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Multivalued Exponentiation Analysis. Part I: Maclaurin Exponentials

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Abstract The exponentiation theory of linear continuous operators on Banach spaces can be extended in manifold ways to a multivalued context. In this paper we explore the Maclaurin exponentiation technique which is based on the use of a suitable power series. More precisely, we discuss about the existence and characterization of the Painlevé–Kuratowski limit

$$[\mathrm{Exp}\ F](x) = \lim_{n\to\infty} \sum_{p=0}^{n} \frac{1}{p!} F^p(x)$$

under different assumptions on the multivalued map $F\colon X \rightrightarrows X$. In Part II of this work we study the so-called recursive exponentiation method which uses as ingredient the set of trajectories associated to a discrete time evolution system governed by F.

Key words exponentiation · multivalued map · differential inclusion · power series · Painlevé–Kuratowski convergence.

Mathematics Subject Classifications 26E25 · 33B10 · 34A60.

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1. Introduction

1.1. Formulation of the Problem

Throughout this work, X is assumed to be a real Banach space equipped with a norm denoted by $|\cdot|$. The closed unit ball in X is represented by the symbol $\mathbb{B}_X$. In a couple of occasions we will ask X to be a Hilbert space or even a finite dimensional Euclidean space, but this will be explicitly mentioned in the appropriate place.

What does it mean exponentiating a multivalued operator $F\colon X \rightrightarrows X$? More than the nature of the underlying space X, what is important to stress here is the multivalued character of F. We are using the double arrow notation for emphasizing that $F(x)$ is a subset of X and not just a single point.

The above question arises, for instance, when it comes to study a Cauchy problem of the form

$$\begin{cases} \dot{z}(t) \in F(z(t)) & \text{for a.e. } t \in [0,1] \\ z(0) = x, \end{cases} \tag{1}$$

with trajectories being sought in a suitable space of functions, say

$$AC([0,1], X) = \{z\colon [0,1] \to X \mid z \text{ is absolutely continuous}\}.$$

By analogy with the concept of velocity field employed in the context of ordinary differential equations, the operator F on the right-hand side of (1) is sometimes referred to as a *velocity map*.

1.2. The Case of Linear Continuous Operators

For the sake of the exposition, we start with a few remarks concerning the exponentiation of linear continuous operators. As usual, we equip the vector space

$$\mathcal{L}(X) = \{A\colon X \to X \mid A \text{ is linear continuous}\}$$

with the operator norm $\|A\| = \sup_{|x|=1} |Ax|$.

The commonest way of defining the exponential of an operator $A \in \mathcal{L}(X)$ is by means of the Maclaurin series expansion

$$e^A = \lim_{n\to\infty} \sum_{p=0}^{n} \frac{1}{p!} A^p, \tag{2}$$

with convergence taking place in the space $(\mathcal{L}(X), \|\cdot\|)$. This definition turns out to be equivalent to

$$e^A = \lim_{n\to\infty} \left(I + \frac{1}{n} A\right)^n, \tag{3}$$

but, in practice, evaluating (2) or (3) doesn't involve the same amount of computational effort. Each formula has its own advantages and inconveniences.

There is yet another equivalent way of introducing the exponential operator e^A. The Maclaurin series approach can be reformulated as

$$e^A x = \lim_{n\to\infty} \sum_{p=0}^{n} \frac{1}{p!} z_p\ , \tag{4}$$

where the sequence $\{z_p\}_{p\geqslant 0}$ is defined recursively by

$$\begin{cases} z_{k+1} = Az_k & \text{for } k = 0, 1, \ldots \\ z_0 = x. \end{cases}$$

This formulation gives greater insight into how to interpret (2) in a multivalued setting and it explains better some properties of the exponentiation operation.

1.3. The Multivalued Setting

In order to define the exponential of a multivalued operator $F\colon X \rightrightarrows X$ one could adopt any of the three approaches mentioned in Section 1.2. From the outset of the discussion we warn the reader of the fact that each approach leads now to a different concept of exponentiation. Multivaluedness of F introduces a number of technical difficulties which don't show up in the single-value case.

The extension of the forward exponentiation method (3) to a multivalued setting has been explored by Wolenski [14, 15] and Amri and Seeger [1]. In this paper we concentrate on Maclaurin exponentials only. The recursive exponentiation method (4) will be treated in the companion paper [6].

2. Preliminaries on Painlevé–Kuratowski Limits

In relation to the theory of Painlevé–Kuratowski set-convergence, we use the material and terminology found in the books by Aubin and Frankowska [2] and Rockafellar and Wets [13]. An equality of the form

$$C = \lim_{n\to\infty} C_n$$

indicates that $\{C_n\}_{n\in\mathbb{N}}$ is a sequence of sets C_n in X converging in the Painlevé–Kuratowski sense toward the set $C \subset X$. The notation

$$\liminf_{n\to\infty} C_n = \{x \in X \mid \lim_{n\to\infty} \operatorname{dist}[x, C_n] = 0\},$$

$$\limsup_{n\to\infty} C_n = \{x \in X \mid \liminf_{n\to\infty} \operatorname{dist}[x, C_n] = 0\}$$

refers, respectively, to the lower and upper Painlevé–Kuratowski limit of the sequence $\{C_n\}_{n\in\mathbb{N}}$. By construction, these limits are always closed sets and

$$\liminf_{n\to\infty} C_n \subset \limsup_{n\to\infty} C_n.$$

A sequence of nonempty sets may converge to the empty set. This is perfectly acceptable. In most cases, however, we will ask the Painlevé–Kuratowski limit to be a nonempty set. We will mention explicitly this requirement whenever a doubt may arise.

The next lemma will be extensively used in the sequel. It is a well known result, but we state it below for the sake of convenience.

LEMMA 1. *Let* $\{C_n\}_{n\in\mathbb{N}}$ *be a sequence of sets* C_n in X. *If this sequence is nondecreasing with respect to set-inclusion, i.e.* $C_n \subset C_{n+1}$ *for all* $n \in \mathbb{N}$, *then it is Painlevé–Kuratowski convergent and*

$$\lim_{n\to\infty} C_n = \mathrm{cl}\big[\cup_{n\in\mathbb{N}} C_n\big], \tag{5}$$

with 'cl' *standing for the topological closure operation.*

Proof. See, for instance, [4, Section 2] or [13, Section 4.B]. □

Remark 1. In general, Painlevé–Kuratowski convergence is not stable with respect to set-addition. However, $\lim_{n\to\infty}\big(v_n + C_n\big) = v + \mathrm{cl}\big[\cup_{n\in\mathbb{N}} C_n\big]$ whenever $\{C_n\}_{n\in\mathbb{N}}$ is nondecreasing and $|v_n - v| \to 0$. To see this, take an arbitrary $x \in X$ and write

$$|\mathrm{dist}[x, v_n + C_n] - \mathrm{dist}[x, v + C]| \leqslant |v_n - v| + |\mathrm{dist}[x - v, C_n] - \mathrm{dist}[x - v, C]|,$$

where C denotes the set on the right-hand side of (5).

The next result has to do with the convergence of an infinite sum (or series) whose general term is an arbitrary nonempty set. Such a situation occurs quite often in practice and possibly Lemma 2 is to be found somewhere in the literature. Recall that a *selection* of $\{\Omega_p\}_{p\geqslant 1}$ is understood as a sequence $\{y_p\}_{p\geqslant 1}$ such that $y_p \in \Omega_p$ for all $p \geqslant 1$.

LEMMA 2. *Consider a sequence* $\{\Omega_p\}_{p\geqslant 1}$ *of sets in* X. *If*

$$\begin{cases} \{\Omega_p\}_{p\geqslant 1} \text{ admits a selection } \{y_p\}_{p\geqslant 1} \text{ such that} \\ \{\sum_{p=1}^n y_p\}_{n\geqslant 1} \text{ converges in the space } (X, |\cdot|), \end{cases} \tag{6}$$

then $\sum_{p=1}^n \Omega_p$ *Painlevé–Kuratowski converges to a nonempty set as* $n \to \infty$.

Proof. Consider $\{y_p\}_{p\geqslant 1}$ as in (6) and write

$$\sum_{p=1}^n \Omega_p = \sum_{p=1}^n \{y_p + \Omega_p - y_p\} = \Big(\sum_{p=1}^n y_p\Big) + \widehat{C}_n,$$

where $\widehat{C}_n = \sum_{p=1}^n \{\Omega_p - y_p\}$. Since $0 \in \Omega_p - y_p$ for every integer $p \geqslant 1$, the sequence $\{\widehat{C}_n\}_{n\geqslant 1}$ is nondecreasing. By applying Lemma 1 (in fact, Remark 1) one concludes that

$$\sum_{p=1}^\infty \Omega_p = \sum_{p=1}^\infty y_p + \mathrm{cl}\big[\cup_{n\geqslant 1} \widehat{C}_n\big], \tag{7}$$

where, contrary to appearances, the expression on the right-hand side of (7) doesn't depend on the specific choice of the selection $\{y_p\}_{p\geqslant 1}$. □

Remark 2. Chou and Penot [7] wrote an interesting paper on series of nonempty sets. These authors declare $\{\sum_{p=1}^n \Omega_p\}_{n\geqslant 1}$ to be *selectionable convergent* if the sequence $\{\sum_{p=1}^n y_p\}_{n\geqslant 1}$ converges in $(X, |\cdot|)$ for every selection $\{y_p\}_{p\geqslant 1}$ of $\{\Omega_p\}_{p\geqslant 1}$. As one can see from Lemma 2, selectionable convergence is a much stronger requirement than Painlevé–Kuratowski convergence.

Assumption (6) holds, for instance, under any of the following equivalent conditions:

$$\{\Omega_p\}_{p\geqslant 1} \text{ has a selection } \{y_p\}_{p\geqslant 1} \text{ such that } \textstyle\sum_{p=1}^{\infty} |y_p| < \infty, \tag{8}$$

$$\textstyle\sum_{p=1}^{\infty} \operatorname{dist}[0, \Omega_p] < \infty. \tag{9}$$

Needless to say, formula (7) is quite useless when it comes to practical computations. Easily computable limits can be derived only under special circumstances.

We now state a more elaborate version of Lemma 2, namely, the infinite sum is formed by averaging the Ω_p's with suitable weight coefficients.

LEMMA 3. *Consider a sequence $\{\Omega_p\}_{p\geqslant 1}$ of sets in X and a sequence $\{\mu_p\}_{p\geqslant 1}$ of positive scalars such that $\sum_{p=1}^{\infty} \mu_p < \infty$. If*

$$\begin{cases} \{\Omega_p\}_{p\geqslant 1} \textit{ admits a selection } \{y_p\}_{p\geqslant 1} \textit{ such that} \\ \{\sum_{p=1}^{n} \mu_p y_p\}_{n\geqslant 1} \textit{ converges in the space } (X, |\cdot|), \end{cases} \tag{10}$$

then $\sum_{p=1}^{n} \mu_p \Omega_p$ Painlevé–Kuratowski converges to a nonempty set as $n \to \infty$.

Proof. Apply Lemma 2 to the sequence $\{\mu_p \Omega_p\}_{p\geqslant 1}$. □

Assumption (10) holds, for instance, under any of the following equivalent conditions:

$$\{\Omega_p\}_{p\geqslant 1} \text{ has a selection } \{y_p\}_{p\geqslant 1} \text{ such that } \textstyle\sum_{p=1}^{\infty} \mu_p |y_p| < \infty, \tag{11}$$

$$\textstyle\sum_{p=1}^{\infty} \mu_p \operatorname{dist}[0, \Omega_p] < \infty, \tag{12}$$

$$\textstyle\sum_{p=1}^{\infty} \mu_p \operatorname{dist}[v, \Omega_p] < \infty \text{ for some } v \in X, \tag{13}$$

$$\textstyle\sum_{p=1}^{\infty} \mu_p \operatorname{dist}[v, \Omega_p] < \infty \text{ for every } v \in X. \tag{14}$$

3. Maclaurin Exponentiation

3.1. The Maclaurin Series Approach

In what follows we use the symbol $D(F)$ to refer to the domain of a multivalued map $F\colon X \rightrightarrows X$, that is to say, $D(F) = \{x \in X \mid F(x) \neq \emptyset\}$.

