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Inertial Iterative Process for Fixed Points of Certain Quasi-nonexpansive Mappings

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Abstract This paper deals with a general formalism which consists in approximating a point in a nonempty set S, in a real Hilbert space H, by a sequence $(x_n) \subset H$ such that $x_{n+1} := \mathcal{T}_n(x_n + \theta_n(x_n - x_{n-1}))$, where $(\theta_n) \subset [0, 1)$, x_0 x_1 are in H and $(\mathcal{T}_n)_{n\geqslant 0}$ is a sequence included in a certain class of self-mappings on H, such that every fixed point set of $\mathcal{T}_n$ contains S. This iteration method is inspired by an implicit discretization of the second order 'heavy ball with friction' dynamical system. Under suitable conditions on the parameters and the operators $(\mathcal{T}_n)$, we prove that this scheme generates a sequence which converges weakly to an element of S. In particular, by appropriate choices of $(\mathcal{T}_n)$, this algorithm works for approximating common fixed points of infinite countable families of a wide class of operators which includes α-averaged quasi-nonexpansive mappings for $\alpha \in (0, 1)$.

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Key words nonexpansive mapping · quasi-nonexpansive mapping · common fixed point · convex optimization · subgradient projection · heavy ball dynamical system.

1. Introduction

Throughout, H is a real Hilbert space endowed with inner product $\langle ., . \rangle$ and induced norm $| \, . \, |$. For any mapping $T \colon H \to H$, we denote by $\mathrm{Fix}(T)$ the set of fixed points of T, that is $\mathrm{Fix}(T) := \{x \in H \quad | \quad Tx = x\}$. It is well known that $\mathrm{Fix}(T)$ is a closed

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convex set of H when T is a quasi-nonexpansive operator (see for instance [18]). Let $\mathcal{F}_\alpha$ be the class of self-mappings on H defined for $\alpha \in (0, 1]$ by

$$\mathcal{F}_\alpha := \left\{ T \colon H \to H \quad | \quad \langle x - Tx, x - q \rangle \geqslant \frac{1}{2\alpha} |x - Tx|^2, \quad \forall (x, q) \in H \times \mathrm{Fix}(T) \right\}.$$

It is worth noting that, for $\alpha \in (0, 1]$, the set of α-averaged quasi-nonexpansive mappings on H is included in $\mathcal{F}_\alpha$ (see Remark 2.2). In particular, the following interesting results hold: (1) $\mathcal{F}_{\frac{1}{2}}$ contains all $\frac{1}{2}$-averaged quasi-nonexpansive mappings also called *firmly quasinonexpansive mappings*, which attracts great attention and includes *subgradient projection* (see [3, 18–20]); (2) $\mathcal{F}_\alpha$ contains all α-averaged nonexpansive mappings for $\alpha \in (0, 1]$, so that the set of firmly nonexpansive mappings is included in $\mathcal{F}_{\frac{1}{2}}$; (3) $\mathcal{F}_1$ contains the set of quasi-nonexpansive mappings, namely the set of 1-averaged (or 0-attracting) quasi-nonexpansive mappings. In this paper, we are interested in finding common fixed points of infinitely many operators included in the previously defined class of mappings. To this end, in a more general frame, we examine the following iteration method

$$\left[\begin{array}{l} x_{n+1} := \mathcal{T}_n v_n, \quad v_n = x_n + \theta_n (x_n - x_{n-1}), \quad \text{for all } n \geqslant 1, \\ x_0, \quad x_1 \in H, \quad (\theta_n) \subset [0, 1), \end{array} \right. \tag{1.1}$$

where the operators $(\mathcal{T}_n)$ relatively to S, a nonempty subset of H, satisfy:

$$\begin{array}{ll} (\mathrm{C1}): & (\mathcal{T}_n)_{n \geqslant 0} \subset \mathcal{F}_\alpha, \quad \text{where} \quad \alpha \in (0, 1). \\ (\mathrm{C2}): & \forall n \geqslant 0, \quad S \subset \mathrm{Fix}(\mathcal{T}_n). \\ (\mathrm{C3}): & \forall (\xi_n) \subset H, \forall \xi \in H, \\ & \xi \text{ is a weak cluster point of } (\xi_n) \text{ and } \xi_n - \mathcal{T}_n \xi_n \to 0 \text{ strongly } \Rightarrow \xi \in S. \end{array}$$

More precisely, we will focus our attention on finding sufficient conditions on the parameter (θ_n) so that the sequence (x_n) converges weakly to a point in S. Observe that the condition (C3) can be regarded as a sort of demi-closedness of the sequence $(\mathcal{T}_n)$. It reduces to the classical demi-closedness property when $\mathcal{T}_n$ is a constant sequence. Given a demi-closed mapping T in $\mathcal{F}_\alpha$ for some $\alpha \in (0, 1)$, it is then immediate that the conditions (C1–C3) are satisfied, for instance, with $\mathcal{T}_n = T$ and $S = \mathrm{Fix}(T)$. Clearly, by appropriate choices of the operators $(\mathcal{T}_n)$, the considered formalism covers some numerical approaches to solving monotone inclusion and fixed point problems, e.g.:

