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Some Properties of General Minimization Problems with Constraints

Vy K. Le · Dumitru Motreanu

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Abstract The paper studies the existence of solutions and necessary conditions of optimality for a general minimization problem with constraints. Although we focus mainly on the case where the cost functional is locally Lipschitz, a general Palais–Smale condition is proposed and some of its properties are studied. Applications to an optimal control problem and a Lagrange multiplier rule are also given.

Key words minimization problems · constraints · Palais–Smale condition

1. Introduction

The paper deals with the following general minimization problem with constraints:

$$\text{(P)} \qquad \inf_{v \in S} \Phi(v).$$

Here, $\Phi : X \to \mathbb{R} \cup \{+\infty\}$ is a function on a Banach space X and S is an arbitrary nonempty subset of X. We suppose that $S \cap \text{dom}(\Phi) \neq \emptyset$, where the notation $\text{dom}(\Phi)$ stands for the effective domain of Φ, that is

$$\text{dom}(\Phi) = \{x \in X : \ \Phi(x) < +\infty\}.$$

First, we discuss the existence of solutions to problem (P). Precisely, we give an existence result making use of a new type of Palais–Smale condition formulated in terms of the tangent cone to the set S and of the contingent derivative for the function Φ. As a particular case, one recovers the global minimization result for a locally Lipschitz functional satisfying the Palais–Smale condition in the sense of

V. K. Le (✉)
Department of Mathematics and Statistics, University of Missouri-Rolla, Rolla, MO 65401, USA
e-mail: vy@umr.edu

D. Motreanu
Département de Mathématiques, Université de Perpignan, 52 Avenue Paul Alduy,
66860 Perpignan, France
e-mail: motreanu@univ-perp.fr

Chang (cf. [8]). Then, by means of the notion of generalized gradient (see Clarke [9]), we obtain necessary conditions of optimality for problem (P) in the case where the cost functional Φ is locally Lipschitz. A specific feature of our optimality conditions consists in the fact that the set of constraints S is basically involved through its tangent cone. In addition, the co-state variable provided by the given necessary conditions makes use essentially of the imposed tangency hypothesis. Finally, we present two applications of the necessary conditions of optimality that demonstrate the generality of our results. The first application concerns the minimization of a locally Lipschitz functional subject to a boundary value problem for semilinear elliptic equations depending on a parameter that runs in a function space. If this parameter is a control variable, the result can be interpreted as a maximum principle for the stated optimal control problem. The second application of the abstract result concerning the necessary conditions of optimality shows that the Lagrange multiplier rule fits into this setting. In particular, the Lagrange multiplier rule for locally Lipschitz functionals is derived. We also present a simple relation of this new type of Palais–Smale condition of the cost functional Φ on the set of constraints S with the classical coercivity property of Φ on S.

The approach relies on various methods including Ekeland's variational principle, Palais–Smale condition, tangency, generalized subdifferential calculus, orthogonality relations, Nemytskii operators, semilinear elliptic equations. In this respect it is worth to mention that a related work has been developed in [2–4] in the context of nonlinear mathematical programming problems. Here, the basic idea is represented by a kind of linearizing for the set of constraints S which allows to handle S locally by taking advantage of a continuous linear operator related to the tangent cone. This treatment has a unifying effect and can be applied to different problems in the optimization theory.

The rest of the paper is organized as follows. Section 2 is devoted to the existence of solutions to problem (P). A relation between the Palais–Smale condition introduced in Section 2 and the coercivity of Φ is established in Section 3. Section 4 presents our necessary conditions of optimality. Section 5 contains an example in solving an optimal control problem subject to a semilinear elliptic equation. Section 6 deals with an application to the Lagrange multiplier rule. We note that part of our presentation here has been described in the Proceedings paper [14].