DEFINITION 1. One says that $F\colon X \rightrightarrows X$ is Maclaurin exponentiable at $x \in D(F)$ if the limit

$$[\operatorname{Exp} F](x) = \lim_{n\to\infty} \sum_{p=0}^{n} \frac{1}{p!} F^p(x) \tag{15}$$

exists in the Painlevé–Kuratowski sense and it is a nonempty set. Maclaurin exponentiability of F simply means that (15) exists nonvacuously for every $x \in D(F)$.

Although the notation that we are employing is self-explanatory, recall that the pth power of a multivalued map F is understood as an iterated composition:

$$F^p = F \circ F \circ \cdots \circ F \quad (p\ \text{–fold}).$$

By convention, raising $F: X \rightrightarrows X$ to the power 0 yields the identity operator $I: X \to X$.

It is not difficult to construct an example of a map which is not Maclaurin exponentiable. As shown below, this can be done already in a nonlinear single-value context.

EXAMPLE 1. Consider the nonlinear single-value map $F: \mathbb{R} \to \mathbb{R}$ defined as follows. For points lying in the set $\Lambda = \{(-1)^k k! \mid k = 0, 1, 2 \ldots\}$, we simply write

$$F((-1)^k k!) = (-1)^{k+1}(k+1)!\,.$$

Then we complete the definition of F so as to get a piecewise-affine continuous function over the real line $\mathbb{R}$. Consider for instance the point $x = 1$. A simple computation shows that

$$\sum_{p=0}^{n} \frac{1}{p!} F^p(x) = \frac{1 + (-1)^n}{2}, \tag{16}$$

and therefore

$$\liminf_{n\to\infty} \sum_{p=0}^{n} \frac{1}{p!} F^p(x) = \emptyset, \quad \limsup_{n\to\infty} \sum_{p=0}^{n} \frac{1}{p!} F^p(x) = \{0, 1\}.$$

Hence, Maclaurin exponentiability fails at the reference point $x = 1$ because the lower limit is strictly included in the upper limit. This example shows also that the upper limit doesn't need to be a convex set, even if F has convex values! Observe, incidentally, that the definition of F outside Λ is irrelevant for the computation of (16).

The next theorem provides a necessary condition for Maclaurin exponentiability at a given point. As the reader can see, Maclaurin exponentiability is guaranteed under a very mild assumption. Example 1 is somewhat pathological and should not be considered as a serious nuisance to the theory of Maclaurin exponentials.

THEOREM 1. *$F: X \rightrightarrows X$ is Maclaurin exponentiable at $x \in D(F)$ if*

$$\begin{cases} \{F^p(x)\}_{p\geqslant 1} \text{ admits a selection } \{y_p\}_{p\geqslant 1} \text{ such that} \\ \{\sum_{p=1}^{n} \frac{1}{p!} y_p\}_{n\geqslant 1} \text{ converges in the space } (X, |\cdot|). \end{cases} \tag{17}$$

Proof. Apply Lemma 3 with $\mu_p = 1/p!$ and $\Omega_p = F^p(x)$. □

Assumption (17) holds for instance if

$$\sum_{p=1}^{\infty} \frac{1}{p!} \operatorname{dist}[0, F^p(x)] < \infty. \tag{18}$$

The following proposition is stated as complement to Theorem 1. Its proof is immediate and therefore omitted.

PROPOSITION 1. *Consider a map* $F\colon X \rightrightarrows X$ *and a point* $x \in D(F)$. *The convergence criterion* (18) *holds under any of the following successively weaker conditions*:

(a) $\cap_{p\geqslant 1} F^p(x) \neq \emptyset$,
(b) *there is an integer* $N \geqslant 1$ *such that* $\cap_{p\geqslant N} F^p(x) \neq \emptyset$,
(c) $\{F^p(x)\}_{p\geqslant 1}$ *has a bounded selection*,
(d) *there exist a selection* $\{y_p\}_{p\geqslant 1}$ *of* $\{F^p(x)\}_{p\geqslant 1}$ *and positive constants* M, β *such that* $|y_p| \leqslant M\beta^p$ *for all* $p \geqslant 1$.

Remark 3. Condition (18) can be checked, of course, by using any classical criteria ensuring convergence of a numerical series with nonnegative terms. For instance, Cauchy's rule takes here the form

$$\limsup_{p\to\infty} \left\{ \frac{1}{p!} \operatorname{dist}[0, F^p(x)] \right\}^{1/p} < 1, \tag{19}$$

while d'Alembert's rule requires

$$\lim_{p\to\infty} \frac{1}{p+1} \frac{\operatorname{dist}[0, F^{p+1}(x)]}{\operatorname{dist}[0, F^p(x)]} < 1 \tag{20}$$

with the understanding that the denominator doesn't vanish. If the limit in (20) happens to be equal to 1, then Duhamel's rule can be brought into the picture: Convergence will take place if one can write

$$\frac{1}{p+1} \frac{\operatorname{dist}[0, F^{p+1}(x)]}{\operatorname{dist}[0, F^p(x)]} = 1 - \frac{\beta}{p} + o\left(\frac{1}{p}\right), \tag{21}$$

with $\beta > 1$ and $o(t) \to 0$ as $t \to 0$. What is bothersome about (19), (20), and (21), is that one still needs to estimate the expression $\operatorname{dist}[0, F^p(x)]$, a task which turns out to be quite cumbersome in practice.

3.2. The Weak Affine Growth Hypothesis

The next theorem has the merit of displaying a sufficient criterion for Maclaurin exponentiability that involves only the original map F and not the successive powers. What Theorem 2 essentially says is that the Maclaurin exponentiability of F is guaranteed if the real-valued function $x \in X \mapsto \operatorname{dist}[0, F(x)]$ doesn't grow too fast with respect to $|x|$.

THEOREM 2. *A nonempty-valued map* $F\colon X \rightrightarrows X$ *is Maclaurin exponentiable if the following 'Weak Affine Growth Hypothesis' holds*:

$$\begin{cases} \text{there are nonnegative constants a } \textit{and} \text{ b such that} \\ \operatorname{dist}[0, F(x)] \leqslant a|x| + b \quad \text{for all } \; x \in X. \end{cases} \tag{22}$$

Furthermore, under the above assumption, the Maclaurin exponential Exp $F: X \rightrightarrows X$ *satisfies the weak affine growth condition*

$$\text{dist}\big[0, [\text{Exp } F](x)\big] \leqslant \alpha|x| + \beta \quad \forall x \in X$$

with constants

$$\alpha = e^a \quad \textit{and} \quad \beta = \left(\sum_{p=1}^{\infty} \frac{1 + a + \ldots + a^{p-1}}{p\,!} \right) b.$$

Proof. Take $x \in X$. Choose any $\eta > 0$ and construct a sequence $\{y_p\}_{p \geqslant 1}$ such that

$$y_{k+1} \in F(y_k) \quad \text{and} \quad |y_{k+1}| \leqslant \text{dist}[0, F(y_k)] + \eta \quad \text{for } k = 0, 1, \ldots,$$

with the convention $y_0 = x$. One can easily see that $\{y_p\}_{p \geqslant 1}$ is a selection of $\{F^p(x)\}_{p \geqslant 1}$. Let us show that $\{y_p\}_{p \geqslant 1}$ doesn't grow too fast. By using (22), one gets

$$|y_1| \leqslant a|x| + b + \eta,$$

$$|y_2| \leqslant a|y_1| + b + \eta \leqslant a^2|x| + ab + a\eta + b + \eta,$$

and, in general,

$$|y_p| \leqslant a^p|x| + [1 + a + \ldots + a^{p-1}](b + \eta).$$

If $a < 1$, then $\{y_p\}_{p \geqslant 1}$ is bounded. When $a \geqslant 1$, the sequence $\{|y_p|\}_{p \geqslant 1}$ may go up to ∞ but not as fast as $p!$. In any case one has $\sum_{p=1}^{\infty} \frac{1}{p\,!}|y_p| < \infty$. Theorem 1 yields then the Maclaurin exponentiability of F at x. For the second part of the theorem, observe that

$$\sum_{p=0}^{\infty} \frac{1}{p\,!} y_p \in [\text{Exp } F](x)$$

and

$$\left| \sum_{p=0}^{\infty} \frac{1}{p\,!} y_p \right| \leqslant \sum_{p=0}^{\infty} \frac{1}{p\,!} |y_p| \leqslant \left(\sum_{p=0}^{\infty} \frac{1}{p\,!} a^p \right) |x| + \left(\sum_{p=1}^{\infty} \frac{1 + a + \ldots + a^{p-1}}{p\,!} \right) (b + \eta)$$

To arrive at the desired conclusion it suffices to let $\eta \to 0^+$. □

Remark 4. Perhaps the most bothering aspect of (22) is that it forces F to be nonempty-valued. If one has to deal with an operator F taking possibly empty values, then one can invoke the generalized hypothesis

$$\begin{cases} \text{there is a set } K \subset D(F) \text{ and nonnegative constants } a \text{ and } b \text{ such that} \\ F(K) \subset K \text{ and dist}[0, F(x)] \leqslant a|x| + b \ \text{ for all } \ x \in K. \end{cases} \tag{23}$$

Under (23) the conclusion is that F is Maclaurin exponentiable at every point in K. The proof of this fact follows the same line of argument as in Theorem 2. The invariance condition $F(K) \subset K$ is important for ensuring that the sequence $\{y_p\}_{p \geqslant 1}$ remains in the domain where F has an affine growth.

COROLLARY 1. *Suppose that* $F\colon X \rightrightarrows X$ *is a nonempty-valued map satisfying the Weak Affine Growth Hypothesis* (22). *Then,*

(a) *For all* $t \in \mathbb{R}$, tF *is Maclaurin exponentiable.*
(b) *For all integer* $m \geqslant 1$, F^m *is Maclaurin exponentiable.*

More generally, any polynomial expression $t_0 I + t_1 F + \ldots + t_m F^m$ *is Maclaurin exponentiable.*

Proof. The class of nonempty-valued maps satisfying the Weak Affine Growth Hypothesis is stable with respect to scalar multiplication, composition, and addition. It suffices then to apply Theorem 2. □

3.3. Positive Homogeneity

Positive homogeneity is one of the most fruitful ways of extending the concept of linearity. A multivalued map $F\colon X \rightrightarrows X$ is declared positively homogeneous if the following condition holds:

$$F(\alpha x) = \alpha F(x) \quad \forall \alpha > 0,\ \forall x \in X.$$

As we will see next, maps of this kind behave very nicely with respect to the operation of Maclaurin exponentiation, specially when positive homogeneity is coupled with some kind of boundedness property. The meaning of boundedness in a multivalued setting can be interpreted in manifold ways. In any case, it will be clear in a moment that the inner norm

$$\|F\|_{\text{inn}} = \sup_{x \in \mathbb{B}_X \cap D(F)} \ \inf_{v \in F(x)} |v|$$

and the outer norm

$$\|F\|_{\text{out}} = \sup_{x \in \mathbb{B}_X \cap D(F)} \ \sup_{v \in F(x)} |v|$$

will both play an important role in the discussion. For the general theory concerning these ‘norms’ the reader is conveyed to the references [2, 8, 13].

Without extra effort one can state a number of Maclaurin exponentiability results in the more general context of positive homogeneity of arbitrary degree. That $F\colon X \rightrightarrows X$ is positively homogeneous of degree $r \in \mathbb{R}_+$ simply means that

$$F(\alpha x) = \alpha^r F(x) \quad \forall \alpha > 0,\ \forall x \in X.$$

The Maclaurin exponentiability analysis of F is not quite the same if the degree of homogeneity r is smaller than 1 or bigger than 1. For the sake of readability, we state each case in a separate proposition.