(1) Let us mention the inertial proximal algorithm (IPA) studied by Alvarez and Attouch [1] (see also Jules and Maingé [11], Moudafi and Elisabeth [13]), for computing zeroes of a maximal monotone set-valued mapping $A \colon H \to \mathcal{P}(H)$. Alvarez and Attouch's paper is covered by our formalism with $S = A^{-1}(0)$ and $\mathcal{T}_n = J^A_{\lambda_n}$, where $(\lambda_n) \subset (\lambda, +\infty)$ (for some positive λ) and $J^A_{\lambda_n} := (I + \lambda_n A)^{-1}$ is the resolvent of A of parameter λ_n (see Brezis [4] for details on resolvents). Indeed, in this special case, it is obvious that (C1) is satisfied with $\alpha = \frac{1}{2}$, (C2) holds since $\mathrm{Fix}(J^A_{\lambda_n}) = A^{-1}(0)$ and (C3) is deduced from the fact that the graph of a maximal monotone mapping is weakly strongly closed (see, for instance, [4]).
(2) Consider the common fixed point problem related to an infinite countable family $(T_i)_{i \geqslant 0} \subset \mathcal{F}_\alpha$ (where $\alpha \in (0, 1)$) such that $\mathrm{Fix}(T_i) \neq \emptyset$ for all $i \geqslant 0$, that is:

$$\text{find } \tilde{x} \in H \text{ such that } T_i \tilde{x} = \tilde{x}, \forall i \geqslant 0. \tag{1.2}$$

In view of solving (1.2), we propose the following algorithm:

$$x_{n+1} := \sum_{i\geqslant 0} w_{i,n} T_i \left(x_n + \theta_n(x_n - x_{n-1})\right), \qquad \text{for all } n \geqslant 1, \tag{1.3}$$

where $(\theta_n) \subset (0, 1)$ and $(w_{i,n}) \subset [0, +\infty)$ are real numbers such that:

(1) $\forall n \geqslant 0$, $\sum_{i\geqslant 0} w_{i,n} = 1$;
(2) $\forall i \geqslant 0$, $(w_{i,n})_{n\geqslant 0}$ is bounded away from zero for n large enough (that is: $\forall i \geqslant 0$, $\exists N_i \in \mathbb{N}$ and $\exists w_i > 0$ such that $\forall n \geqslant N_i$, $w_{i,n} \geqslant w_i$).

The operator $\sum_{i\geqslant 0} w_{i,n} T_i$ makes sense (see Lemma 4.1) and using (1.3) for solving (1.2) is covered by our formalism with $\mathcal{T}_n = \sum_{i\geqslant 0} w_{i,n} T_i$ and $S = \bigcap_{i\geqslant 0} \mathrm{Fix}(T_i)$, provided that each operator T_i is demi-closed. Indeed, we will prove in this setting that the conditions (C1–C3) are satisfied. As an interesting special case of (1.3), we also consider the following process:

$$x_{n+1} = \left(\sum_{k=1}^{n} \gamma_k\right)^{-1} \sum_{i=1}^{n} \gamma_i T_i(x_n + \theta_n(x_n - x_{n-1})), \qquad \forall n \geqslant 1, \tag{1.4}$$

where $(\theta_n) \subset (0, 1)$, $(\gamma_n) \subset (0, +\infty)$ and $\sum_{n\geqslant 0} \gamma_n < +\infty$.

Recall that the standard proximal point algorithm (PPA) comes from an implicit discretization of the first order steepest descent method, while IPA (see [1]) is a discrete version of a second order dissipative dynamical system. This latter system is usually called 'heavy ball with friction' and may be exploited in certain situations in order to accelerate the convergence of the trajectories (see [1, 15]). Numerical simulations are presented in [11], comparing the behavior of PPA, IPA and the gradient method. It turns out that the introduction of the term θ_n and the two iterates x_{n-1}, x_n considerably improves the speed of convergence for IPA. This can be explained since the vector $x_n - x_{n-1}$ is acting as an impulsion term and since θ_n is acting as a speed regulator. Then it seems natural to consider the case when the resolvent in (IPA) is replaced by a more general self-mapping in view of constructing fast and stable algorithms for fixed points problems.

Let us mention that, in the framework of Hilbert or Banach spaces, there are already several iteration processes for finding fixed points or common fixed points of self-mappings, e.g.: (1) the method of successive approximations and its regularized variants for nonexpansive mappings (see Browder [5], Halpern [10], Lions [12], Wittman [17], Bauschke [2]); (2) the Ishikawa iterates for two nonexpansive mappings (see Cirik et al. [9]); (3) the hybrid steepest descent method for certain quasi-nonexpansive operators called quasi-shrinking mappings (see Yamada and Ogura [18]). Most of them are cyclic-like algorithms for finding common fixed points of many finitely operators. Even though interesting strong convergence results are obtained for some of these algorithms, the proposed method (1.1) can be regarded as a procedure of speeding up their convergence properties.

Note also that when $(T_i)_{i\geqslant 0} \subset \mathcal{F}_\alpha$, where $\alpha \in (0, 1]$, our problem (1.2) is supposed to be solved by any process for finding fixed points of a given mapping in $\mathcal{F}_\alpha$. Indeed, if we denote $T := \sum_{i\geqslant 0} w_i T_i$, where $(w_i)_{i\geqslant 0} \subset (0, +\infty)$ and $\sum_{i\geqslant 0} w_i = 1$, then T belongs to $\mathcal{F}_\alpha$ and $\mathrm{Fix}(T) = \cap_{i\geqslant 0}\mathrm{Fix}(T_i)$ (see Lemma 4.1). In the special case when $(T_i)_{i\geqslant 0}$ is a family of nonexpansive mappings (hence $(T_i)_{i\geqslant 0} \subset \mathcal{F}_1$), it is easily checked that the above operator T is additionally nonexpansive, so that the considered problem (1.2) can be solved by any existing algorithm for finding fixed points of a given nonexpansive mapping. Nevertheless this strategy does not seem really realistic from the numerical point of view, because of the infinite sum. To the best of our knowledge, the most significant attempt to solve the proposed problem is due to Combettes [6]. This author suggested a Mann-like iteration process with variable blocks which is applicable to infinite countable families of firmly nonexpansive mappings.