2. Existence of Solutions

In the following we make use of the notion of tangent vector to the set S at a given point $v \in S$. Precisely, the tangent cone $T_v S$ to S at $v \in S$ ($T_v S$ is sometimes called the contingent cone to S at v) is defined as

$$T_v S = \left\{ w \in X : \liminf_{t \downarrow 0} \frac{1}{t} d(v + tw, S) = 0 \right\}, \tag{2.1}$$

where the notation $d(\cdot, S)$ stands for the distance function to the subset S in X. It is well-known that $T_v S$ is a closed cone in X. If S is a convex subset of X, then for every $v \in S$ a very convenient description for $T_v S$ holds:

$$T_v S = \mathrm{cl}\left(\bigcup_{t>0} \frac{1}{t}(S - v) \right),$$

where cl means the strong closure of a set in X. For further information on the tangent cone we refer to [5], Chapter 6.

Another useful tool in our approach is the contingent derivative $\Phi^D(u; v)$ of a function $\Phi : X \to \mathbb{R} \cup \{+\infty\}$ at a point $u \in \mathrm{dom}(\Phi)$ in any direction $v \in X$ which is defined by

$$\Phi^D(u; v) = \limsup_{\substack{t \downarrow 0 \\ w \to 0}} \frac{1}{t}(\Phi(u + t(v + w)) - \Phi(u)). \tag{2.2}$$

The following example points out a significant particular case.

EXAMPLE 2.1. If the function $\Phi : X \to \mathbb{R} \cup \{+\infty\}$ is locally Lipschitz at a point $u \in X$ then one has

$$\Phi^D(u; v) = \Phi^o(u; v), \quad \forall v \in X,$$

where Φ^o means the generalized directional derivative in the sense of Clarke (cf. [9]), i.e.,

$$\Phi^o(u; v) = \limsup_{\substack{t \downarrow 0 \\ x \to u}} \frac{1}{t}(\Phi(x + tv) - \Phi(x)). \tag{2.3}$$

This is clearly seen due to the locally Lipschitz property of Φ near u because then we may write

$$\begin{aligned} &\limsup_{\substack{t \downarrow 0 \\ w \to 0}} \frac{1}{t}(\Phi(u + t(v + w)) - \Phi(u)) \\ &\quad = \limsup_{\substack{t \downarrow 0 \\ w \to 0}} \frac{1}{t}(\Phi(u + tw + tv) - \Phi(u + tw)) + \lim_{\substack{t \downarrow 0 \\ w \to 0}} \frac{1}{t}(\Phi(u + tw) - \Phi(u)) \\ &\quad = \limsup_{\substack{t \downarrow 0 \\ x \to u}} \frac{1}{t}(\Phi(x + tv) - \Phi(x)). \end{aligned}$$

We now introduce a new type of Palais–Smale condition for nonsmooth functionals involving the tangent cone and contingent derivative.

DEFINITION 2.2. The functional $\Phi : X \to \mathbb{R} \cup \{+\infty\}$ is said to satisfy the Palais–Smale condition (for short, (PS)) at the level c ($c \in \mathbb{R}$) on the subset S of X if every sequence $(u_n) \subset S$ such that

$$\Phi(u_n) \to c \tag{2.4}$$

and

$$\Phi^D(u_n; v) \geqslant -\varepsilon_n \|v\|, \quad \forall v \in T_{u_n} S, \tag{2.5}$$

for a sequence $\varepsilon_n \to 0^+$, contains a strongly convergent subsequence in X.

Note that in our existence result below (Theorem 2.4), we only need the (PS) condition at the level $c = \inf_S \Phi$. The next example establishes that the (PS) condition in Definition 2.2 reduces to the usual Palais–Smale condition in the case of locally Lipschitz functionals.