PROPOSITION 2. *Let* $F\colon X \rightrightarrows X$ *be nonempty-valued and positively homogeneous of degree* $r \in [0, 1]$. *Suppose that* $\|F\|_{\text{inn}}$ *is finite. Then, for all* $t \in \mathbb{R}_+$, *the map* tF *is Maclaurin exponentiable with a Maclaurin exponential admitting the representation formula*

$$[\mathrm{Exp}(tF)](x) = x + \sum_{p=1}^{\infty} \frac{t^{1+r+\ldots+r^{p-1}}}{p!} F^p(x) \qquad \forall x \in X. \tag{24}$$

For the particular case $r = 1$, *the map* $\mathrm{Exp}(tF)$ *is positively homogeneous with*

$$[\mathrm{Exp}(tF)](x) \quad = \quad x + \sum_{p=1}^{\infty} \frac{t^p}{p!} F^p(x) \qquad \forall x \in X, \tag{25}$$

$$\|\mathrm{Exp}(tF)\|_{\mathrm{inn}} \quad \leqslant \quad e^{t\,\|F\|_{\mathrm{inn}}}. \tag{26}$$

Proof. Pick up any $x \neq 0$. Since F is positively homogeneous of degree r, we have $F(x) = |x|^r\, F(x/|x|)$ and

$$\mathrm{dist}[0,\, F(x)] = |x|^r \,\mathrm{dist}\left[0,\, F\left(x/|x|\right)\right] \leqslant |x|^r\, \|F\|_{\mathrm{inn}}.$$

But $r \in [0, 1]$, so we can write $|x|^r \leqslant |x| + 1$, and therefore

$$\mathrm{dist}(0,\, F(x)) \leqslant (|x| + 1)\, \|F\|_{\mathrm{inn}}.$$

The above inequality holds also for $x = 0$. In short, F satisfies the Weak Affine Growth Hypothesis (22). As a consequence, not only F but also every tF is Maclaurin exponentiable. For proving the representation formula (24) we start by writing

$$(tF)^2(x) = \bigcup_{y \in F(x)} tF(ty) = \bigcup_{y \in F(x)} t^{1+r} F(y) = t^{1+r} F^2(x).$$

By iterating the previous computation, one gets $(tF)^p(x) = t^{1+r+\ldots+r^{p-1}} F^p(x)$ for every $p \geqslant 1$. Hence,

$$\sum_{p=0}^{n} \frac{1}{p!} (tF)^p(x) = x + \sum_{p=1}^{n} \frac{t^{1+r+\ldots+r^{p-1}}}{p!} F^p(x), \tag{27}$$

and (24) follows by letting $n \to \infty$. As far as (26) is concerned, standard calculus rules on inner norms for sums and compositions of positively homogeneous maps yield

$$\Big\| \sum_{p=0}^{n} \frac{t^p}{p!} F^p \Big\|_{\mathrm{inn}} \leqslant \sum_{p=0}^{n} \frac{t^p}{p!} \|F^p\|_{\mathrm{inn}} \leqslant e^{t\,\|F\|_{\mathrm{inn}}}.$$

A careful passage to the limit completes the proof of the desired inequality. □

PROPOSITION 3. *Let* $F \colon X \rightrightarrows X$ *be nonempty-valued and positively homogeneous of degree* $r > 1$. *Suppose that* $\|F\|_{\mathrm{inn}}$ *is finite. Consider any* $x \in X$ *and write*

$$t_*(x) = \begin{cases} \left(\|F\|_{\mathrm{inn}}\, |x|^{r-1}\right)^{-1} & \text{if } \|F\|_{\mathrm{inn}} \neq 0,\, x \neq 0, \\ \infty & \text{otherwise} \end{cases} \tag{28}$$

Then, for every $t \in]0, t_*(x)[$, *the map* tF *is Maclaurin exponentiable at* x *and the Maclaurin exponential* $[\mathrm{Exp}(tF)](x)$ *admits the representation formula* (24).

Proof. Observe that the equality (27) holds for an arbitrary degree $r \in \mathbb{R}_+$, and not just for $r \in [0, 1]$ as in Proposition 2. However, for $r > 1$ we have to be more careful while passing to the limit as $n \to \infty$. By using the assumptions made on F one obtains the upper estimate

$$\mathrm{dist}[0,\, (tF)^p(x)] \leqslant (t\, \|F\|_{\mathrm{inn}})^{1+r+\ldots+r^{p-1}} |x|^{r^p} \qquad \forall x \in X,\ \forall p \geqslant 1. \tag{29}$$

The proof of (29) is not difficult, so we omit the details. According to Theorem 1, Maclaurin exponentiability of tF at x is guaranteed if

$$\sum_{p=1}^{\infty} \frac{(t\ \|F\|_{\text{inn}})^{1+r+\ldots+r^{p-1}}}{p!}\ |x|^{r^p} < \infty.$$

By invoking the classical d'Alembert's criterion for checking the convergence of a series of positive terms, one sees that the above series converges if

$$\lim_{p\to\infty} \frac{1}{p+1}\left\{t\ \|F\|_{\text{inn}}\ |x|^{r-1}\right\}^{r^p} \leqslant 1,$$

that is to say, if $t\ \|F\|_{\text{inn}}\ |x|^{r-1} \leqslant 1$. This explains why we are choosing t smaller than the value $t_*(x)$ given by (28). □

Remark 5. Positive homogeneity of degree $r = 2$ is somewhat special. In this particular case one can check that tF is Maclaurin exponentiable at x if and only if F is Maclaurin exponentiable at tx. Moreover, one can write

$$[\text{Exp}(tF)](x) = 1/t\ [\text{Exp}F](tx).$$

This is a quite surprising relation indeed. There is no counter-part for this relation in the classical case of linear continuous operators, nor in the case of positive homogeneity of degree $r \neq 2$ for that matter.

4. Computation of Maclaurin Exponentials

4.1. Maclaurin Exponentials of Projectors

In the next proposition we present a nontrivial example of a Maclaurin exponentiable operator admitting an easily computable Maclaurin exponential. Recall that a multi-valued map $F\colon X \rightrightarrows X$ is called a *projector* if $F^2 = F$. In what follows, the notation $\overline{\text{co}}$ stands for the operation which consists in taking the closure of the convex hull, and $e \approx 2.718...$ refers to the Neperian constant.

PROPOSITION 4. *If $F\colon X \rightrightarrows X$ is a projector, then F is Maclaurin exponentiable and*

$$x + (e-1)\text{cl}[F(x)] \subset [\text{Exp}\ F](x) \subset x + (e-1)\overline{\text{co}}[F(x)] \qquad \forall x \in D(F). \tag{30}$$

In particular, if F is a projector with closed convex values, then

$$[\text{Exp}\ F](x) = x + (e-1)F(x) \qquad \forall x \in D(F). \tag{31}$$

Proof. The assumption $F^2 = F$ implies that $F^p = F$ for all $p \geqslant 1$, so any $x \in D(F)$ satisfies the intersection property (a) of Proposition 1. The Maclaurin exponentiability of F is then a consequence of Theorem 1. In order to estimate the Maclaurin exponential of F, we start by writing

$$\sum_{p=0}^{n} \frac{1}{p!} F^p(x) = x + \sum_{p=1}^{n} \frac{1}{p!} F(x),$$

and then we observe that

$$\Big[\sum_{p=1}^{n}\frac{1}{p!}\Big]F(x)\ \subset\ \sum_{p=1}^{n}\frac{1}{p!}F(x)\ \subset\ \sum_{p=1}^{n}\frac{1}{p!}\mathrm{co}[F(x)]=\Big[\sum_{p=1}^{n}\frac{1}{p!}\Big]\mathrm{co}[F(x)],$$

the last equality being due to the fact that $\mathrm{co}[F(x)]$ is a convex set. We are invoking here a general factorization result stated in [12, Theorem 3.2]. By passing to the limit as $n\to\infty$, one ends up with the inclusions stated in (30). Notice that these inclusions also hold for $x\notin D(F)$, but in such a case one gets the empty set everywhere. □

Examples of projectors are quite numerous and include the following:

$$\begin{cases}\text{Constant operator : } F(x)=\Omega \text{ for all } x\in X.\\ \text{Metric projector over a closed set : } F(x)=P_\Omega(x)=\mathrm{argmin}_{z\in\Omega}|z-x|.\\ F=I+N_C \text{ with } N_C \text{ denoting the normal map of a convex set } C.\\ F=I-T_C \text{ with } T_C \text{ denoting the tangent map of a convex set } C.\end{cases}$$

It is also good to know that if F is a projector, then F^{-1} is a projector as well. Inversion of F is understood of course in the multivalued sense. By way of illustration of Proposition 4, let us work out in detail the last case listed above.

COROLLARY 2. *Let C be a closed convex set in X. Then, $I-T_C$ is Maclaurin exponentiable and*

$$[\mathrm{Exp}(I-T_C)](x)=e\,x-T_C(x)\qquad\forall x\in C.$$

Proof. Recall that the tangent map of C is defined by $T_C(x)=\mathrm{cl}\big[\mathbb{R}_+(C-x)\big]$ for $x\in C$, and $T_C(x)=\emptyset$ for $x\notin C$. So, $I-T_C$ has closed convex values. For checking that $I-T_C$ is a projector, we start by observing that

$$(I-T_C)^2(x)=\bigcup_{\xi\in(I-T_C)(x)}(I-T_C)(\xi)\ \supset\ (I-T_C)(x)$$

because one can choose $\xi=x$ in the above union. For the reverse inclusion, we prove first that

$$T_C(\xi)\subset T_C(x)\qquad\forall\xi\in(I-T_C)(x).\tag{32}$$

Take $c\in C$ and write $c-\xi=c-x+x-\xi$. Clearly $c-x\in C-x\subset T_C(x)$. On the other hand, $x-\xi$ belongs to $T_C(x)$ by assumption. It ensues that $c-\xi\in T_C(x)$. Since this is true for any $c\in C$, we have $C-\xi\subset T_C(x)$, and therefore $T_C(\xi)\subset T_C(x)$ as desired. In view of (32), one can write

$$\begin{aligned}(I-T_C)^2(x)&\subset\cup_{\xi\in(I-T_C)(x)}\,\xi-T_C(x)\\&=x-T_C(x)-T_C(x)\\&=x-T_C(x).\end{aligned}$$

This confirms that $(I-T_C)^2=I-T_C$. It remains now to write down formula (31). □

Remark 6. The case $I+N_C$ can be treated in a similar way. This time, however, we must ask X to be a Hilbert space. In such a way one can define N_C as a map from X into itself by means of the expression $N_C(x)=\{y\in X\mid\langle y,x'-x\rangle\leqslant 0\ \ \forall x'\in C\}$ if $x\in C$. As usual, one sets $N_C(x)=\emptyset$ if $x\notin C$. Since $I+N_C$ is a projector with closed convex values, Proposition 4 yields the formula $[\mathrm{Exp}(I + N_C)](x) = e\, x + N_C(x)$ for all $x \in C$.

4.2. Stationarity and Cyclicity Assumptions

Proposition 4 can be extended in at least two different ways. The first way corresponds to the case in which the sequence $\{F^p\}_{p\geqslant 1}$ becomes stationary after a certain range. Said in other words, there is no need to compose F with itself more than a certain number of times because after a while the composition does not change any longer.