The purpose of our work is to prove that the sequence (x_n) generated by the formalism (1.1) – (C1)–(C3) weakly converges to a common fixed point of the operators $(\mathcal{T}_n)$ under suitable conditions on the parameter (θ_n). As a direct application, we provide by (1.3) and (1.4) iterative processes of practical interest for solving the common fixed point problem (1.2). The proposed methods are also complementary to the known ones since the techniques used are completely different.

2. Preliminaries

For the convenience of the reader, we recall some definitions related to nonexpansive and quasi-nonexpansive mappings:

- A mapping $T\colon H \to H$ is called *nonexpansive* if $|Tx - Ty| \leqslant |x - y|$ for all x, $y \in H$. In particular, T is said to be *α-averaged nonexpansive* (where $\alpha \in [0, 1)$) if there exists a nonexpansive mapping $N\colon H \to H$ such that $T = (1-\alpha)I + \alpha N$; *firmly nonexpansive* if $\langle Tx - Ty, x - y\rangle \geqslant |Tx - Ty|^2$ for all x, y in H. A firmly nonexpansive mapping is alternatively characterized as $\frac{1}{2}$-averaged nonexpansive mapping.
- A mapping $T\colon H \to H$ is called *quasi-nonexpansive* if $|Tx - q| \leqslant |x - q|$ for all $(x, q) \in H \times \mathrm{Fix}(T)$. In particular, a mapping $T\colon H \to H$ is called *α-averaged quasi-nonexpansive* (where $\alpha \in [0, 1)$) if there exists a quasi-nonexpansive mapping $N\colon H \to H$ such that $T = (1-\alpha)I + \alpha N$.
- A mapping $T\colon H \to H$ is called *δ-attracting quasi-nonexpansive* $(\delta \geqslant 0)$ if $|x - q|^2 - |Tx - q|^2 \geqslant \delta |x - Tx|^2$ for all $(x, q) \in H \times \mathrm{Fix}(T)$.
- A self-mapping $T\colon D \to D$ (where $D \subset H$) satisfies the demi-closedness principle means that if (x_n) converges weakly to $q \in D$ and $(x_n - Tx_n)$ converges strongly to 0, then q is a fixed point of T. When D is a closed convex set in H, it is well known that any nonexpansive mapping $T\colon D \to D$ is demi-closed on D.

REMARK 2.1. It is easily seen that a δ-attracting quasi-nonexpansive mapping (for $\delta \geqslant 0$) satisfies

$$\langle x - Tx, x - q\rangle \geqslant \frac{1}{2}(\delta + 1)|x - Tx|^2, \qquad \text{for all} \quad (x, q) \in H \times \mathrm{Fix}(T). \tag{2.1}$$

Indeed, for $(x, q) \in H \times \mathrm{Fix}(T)$ we obviously have

$$\begin{aligned} |x - q|^2 &\geqslant |Tx - q|^2 + \delta |x - Tx|^2 \\ &= |Tx - x|^2 + |x - q|^2 + 2\langle Tx - x, x - q\rangle + \delta |x - Tx|^2, \end{aligned}$$

which yields (2.1).

REMARK 2.2. For $\alpha \in (0, 1]$, a mapping $T\colon H \to H$ is α-averaged quasi-nonexpansive if and only if T is $\frac{1-\alpha}{\alpha}$-attracting quasi-nonexpansive (see 1[8], Proposition 1). By Remark 2.1, it is easily deduced that all α-averaged quasi-nonexpansive mappings are included in $\mathcal{F}_\alpha$ when $\alpha \in (0, 1]$.

We also recall the well-known Opial lemma which provides a criterion for weak convergence that does not require the knowledge of the limit point.

LEMMA 2.1 ([14]). *Let H be a Hilbert space and $\{x_n\}$ a sequence such that there exists a nonempty set $S \subset H$ verifying:*

- *For every $q \in S$, $\lim_{n\to+\infty} |x_n - q|$ exists.*
- *Every weak limit point of $\{x_n\}$ belongs to S.*

Then, there exists $\bar{x} \in S$ such that $\{x_n\}$ converges weakly to $\bar{x}$ in H.

3. Asymptotic Convergence of the Method

In this section we establish a weak convergence result of (x_n) generated by (1.1) under the conditions (C1)–(C3).

The following two lemmas are needed to state our convergence result.

LEMMA 3.1. *For any $q \in H$ and any sequences $(x_n) \subset H$, $(\theta_n) \subset \mathbb{R}$, we have*

$$\begin{aligned}\langle v_n - x_{n+1}, v_n - q\rangle &= -\phi_{n+1} + \phi_n + \theta_n\left(\phi_n - \phi_{n-1}\right)\\ &\quad + \frac{1}{2}|x_{n+1} - v_n|^2 + \frac{1}{2}\left(\theta_n + \theta_n^2\right)|x_n - x_{n-1}|^2,\end{aligned}$$

where $\phi_j := \frac{1}{2}|x_j - q|^2$, $v_j := x_j + \theta_j(x_j - x_{j-1})$.

Proof. Thanks to the relation $v_n = x_n + \theta_n(x_n - x_{n-1})$, we obtain

$$\begin{aligned}\langle v_n - x_{n+1}, v_n - q\rangle &= \langle x_n - x_{n+1} + \theta_n(x_n - x_{n-1}), x_n - q + \theta_n(x_n - x_{n-1})\rangle\\ &= -\langle x_{n+1} - x_n, x_{n+1} - q\rangle + |x_n - x_{n+1}|^2\\ &\quad + \theta_n\langle x_n - x_{n+1}, x_n - x_{n-1}\rangle\\ &\quad + \theta_n\langle x_n - x_{n-1}, x_n - q\rangle\\ &\quad + \theta_n^2|x_n - x_{n-1}|^2.\end{aligned}$$