EXAMPLE 2.3. If $\Phi : X \to \mathbb{R} \cup \{+\infty\}$ is locally Lipschitz and $S = X$, then Definition 2.2 becomes as follows: Every sequence (u_n) in X such that (2.4) holds and

$$\lambda(u_n) := \inf_{z \in \partial\Phi(u_n)} \|z\| \to 0 \quad \text{as } n \to \infty$$

possesses a strongly convergent subsequence. This equivalence follows readily from Definition 2.2, Example 2.1 and the definition of generalized gradient

$$\partial\Phi(u_n) = \left\{z \in X^* : \langle z, v\rangle \leqslant \Phi^o(u_n; v), \quad \forall v \in X\right\}.$$

We present our existence result in solving problem (P).

THEOREM 2.4. *Let S be a closed subset of X and $\Phi : X \to \mathbb{R} \cup \{+\infty\}$ a function such that $S \cap \operatorname{dom}(\Phi) \neq \emptyset$. Assume that $\Phi|_S$ is lower semicontinuous and bounded from below, and Φ satisfies the* (PS)*-condition in Definition 2.2 on S at the level $c = \inf_S \Phi$. Then problem (P) has at least a solution $u \in S$ which is a critical point of Φ on S in the following sense*

$$\Phi^D(u; v) \geqslant 0, \quad \forall v \in T_u S. \tag{2.6}$$

Proof. Applying Ekeland's variational principle (cf. [11] or Section 3 below) to the function $\Phi|_S$ yields a sequence $(u_n) \subset S$ such that (2.4) and

$$\Phi(y) \geqslant \Phi(u_n) - 1/n\|y - u_n\|, \quad \forall y \in S. \tag{2.7}$$

hold. Fix any $v \in T_{u_n}S$. By (2.1) there exist sequences $t_k \to 0^+$ in $\mathbb{R}$ and $w_k \to 0$ in X as $k \to \infty$ such that $u_n + t_k(v + w_k) \in S$ for all k. Plugging in (2.7) gives

$$\frac{1}{t_k}(\Phi(u_n + t_k(v + w_k)) - \Phi(u_n)) \geqslant -1/n\|v + w_k\|.$$

Letting $k \to \infty$ shows that

$$\liminf_{k\to\infty} \frac{1}{t_k}(\Phi(u_n + t_k(v + w_k)) - \Phi(u_n)) \geqslant -1/n\|v\|, \quad \forall v \in T_{u_n}S.$$

It turns out that (2.5) is verified with $\varepsilon_n = 1/n$. Therefore the (PS) condition as formulated in Definition 2.2 provides a relabelled subsequence satisfying $u_n \to u$ in X. Moreover, we have that $u \in S$ because S is closed. Taking into account that Φ is lower semicontinuous on S, we conclude $\Phi(u) = \inf_S \Phi$.

In order to check (2.6), let $v \in T_u S$. By (2.1) there exist sequences $t_k \to 0^+$ in $\mathbb{R}$ and $w_k \to 0$ in X as $k \to \infty$ such that $u + t_k(v + w_k) \in S$ for all k. Since $\Phi(u + t_k(v + w_k)) \geqslant \Phi(u)$ we readily obtain (2.6) that completes the proof. □

We illustrate the applicability of Theorem 2.4 by deriving the existence result of Chang [8], Theorem 3.5.

COROLLARY 2.5. *Assume that $\Phi : X \to \mathbb{R}$ is a locally Lipschitz function on a Banach space X, Φ is bounded from below and satisfies the Palais–Smale condition in the sense of Chang in* [8]. *Then there exists $u \in X$ such that $\Phi(u) = \inf_X \Phi$ and u is a critical point of Φ, i.e., it solves the inclusion problem*

$$0 \in \partial\Phi(u),$$

where $\partial\Phi(u)$ stands for the generalized gradient of Φ at u.

Proof. By Example 2.3 we know that the (PS) condition in the sense of Definition 2.2 is fulfilled with $S = X$. Then it is straightforward to deduce the result by applying Theorem 2.4. □

3. A Property of the Generalized Palais–Smale Condition

We prove here a simple relation between the (PS) condition just introduced and the classical coercivity condition of Φ on S. First, let us recall the definition of the coercivity of a functional on a set.