PROPOSITION 5. *Let* $F\colon X \rightrightarrows X$ *be a map such that* $F^{q+1} = F^q$ *for some integer* $q \geqslant 1$. *Then, F is Maclaurin exponentiable and*

$$\mathrm{cl}\left\{\sum_{p=0}^{q-1} \frac{1}{p!} F^p(x) + \left(\sum_{p=q}^{\infty} \frac{1}{p!}\right) F^q(x)\right\} \subset [\mathrm{Exp}\ F](x)$$

$$\subset \mathrm{cl}\left\{\sum_{p=0}^{q-1} \frac{1}{p!} F^p(x) + \left(\sum_{p=q}^{\infty} \frac{1}{p!}\right) \mathrm{co}[F^q(x)]\right\} \tag{33}$$

for all $x \in D(F)$. When $F^q(x)$ is convex, the above inclusions yield

$$[\mathrm{Exp}\ F](x) = \mathrm{cl}\left\{\sum_{p=0}^{q-1} \frac{1}{p!} F^p(x) + \left(\sum_{p=q}^{\infty} \frac{1}{p!}\right) F^q(x)\right\} \qquad \forall x \in D(F). \tag{34}$$

Proof. Everything is essentially the same as in the proof of Proposition 4, except that now one starts with the inclusions

$$\sum_{p=0}^{q-1} \frac{1}{p!} F^p(x) + \left(\sum_{p=q}^{n} \frac{1}{p!}\right) F^q(x) \subset \sum_{p=0}^{n} \frac{1}{p!} F^p(x), \tag{35}$$

$$\sum_{p=0}^{n} \frac{1}{p!} F^p(x) \subset \sum_{p=0}^{q-1} \frac{1}{p!} F^p(x) + \left(\sum_{p=q}^{n} \frac{1}{p!}\right) \mathrm{co}[F^q(x)], \tag{36}$$

and one needs to be a bit more careful with the process of passing to the limit as $n \to \infty$. There is no need to worry about the existence of the limit $[\mathrm{Exp}\ F](x)$ because this is being taken care of by Proposition 1(b) and Theorem 1. However, as mentioned before, Painlevé–Kuratowski convergence is not stable with respect to set-addition. Passing to lower limits in (35) offers no difficulty and this leads to the left inclusion in (33). The set on the right-hand side of (36) is of the form $P + \alpha_n Q$, with Q convex and

$$\alpha_n = \sum_{p=q}^{n} \frac{1}{p!} \to \alpha = \sum_{p=q}^{\infty} \frac{1}{p!} \qquad \text{as } n \to \infty.$$

Given the convexity of Q and the upward monotonicity of $\{\alpha_n\}_{n\geqslant q}$, one can check that

$$P + \alpha_n Q = P + \alpha_n(Q - y) + \alpha_n y \subset P + \alpha(Q - y) + \alpha_n y = P + \alpha Q - (\alpha - \alpha_n) y,$$

with y being an arbitrary element of Q. This leads to $\limsup_{n\to\infty} (P + \alpha_n Q) \subset \mathrm{cl}\,(P + \alpha Q)$, which is what we need in order to obtain the second inclusion in (33). □

It is worth mentioning that the multivalued map $I + T_C$ is not a projector, but it falls within the context of the above proposition. The details are explained in the next corollary. The *pseudo-interior* of C is defined as the set of all points $\xi \in C$ such that $\xi + T_C(\xi) = \mathrm{cl}[\mathrm{aff(C)}]$, with $\mathrm{aff}(C)$ denoting the smallest affine space containing C.

COROLLARY 3. *Consider a closed convex set $C \subset X$ with nonempty pseudo-interior. Then, $I + T_C$ is Maclaurin exponentiable and*

$$[\mathrm{Exp}(I + T_C)](x) = e\,x + \mathrm{cl}[\mathbb{R}(C - x)] \qquad \forall x \in C. \tag{37}$$

Proof. We will check that $F = I + T_C$ satisfies the relation $F^3 = F^2$. In fact, we claim that

$$(I + T_C)^n(x) = \mathrm{cl}[\mathrm{aff}(C)] \qquad \forall x \in C, \forall n \geqslant 2. \tag{38}$$

To see this, take $x \in C$ and write

$$(I + T_C)^2(x) = \cup_{\xi \in (I+T_C)(x)}(I + T_C)(\xi) \;=\; \cup_{\xi \in C}(I + T_C)(\xi),$$

the second equality being due to the fact that $C \subset x + T_C(x)$. Notice that $\xi + T_C(\xi) \subset \mathrm{cl}[\mathrm{aff}(C)]$ for every $\xi \in C$, with equality in case ξ belongs to the pseudo-interior of C. As a consequence, (38) holds for $n = 2$. A simple recursion argument shows that (38) is true for any integer $n \geqslant 2$. In short, the relation $F^3 = F^2$ holds true and the general formula (34) leads to

$$[\mathrm{Exp}(I + T_C)](x) = \mathrm{cl}\Big\{x + x + T_C(x) + (e - 2)\mathrm{cl}[\mathrm{aff}(C)]\Big\}.$$

After a short simplification one arrives at the announced expression (37). □

Remark 7. In a finite dimensional setting the pseudo-interiority assumption holds automatically and the closure operation in (37) is superfluous.

The second way of extending Proposition 4 corresponds to the case in which $\{F^p\}_{p\geqslant 1}$ exhibits a certain cyclic behavior. This time the extension is less trivial and some notation needs to be introduced beforehand. For a given integer $r \geqslant 1$, consider the coefficients

$$\gamma_p(r) = \sum_{\ell=0}^{\infty} \frac{1}{(\ell r + p)!} \qquad \forall p \in \{1, \ldots, r\}.$$

These coefficients are positive and $\sum_{p=1}^{r} \gamma_p(r) = e - 1$.

PROPOSITION 6. *Let $F\colon X \rightrightarrows X$ be a map such that $F^{r+1} = F$ for some integer $r \geqslant 1$. Then, F is Maclaurin exponentiable and*

$$x + \mathrm{cl}\Big\{\sum_{p=1}^{r} \gamma_p(r) F^p(x)\Big\} \;\subset\; [\mathrm{Exp}\ F](x) \;\subset\; x + \mathrm{cl}\Big\{\sum_{p=1}^{r} \gamma_p(r)\mathrm{co}[F^p(x)]\Big\} \tag{39}$$

for all $x \in D(F)$. If the sets $F(x), \ldots, F^r(x)$ are convex, these inclusions yield

$$[\mathrm{Exp}\ F](x) = x + \mathrm{cl}\Big\{\sum_{p=1}^{r} \gamma_p(r) F^p(x)\Big\}.$$

Proof. The successive powers of F have the following cyclic structure:

$$\underbrace{F, F^2, \ldots, F^r}_{r \text{ terms}}, \underbrace{F, F^2, \ldots, F^r}_{r \text{ terms}}, \ldots$$

For proving Maclaurin exponentiability of F we use our old friend Theorem 1 combined with Proposition 1(c). In order to prove (39), take $x \in D(F)$ and consider an integer $n = kr + j$, with $k \in \mathbb{N}$ and $j \in \{0, \ldots, r-1\}$. We start the proof with the decomposition

$$\sum_{p=0}^{n} \frac{1}{p!} F^p(x) = x + \left\{ F(x) + \frac{1}{2!} F^2(x) + \ldots + \frac{1}{r!} F^r(x) \right\}$$
$$+ \left\{ \frac{1}{(r+1)!} F(x) + \frac{1}{(r+2)!} F^2(x) + \ldots + \frac{1}{(2r)!} F^r(x) \right\}$$
$$+ \ldots \quad + \sum_{p=1}^{j} \frac{1}{(kr+p)!} F^p(x),$$

that is to say,

$$\sum_{p=0}^{n} \frac{1}{p!} F^p(x) = x + \sum_{\ell=0}^{k-1} \left\{ \frac{1}{(\ell r+1)!} F(x) + \frac{1}{(\ell r+2)!} F^2(x) + \ldots + \frac{1}{((\ell+1)r)!} F^r(x) \right\}$$
$$+ \sum_{p=1}^{j} \frac{1}{(kr+p)!} F^p(x).$$

We drop, of course, the last sum in case $j = 0$. Since Minkowski addition of sets is a commutative and associative operation, we can rearrange the above expression in the form

$$\sum_{p=0}^{n} \frac{1}{p!} F^p(x) = x + \sum_{\ell=0}^{k-1} \frac{1}{(\ell r+1)!} F(x) + \sum_{\ell=0}^{k-1} \frac{1}{(\ell r+2)!} F^2(x)$$
$$+ \ldots + \sum_{\ell=0}^{k-1} \frac{1}{((\ell+1)r)!} F^r(x)$$
$$+ \sum_{p=1}^{j} \frac{1}{(kr+p)!} F^p(x).$$

We now get rid off the last sum by distributing the summands into the first j sums, obtaining in this way

$$\sum_{p=0}^{n} \frac{1}{p!} F^p(x) = x + \sum_{\ell=0}^{k} \frac{1}{(\ell r+1)!} F(x) + \ldots + \sum_{\ell=0}^{k} \frac{1}{(\ell r+j)!} F^j(x)$$
$$+ \sum_{\ell=0}^{k-1} \frac{1}{(\ell r+j+1)!} F^{j+1}(x) + \ldots + \sum_{\ell=0}^{k-1} \frac{1}{((\ell+1)r)!} F^r(x)$$

The remaining part of the proof is standard: We write down the corresponding lower and upper estimates for the above set, and then we pass to the lower limit on the left-hand side and to the upper limit on the right-hand side. The upper limit case is a bit more delicate, but one can adjust the proof technique used in Proposition 5: One cleverly adds and subtracts a point y_1 taken from co$[F(x)]$, another point y_2 from co$[F^2(x)]$, and so on until $y_r \in$ co$[F^r(x)]$. There is no need to enter into the details again. □

Remark 8. The proof techniques of Propositions 5 and 6 lead to the following existential result:

$$\begin{cases} \text{if } F\colon X \rightrightarrows X \text{ is a map satisfying } F^{m_1} = F^{m_2} \text{ for a couple of integers} \\ m_1, m_2 \geqslant 0 \text{ such that } m_1 \neq m_2, \text{ then } F \text{ is Maclaurin exponentiable.} \end{cases} \tag{40}$$

Writing down the explicit formula of the corresponding Maclaurin exponential is now a very tedious task, so we omit entering into the details. To check (40), consider for instance the case $1 \leqslant m_1 < m_2$. The powers of F have the structure

$$\overbrace{F, F^2, \ldots, F^{m_1-1}}^{m_1-1 \text{ terms}}, \underbrace{F^{m_1}, F^{m_1+1}, \ldots, F^{m_2-1}}_{m_2-m_1 \text{ terms}}, \underbrace{F^{m_1}, F^{m_1+1}, \ldots, F^{m_2-1}}_{m_2-m_1 \text{ terms}}, \ldots$$

The announced result follows from Theorem 1 combined with Proposition 1(c). By-the-way, one arrives at the same conclusion as in (40) if one assumes a more general relation of the form $F^{m_1} = \sigma F^{m_2}$, where $\sigma \in \mathbb{R}$ is arbitrary and $m_1, m_2 \geqslant 0$ are integers such that $m_1 \neq m_2$. The formula for the Maclaurin exponential of F incorporates the pair m_1, m_2 as well as the constant σ.

Projectors, or more generally maps as in Propositions 5 or 6, are 'simple' objects in the sense that their successive powers follow a pattern that is possible to identify. We present next a different class of multivalued maps that admit easily computable powers.

4.3. Velocity Maps in Constrained Linear Control Problems

One important problem in linear control theory is characterizing the set of all states that a system

$$\begin{cases} \dot{z}(t) = Az(t) + Bu(t) \\ u(t) \in P \\ z(0) = x \end{cases} \tag{41}$$

can reach if one chooses an appropriate input function u. To fix the ideas, suppose that the space of input functions is $L^1([0, 1], U)$, with U denoting a certain real Banach space. Assume also that

$$\begin{cases} A\colon X \to X \text{ and } B\colon U \to X \text{ are linear continuous,} \\ P \text{ is a nonempty set in } U. \end{cases}$$

To see that (41) can be written in the form of a differential inclusion as in (1), it suffices to take $F(x) = Ax + B(P)$. The linear control model (41) has been studied under different angles by authors like Brammer [5], Korobov [11], Aubin et al. [3], and Henrion et al. [10], just to mention a few names.

In the next proposition we discuss the Maclaurin exponentiability of an operator $F: X \rightrightarrows X$ having the structure

$$F_{A,K}(x) = Ax + K.$$

By obvious reasons, $F_{A,K}$ is referred to as the *affine-like* operator associated to the pair (A, K). It will be helpful to introduce the sets

$$\Gamma_n(A, K) = K + \frac{1}{2!}(K + AK) + \ldots + \frac{1}{n!}(K + AK + \ldots + A^{n-1}K)$$

whose role will be clear in a moment. For simplicity, we are omitting parentheses while writing $AK, A^2K, \ldots$ since is clear that $A^pK = \{A^px : x \in K\}$.

