The Journal of Supercomputing (2024) 80:24738–24757
https://doi.org/10.1007/s11227-024-06342-z

Link fault tolerability of 3-ary *n*-cube based on *g*-good-neighbor *r*-component edge-connectivity

Qifan Zhang[1] · Shuming Zhou[1,2,3] · Lulu Yang[1]

Accepted: 2 July 2024 / Published online: 29 July 2024

Abstract
High-performance computing relies heavily on parallel and distributed systems, which promptes us to establish both qualitative and quantitative criteria to assess the fault tolerability and vulnerability of the system's underlying interconnection networks. Consider the scenario in which large-scale link failures split the interconnection network into several components and each processor has multiple good neighboring processors. In this scenario, the fault tolerability of the system can be measured by g-good-neighbor r-component edge-connectivity, denoted by $\lambda_{g,r}(G)$, which is defined as the minimum number of edges whose removal results in a disconnected network with at least r connected components and each vertex has at least g good neighbors. It combines the strategies of g-good-neighbor edge-connectivity and component edge-connectivity. In this paper, the g-good-neighbor $(r+1)$-component edge-connectivity of 3-ary n-cube is investigated. This work is the first attempt enhancing link fault tolerability for 3-ary n-cube under double constraints in the presence of the large-scale faulty links, which breaks down the inherent idea that poses one limitation on the resulting network. In addition, our results cover the work of Xu et al. (IEEE Trans Reliab, 71(3):1230–1240, 2022) and Li et al. (J Parallel Distrib Comput, 27:104886, 2024). Finally, the compared results reveal that the g-good-neighbor $(r+1)$-component edge-connectivity is almost r times the size of g-good-neighbor edge-connectivity and much larger than $(r+1)$-component edge-connectivity in 3-ary n-cube.

Keywords Parallel and distributed systems · Link fault tolerability · g-good-neighbor r-component edge-connectivity · 3-ary n-cube

✉ Shuming Zhou
zhoushuming@fjnu.edu.cn

1 School of Mathematics and Statistics, Fujian Normal University, Fuzhou 350117, Fujian, China

2 Center for Applied Mathematics of Fujian Province (Fujian Normal University), Fuzhou, China

3 Key Laboratory of Analytical Mathematics and Applications (Ministry of Education), Fujian Normal University, Fuzhou, China

1 Introduction

Massively parallel computing systems employing hundreds to thousands of processors are commercially available today and offer substantially higher raw computing power than the fastest sequential supercomputers. Availability of such systems has fueled interest in investigating the performance of parallel and distributed systems containing a large number of processors. Parallel computing architectures for large-scale parallel and distributed systems have advanced significantly during the last decades. The interconnection network is well recognized to be critical in large-scale parallel and distributed systems, since its design has a direct impact on the system's performance and cost-effectiveness [3]. Topological structure is generally deemed a key design concern for interconnection networks. A large number of topological structures have been proposed and explored, most notably the hypercube structure and its variants, such as k-ary n-cube, folded hypercube, crossed cube, and so on. The k-ary n-cube, especially the 3-ary n-cube, has attracted a great deal of interest due to its many appealing attributes [11], including its capacity to minimize message latency and ease of implementation. A variety of parallel and distributed systems, including the J-machine [18], the Cray T3D [12], the Cray T3E [2], and the IBM supercomputer BlueGene/L [1], have been constructed with the k-ary n-cube as the underlying topological structure. The IBM supercomputer BlueGene/L's fundamental architecture is the 3-ary n-cube.

With the increasing of network scale and business traffic, interconnection network is facing a lot of challenges, among which fault tolerability is of the most importance. Actually, the network malfunction, especially for link failure, is common and always inevitable. In a large-scale parallel and distributed system, the link failure on the data plane occurs more occasionally and frequently, and some links are likely to fail every half hour on average [9]. Link failure results in the interruption of traffic transmission and further degrades the network performance. Therefore, considering the seriousness and inevitability of link failure, both academia and industry pay great attention to the link fault tolerability.

There are several metrics that characterize the link fault tolerability of interconnection networks, such as classical edge-connectivity, extra edge-connectivity, component edge-connectivity, g-good-neighbor edge-connectivity, cycle edge-connectivity, embedded edge-connectivity, and so on. Among these metrics, extra edge-connectivity, component edge-connectivity and g-good-neighbor edge-connectivity of many prominent networks have been widely investigated over the years. The h-extra edge-connectivity $\lambda_h(G)$ was proposed by Fàbrega et al. [8], which is defined as the minimum number of edges whose removal results in a disconnected network and each component has at least $h+1$ vertices. Many scholars have studied its h-extra edge-connectivity in diverse networks, such as bijective connection networks B_n [28], folded hypercube FQ_n