DEFINITION 3.1. Φ is said to be coercive on S if and only if $\Phi(u) \to +\infty$ as $\|u\| \to +\infty$, $u \in S$.

We also recall the following version of Ekeland's variational principle (cf. [11]):

THEOREM 3.2. *Let* (X, d) *be a complete metric space and* $\Phi : X \to \mathbb{R} \cup \{+\infty\}$ *be a lower semicontinuous functional such that* $\inf_{u\in X} \Phi(u) \in \mathbb{R}$. Let $\varepsilon > 0$ *and let* $u \in X$ *be such that*

$$\Phi(u) \leqslant \inf_X \Phi + \varepsilon.$$

Then, for every $\lambda > 0$*, there exists* $v \in X$ *such that*

$$\begin{aligned} &\text{(i)}\ \Phi(v) \leqslant \Phi(u)\\ &\text{(ii)}\ d(u, v) \leqslant \lambda^{-1}\\ &\text{(iii)}\ \Phi(w) > \Phi(v) - \lambda\varepsilon d(w, v),\ \ \forall w \in X \setminus \{v\}. \end{aligned}$$

We are now ready to state and prove the following result:

THEOREM 3.3. *Let S be a closed and unbounded subset of X. Assume* Φ *is lower semicontinuous, bounded below on S and satisfies the (PS) condition (at any level) on S. Then* Φ *is coercive on S.*

Proof. Assume by contradiction that Φ is not coercive on S, that is

$$c_0 := \liminf_{\substack{\|u\|\to\infty\\ u\in S}} \Phi(u) \in \mathbb{R}$$

(note that Φ is bounded below on S). Therefore, for each $n \in \mathbb{N}$, there exists $u_n \in S$ such that

$$\Phi(u_n) \leqslant c_0 + \frac{1}{n}$$

and

$$\|u_n\| \geqslant 2n.$$

Let $c_1 = \inf_S \Phi$. Since S is clearly a complete metric space with $d(u, v) = \|u - v\|$, by choosing $\varepsilon = c_0 - c_1 + n^{-1} (> 0)$ and $\lambda = n^{-1}$ in Theorem 3.2, we see that there exists $v_n \in S$ such that

$$\begin{aligned} &\text{(i)}\ \Phi(v_n) \leqslant \Phi(u_n) \leqslant c_0 + n^{-1}\\ &\text{(ii)}\ \|u_n - v_n\| \leqslant n\\ &\text{(iii)}\ \Phi(w) \geqslant \Phi(v_n) - n^{-1}(c_0 - c_1 + n^{-1})\|w - u_n\|,\ \ \forall w \in S. \end{aligned} \tag{3.1}$$

From (ii) and the choice of u_n, we have that

$$\|v_n\| \geqslant \|u_n\| - \|u_n - v_n\| \geqslant \|u_n\| - n \geqslant n. \tag{3.2}$$

Let us show that

$$\Phi^D(v_n; v) \geqslant -\varepsilon_n \|v\|, \ \forall v \in T_{v_n} S, \tag{3.3}$$

for some sequence (ε_n) converging to 0^+. In fact, let $v \in T_{v_n} S$. From the definition of the tangent cone in (2.1), there exist sequences $(t_k) \subset (0, +\infty)$ and $(w_k) \subset X$ such that $t_k \to 0$, $w_k \to 0$, and $s_k := v_n + t_k(v + w_k) \in S$, $\forall k \in \mathbb{N}$. From (3.1)(iii) with $w = s_k \in S$, we have

$$\begin{aligned}\Phi(s_k) - \Phi(v_n) &\geqslant -n^{-1}(c_0 - c_1 + n^{-1})\|s_k - v_n\| \\ &= -n^{-1}(c_0 - c_1 + n^{-1}) t_k \|v + w_k\|, \quad \forall k \in \mathbb{N}.\end{aligned} \tag{3.4}$$