PROPOSITION 7. *For any operator $A \in \mathcal{L}(X)$ and nonempty set $K \subset X$, the Painlevé–Kuratowski limit $\Gamma(A, K) = \lim_{n\to\infty} \Gamma_n(A, K)$ exists and is nonempty. Furthermore, the affine-like operator $F_{A,K}: X \rightrightarrows X$ is Maclaurin exponentiable and*

$$[\mathrm{Exp}\ F_{A,K}](x) = e^A x + \Gamma(A, K) \qquad \forall x \in X. \tag{42}$$

In other words, $\mathrm{Exp} F_{A,K}$ *is the affine-like operator associated to the pair* $(e^A, \Gamma(A, K))$.

Proof. A matter of computation shows that

$$F^p_{A,K}(x) = A^p x + K + AK + \ldots + A^{p-1}K \qquad \forall p \geqslant 1.$$

By plugging this term in the Maclaurin series for $F_{A,K}$, one obtains

$$\sum_{p=0}^{n} \frac{1}{p!} F^p_{A,K}(x) = \sum_{p=0}^{n} \frac{1}{p!} A^p x + \Gamma_n(A, K). \tag{43}$$

For getting (42) it suffices to pass to the limit on both sides of (43). First, however, we must make sure that the limit $\Gamma(A, K)$ exists. To do this, we use Lemma 3 with $\mu_p = 1/p!$ and $\Omega_p = K + AK + \ldots + A^{p-1}K$. Take an arbitrary vector s in K and form

$$y_p = s + As + \ldots + A^{p-1}s.$$

Clearly $\{y_p\}_{p\geqslant 1}$ is a selection of $\{\Omega_p\}_{p\geqslant 1}$ and

$$|y_p| \leqslant \left\{1 + \|A\| + \ldots + \|A\|^{p-1}\right\}|s|.$$

Regardless of the value of $\|A\|$, the sequence $\{y_p\}_{p\geqslant 1}$ satisfies (11). □

Remark 9. Maclaurin exponentiability of $F_{A,K}$ follows directly from Theorem 2 because any affine-like operator satisfies the Weak Affine Growth Hypothesis (22). Theorem 2 doesn't provide however a formula for the Maclaurin exponential of $F_{A,K}$.

It is interesting to observe that Exp $F_{A,K}$ has the same structure as $F_{A,K}$. To render Proposition 7 really useful, it would be nice having a more explicit expression for the set $\Gamma(A, K)$. As shown next, this can be done in at least two special situations.

COROLLARY 4. *Let* $A \in \mathcal{L}(X)$ *and* $K \subset X$ *be a convex cone containing the origin. Then*,

$$(\mathrm{Exp}\ F_{A,K})(x) = e^{Ax} + \lim_{n\to\infty}[K + AK + \cdots + A^{n-1}K] \\ = e^{Ax} + \mathrm{cl}\left\{\cup_{n\geqslant 1}[K + AK + \ldots + A^{n-1}K]\right\}.$$

Proof. For every $p \in \mathbb{N}$, $A^p K$ is a convex cone containing the origin, and so is the set $K + AK + \ldots + A^p K$. On the other hand, it is clear that the sequence $\{K + AK + \ldots + A^p K\}_{p\in\mathbb{N}}$ is nondecreasing. It follows that

$$\Gamma_n(A, K) = K + AK + \ldots + A^{n-1}K.$$

The sequence $\{\Gamma_n(A, K)\}_{n\in\mathbb{N}}$ is Painlevé–Kuratowski convergent due to its monotonicity. We deduce from Lemma 1 and Proposition 7 that Exp $F_{A,K}$ has the announced form. □

The fact that K is a convex cone simplifies enormously the computation of $\Gamma_n(A, K)$. In the next proposition we consider the case in which K is just a (closed) convex set. However, we will ask A to be a multiple of the identity operator.

PROPOSITION 8. *Let* $K \subset X$ *be a closed convex set and* $A \in \mathcal{L}(X)$ *be a multiple of the identity operator, say* $A = aI$, *with* $a \in \mathbb{R}$. *Then,* $[\mathrm{Exp}\ F_{A,K}](x)$ *has the following form depending on the value of* a:

$$e^a\, x + \frac{e^a - e}{a-1}\, K \qquad \text{if} \quad a \geqslant 0, a \neq 1, \tag{44}$$

$$e\, x + e\, K \qquad \text{if} \quad a = 1, \tag{45}$$

$$e^a\, x + \frac{\mathrm{sh}\ a}{a}\, K + \frac{1}{a^2-1}\left(\mathrm{ch}\ a + \frac{\mathrm{sh}\ a}{a} - e\right)(K + a\, K) \quad \text{if} \quad a < 0, a \neq -1, \tag{46}$$

$$e^{-1}\, x + (\mathrm{sh}\ 1)\, K + \frac{\mathrm{ch}\ 1}{2}\,(K - K) \qquad \text{if} \quad a = -1. \tag{47}$$

Proof. We rely again on Proposition 7. For notational simplicity, let us write

$$\Gamma_n(a, K) := \Gamma_n(a\, I, K) \\ = K + \frac{1}{2!}(K + aK) + \ldots + \frac{1}{n!}(K + aK + \ldots + a^{n-1}K).$$

Consider first the case $a \geqslant 0$. By convexity of K, one has

$$\Gamma_n(a, K) = K + \frac{1+a}{2!}K + \ldots + \frac{1 + a + \ldots + a^{n-1}}{n!}K \\ = \left[1 + \frac{1+a}{2!} + \ldots + \frac{1 + a + \ldots + a^{n-1}}{n!}\right] K. \tag{48}$$

If $a \neq 1$, the previous expression can be rewritten as

$$\Gamma_n(a, K) = \frac{1}{a-1}\left[\frac{a^0-1}{0!} + \frac{a^1-1}{1!} + \ldots + \frac{a^n-1}{n!}\right] K$$
$$= \frac{1}{a-1}\left[\sum_{k=0}^{n}\frac{a^k}{k!} - \sum_{k=0}^{n}\frac{1}{k!}\right] K.$$

By passing to the Painlevé–Kuratowski limit as $n \to +\infty$, one obtains $\lim_{n\to\infty} \Gamma_n(a, K) = \frac{e^a - e}{a-1} K$. This takes care of the formula (44). If $a = 1$, we deduce from (48) that

$$\Gamma_n(1, K) = \left(\sum_{k=0}^{n-1}\frac{1}{k!}\right) K,$$

and therefore $\lim_{n\to\infty} \Gamma_n(1, K) = e\, K$, obtaining in this way (45). Let us consider now the remaining case $a < 0$. After some careful computation one can check that

$$\lim_{n\to\infty} \Gamma_n(a, K) = \left[1 + \frac{a^2}{3!} + \frac{a^4}{5!} + \ldots\right] K + \mu_a(K + a\, K),$$

with μ_a being the scalar given by

$$\mu_a = \frac{1}{2!} + \frac{1}{3!} + \frac{1+a^2}{4!} + \frac{1+a^2}{5!} + \frac{1+a^2+a^4}{6!} + \frac{1+a^2+a^4}{7!} + \ldots$$

Notice that the sum inside square brackets corresponds simply to $(\mathrm{sh}\, a)/a$. Let us examine more closely the term μ_a. If $a \neq -1$, then one can write

$$\mu_a = \frac{1}{a^2-1}\left[\frac{a^0-1}{0!} + \frac{a^0-1}{1!} + \frac{a^2-1}{2!} + \frac{a^2-1}{3!} + \frac{a^4-1}{4!} + \frac{a^4-1}{5!} + \ldots\right]$$
$$= \frac{1}{a^2-1}\left(\mathrm{ch}\, a + \frac{\mathrm{sh}\, a}{a} - e\right).$$

On the other hand, when $a = -1$ one gets

$$\mu_{-1} = \frac{1}{2!} + \frac{1}{3!} + \frac{2}{4!} + \frac{2}{5!} + \frac{3}{6!} + \frac{3}{7!} + \ldots$$
$$= \frac{1}{2}\sum_{k=1}^{\infty}\left[\frac{2k}{(2k)!} + \frac{2k}{(2k+1)!}\right]$$
$$= \frac{1}{2}\sum_{k=1}^{\infty}\left[\frac{1}{(2k-1)!} + \frac{1}{(2k)!} - \frac{1}{(2k+1)!}\right]$$
$$= \frac{\mathrm{ch}\, 1}{2}.$$

We derive in this way (46) or (47), depending on whether a is different from -1 or not. □

Although Proposition 8 is not a very general and useful result, it has however the merit of showing that the Maclaurin exponential of a multivalued map can take various forms depending on the structure of the map itself. Maclaurin exponentiation turns out to be a rather complex operation involving the possible use of hyperbolic functions (on the real line) and not just the usual exponential function. As we will see in Section 4.4, other examples lead to the use of more elaborate functional operations.

4.4. Velocity Maps in Differential Inequalities

The general form of an autonomous differential inequality is

$$g(z(t), \dot{z}(t)) \leqslant 0,$$

where $g\colon X \times X \to \mathbb{R}$ is, in principle, an arbitrary function. Such an evolution system can be written as a differential inclusion with right-hand side $F\colon X \rightrightarrows X$ given by

$$F(x) = \{y \in X \mid \; g(x, y) \leqslant 0\} \qquad \forall x \in X. \tag{49}$$

In order to compute the successive powers of the map (49), one needs to specify the structure of g. In the next theorem we state a Maclaurin exponentiability result for the case in which g takes the particular form $g(x, y) = |y| - \Psi(x)$, that is to say, we concentrate on differential inequality

$$|\dot{z}(t)| \leqslant \Psi(z(t)).$$

With such a choice of g, the multivalued map F is given by

$$F(x) = \Psi(x)\, \mathbb{B}_X \quad \forall x \in X,$$

i.e. the set $F(x)$ corresponds to a 'dilatation' of the closed unit ball $\mathbb{B}_X$, the dilatation factor being given by $\Psi(x)$. As we will see in a moment, exponentiating the map $\Psi(\cdot)\, \mathbb{B}_X$ amounts essentially in carrying out a certain weird operation involving the function $\Psi_{\max}\colon \mathbb{R}_+ \mapsto \mathbb{R}_+$ defined by

$$\Psi_{\max}(s) = \sup_{|w| \leqslant s} \Psi(w). \tag{50}$$

In fact, we will need to evaluate a series of the form

$$f_\Psi(s) = \sum_{p=0}^{\infty} \frac{1}{(p+1)!}\, \Psi^p_{\max}(s). \tag{51}$$

For convenience we introduce the notation

$$\mathrm{dom}(f_\Psi) = \{s \in \mathbb{R}_+ \mid f_\Psi(s) < \infty\}$$

to indicate the domain of convergence of the function f_Ψ.

THEOREM 3. *Suppose that $\Psi\colon X \to \mathbb{R}$ is a nonnegative function attaining its maximum on each closed ball centered at the origin. Then, the multivalued map $F = \Psi(\cdot)\, \mathbb{B}_X$ is Maclaurin exponentiable and one can write*

$$[\mathrm{Exp}\; F](x) = x + f_\Psi(\Psi(x))\, \mathbb{B}_X, \tag{52}$$

where the convention $\infty\, \mathbb{B}_X = X$ is in order. In particular, for $x \in X$ such that $\Psi(x) \in \mathrm{dom}(f_\Psi)$, the Maclaurin exponential $[\mathrm{Exp}\; F](x)$ is a closed ball of center x and radius $f_\Psi(\Psi(x))$.

