen sequence $\{t_j\}$.

Let $y \in \omega(\varphi)$. Then $y = \lim_{j\to+\infty} \varphi(t_j)$, $t_j \to +\infty$. So $\lim_{j\to+\infty} V(\varphi(t_j)) = V(y) = c$. Since $\omega(\varphi)$ is quasi-invariant, then there is a complete orbit $\widetilde{\psi}$ through y such that $\widetilde{\psi}(t) \in \omega(\varphi)$, $\forall\, t \in \mathbb{R}$ and $V(\widetilde{\psi}(t)) = c = V(y)$, $\forall\, t \in \mathbb{R}$, then $\widetilde{\psi}$ is stationary. Therefore $y \in Z(\mathscr{G})$ and $\omega(\varphi) \subset Z(\mathscr{G})$.

As we have $\lim_{t\to+\infty} d(\varphi(t), \omega(\varphi)) = 0$ and $\omega(\varphi) \subset Z(\mathscr{G})$, $\forall\, \varphi \in \mathscr{G}$, $Z(\mathscr{G})$ is a global φ-attractor. By Theorem 20, $\emptyset \neq \widehat{N} \subset Z(\mathscr{G})$.

On the other hand, if $z \in Z(\mathscr{G})$ then there is a complete orbit ψ with $\psi(t) = z$, $\forall\, t \in \mathbb{R}$. Consider $\varphi \doteq \psi_{|\mathbb{R}^+} \in \mathscr{G}$. Since $\widehat{N}$ is a global φ-attractor, given $\varepsilon > 0$, there exists a $t_0 = t_0(\varepsilon, \varphi) \geqslant 0$ such that $z = \varphi(t) \in O_\varepsilon(\widehat{N})$, $\forall\, t \geqslant t_0$. Then $z \in \overline{\widehat{N}} = \widehat{N}$. Therefore $Z(\mathscr{G}) \subset \widehat{N}$. □

Remark 15 When dealing with a parabolic problem without uniqueness such that the generalized semiflow associated with it satisfies hypothesis on Theorem 21, we conclude that the associated elliptic problem has at minimum one solution and the set of all stationary solutions $Z(\mathscr{G}) = \widehat{N} = \overline{\omega_\varphi(X)}$.

Theorem 22 *If $\mathscr{G}$ is an asymptotically compact generalized semiflow, then there exist the minimal closed global φ-attractor $\widehat{N}$, the minimal closed global point attractor $\widehat{M}$, and the minimal closed global B-attractor M and $\widehat{N} \subset \widehat{M} \subset M$. If $\widehat{N} \in B(X)$, then for any $\delta > 0$ we have $M = \omega(O_\delta(\widehat{N}))$, M is the maximal compact invariant subset of X and $\widehat{N}$ and $\widehat{M}$ are compact sets. Moreover, if $\mathscr{G}$ possesses a Lyapunov function $V: X \to \mathbb{R}$, then for any $\delta > 0$, $M = \omega\Big(O_\delta(Z(\mathscr{G}))\Big)$.*

Proof Since $\mathscr{G}$ is asymptotically compact, by Lemma 5, for each $B \in B(X)$, $\omega(B)$ is non-empty, compact, quasi-invariant and attracts B. So, by Theorem 1 (*ii*) and Theorem 2 (*ii*), $\mathscr{G}$ has the unique minimal closed global point attractor $\widehat{M}$ and has the unique minimal closed global B-attractor M and $\widehat{M} = \overline{\bigcup_{x\in X} \omega(x)}$, $M = \overline{\bigcup_{B\in B(X)} \omega(B)}$. Since $\mathscr{G}$ asymptotically compact implies $\mathscr{G}$ φ-asymptotically compact, by Theorem 20, $\mathscr{G}$ has the unique non-empty minimal closed global φ-attractor $\widehat{N} = \overline{\bigcup_{x\in X} \omega_\varphi(x)}$, and $\widehat{N} \subset \widehat{M} \subset M$.

If $\widehat{N} \in B(X)$, then $\mathscr{G}$ is φ-dissipative, so if we set $B_1 \doteq \gamma^+_{\tau(\widehat{N},\delta)}(O_\delta(\widehat{N}))$ we conclude by Remark 7, as in Theorem 12, that $M = \omega(B_1)$ is the maximal compact invariant subset of X and $M = \omega(O_\delta(\widehat{N}))$. Moreover, if $\mathscr{G}$ possesses a Lyapunov function, by Theorem 21, $\widehat{N} = Z(\mathscr{G})$, so $M = \omega(O_\delta(\widehat{N})) = \omega\Big(O_\delta(Z(\mathscr{G}))\Big)$. □

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