From the definition (2.2) and (3.4), one gets

$$\begin{aligned}\Phi^D(v_n; v) &= \limsup_{\substack{t \downarrow 0 \\ w \to 0}} \frac{1}{t} \left[\Phi(v_n + t(v + w)) - \Phi(v_n)\right] \\ &\geqslant \limsup_{k \to \infty} \frac{1}{t_k} \left[\Phi(v_n + t_k(v + w_k)) - \Phi(v_n)\right] \\ &\geqslant \limsup_{k \to \infty} \left[-n^{-1}(c_0 - c_1 + n^{-1})\|v + w_k\|\right] \\ &= -n^{-1}\left(c_0 - c_1 + n^{-1}\right)\|v\|.\end{aligned}$$

Therefore, (3.3) holds with $\varepsilon_n = n^{-1}(c_0 - c_1 + n^{-1}) \to 0^+$ as $n \to \infty$. Since $\|v_n\| \to \infty$ as seen from (3.2), we have $\liminf \Phi(v_n) \geqslant c_0$. By (3.1)(i), this gives

$$\lim_{n \to \infty} \Phi(v_n) = c_0. \tag{3.5}$$

Since Φ satisfies the (PS) condition at level c_0, it follows from (3.3) and (3.5) that there exists a convergent subsequence (v_{n_k}) of (v_n), which contradicts (3.2) and completes our proof. □

Remark 3.4. (a) As used in the proof, we need the (PS) condition for Φ on S only at level c_0 for Theorem 3.3 to hold.

(b) Relations such as that presented in Theorem 3.3 were initiated and studied for smooth functionals in [7, 10, 15, 22] and have been extended later to other kinds of functionals in, e.g., [12, 13, 16, 17, 23].

As noted above, in the particular case where $S = X$ and Φ is locally Lipschitz, the definition above of the generalized (PS) condition reduces to the classical condition for locally Lipschitz functionals. Hence, Theorem 3.3 is a generalization of the corresponding property for locally Lipschitz functionals, as proved in [23].

We discuss a situation where S is not the whole space and the tangent cones are essentially taken into account.

COROLLARY 3.5. *Let S be a closed and unbounded C^1 submanifold of the Banach space X. Assume Φ is lower semicontinuous, bounded below on S and satisfies (at any level) on S the following condition of (PS) type: If (u_n) is a sequence such that $(u_n) \subset S$, $\Phi(u_n) \to c$ for some real number c, and*

$$\Phi^D(u_n; v) \geqslant -\varepsilon_n \|v\|, \quad \forall v \in \mathcal{T}_{u_n} S,$$

for a sequence $\varepsilon_n \to 0^+$, where $\mathcal{T}_{u_n} S$ is the tangent space to S at u_n in the sense of manifolds, then (u_n) contains a strongly convergent subsequence in X. Then Φ is coercive on S.

Proof. The conclusion follows from Theorem 3.3 provided we show that

$$T_v S = \mathcal{T}_v S, \quad \forall v \in S. \tag{3.6}$$

Fix $v \in S$. Since S is a C^1 submanifold of X there exists a C^1 diffeomorphism φ of an open neighborhood U of v in X onto an open subset $\varphi(U)$ of $X = X_1 \times X_2$ such that $\varphi(U \cap S) = \varphi(U) \cap (X_1 \times \{0\})$. Denoting by $\varphi'(v)$ the (Fréchet) differential of φ at v, relation (3.6) is equivalent to

$$\varphi'(v)(T_v S) = X_1 \times \{0\}. \tag{3.7}$$

In order to justify (3.7) let $w \in T_v S$. The definition of the tangent cone in (2.1) ensures that there exist sequences $(t_k) \subset (0, +\infty)$ and $(w_k) \subset X$ such that $t_k \to 0$, $w_k \to 0$, and $v + t_k(w + w_k) \in U \cap S$, $\forall k \in \mathbb{N}$. This leads to

$$\frac{1}{t_k}(\varphi(v + t_k(w + w_k)) - \varphi(v)) \in X_1 \times \{0\},$$

so thanks to the mean value theorem we find

$$\varphi'(v + \theta_k t_k(w + w_k))w \in X_1 \times \{0\},$$

for some number $\theta_k \in (0, 1)$. Then the continuity of φ' implies $\varphi'(v)w \in X_1 \times \{0\}$.