Moreover, for any $a, b \in H$, it is easily checked that

$$\langle a, b\rangle = -\frac{1}{2}|a - b|^2 + \frac{1}{2}|a|^2 + \frac{1}{2}|b|^2. \tag{3.1}$$

Consequently, we get

$$\begin{aligned}\langle v_n - x_{n+1}, v_n - q\rangle = &-\left(-\phi_n + \phi_{n+1} + \frac{1}{2}|x_{n+1} - x_n|^2\right) + |x_{n+1} - x_n|^2\\ &+ \theta_n\langle x_n - x_{n+1}, x_n - x_{n-1}\rangle\\ &+ \theta_n\left(-\phi_{n-1} + \phi_n + \frac{1}{2}|x_n - x_{n-1}|^2\right)\\ &+ \theta_n^2|x_n - x_{n-1}|^2,\end{aligned}$$

namely

$$\langle v_n - x_{n+1}, v_n - q\rangle = -\phi_{n+1} + \phi_n + \frac{1}{2}|x_{n+1} - x_n|^2 + \theta_n\langle x_n - x_{n+1}, x_n - x_{n-1}\rangle + \theta_n(\phi_n - \phi_{n-1}) + \left(\frac{\theta_n}{2} + \theta_n^2\right)|x_n - x_{n-1}|^2. \quad (3.2)$$

By the definition of v_n, we also have

$$|v_n - x_{n+1}|^2 = |x_n - x_{n+1} + \theta_n(x_n - x_{n-1})|^2 = |x_n - x_{n+1}|^2 + \theta_n^2|x_n - x_{n-1}|^2 + 2\theta_n\langle x_n - x_{n+1}, x_n - x_{n-1}\rangle, \quad (3.3)$$

or equivalently

$$\frac{1}{2}|x_{n+1} - x_n|^2 + \theta_n\langle x_n - x_{n+1}, x_n - x_{n-1}\rangle = \frac{1}{2}\left(|x_{n+1} - v_n|^2 - \theta_n^2|x_n - x_{n-1}|^2\right). \quad (3.4)$$

From (3.2) and (3.4), we then obtain the desired result □

THEOREM 3.2. *Let* $(\mathcal{T}_n)$ *and* $S \neq \emptyset$ *satisfy the conditions* (C1)–(C3)*, let* $(\theta_n) \subset \mathbb{R}_+$ *verify*

(H1): $\exists\theta \in [0, 1)$ *such that* $\forall n \geqslant 0, \quad \theta_n \in [0, \theta]$,

and let $(x_n) \subset H$ *be a sequence such that*

$$x_{n+1} := \mathcal{T}_n(x_n + \theta_n(x_n - x_{n-1})), \quad \forall n \geqslant 1.$$

If the following condition holds

$$\sum_{n\geqslant 1}\theta_n|x_n - x_{n-1}|^2 < \infty, \quad (3.5)$$

then there exists $\bar{x}$ in S *such that* $x_n \rightharpoonup \bar{x}$ *weakly in* H *as* $n \to \infty$.

Proof. Taking $q \in S$, by (C2) we have $q \in \mathrm{Fix}(\mathcal{T}_n)$ for all $n \geqslant 0$, which by (C1) (that is $(\mathcal{T}_n)_{n\geqslant 0} \subset \mathcal{F}_\alpha$) leads to

$$\frac{1}{2\alpha}|v_n - \mathcal{T}_n v_n|^2 \leqslant \langle v_n - \mathcal{T}_n v_n, v_n - q\rangle,$$

where $v_n := x_n + \theta_n(x_n - x_{n-1})$, or equivalently

$$\frac{1}{2\alpha}|v_n - x_{n+1}|^2 \leqslant \langle v_n - x_{n+1}, v_n - q\rangle,$$

because $x_{n+1} = \mathcal{T}_n v_n$, which by Lemma 3.1 yields

$$\frac{1}{2\alpha}|v_n - x_{n+1}|^2 \leqslant -\phi_{n+1} + \phi_n + \theta_n(\phi_n - \phi_{n-1}) + \frac{1}{2}|x_{n+1} - v_n|^2 + \frac{1}{2}\left(\theta_n + \theta_n^2\right)|x_n - x_{n-1}|^2.$$

Therefore, setting $\eta := \frac{1}{2}(\frac{1}{\alpha} - 1)$ (hence $\eta > 0$, since $\alpha \in (0, 1)$), we get

$$\phi_{n+1} - \phi_n - \theta_n(\phi_n - \phi_{n-1}) \leqslant -\eta|x_{n+1} - v_n|^2 + \frac{1}{2}(\theta_n + \theta_n^2)|x_n - x_{n-1}|^2. \tag{3.6}$$

Moreover, from (3.3) and using Young's inequality we obtain

$$|x_{n+1} - v_n|^2 \geqslant (1 - \theta_n)|x_n - x_{n+1}|^2 + (\theta_n^2 - \theta_n)|x_n - x_{n-1}|^2;$$

by (3.6) and setting $d_j := |x_j - x_{j-1}|^2$, we deduce

$$\phi_{n+1} - \phi_n - \theta_n(\phi_n - \phi_{n-1}) \leqslant -\eta(1 - \theta_n)d_{n+1} + \left(\eta(\theta_n - \theta_n^2) + \frac{1}{2}(\theta_n + \theta_n^2)\right) d_n.$$

Regarding the right-hand side of this last inequality, since $(\theta_n) \subset [0, 1]$ it is easily seen that

$$\left(\eta(1 - \theta_n) + \frac{1}{2}(1 + \theta_n)\right) \leqslant \mu,$$

where $\mu := 2\max\{\frac{1}{2}, \eta\}$, which yields

$$\phi_{n+1} - \phi_n - \theta_n(\phi_n - \phi_{n-1}) \leqslant -\eta(1 - \theta_n)d_{n+1} + \mu\theta_n d_n. \tag{3.7}$$