Proof. First of all, observe that for each $s \in \mathbb{R}_+$ one can write

$$\bigcup_{|w| \leqslant s} \Psi(w)\, \mathbb{B}_X = \Psi_{\max}(s)\, \mathbb{B}_X.$$

The above relation follows from the very definition of $\Psi_{\max}(s)$ and the fact that Ψ attains its maximum over the closed ball $s\mathbb{B}_X$. Let us compute the successive powers of F at x. We start by writing

$$F^2(x) = F(F(x)) = \bigcup_{|w|\leqslant\Psi(x)} F(w) = \bigcup_{|w|\leqslant\Psi(x)} \Psi(w)\,\mathbb{B}_X = \Psi_{\max}(\Psi(x))\,\mathbb{B}_X,$$

and then we get

$$F^3(x) = F(F^2(x)) = \bigcup_{|w|\leqslant\Psi_{\max}(\Psi(x))} F(w) = \bigcup_{|w|\leqslant\Psi_{\max}(\Psi(x))} \Psi(w)\,\mathbb{B}_X = \Psi^2_{\max}(\Psi(x))\,\mathbb{B}_X$$

and the higher-order powers. Thus,

$$\begin{aligned}\sum_{p=0}^{n}\frac{1}{p!}F^p(x) &= x + \Psi(x)\,\mathbb{B}_X + \frac{1}{2!}\Psi_{\max}(\Psi(x))\,\mathbb{B}_X + \cdots + \frac{1}{n!}\Psi^{n-1}_{\max}(\Psi(x))\,\mathbb{B}_X\\ &= x + \left[\Psi(x) + \frac{1}{2!}\Psi_{\max}(\Psi(x)) + \cdots + \frac{1}{n!}\Psi^{n-1}_{\max}(\Psi(x))\right]\mathbb{B}_X.\end{aligned}$$

The announced formula (52) is obtained by letting $n \to \infty$, and this argument applies whether $\Psi(x)$ is in $\mathrm{dom}(f_\Psi)$ or not. □

Computing the series (51) can be quite involved in practice. Everything depends essentially on the complexity of the nonlinear function $\Psi_{\max}(\cdot)$ and the successive compositions $\Psi^2_{\max}(\cdot)$, $\Psi^3_{\max}(\cdot)\ldots$ Observe that (51) looks quite similar to the series

$$[\mathrm{Exp}\ \Psi_{\max}](s) = \sum_{p=0}^{\infty}\frac{1}{p!}\ \Psi^p_{\max}(s) \tag{53}$$

defining the Maclaurin exponential at s of the single-value function $\Psi_{\max}: \mathbb{R}_+ \to \mathbb{R}_+$. In fact, when $\Psi(x)$ belongs to $\Psi_{\max}(\mathbb{R}_+) = \{\Psi_{\max}(s) \mid s \in \mathbb{R}_+\}$, one can write

$$f_\Psi(\Psi(x)) = [\mathrm{Exp}\ \Psi_{\max}](s) - s, \tag{54}$$

where $s \in \mathbb{R}_+$ is any solution of the nonlinear equation

$$\Psi_{\max}(s) = \Psi(x). \tag{55}$$

Contrary to appearances, the term on the right-hand side of (54) remains the same if we change one solution s by another one, say s'. Of course, if the function $\Psi_{\max}: \mathbb{R}_+ \to \mathbb{R}_+$ is invertible, then $s = \Psi^{-1}_{\max}(\Psi(x))$ is the unique solution of (55).

Computing (51), or the Maclaurin exponential (53), is a much simpler task when $\Psi: X \to \mathbb{R}$ enjoys a certain homogeneity property. As seen in the next corollary, if Ψ is positively homogeneous of degree $r > 0$, then f_Ψ takes the form of a power series whose coefficients can be explicitly calculated.

COROLLARY 5. *Let $r > 0$ and $\Psi: X \to \mathbb{R}$ be a nonnegative function such that:*

(1) $\Psi(\lambda w) = \lambda^r\Psi(w)$ *for all* $\lambda > 0$ *and* $w \in X$.
(2) $\Psi(0) = 0$.
(3) *the supremum* $a = \sup_{|w|\leqslant 1}\Psi(w)$ *is positive and it is attained.*

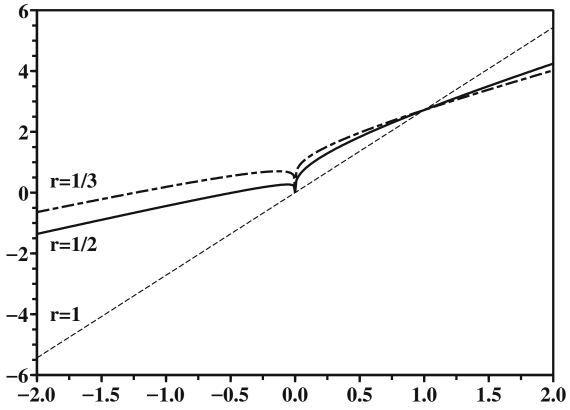


Figure 1 $[\mathrm{Exp}\ \Psi_{\max}](\cdot)$ for the choices $r = 1, r = 1/2$, and $r = 1/3$.

Then, the conclusions of Theorem 3 are true with f_Ψ *having the special form*

$$f_\Psi(s) = s + \sum_{p=1}^{\infty} \frac{a^{1+r+\ldots+r^{p-1}}}{(p+1)!}\, s^{r^p}. \tag{56}$$

In particular, the equality (54) *holds with* $s = (\Psi(x)/a)^{1/r}$ *and*

$$[\mathrm{Exp}\ \Psi_{\max}](s) = s + \sum_{p=1}^{\infty} \frac{a^{1+r+\ldots+r^{p-1}}}{p!}\, s^{r^p}. \tag{57}$$

Proof. Under the present assumptions, the supremum (50) is attained for any $s \in \mathbb{R}_+$ and

$$\Psi_{\max}(s) = s^r\ \Psi_{\max}(1) = a\ s^r.$$

Repeated compositions of $\Psi_{\max}(\cdot)$ with itself yield

$$\Psi^p_{\max}(s) = a^{1+r+\ldots+r^{p-1}} s^{r^p} \qquad \forall p \geqslant 1,$$

from where one derives the announced expressions of f_Ψ and Exp $\Psi_{\max}$. □

It is worth mentioning that the power series (57) has nothing to do with the usual exponential function $s \mapsto e^s$, nor with the hyperbolic functions $s \mapsto \mathrm{ch}(s)$ or $s \mapsto \mathrm{sh}(s)$ for that matter. The special choice $r = 1$ produces the linear expression

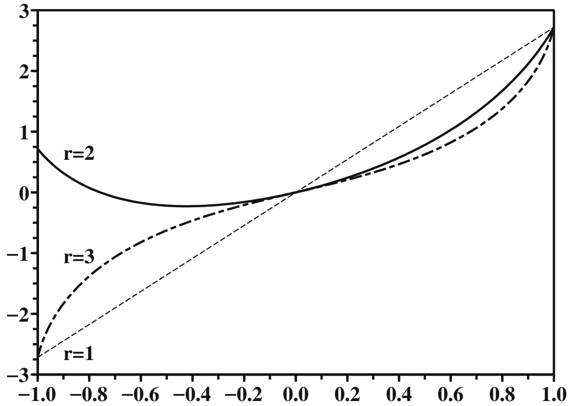
$$[\mathrm{Exp}\ \Psi_{\max}](s) = e^a\ s,$$

whereas the choice $r \neq 1$ yields

$$[\mathrm{Exp}\ \Psi_{\max}](s) = s + \sum_{p=1}^{\infty} \frac{a^{\frac{r^p-1}{r-1}}}{p!}\, s^{r^p} \tag{58}$$

Notice, incidentally, that the power series (58) converges for any $s \in \mathbb{R}_+$ when $r < 1$, and it converges only for s in the compact interval $\left[0, (1/a)^{1/(r-1)}\right]$ when $r > 1$. Anyhow, the power series (58) doesn't yield an easily recognizable function. Figure 1 illustrates the shape of the function $[\mathrm{Exp}\ \Psi_{\max}](\cdot)$ when $r = 1/2$ and $r = 1/3$. The

Figure 2 [Exp $\Psi_{\max}$]$(\cdot)$ for $r = 1, r = 2$, and $r = 3$.



cases $r = 2$ and $r = 3$ are illustrated in Figure 2. In both figures we take $a = 1$ and include the case $r = 1$ for the sake of comparison.

As we have learned from Proposition 8 and Corollary 5, the Maclaurin exponential of a multivalued operator can take all kinds of forms and it is quite difficult to detect a general pattern.

5. Miscellaneous Results on Maclaurin Exponentials

5.1. Topological Properties

Under suitable convexity and 'interiority' assumptions at the reference point x, it is possible to guarantee not only the existence of the Maclaurin exponential $[\text{Exp } F](x)$, but also the existence of interior points in this set.

PROPOSITION 9. *Consider a map* $F : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ *and a point* $x \in D(F)$ *such that*

(a) $F^p(x)$ *is convex for every* p *large enough*,
(b) $\liminf_{p\to\infty} F^p(x)$ *has nonempty interior*.

Then, F *is Maclaurin exponentiable at* x *and* $[\text{Exp } F](x)$ *has nonempty interior*.

Proof. Suppose that the set $F^p(x)$ is convex starting from $p \geqslant N_1$. In view of the assumption (b), there is a point $y \in \mathbb{R}^d$ and a scalar $\varepsilon > 0$ such that

$$y + \varepsilon\mathbb{B} \subset \text{int}\left[\liminf_{p\to\infty} F^p(x)\right],$$

with $\mathbb{B}$ denoting the closed unit ball in $\mathbb{R}^d$. By compactness of the unit ball (here is where finite dimensionality enters into the picture), we can apply [13, Proposition 4.15] and deduce that there is an integer $N \geqslant N_1$ such that

$$y + \varepsilon\mathbb{B} \subset F^p(x) \quad \forall p \geqslant N. \tag{59}$$

The above condition yields the intersection property (b) considered in Proposition 1, which in turn yields the exponentiability of F at x. The second part of the proposition is also a consequence of (59). One obtains the lower estimate

$$\sum_{p=0}^{N-1} \frac{1}{p!} F^p(x) + \sum_{p=N}^{n} \frac{1}{p!} \left[y + \varepsilon \mathbb{B}\right] \subset \sum_{p=0}^{n} \frac{1}{p!} F^p(x),$$

from where one gets

$$\gamma + \left[\sum_{p=N}^{n} \frac{1}{p!}\right] y + \varepsilon \left[\sum_{p=N}^{n} \frac{1}{p!}\right] \mathbb{B} \subset \sum_{p=0}^{n} \frac{1}{p!} F^p(x),$$

with γ denoting an arbitrary point taken from $\sum_{p=0}^{N-1} \frac{1}{p!} F^p(x)$. By letting $n \to \infty$, one arrives at

$$\gamma + \left[\sum_{p=N}^{\infty} \frac{1}{p!}\right] y + \varepsilon \left[\sum_{p=N}^{\infty} \frac{1}{p!}\right] \mathbb{B} \subset [\text{Exp } F](x).$$

This proves that $[\text{Exp } F](x)$ has nonempty interior because it contains a ball of radius $\varepsilon \left[\sum_{p=N}^{\infty} 1/p!\right]$. □

Remark 10. We have shown in fact that (59) yields the lower estimate

$$\sum_{p=0}^{N-1} \frac{1}{p!} F^p(x) + \left[\sum_{p=N}^{\infty} \frac{1}{p!}\right] y + \varepsilon \left[\sum_{p=N}^{\infty} \frac{1}{p!}\right] \mathbb{B} \subset [\text{Exp } F](x).$$

In principle the integer N is unknown and it could be very large. This means that the ball contained in $[\text{Exp } F](x)$ could have a very small radius. Notice that Proposition 9 makes sense only in a multivalued setting. The hypothesis (b) has no chance of being true if F is a single-value map.