Conversely, let $y \in X_1 \times \{0\}$. Corresponding to a sequence $t_k \to 0^+$, we introduce

$$w_k = \frac{1}{t_k}[\varphi^{-1}(\varphi(v) + t_k y) - v] - \varphi'(v)^{-1} y$$

for all sufficiently large k. Using that φ^{-1} is continuously differentiable, it follows that $w_k \to 0$ in X as $k \to \infty$. Moreover, we note that $v + t_k(\varphi'(v)^{-1} y + w_k) \in U \cap S$ which guarantees $\varphi'(v)^{-1} y \in T_y S$. Thus it is seen that (3.7) holds true. This completes the proof. □

4. Necessary Conditions of Optimality

From now on we assume that the function $\Phi : X \to \mathbb{R}$ in problem (P) is locally Lipschitz on a Banach space X and S is an arbitrary nonempty subset of X. We formulate the following condition on the set S:

(H) For every $v \in S$, there exists a (possibly unbounded) linear operator

$$A_v : D(A_v) \subset X \to Y_v\,,$$

where Y_v is a Banach space, such that the domain $D(A_v)$ of A_v is dense in X, A_v is a closed operator (i.e., its graph is closed in $X \times Y_v$), and

$$\text{the range } R(A_v) \text{ is closed in } Y_v\,. \tag{4.1}$$

Moreover, the null space $N(A_v)$ of A_v satisfies

$$N(A_v) \subset T_v S, \tag{4.2}$$

where $T_v S$ stands for the tangent cone to S at v as introduced in (2.1).

THEOREM 4.1. *Under hypothesis* (H), *if* $u \in S$ *is a solution of problem* (P) *(at least locally), then the following necessary condition of optimality holds: There exists an element* $p \in D(A_u^*)$ *such that*

$$A_u^*(p) \in \partial\Phi(u),$$

where $A_u^* : D(A_u^*) \subset Y_u^* \to X^*$ *is the adjoint operator of* A_u *and* $\partial\Phi(u)$ *denotes the generalized gradient of* Φ *at* u.

Proof. Fix any $z \in N(A_u)$. It follows from (4.2) that

$$z \in T_u S. \tag{4.3}$$

Taking into account (2.1), we deduce from (4.3) that there exist sequences $t_n \to 0^+$ in $\mathbb{R}$ and $x_n \to 0$ in X such that

$$u + t_n(z + x_n) \in S, \quad \forall n.$$

Using the optimality of $u \in S$, we obtain

$$\Phi(u + t_n(z + x_n)) \geqslant \Phi(u), \quad \forall n,$$

that leads to

$$\liminf_{n\to\infty} \frac{1}{t_n} [\Phi(u + t_n(z + x_n)) - \Phi(u)] \geqslant 0. \tag{4.4}$$

In particular, according to (2.3), inequality (4.4) implies that

$$\Phi^o(u; z) \geqslant 0, \quad \forall z \in N(A_u). \tag{4.5}$$

On the basis of (4.5), we may apply the Hahn–Banach theorem to obtain the existence of some $\xi \in X^*$ with the properties

$$\langle \xi, z\rangle_{X^*,X} = 0, \quad \forall z \in N(A_u), \tag{4.6}$$

and

$$\langle \xi, y\rangle_{X^*,X} \leqslant \Phi^o(u; y), \quad \forall y \in X. \tag{4.7}$$

We see from (4.7) that

$$\xi \in \partial\Phi(u), \tag{4.8}$$

while (4.6) ensures that

$$\xi \in [N(A_u)]^\perp. \tag{4.9}$$

Since $[N(A_u)]^\perp = \overline{R(A_u^*)}$ (see for example [6]), in view of (4.1), relation (4.9) reads

$$\xi \in \overline{R(A_u^*)} = R(A_u^*), \tag{4.10}$$

(note that because $\overline{R(A_u)} = R(A_u)$, we also have $\overline{R(A_u^*)} = R(A_u^*)$, see again [6]).