The rest of the proof follows a same process as in [1] and can be divided into two parts:

(1) We prove that $\lim_{n\to\infty} \phi_n$ exists. Set $u_n := \phi_n - \phi_{n-1}$, $\delta_n := \mu\theta_n|x_n - x_{n-1}|^2$ and define $[t]_+ := \max\{t, 0\}$ (for any $t \in \mathbb{R}$). By (H1) (hence $(\theta_n) \subset [0, \theta]$) and by (3.7), we easily obtain $[u_{n+1}]_+ \leqslant \theta[u_n]_+ + \delta_n$. Furthermore, a simple calculation gives $[u_{n+1}]_+ \leqslant \theta^n[u_1]_+ + \sum_{j=0}^{n-1} \theta^j\delta_{n-j}$, hence by (3.5) and since $\theta \in [0, 1)$ we deduce

$$\sum_{n\geqslant 0}[u_{n+1}]_+ \leqslant \frac{1}{1-\theta}\left([u_1]_+ + \sum_{n\geqslant 1}\delta_n\right) < \infty.$$

Consequently the sequence defined by $w_n := \phi_n - \sum_{j=1}^{n}[u_j]_+$ is bounded from below and also satisfies

$$w_{n+1} = \phi_{n+1} - [u_{n+1}]_+ - \sum_{j=1}^{n}[u_j]_+ \leqslant w_n.$$

It turns out that (w_n) is nonincreasing, hence (w_n) is convergent as well as (ϕ_n), which proves that $\lim_{n\to\infty} |x_n - q|$ exists for any $q \in S$, so that (x_n) is a bounded sequence.

(2) Let us prove that any weak cluster point of (x_n) is in S. Let (x_{n_k}) be a subsequence of (x_n) that converges weakly to a point $\tilde{x}$ in H. By (3.7), since $(\theta_n) \subset [0, \theta]$ we obviously have

$$\begin{aligned}\eta(1-\theta)d_{n+1} &\leqslant \phi_n - \phi_{n+1} + \theta_n(\phi_n - \phi_{n-1}) + \mu\theta_n d_n \\ &\leqslant \phi_n - \phi_{n+1} + \theta[u_n]_+ + \mu\theta_n d_n,\end{aligned}$$

hence

$$\eta(1-\theta)\sum_{n\geqslant 1} d_{n+1} \leqslant \phi_1 + \theta\sum_{n\geqslant 1}[u_n]_+ + \mu\sum_{n\geqslant 1}\theta_n d_n < \infty,$$

so that $\lim_{n\to\infty}|x_{n+1}-x_n| = 0$ (since $\eta(1-\theta) > 0$). Recalling that $x_{n+1} = \mathcal{T}_n v_n$ and $v_n = x_n + \theta_n(x_n - x_{n-1})$, we get

$$|\mathcal{T}_n v_n - v_n| = |x_{n+1} - v_n| = |(x_{n+1}-x_n) - \theta_n(x_n - x_{n-1})|,$$

so that $\lim_{n\to\infty}|\mathcal{T}_n v_n - v_n| = 0$. It is also immediate that (v_{n_k}) converges weakly to $\tilde{x}$. Invoking the condition (C3), we conclude that $\tilde{x}$ belongs to S.

As a straightforward consequence, Opial's lemma ensures the desired result. □

THEOREM 3.3. *Under the assumptions of Theorem 3.2 with* (H1) *replaced by*

(H2): $\exists\theta \in [0, c)$, *where* $c := \dfrac{\frac{1}{2}(\frac{1}{\alpha}-1)}{\frac{1}{2}(\frac{1}{\alpha}-1) + \max\{1, (\frac{1}{\alpha}-1)\}}$, *such that* $(\theta_n) \subset [0, \theta]$ *and* (θ_n) *is nondecreasing,*

we have

$$\sum_{n\geqslant 1}|x_n - x_{n-1}|^2 < \infty, \tag{3.8}$$

hence there exists $\bar{x}$ *in* S *such that* $x_n \rightharpoonup \bar{x}$ *weakly in* H *as* $n \to \infty$.

Proof. By (H2) (hence (θ_n) is nondecreasing) and by (3.7), we obtain

$$\begin{aligned}\phi_{n+1} - \phi_n - (\theta_n\phi_n - \theta_{n-1}\phi_{n-1}) &\leqslant -\eta(1-\theta_{n+1})d_{n+1} + \mu\theta_n d_n \\ &= -\mu\theta_{n+1}d_{n+1} + \mu\theta_n d_n - (\eta - (\eta+\mu)\theta_{n+1})d_{n+1},\end{aligned} \tag{3.9}$$

where $\eta := \frac{1}{2}\left(\frac{1}{\alpha}-1\right)$ and $\mu := \max\{1, 2\eta\}$ (so that $\eta > 0$, $\mu \geqslant 1$). Let $\theta \in [0, c)$, where $c := \dfrac{\eta}{\eta+\mu}$, and set $\Gamma_n := \phi_n - \theta_{n-1}\phi_{n-1} + \mu\theta_n d_n$. Again with (H2) (hence $(\theta_n) \subset [0, \theta]$) and by (3.9), we obtain

$$\Gamma_{n+1} - \Gamma_n \leqslant -\gamma d_{n+1}, \tag{3.10}$$

where $\gamma := (\eta - (\eta+\lambda)\theta)$ (hence $\gamma > 0$). As a consequence, (Γ_n) is nonincreasing, so that

$$\phi_n - \theta\phi_{n-1} \leqslant \Gamma_n \leqslant \Gamma_1, \tag{3.11}$$

which entails

$$\phi_n \leqslant \theta^n\phi_0 + \frac{\Gamma_1}{1-\theta}. \tag{3.12}$$

Again with (3.10), we get $\gamma\sum_{k=1}^{n} d_{k+1} \leqslant \Gamma_1 - \Gamma_{n+1}$; by (3.11) we also have $-\Gamma_n \leqslant \theta\phi_n$ for all $n \geqslant 0$, hence $\gamma\sum_{k=1}^{n} d_{k+1} \leqslant \Gamma_1 + \theta\phi_n$, which by (3.12) and since $\theta \in [0, 1)$ leads to $\gamma\sum_{k\geqslant 1} d_{k+1} \leqslant \Gamma_1 + \theta\phi_0 + \frac{\Gamma_1}{1-\theta}$. Consequently, we get (3.8), which by Theorem 3.2 ends the proof. □

4. Application to Common Fixed Point Problems

In this section, we prove that (1.3) works for approximating a solution of the common fixed point problem (1.2).