5.2. Maclaurin Exponentiability at Absorption States

We start with the following observation:

LEMMA 4. *A map $F: X \rightrightarrows X$ is Maclaurin exponentiable at each one of its fixed points. More precisely, if $x \in X$ satisfies the fixed point condition*

$$x \in F(x), \tag{60}$$

then F is Maclaurin exponentiable at x and, in addition, one can write

$$e\, x \in [\text{Exp} F](x). \tag{61}$$

Proof. Condition (60) implies that $x \in F^p(x)$ for every integer $p \geqslant 1$. Hence, the intersection property (a) of Proposition 1 is in force and we obtain the Maclaurin exponentiability of F at x. We also have the relation

$$\sum_{p=0}^{n} \frac{1}{p!} x \in \sum_{p=0}^{n} \frac{1}{p!} F^p(x),$$

from where we get (61) by passing to the limit. □

The Maclaurin exponentiability result stated in Lemma 4 can be extended from

$$\text{fix}(F) = \{x \in X \,|\, x \in F(x)\}$$

to a much larger set. One says that $x \in X$ is an *absorption state* of F if there are an integer $n \geqslant 0$ and points $\{z_0, z_1 \ldots, z_n\}$ such that

$$\begin{cases} z_{p+1} \in F(z_p) & \text{for } p \in \{0, 1, \ldots, n-1\} \\ z_0 = x \\ z_n \in \text{fix}(F). \end{cases} \tag{62}$$

So, by definition, an absorption state is an initial point $x \in X$ that can be brought in a finite number of iterations to a fixed point of F. One can easily check that the set of absorption states of F is given by

$$\text{abs}(F) = \bigcup_{\xi \in \text{fix}(F)} \bigcup_{n \geqslant 0} F^{-n}(\xi),$$

where $F^{-n} = (F^{-1})^n = (F^n)^{-1}$.

PROPOSITION 10. *A map $F\colon X \rightrightarrows X$ is Maclaurin exponentiable at each one of its absorption states.*

Proof. Let x be an absorption state of F. One constructs $\{z_0, z_1 \ldots, z_n\}$ as in (62), with $z_n = \xi$ being a fixed point of F. Then we set $z_p = \xi$ for all $p \geqslant n+1$. The stationary sequence $\{z_p\}_{p \geqslant 1}$ constructed in this way is a selection of $\{F^p(x)\}_{p \geqslant 1}$. Theorem 1 does the rest of the job. □

5.3. The Use of Selectors

By a *selector* of a nonempty-valued map $F\colon X \rightrightarrows X$ one understands a single-value function $f\colon X \to X$ such that $f(x) \in F(x)$ for every $x \in X$. Maclaurin exponentiability for f simply means that, for all $x \in X$, the limit

$$[\text{Exp } f](x) = \sum_{p=0}^{\infty} \frac{1}{p!} f^p(x)$$

exists in the space $(X, |\cdot|)$.

In principle, the evaluation of the single-value function $\text{Exp } f\colon X \to X$ is a simple task while compared with the evaluation of the multivalued map $\text{Exp } F\colon X \rightrightarrows X$. An interesting and natural question is then the following:

$$\begin{cases} \text{how much information on Exp } F \text{ can be drawn if one knows} \\ \text{Exp } f \text{ for every Maclaurin exponentiable selector } f \text{ of } F? \end{cases}$$

In what follows we use the notation

$$\Theta_F = \{f\colon X \to X \,|\, f \text{ is a Maclaurin exponentiable selector of } F\}.$$

PROPOSITION 11. *Let $F\colon X \rightrightarrows X$ be a nonempty-valued map admitting a Maclaurin exponentiable selector. Then,*

(a) *F is Maclaurin exponentiable,*
(b) *for every $f \in \Theta_F$, the function* Exp f *is a selector of* Exp F.

Proof. Take $f \in \Theta_F$ and $x \in X$. Clearly $f^p(x) \in F^p(x)$ for every integer $p \geqslant 1$. Everything follows then directly from Theorem 1. □

Proposition 11 says that, for each $x \in X$, the set $[\text{Exp } F]_{\oplus}(x) = \{[\text{Exp } f](x) \mid f \in \Theta_F\}$ is a subset of $[\text{Exp } F](x)$. For convenience, we call $[\text{Exp } F]_{\oplus}(x)$ the *intrinsic* Maclaurin exponential of F at x. Unfortunately, $[\text{Exp } F]_{\oplus}(x)$ may contain only a very small portion of the whole Maclaurin exponential $[\text{Exp } F](x)$ and, what is more dramatic, it is not clear at all how to recover the missing part. The following example explains better what we have in mind.

EXAMPLE 2. Suppose that $F\colon X \rightrightarrows X$ is a constant map, say $F(x) = \Omega$ for all $x \in X$. If Ω is finite, then $[\text{Exp } F]_{\oplus}$ is finite-valued. More precisely, if $\text{card}[\Omega] \leqslant m$, then $[\text{Exp } F]_{\oplus}(x)$ contains at most

$$\pi_m = \sum_{k=1}^{m} k \, \frac{m!}{(m-k)!} \tag{63}$$

elements. The proof of this fact runs as follows. The integer π_m increases with respect to m, so we can take $\Omega = \{a_1, \ldots a_m\}$ formed with exactly m different points. Let $x \in X$ and $f \in \Theta_F$. Since, $f(x) \in \Omega$, there are m possible ways of defining the value of f at x. Suppose, for instance, $f(x) = a_{i_1}$. Now, we must decide how to define $f^2(x) = f(a_{i_1})$. Since $f(a_{i_1}) \in \Omega$, we have again m possibilities. Consider, for instance, $f(a_{i_1}) = a_{i_2}$. If $i_2 = i_1$, then $f^p(x) = a_{i_1}$ for all $p \geqslant 1$. Otherwise, we have to decide how to define $f^3(x) = f(a_{i_2})$. We take for instance $f(a_{i_2}) = a_{i_3}$. If $i_3 \in \{i_1, i_2\}$, then $f^p(x)$ is fully determined, otherwise we must continue with a new decision stage. This line of argument leads to the upper bound (63).

As we shall see in [6], for the special case $F(x) = \{0, 1\}$, one gets a Maclaurin exponential $[\text{Exp } F](x)$ which is noncountable. On the other hand, as explained in Example 2, the amount of information provided by the intrinsic Maclaurin exponential $[\text{Exp } F]_{\oplus}(x)$ is only finite. To say the least, intrinsic Maclaurin exponentiation provides in this example a very small portion of the whole picture.

For the sake of completeness we state below a generalized version of Proposition 11. This extended version is specially useful when $F\colon X \rightrightarrows X$ takes possibly empty values, or when we wish to concentrate on Maclaurin exponentiability over a certain region of X.

PROPOSITION 12. *Let $F\colon X \rightrightarrows X$ be an arbitrary multivalued map. Suppose there are a nonempty set $K \subset D(F)$ and a single-value function $f\colon K \to X$ such that*

(a) $f(K) \subset K$,
(b) *the limit* $[\text{Exp } f](x)$ *exists for all* $x \in K$,
(c) $f(x) \in F(x)$ *for all* $x \in K$.

Then, F is Maclaurin exponentiable at each $x \in K$ and $[\text{Exp } f](x) \in [\text{Exp } F](x)$.

Proof. The proof is essentially the same as in Proposition 11. The invariance property (a) ensures that the successive powers $f^2(x), f^3(x), \ldots$ are well defined. □

COROLLARY 6. *Consider a map $F\colon X \rightrightarrows X$ of the form $F = \psi + G$, where*

(a) $\psi : X \to X$ *is a Maclaurin exponentiable single-value function*,
(b) $0 \in G(x)$ *for all* $x \in D(G)$,
(c) $\psi(D(G)) \subset D(G)$.

Then, F *is Maclaurin exponentiable and* $[\mathrm{Exp}\ \psi](x) \in [\mathrm{Exp}\ F](x)$ for all $x \in D(F)$.

Proof. One just needs to apply Proposition 12 with $K = D(G) = D(F)$ and $f = \psi$. □

Remark 11. Observe that Corollary 6 implies the Maclaurin exponentiability of maps like $I + T_C$ and $I - T_C$, even when the set $C \subset X$ is closed but not necessarily convex. In a nonconvex setting, the tangent cone map T_C can be interpreted in the sense of Bouligand ([2, Def. 4.1.1]), in the sense of Clarke ([2, Def. 4.1.5]), or in any other sense as long as we respect the condition $0 \in T_C(x)$ for all $x \in C$. A similar remark applies to maps of the form $I + N_C$ and $I - N_C$ when X is a Hilbert space.

6. Infinitesimal Generator Formula

A fundamental result concerning the exponentiation of a linear continuous operator $A: X \to X$ asserts that

$$\lim_{t \to 0^+} \frac{e^{tA}x - x}{t} = Ax \qquad \forall x \in X, \tag{64}$$

with a limit taking place in the space $(X, |\cdot|)$. The 'infinitesimal generator' formula (64) is at the origin of a line of thought which consists in recovering the operator $A \in \mathcal{L}(X)$ by starting from a semigroup $\{e^{tA}\}_{t \in \mathbb{R}_+}$.

6.1. The General Case

We explore next this path of thought in a multivalued context. Although one should not be too optimistic concerning the semi-group character of the family $\{\mathrm{Exp}(tF)\}_{t \in \mathbb{R}_+}$, it is reasonable to expect drawing some information on the map F when the Maclaurin exponential $\mathrm{Exp}(tF)$ is known for every $t \in \mathbb{R}_+$. The theorem stated below corroborates this point-of-view.

THEOREM 4. *Consider a nonempty-valued map* $F : X \rightrightarrows X$ *satisfying the following regularity requirement at the origin*:

$$\lim_{w \to 0} \sup_{y \in F(w)} |y| = 0. \tag{65}$$

Let $x \in X$ *be a point such that* $F(x)$ *is bounded. Then*,

(a) *there exists* $t_* > 0$ *such that, for every* $t \in]0, t_*[$, *the map* tF *is Maclaurin exponentiable at* x *and* $[\mathrm{Exp}(tF)](x)$ *is bounded*;
(b) *the infinitesimal generator formula*

$$\lim_{t \to 0^+} \frac{[\mathrm{Exp}(tF)](x) - x}{t} = \mathrm{cl}[F(x)], \tag{66}$$

holds with a limit in the left-hand side understood in the Painlevé–Kuratowski sense.

Proof. We start with the proof of (a). Since the set $F(x)$ is bounded, there exists a radius $M > 0$ such that $F(x) \subset M\mathbb{B}_X$. Take any $\varepsilon \in]0, M]$. From assumption (65), we know there exists $\eta_\varepsilon > 0$ such that

$$F(\eta_\varepsilon \mathbb{B}_X) \subset \varepsilon \mathbb{B}_X. \tag{67}$$

Let us estimate $(tF)^p(x)$ for an arbitrary $t \in]0, \eta_\varepsilon / M]$. We start by writing

$$(tF)^2(x) = \bigcup_{y \in tF(x)} tF(y) \subset \bigcup_{y \in tM\mathbb{B}_X} tF(y) \subset \bigcup_{y \in \eta_\varepsilon \mathbb{B}_X} tF(y).$$

In view of (67), we get the inclusion $(tF)^2(x) \subset t\varepsilon\mathbb{B}_X$. Similarly,

$$(tF)^3(x) = \bigcup_{y \in (tF)^2(x)} tF(y) \subset \bigcup_{y \in t\varepsilon\mathbb{B}_X} tF(y) \subset \bigcup_{y \in \eta_\varepsilon \mathbb{B}_X} tF(y),$$

and we deduce as above that $(tF)^3(x) \subset t\varepsilon\mathbb{B}_X$. One ends up with

$$(tF)^p(x) \subset t\varepsilon\mathbb{B}_X \qquad \forall p \geqslant 2. \tag{68}$$

With the help of (68) and Theorem 1 one deduces that, for all $t \in]0, \eta_\varepsilon / M]$, the map tF is Maclaurin exponentiable at x and

$$[\mathrm{Exp}(tF)](x) \subset x + \sum_{p=1}^{\infty} \frac{1}{p!}\, tM\mathbb{B}_X = x + tM(e-1)\mathbb{B}_X.$$

This takes care of part (a). The proof of part (b) is a bit more delicate. We start by introducing the notation

$$C_n(t) := \frac{1}{t} \sum_{p=2}^{n} \frac{1}{p!} (tF)^p(x),$$

$$\rho(t) \;\; := \sup_{n \geqslant 2} \sup_{y \in C_n(t)} |y|.$$

In view of (68), we infer that

$$\forall t \in]0, \eta_\varepsilon / M], \quad \forall n \geqslant 2, \quad C_n(t) \subset (e-2)\varepsilon\mathbb{B}_X$$

or, what is equivalent,

$$\forall t \in]0, \eta_\varepsilon / M], \quad \rho(t) \leqslant (e-2)\varepsilon.$$