[29], augmented cube AQ_n [23, 31], 3-ary n-cube Q_n^3 [22], folded crossed cube FCQ_n [19]. The g-good-neighbor edge-connectivity $\lambda^g(G)$ was proposed by Latifi [13], which is defined as the minimum number of edges whose removal results in a disconnected network and each vertex has at least g neighbors. The g-good-neighbor edge-connectivities of many networks, such as bijective connection networks B_n [14], modified bubble-sort networks MB_n [5], augmented cube AQ_n [30], k-ary n-cube [15], have been determined. By the minimality of edge-cuts, the two types of conditional edge-connectivities mentioned above allow only two components to be generated after deleting the smallest conditional edge-cut. Generally speaking, a disconnected network with two components may not be as bad as a disconnected network with many more components. By constraining the number of components of the disconnected network, the r-component edge-connectivity $c\lambda_r(G)$ was proposed by Sampathkumar [20], which is defined as the minimum number of edges whose removal results in a disconnected network with at least r connected components. A lot of related work on specific networks have been studied, including hypercube Q_n [33], locally twisted cube LTQ_n [21], augmented cube AQ_n [31], bijective connection networks B_n [16], 3-ary n-cube Q_n^3 [22], hamming graph K_L^n [25], folded Petersen networks P^n [24]. Note that this metric only puts a limit on the number of components but not on the structure of each component, i.e., it may produce sub-networks that have only one vertex, but for which it is not able to perform any of the tasks assigned by the system.

As a result, in order to establish a balance between the number of components and their structure, Yang et al. [26] and Liu et al. [17] proposed two new concepts, namely h-extra r-component edge-connectivity and g-good-neighbor r-component edge-connectivity, by integrating the strategies of r-component edge-connectivity and one of h-extra edge-connectivity and g-good-neighbor edge-connectivity, respectively. Both emphasize the double constraints on the resulting network after removing the smallest edge-cut. The h-extra r-component edge-connectivity, denoted by $c\lambda_r^h(G)$, is defined as the minimum number of edges whose removal results in a disconnected network with at least r connected components and each component has at least $h+1$ vertices. The g-good-neighbor r-component edge-connectivity, denoted by $\lambda_{g,r}(G)$, is defined as the minimum number of edges whose removal results in a disconnected network with at least r connected components and each vertex has at least g neighbors. In particular, $c\lambda_r^0(G)$ and $\lambda_{0,r}(G)$ reduce to r-component edge-connectivity $c\lambda_r(G)$, while $c\lambda_2^h(G)$ and $\lambda_{g,2}(G)$ degrade into h-extra edge-connectivity $\lambda_h(G)$ and g-good-neighbor edge-connectivity $\lambda^g(G)$, respectively. As applications, Yang et al. [26] explored the h-extra 3-component edge-connectivities of bijective connection networks B_n and folded hypercube FQ_n for specific ranges of h. Moreover, Liu et al. [17] investigated the g-good-neighbor r-component edge-connectivity of hypercube in the same specific ranges with regarded to g and r. For all the related work mentioned above, we present all the results in Table 1 to facilitate the knowledge of the progress of the related work.