LEMMA 4.1. *Let* $(T_i)_{i\geqslant 0} \subset \mathcal{F}_\alpha$ *for some* $\alpha \in (0, 1]$ *be such that* $\bigcap_{i\geqslant 0} \mathrm{Fix}(T_i) \neq \emptyset$ *and let* $(w_i)_{i\geqslant 0} \subset [0, +\infty)$ *be satisfying* $\sum_{i\geqslant 0} w_i = 1$. *Then the following results hold:*

(1) $\sum_{i\geqslant 0} w_i T_i$ *is a welldefined mapping on* H .
(2) $\mathrm{Fix}(\sum_{i\geqslant 0} w_i T_i) = \bigcap_{i\in I} \mathrm{Fix}(T_i)$, *where* $I := \{i \geqslant 0 \quad | \quad w_i \neq 0\}$.
(3) $\sum_{i\geqslant 0} w_i T_i$ *belongs to* $\mathcal{F}_\alpha$.

Proof. Let us prove (1). Set $S := \bigcap_{i\geqslant 0} \mathrm{Fix}(T_i) \neq \emptyset$ and let $(x, q) \in H \times S$, so that $q \in \mathrm{Fix}(T_i)$ for all $i \geqslant 0$. Assuming that each T_i belongs to $\mathcal{F}_\alpha$, we then have $\frac{1}{2\alpha}|x - T_i x|^2 \leqslant \langle x - T_i x, x - q\rangle$, hence $|x - T_i x| \leqslant 2\alpha|x - q|$. Consequently, we obviously get $|T_i x| \leqslant 2\alpha|x - q| + |x|$, which entails $\sum_{i\geqslant 0} |w_i T_i(x)| \leqslant 2\alpha|x - q| + |x|$, provided that $\sum_{i\geqslant 0} w_i = 1$. It follows that $\sum_{i\geqslant 0} w_i T_i(x)$ makes sense, that is (1).

In order to prove (2), we set $\mathcal{T} := \sum_{i\geqslant 0} w_i T_i$. It is clear that $S \subset \mathrm{Fix}(\mathcal{T})$, so that $\mathrm{Fix}(\mathcal{T}) \neq \emptyset$. Let $q \in \mathrm{Fix}(\mathcal{T})$ and let $p \in S$. It is easily seen that $\sum_{i\geqslant 0} w_i(q - T_i q) = 0$, because $\sum_{i\geqslant 0} w_i = 1$. Consequently, since $(T_i)_{i\geqslant 0} \subset \mathcal{F}_\alpha$ and since p belongs to each $\mathrm{Fix}(T_i)$, we have

$$0 = \sum_{i\geqslant 0} w_i\langle q - T_i q, q - p\rangle \geqslant \frac{1}{2\alpha} \sum_{i\geqslant 0} w_i|q - T_i q|^2.$$

We then obtain $q - T_i q = 0$ for each $i \in I$, which leads to $\mathrm{Fix}(\mathcal{T}) \subset \cap_{i\in I}\mathrm{Fix}(T_i)$, while the converse is obvious. Hence $\mathrm{Fix}(\mathcal{T}) = \cap_{i\in I}\mathrm{Fix}(T_i)$, which proves (2).

Let us prove (3). For any $(x, q) \in H \times \mathrm{Fix}(\mathcal{T})$, we easily observe that

$$\langle x - \mathcal{T}x, x - q\rangle = \left\langle x - \sum_{i\geqslant 0} w_i T_i x, x - q \right\rangle = \sum_{i\geqslant 0} w_i\langle x - T_i x, x - q\rangle,$$

hence, as $(T_i)_{i\geqslant 0} \subset \mathcal{F}_\alpha$, we obtain

$$\langle x - \mathcal{T}x, x - q\rangle \geqslant \frac{1}{2\alpha} \sum_{i\geqslant 0} w_i|x - T_i x|^2. \tag{4.1}$$

Moreover, we obviously have $|x - \mathcal{T}x| = |\sum_{i\geqslant 0} w_i(x - T_i x)| \leqslant \sum_{i\geqslant 0} w_i|x - T_i x|$, which by the fact that $\sum_{i\geqslant 0} w_i = 1$ and thanks to Young's inequality leads to

$$|x - \mathcal{T}x|^2 \leqslant \left(\sum_{i\geqslant 0} w_i\right) \left(\sum_{i\geqslant 0} w_i|x - T_i x|^2\right) = \sum_{i\geqslant 0} w_i|x - T_i x|^2.$$

By joining this last inequality to (4.1), we get $\langle x - \mathcal{T}x, x - q\rangle \geqslant \frac{1}{2\alpha}|x - \mathcal{T}x|^2$, so that $\mathcal{T} \in \mathcal{F}_\alpha$, which completes the proof. □