Combining relations (4.8) and (4.10), we arrive at the desired conclusion. □

Remark 4.2. (a). By Theorem 4.1, we have the system

$$u \in S,$$
$$A_u^*(p) \in \partial\Phi(u)$$

formed by two relations with two unknowns u and p which eventually enables one to determine the optimal solution u.

(b). We consider here the problem with the functional Φ being locally Lipschitz, which is a good and convenient model for our calculations. However, it could be possible to study such problem with other types of functionals and subdifferentials (see, e.g., [9, 18, 19, 21]).

5. An Example

For an example of the general result in Theorem 3.1, let us consider the problem of minimizing the functional $\Phi(v, w)$ subject to the following conditions expressed as a Dirichlet problem:

$$\begin{aligned}&(v, w) \in \left[H^2(\Omega) \cap H_0^1(\Omega)\right] \times L^2(\Omega)\\&-\Delta v = f(x, v) + w \text{ in } \Omega\\&v = 0 \text{ on } \partial\Omega.\end{aligned}$$

Interpreting the parameter w as a control variable, this is in fact an optimal control problem. Here, Φ is a locally Lipschitz functional defined on $X = L^2(\Omega) \times L^2(\Omega)$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with $f(\cdot, 0) \in L^2(\Omega)$. Moreover, the partial derivative $\frac{\partial f}{\partial v}(x, v)$ exists for a.e. $x \in \Omega$, all $v \in \mathbb{R}$ with $\frac{\partial f}{\partial v}$ being a bounded Carathéodory function, that is,

$$\left|\frac{\partial f}{\partial v}(x, v)\right| \leqslant c, \quad \text{for a.e. } x \in \Omega, \text{all } v \in \mathbb{R}, \tag{5.1}$$

for some constant $c > 0$. Notice that the considered problem is of the general form (P) in Section 1 with

$$S = \{(v, w) \in X : -\Delta v = f(x, v) + w \text{ in } \Omega \text{ and } v = 0 \text{ on } \partial\Omega\}.$$

Let us prove that under the above conditions, hypothesis (H) holds. Specifically, for any $(v, w) \in S$, let

$$Y_{(v,w)} = L^2(\Omega) \times L^2(\Omega),$$

and let

$$A_{(v,w)} : \left[H^2(\Omega) \cap H_0^1(\Omega)\right] \times L^2(\Omega) \to L^2(\Omega) \times L^2(\Omega)$$

be given by

$$A_{(v,w)}(z, q) = \left(-\Delta z - \frac{\partial f}{\partial v}(\cdot, v)z, q\right)$$

whenever $(z, q) \in \left[H^2(\Omega) \cap H_0^1(\Omega)\right] \times L^2(\Omega)$. From (5.1) and the classical results of Agmon–Douglis–Nirenberg in [1] on linear elliptic operators, we see that the range of the operator

$$z \mapsto -\Delta z - \frac{\partial f}{\partial v}(\cdot, v)z \left(\in L^2(\Omega)\right)$$

is closed in $L^2(\Omega)$ (cf., e.g., [20], Theorem 8.41). It follows immediately that the range $R(A_{(v,w)})$ of $A_{(v,w)}$ is closed in $L^2(\Omega) \times L^2(\Omega)$. Condition (4.1) is thus verified.