Since for all $\varepsilon \in]0, M]$, we have proved the existence of $\eta_\varepsilon > 0$ such that the above inequality holds, it follows that

$$\lim_{t \to 0} \rho(t) = 0. \tag{69}$$

In order to prove (66), take an arbitrary sequence $\{t_k\}_{k \in \mathbb{N}}$ converging to 0^+. We must show that

$$\mathrm{cl}[F(x)] \subset \liminf_{k \to \infty} \Delta(t_k) \quad \text{and} \quad \limsup_{k \to \infty} \Delta(t_k) \subset \mathrm{cl}[F(x)], \tag{70}$$

where $\Delta(t) := t^{-1} \left\{ [\mathrm{Exp}(tF)](x) - x \right\}$. From the definition of $C_n(t)$, one has

$$\Delta(t) = \lim_{n \to \infty} [F(x) + C_n(t)].$$

The definition of $\rho(t)$ shows that

$$\forall n \geqslant 2, \qquad C_n(t) \subset \rho(t)\mathbb{B}_X. \tag{71}$$

Thus, $\Delta(t) \subset \mathrm{cl}\,[F(x) + \rho(t)\mathbb{B}_X]$, and

$$\limsup_{k\to\infty} \Delta(t_k) \subset \limsup_{k\to\infty}\ \mathrm{cl}\,[F(x) + \rho(t_k)\mathbb{B}_X] = \mathrm{cl}[F(x)].$$

This takes care of the second inclusion in (70). On the other hand, one has

$$\mathrm{cl}[F(x)] + \liminf_{n\to\infty} C_n(t) \subset \Delta(t),$$

and therefore

$$\mathrm{cl}[F(x)] + \liminf_{k\to\infty}\liminf_{n\to\infty}\ C_n(t_k) \subset \liminf_{k\to\infty} \Delta(t_k).$$

For proving the first inclusion in (70), it is now enough to check that

$$0 \in \liminf_{k\to\infty}\liminf_{n\to\infty}\ C_n(t_k). \tag{72}$$

Consider, for each $k \in \mathbb{N}$, the sequence $\{z_{k,p}\}_{p\geqslant 0}$ defined recursively by

$$\begin{cases} z_{k,p+1} \in t_k F(z_{k,p}) & \text{for } p = 0, 1, \ldots \\ z_{k,0} = x. \end{cases}$$

Observe that

$$\frac{1}{t_k}\sum_{p=2}^{n} \frac{1}{p!} z_{k,p} \in C_n(t_k)$$

and therefore

$$z_k := \frac{1}{t_k}\sum_{p=2}^{\infty} \frac{1}{p!} z_{k,p} \in \liminf_{n\to\infty}\ C_n(t_k)$$

By using the upper estimate (71) one sees that $|z_k| \leqslant \rho(t_k)$. In view of (69), we deduce that $\{z_k\}_{k\in\mathbb{N}} \to 0$. This yields (72) and completes the proof of the theorem. □

If F and G are Maclaurin exponentiable maps such that $\mathrm{Exp}\ F = \mathrm{Exp}\ G$, it does not follow that F and G are the same, not even up to a closure operation. However, if F and G are two bounded-valued maps satisfying the regularity requirement (65), and such that $\mathrm{Exp}(tF) = \mathrm{Exp}(tG)$ for all $t > 0$ small enough, then $\mathrm{cl}[F(\cdot)]$ and $\mathrm{cl}[G(\cdot)]$ are the same. This is just one of the many conclusions that one can draw from Theorem 4.

Remark 12. A different type infinitesimal generator formula is obtained by Amri and Seeger [1] in the context of forward exponentiation of bundles of linear continuous operators. Frankowska [9] derives also an infinitesimal generator formula in a multivalued context. However, the ingredients and concepts that she uses are not the same as ours.

6.2. The Concept of Blow-up Time

Due to space restriction, we will not elaborate on the many ramifications of the theory associated to the infinitesimal generator formula. We shall say however a few words on the concept of 'blow-up time' that arises in a natural way from Theorem 4(a).

DEFINITION 2. Consider a nonempty-valued map $F\colon X \rightrightarrows X$. The Maclaurin blow-up time of F at $x \in X$ is defined as the number

$$t_F(x) = \sup\{t_* > 0 \mid \forall t \in]0, t_*[,\ tF \textit{ is Maclaurin exponentiable at } x \textit{ and } [\mathrm{Exp}(tF)](x) \textit{ is bounded}\}.$$

When $t_F(x) = \infty$, one says that F is free of Maclaurin blow-up time at x.

A variant of this concept is obtained by dropping the boundedness requirement on $[\mathrm{Exp}(tF)](x)$. We prefer however to keep the above formulation because unboundedness is at the core of many troubles, be them theoretical or computational.

The general idea behind Definition 2 is that tF can be nicely exponentiated at x as long as the time parameter t doesn't go beyond a certain threshold value. Needless to say, determining the precise moment where the Maclaurin blow-up will occur is a quite difficult task. In the next proposition we provide at least a lower estimate for the number $t_F(x)$.

PROPOSITION 13. *Let $F\colon X \rightrightarrows X$ and $x \in X$ be as in Theorem 4. Then,*

$$t_F(x) \geqslant \frac{1}{M} \sup\{\eta > 0 \mid F(\eta \mathbb{B}_X) \subset M\mathbb{B}_X\} \tag{73}$$

for every M such that $F(x) \subset M\mathbb{B}_X$. The inequality (73) holds in particular for $M = \sup_{y \in F(x)} |y|$.

Proof. This result follows by examining more closely the proof of part (a) in Theorem 4. What we have to do is simply taking $\varepsilon = M$, and then working with the largest $\eta > 0$ such that $F(\eta \mathbb{B}_X) \subset M\mathbb{B}_X$. □

6.3. Exploiting Positive Homogeneity

If F happens to be positively homogeneous of a certain degree, then it is possible to be a bit more precise concerning the moment at which the Maclaurin blow-up occurs.

COROLLARY 7. *Let $F\colon X \rightrightarrows X$ be nonempty-valued and positively homogeneous of degree $r > 0$. Suppose that $\|F\|_{\mathrm{out}}$ is finite. Consider any $x \in X$ and write*

$$t_*(x) = \begin{cases} \left(\|F\|_{\mathrm{out}}\, |x|^{r-1}\right)^{-1} & \text{if } r > 1,\ \|F\|_{\mathrm{out}} \neq 0, x \neq 0, \\ \infty & \text{otherwise} \end{cases} \tag{74}$$

Then, for every $t \in]0, t_*(x)[$, the map tF is Maclaurin exponentiable at x, the Maclaurin exponential $[\mathrm{Exp}(tF)](x)$ is bounded, and the infinitesimal generator formula (66) holds.

Proof. Recall that the equality

$$\sum_{p=0}^{n} \frac{1}{p!}(tF)^p(x) = x + \sum_{p=1}^{n} \frac{t^{1+r+\ldots+r^{p-1}}}{p!} F^p(x) \tag{75}$$

holds for an arbitrary degree $r \in \mathbb{R}_+$. The assumptions made on F lead to the upper estimate

$$F^p(x) \subset \|F\|_{\mathrm{out}}^{1+r+\ldots+r^{p-1}} |x|^{r^p}\, \mathbb{B}_X \quad \forall x \in X,\ \forall p \geqslant 1. \tag{76}$$

By plugging (76) into (75) one gets after some simplification

$$\sum_{p=0}^{n} \frac{1}{p!}(tF)^p(x) \subset x + \left[\sum_{p=1}^{n} \frac{(t\ \|F\|_{\mathrm{out}})^{1+r+\ldots+r^{p-1}}}{p!}\ |x|^{r^p}\right] \mathbb{B}_X. \tag{77}$$

Passing to the limit on both sides of (77) is possible if

$$\sum_{p=1}^{\infty} \frac{(t\ \|F\|_{\mathrm{out}})^{1+r+\ldots+r^{p-1}}}{p!}\ |x|^{r^p} < \infty.$$

According to d'Alembert's test, convergence of the above series occurs if $t\ \|F\|_{\mathrm{out}}\ |x|^{r-1} \leqslant 1$. This is why we are taking t smaller than the value $t_*(x)$ given by (74). In view of the inclusion (77), the Maclaurin exponential $[\mathrm{Exp}(tF)](x)$ not only exists but also it is bounded. Notice that up to this point we have been working with an arbitrary $r \geqslant 0$. For obtaining the infinitesimal generator formula we must rule out the case $r = 0$. Since F is assumed to be positively homogeneous of degree $r > 0$, for any $\eta > 0$ one can write the equality

$$\sup_{w \in \eta \mathbb{B}_X} \sup_{v \in F(w)} |v| \ = \ \eta^r\ \|F\|_{\mathrm{out}}$$

and the inclusion

$$F(\eta \mathbb{B}_X) \subset \eta^r \|F\|_{\mathrm{out}}\, \mathbb{B}_X.$$

Taking into account that $r \neq 0$, it follows that F is bounded-valued and satisfies the regularity condition (65). Theorem 4 ensures then the infinitesimal generator formula (66). □

Remark 13. The case $r = 0$ in Proposition 2 yields the particular representation formula

$$[\mathrm{Exp}(tF)](x) = (1-t)x + t\ [\mathrm{Exp}\ F](x) \quad \forall t > 0,\ \forall x \in X.$$

This case is somewhat pathological because the differential quotient $t^{-1}\big\{[\mathrm{Exp}(tF)](x) - x\big\} = [\mathrm{Exp}\ F](x) - x$ does not converge to $\mathrm{cl}[F(x)]$ as it is happening in Theorem 4.

7. Conclusions

In this long paper we have gathered a wealth of information concerning the operation of Maclaurin exponentiation in a multivalued setting. It is quite hard to summarize all the obtained results and their ramifications. We wish however to mention a few striking features of the Maclaurin exponentiation approach:

(1) Maclaurin exponentials exist more often than not. As seen in Theorem 1, a very mild assumption on F is enough to guarantee the Painlevé–Kuratowski convergence of the partial sums $\sum_{p=0}^{n} \frac{1}{p!} F^p$. From a practical point-of-view, the Weak Affine Growth Hypothesis (22) is a very handy test for checking convergence because we are not asked to evaluate the power map F^p.
(2) Computing a Maclaurin exponential $[\text{Exp } F](x)$ may be a difficult task or not depending on the specific structure of F. Anyhow, we have identified several classes of maps for which $[\text{Exp } F](x)$ can be computed explicitly. It happens quite often that Exp F has the same structure as F. For instance, if F is an affine-like operator, then so is Exp F. In many occasions, however, the structure and properties of Exp F are hard to predict.
(3) A beautiful aspect of Maclaurin exponentiation is that under mild and reasonable assumptions, it is possible to write an extension of the classical infinitesimal generator formula. Theorem 4 is a major result of this paper.
(4) A natural way of finding an element in $[\text{Exp } F](x)$ is by evaluating the limit $\sum_{p=0}^{\infty} \frac{1}{p!} f^p(x)$ for a Maclaurin exponentiable selector f of F. Unfortunately, this computation mechanism doesn't allow us to recover the whole set $[\text{Exp } F](x)$. In fact, selectors produce only a small portion of the whole picture. As we will see in [6], a richer approximation of the Maclaurin exponential can be obtained with the help of the so-called recursive exponentials.

The last but not the least observation is that a large portion of this paper could have been written in the general context of a series

$$[\Phi_\mu(F)](x) = \lim_{n\to\infty} \sum_{p=0}^{n} \mu_p F^p(x)$$

with reals μ_p such that $\sum_{p=0}^{\infty} |\mu_p| < \infty$. In other words, functional operations like $\cos F$, $\sin F$, $\text{ch}F$, $\text{sh}F$, could have been treated in a similar way. We have resisted the temptation to work in such a general context because it is the concept of exponentiation that is at the core of our preoccupations.

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