THEOREM 4.2. Let $(T_i)_{i\geqslant 0} \subset \mathcal{F}_\alpha$, where $\alpha \in (0, 1)$, be such that $\bigcap_{i\geqslant 0} \mathrm{Fix}(T_i) \neq \emptyset$ and suppose each T_i is demi-closed. Let $(x_n) \subset H$ be a sequence such that

$$x_{n+1} := \sum_{i\geqslant 0} w_{i,n} T_i \left(x_n + \theta_n(x_n - x_{n-1})\right), \quad \text{for all } n \geqslant 1,$$

where $(\theta_n) \subset [0, 1]$ and $(w_{i,n}) \subset [0, +\infty)$ are real numbers such that:

(1) $\forall n \geqslant 0, \sum_{i\geqslant 0} w_{i,n} = 1$;
(2) *for all* $i \geqslant 0$, $(w_{i,n})_{n\geqslant 0}$ *is bounded away from zero for n large enough* (*that is:* $\forall i \geqslant 0, \exists N_i \in \mathbb{N}$ *and* $\exists w_i > 0$ *such that* $\forall n \geqslant N_i$, $w_{i,n} \geqslant w_i$).

Assume, in addition, one of the following conditions is satisfied:

(H1): $(\theta_n) \subset [0, \theta)$ *for some* $\theta \in [0, 1)$ *and* $\sum \theta_n |x_n - x_{n-1}|^2 < \infty$.

(H2): $\exists \theta \in [0, c)$, *where* $c := \dfrac{\frac{1}{2}\left(\frac{1}{\alpha} - 1\right)}{\frac{1}{2}\left(\frac{1}{\alpha} - 1\right) + \max\left\{1, \left(\frac{1}{\alpha} - 1\right)\right\}}$, *such that* $(\theta_n) \subset [0, \theta]$ *and* (θ_n) *is nondecreasing.*

Then there exists $\bar{x}$ in $\bigcap_{i\geqslant 0} \mathrm{Fix}(T_i)$ such that $x_n \rightharpoonup \bar{x}$ weakly in H as $n \to \infty$.

Proof. This result is a straightforward consequence of Theorems 3.2 and 3.3. It is just sufficient to prove that the conditions (C1–C3) hold with $\mathcal{T}_n = \sum_{i\geqslant 0} w_{i,n} T_i$ and $S = \bigcap_{i\geqslant 0} \mathrm{Fix}(T_i)$. From Lemma 4.1, we obviously have $(\mathcal{T}_n) \subset \mathcal{F}_\alpha$, so that (C1) holds. Again with Lemma 4.1, we obtain $\mathrm{Fix}(\mathcal{T}_n) = \bigcap_{i\in I_n} \mathrm{Fix}(T_i)$ for all $n \geqslant 0$, where $I_n := \{i \geqslant 0 \mid w_{i,n} \neq 0\}$. Noting that $S \subset \bigcap_{i\in I_n} \mathrm{Fix}(T_i)$, we deduce that $S \subset \mathrm{Fix}(\mathcal{T}_n)$, that is (C2). It just remains to prove that (C3) is true. Let $(\xi_n) \subset H$ be such that $\lim_{n\to 0} |\xi_n - \mathcal{T}_n \xi_n| = 0$ and let ξ be a weak-cluster point of (ξ_n), namely there exists a subsequence (ξ_{n_k}) such that $\xi_{n_k} \rightharpoonup \xi$ weakly as $k \to \infty$. Clearly, $(\xi_{n_k})_{k\geqslant 0}$ is a bounded sequence (thanks to the weak convergence) and $\lim_{k\to 0} |\xi_{n_k} - \mathcal{T}_{n_k} \xi_{n_k}| = 0$. Moreover, by taking into account the fact that each $\mathcal{T}_n$ belongs to $\mathcal{F}_\alpha$, we easily have

$$\langle \xi_n - \mathcal{T}_n \xi_n, \xi_n - q\rangle = \sum_{i\geqslant 0} w_{i,n} \langle \xi_n - T_i \xi_n, \xi_n - q\rangle \geqslant \frac{1}{2\alpha} \sum_{i\geqslant 0} w_{i,n} |\xi_n - T_i \xi_n|^2.$$

In particular, for all $k \geqslant 0$, we obtain

$$\langle \xi_{n_k} - \mathcal{T}_{n_k} \xi_{n_k}, \xi_{n_k} - q\rangle \geqslant \frac{1}{2\alpha} \sum_{i\geqslant 0} w_{n_k,i} |\xi_{n_k} - T_i \xi_{n_k}|^2.$$

Consequently, by the boundedness of (ξ_{n_k}), we easily deduce that

$$\lim_{k\to+\infty} \sum_{i\geqslant 0} w_{n_k,i} |\xi_{n_k} - T_i \xi_{n_k}|^2 = 0,$$

hence, for all $i \geqslant 0$, we obtain $\lim_{k\to+\infty} w_{n_k,i} |\xi_{n_k} - T_i \xi_{n_k}|^2 = 0$, which by (2) leads to $\lim_{k\to+\infty} |\xi_{n_k} - T_i \xi_{n_k}| = 0$. Assuming that each T_i is demi-closed and by the weak convergence of (ξ_{n_k}) to ξ, we conclude that $\xi = T_i \xi$ (for all $i \geqslant 0$), so that $\xi \in S$. It follows that (C3) is satisfied, which completes the proof. □

COROLLARY 4.3. *Let* $(T_i)_{i\geqslant 0} \subset \mathcal{F}_\alpha$, *where* $\alpha \in (0, 1)$, *be such that* $\bigcap_{i\geqslant 0} \mathrm{Fix}(T_i) \neq \emptyset$ *and suppose each* T_i *is demi-closed. Let* $(x_n) \subset H$ *be a sequence such that*

$$x_{n+1} = \left(\sum_{k=1}^{n} \gamma_k\right)^{-1} \sum_{i=1}^{n} \gamma_i T_i (x_n + \theta_n (x_n - x_{n-1})), \qquad \forall n \geqslant 1,$$

where $(\theta_n) \subset [0, 1]$ *and where* $(\gamma_n) \subset (0, +\infty)$ *satisfies*

$$\sum_{k\geqslant 1} \gamma_k < \infty. \tag{4.2}$$

Assume, in addition, one of the following conditions is satisfied:

(H1): $(\theta_n) \subset [0, \theta)$ *for some* $\theta \in [0, 1)$ *and* $\sum_{n\geqslant 1} \theta_n |x_n - x_{n-1}|^2 < \infty$.