In order to check (4.2), let $(z, 0) \in N(A_{(v,w)})$. Since

$$-\Delta v = f(x, v) + w \text{ in } \Omega \tag{5.2}$$

and

$$-\Delta z - \frac{\partial f}{\partial v}(\cdot, v)z = 0 \text{ in } \Omega, \tag{5.3}$$

by multiplying (5.3) by $t > 0$ and adding with (5.2), we obtain

$$-\Delta(v + tz) = f(x, v + tz) + w + tq(t), \tag{5.4}$$

where

$$q(t) = -\frac{1}{t}\left[f(\cdot, v + tz) - f(\cdot, v) - t\frac{\partial f}{\partial v}(\cdot, v)z\right], \quad \forall t > 0.$$

Assumption (5.1) guarantees that

$$q(t) \to 0 \text{ in } L^2(\Omega) \text{ as } t \to 0^+.$$

Consequently, setting

$$p(t) = (0, q(t)) \in L^2(\Omega) \times L^2(\Omega),$$

we see from (5.4) that condition (4.2) is fulfilled since

$$(v, w) + t[(z, 0) + p(t)] = (v, w) + t[(z, 0) + (0, q(t))] \in S, \quad \forall t > 0.$$

We have checked all assumptions of Theorem 4.1. According to that theorem, if Φ has a local minimum at (v, w) then there exists a pair $(p_1, p_2) \in L^2(\Omega) \times L^2(\Omega)$ such that

$$p_1 \in H^2(\Omega) \cap H_0^1(\Omega),$$

and

$$\left(-\Delta p_1 - \frac{\partial f}{\partial v}(\cdot, v)p_1, p_2\right) \in \partial\Phi(v, w),$$

(p_2 is not significant for our purpose).

Remark 5.1. If the constant c in (5.1) is smaller than the first eigenvalue λ_1 of $-\Delta$ on $H_0^1(\Omega)$, then the range of $A_{(v,w)}$ is in fact the whole space $L^2(\Omega) \times L^2(\Omega)$. In this case we observe that the auxiliary variable p_1 can be explicitly determined. Consequently, the necessary condition of optimality can be expressed only in terms of the local solution (v, w).

6. Application to Lagrange Multiplier Rule

Assume now that the subset S in problem (P) is given by

$$S = \bigcup_{j \in J} G_j^{-1}(0).$$

Here, for each $j \in J$, $G_j : X \to Y_j$ is a C^1 mapping with Y_j being a Banach space and 0 a regular value of G_j, i.e., the differential $G_j'(x) : X \to Y_j$ is surjective and $N[G_j'(x)]$

has a topological complement whenever $G_j(x) = 0$. Moreover, assume that the sets $G_j^{-1}(0)$ $(j \in J)$ are mutually disjoint, that is,

$$G_i^{-1}(0) \cap G_j^{-1}(0) = \emptyset \text{ if } i \neq j.$$

We check that hypothesis (H) is satisfied. In fact, for any $v \in S$, there is a unique $j \in J$ such that $v \in G_j^{-1}(0)$. Let $Y_v = Y_j$ and $A_v = G_j'(v)$. Then we have $R(A_v) = Y_v$ so (4.1) is verified. Moreover, we know that

$$N(A_v) = N\left[G_j'|(v)\right] = T_v\left[G_j^{-1}(0)\right] = T_v S,$$

because $G_j^{-1}(0)$ is a C^1-submanifold of X, which implies (4.2). Consequently, Theorem 4.1 can be applied, ensuring the existence of $p \in Y_j^*$ with the property that whenever $u \in S$ is a solution of (P) with $u \in G_j^{-1}(0)$ then

$$\left[G_j'(u)\right]^* (p) \in \partial\Phi(u). \tag{6.1}$$

In the particular case where J is a singleton, i.e.,

$$S = G^{-1}(0),$$

and G is a mapping from X to $\mathbb{R}$ (also 0 is a regular value of G), then (6.1) becomes

$$\lambda G'(u) \in \partial\Phi(u),$$

for some $\lambda \in \mathbb{R}$. This is the classical Lagrange multiplier rule for locally Lipschitz functionals.

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