(H2): $\exists\theta \in [0, c)$, *where* $c := \frac{\frac{1}{2}\left(\frac{1}{\alpha}-1\right)}{\frac{1}{2}\left(\frac{1}{\alpha}-1\right)+\max\left\{1,\left(\frac{1}{\alpha}-1\right)\right\}}$, *such that* $(\theta_n) \subset [0, \theta]$ *and* (θ_n) *is nondecreasing.*

Then (x_n) *converges weakly to a point in* $\bigcap_{i\geqslant 0} \mathrm{Fix}(T_i)$.

Proof. It is clear that scheme (1.4) is the special case of (1.3) when $w_{i,n} = \frac{\gamma_i}{\sum_{k=1}^{n} \gamma_k}$ for $1 \leqslant i \leqslant n$ and $w_{i,n} = 0$ for $i \geqslant n+1$, so that $\sum_{i\geqslant 0} w_{i,n} = 1$. For all $i \geqslant 0$ and large enough n, we also have $w_{i,n} \geqslant \frac{\gamma_i}{\sum_{k\geqslant 1} \gamma_k} > 0$, provided that (4.2) holds. Then Theorem 4.2 guarantees the weak convergence result of (x_n). □

COROLLARY 4.4. *Let* $(T_i)_{i\geqslant 0}$ *be an infinite countable family of firmly nonexpansive mappings defined on* H *and such that* $\bigcap_{i\geqslant 0} \mathrm{Fix}(T_i) \neq \emptyset$. *Let* $(x_n) \subset H$ *be a sequence such that*

$$x_{n+1} = \left(\sum_{k=1}^{n} \gamma_k\right)^{-1} \sum_{i=1}^{n} \gamma_i T_i(x_n + \theta_n(x_n - x_{n-1})), \qquad \forall n \geqslant 1,$$

where $(\theta_n) \subset [0, 1]$ *and where* $(\gamma_n) \subset (0, +\infty)$ *satisfies*

$$\sum_{k\geqslant 1} \gamma_k < \infty.$$

Assume, in addition, one of the following conditions is satisfied:

(H1): $(\theta_n) \subset [0, \theta)$ *for some* $\theta \in [0, 1)$ *and* $\sum_{n\geqslant 1} \theta_n |x_n - x_{n-1}|^2 < \infty$.

(H2'): $\exists\theta \in \left[0, \frac{1}{3}\right)$, *such that* $(\theta_n) \subset [0, \theta]$ *and* (θ_n) *is nondecreasing.*

Then (x_n) *converges weakly to a point in* $\bigcap_{i\geqslant 0} \mathrm{Fix}(T_i)$.

Proof. This result is deduced from Corollary 4.3 since $(T_i)_{i\geqslant 0}$ are assumed to be nonexpansive mappings, hence demi-closed, and included in $\mathcal{F}_{\frac{1}{2}}$. □

REMARK 4.1. Remind that $\mathcal{F}_1$ contains the set of quasi-nonexpansive mappings on H, namely 1-averaged quasi-nonexpansive operators. As $\mathcal{F}_1$ stands as a limit case of the sets $\mathcal{F}_\alpha$ (where $\alpha \in (0, 1)$), this view may reveal to be an interesting way for extending the convergence results of this paper to more general operators.

In view of applications of (1.3) and (1.4), we recall that the subgradient projection occurs for instance in signal and image processing [7, 8, 20] as a low computational approximation of the convex projection, when this latter projection is difficult to compute. Recently, a successive approximation scheme was developed for finding fixed points of certain quasi-nonexpansive mappings [3, 19] with successful applications to image recovery problem [8]. Let us make the following remark on the subgradient projection.

REMARK 4.2 (See [3, 16, 18]). Suppose that a continuous convex function $\Phi\colon H \to \mathbb{R}$ satisfies $\mathrm{lev}_{\leqslant 0} := \{x \in H \mid \Phi(x) \leqslant 0\} \neq \emptyset$. Let $\Phi'\colon H \to H$ be a selection of the Fenchel subdifferential of Φ, $\partial\Phi\colon H \to 2^H$, in the sense that $\Phi'(x) \in \partial\Phi(x)$ for all $x \in H$. Then a mapping $T_{(\Phi)}\colon H \to H$ defined by

$$\forall x \in H, \quad T_{(\Phi)}(x) := \begin{cases} x - \dfrac{\Phi(x)}{|\Phi'(x)|^2}\Phi'(x) & \text{if} \quad \Phi(x) > 0, \\ x & \text{if} \quad \Phi(x) \leqslant 0, \end{cases}$$

is called a subgradient projection relative to Φ. Moreover, $T_{(\Phi)}$ is a firmly quasi-nonexpansive mapping such that $\mathrm{Fix}(T_{(\Phi)}) = \mathrm{lev}_{\leqslant 0}$. If, in addition, the subdifferential (as a set valued-mapping) $\partial\Phi\colon H \to 2^H$ is bounded (that is if it maps bounded sets to bounded sets) then $T_{(\Phi)}$ satisfies the demi-closedness principle on H.

It turns out that Theorem 4.2 and Corollary 4.3 provide an alternative tool for approximating fixed points of a subgradient, but also common fixed points of finitely or infinitely many subgradient projections.

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