Table 1 The summary of the progress of the related work

Network	Value range	Related work and references
B_n	$n \geq 4, 1 \leq h \leq 2^{\lfloor \frac{n}{2} \rfloor +1} - f_1{}^*$ or $\frac{2^{n-1}+2c}{3} \leq h \leq 2^{n-1}{}^\star$	$\lambda_h(B_n)$, $\lambda_{2^{n-1}}(B_n)$ [28]
FQ_n	$n \geq 4, 1 \leq h \leq 2^{\lfloor \frac{n}{2} \rfloor +1} - f_2{}^\ddagger$ or $2^{\lceil \frac{n}{2} \rceil +a} - s_a \leq h \leq 2^{\lceil \frac{n}{2} \rceil +a}{}^\dagger$	$\lambda_h(FQ_n)$, $\lambda_{2^{\lceil \frac{n}{2} \rceil +a}}(FQ_n)$ [29]
AQ_n	$n \geq 4, 1 \leq h \leq 2^{\lfloor \frac{n}{2} \rfloor}$ or $\frac{2^{n-1}+2c}{3} \leq h \leq 2^{n-1}$	$\lambda_h(AQ_n)$, $\lambda_{2^{n-1}}(B_n)$ [23, 31]
Q_n^3	$n \geq 3, 1 \leq h \leq \frac{3^n-1}{2}$	$\lambda_h(Q_n^3)$ [22]
FCQ_n	$n \geq 2, 1 \leq h \leq 2^{n-1}$	$\lambda_h(FCQ_n)$[19]
B_n	$n \geq 1, 0 \leq g \leq n-1$	$\lambda^g(B_n)$ [14]
MB_n	$n \geq 4, 0 \leq g \leq \frac{n}{2}$	$\lambda^g(MB_n)$ [5]
AQ_n	$n \geq 1, 0 \leq g \leq 2t-1, 0 \leq t \leq n-1$	$\lambda^g(AQ_n)$ [30]
Q_n^k	$n \geq 3, 0 \leq g \leq n$	$\lambda^g(Q_n^k)$ [15]
Q_n	$n \geq 7, 1 \leq r \leq 2^{\lfloor \frac{n}{2} \rfloor}$	$c\lambda_{r+1}(Q_n)$ [33]
LTQ_n	$n \geq 7, 1 \leq r \leq 2^{\lfloor \frac{n}{2} \rfloor}$	$c\lambda_{r+1}(LTQ_n)$ [21]
AQ_n	$n \geq 7, 1 \leq r \leq 2^{\lfloor \frac{n}{2} \rfloor}$	$c\lambda_{r+1}(AQ_n)$ [31]
B_n	$n \geq 8, 1 \leq r \leq 2^{\lfloor \frac{n}{2} \rfloor}$	$c\lambda_{r+1}(B_n)$ [16]
Q_n^3	$n \geq 6, 1 \leq r \leq 3^{\lceil \frac{n}{2} \rceil}$	$c\lambda_{r+1}(Q_n^3)$ [22]
K_L^n	$n \geq 7, 1 \leq r \leq L^{\lfloor \frac{n}{2} \rfloor}$	$c\lambda_{r+1}(K_L^n)$ [25]
P^n	$n \geq 2, 1 \leq r \leq 2^{n-1}$	$c\lambda_{r+1}(P^n)$ [24]
Q_n	$n \geq 4, 1 \leq h \leq 2^{\lfloor \frac{n}{2} \rfloor -1}$ or $h = 2^{k_0}, 0 \leq k_0 \leq n-2$	$\lambda_3^h(Q_n)$, $\lambda_3^{2^{k_0}}(Q_n)$ [26]
FQ_n	$n \geq 4, 1 \leq h \leq 2^{\lfloor \frac{n}{2} \rfloor -1}$ or $h = 2^{k_0}, 0 \leq k_0 \leq n-2$	$\lambda_3^h(FQ_n)$,$\lambda_3^{2^{k_0}}(FQ_n)$ [26]
Q_n	$n \geq 4, 1 \leq r \cdot 2^g \leq 2^{\lfloor \frac{n}{2} \rfloor}$ or $r = 2^{k_0}, 0 \leq k_0 \leq \left\lfloor \frac{n}{2} \right\rfloor, 0 \leq g \leq n - 2k_0 - 1$	$\lambda_{g,r+1}(Q_n)$ [17]
Q_n^3	$n \geq 3, 1 \leq (2-d)r \cdot 3^{\lfloor \frac{g}{2} \rfloor} \leq (2-c)3^{\lceil \frac{n}{2} \rceil -1}$ or $r = 3^{k_0}, 0 \leq k_0 \leq \left\lfloor \frac{n}{2} \right\rfloor, 0 \leq g \leq 2(n-2k_0-1)$	$\lambda_{g,r+1}(Q_n^3)$ **our result**

$*$ $f_1 = 2$ if n is odd and $f_1 = 4$ if n is even

$\star$ $c = 1$ if n is odd and $c = 0$ if n is even

$\ddagger$ $f_2 = 4$ if n is odd and $f_2 = 2$ if n is even

$\dagger$ $s_a = \frac{2^{2a}-2}{3}$ if s_a is odd and $s_a = \frac{2^{2a}-1}{3}$ if s_a is even, where $a = 1, 2, \ldots, \lceil \frac{n}{2} \rceil - 1$

In this paper, we focus on the g-good-neighbor $(r+1)$-component edge-connectivity of 3-ary n-cube. More specifically, we obtain $\lambda_{g,r+1}(Q_n^3) = (2-d)(2n-g)r \cdot 3^{\lfloor \frac{g}{2} \rfloor} - [\frac{1}{2} ex_{(2-d)r}(Q_n^3) - (1-d)r] \cdot 3^{\lfloor \frac{g}{2} \rfloor}$ for $n \geq 3$, $1 \leq (2-d)r \cdot 3^{\lfloor \frac{g}{2} \rfloor} \leq (2-c) \cdot 3^{\lceil \frac{n}{2} \rceil -1}$ or $r = 3^{k_0}, 0 \leq k_0 \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$ and $0 \leq g \leq 2(n - 2k_0 - 1)$, where $d \equiv g \pmod 2$, $ex_{(2-d)r}(Q_n^3)$ represents the maximum degree sum of the subgraph induced by $(2-d)r$ vertices of Q_n^3 whose exact value is determined by Xu et al. [22].

The rest of this paper is organized as follows. Section 2 reviews some notations and the structure and properties of the 3-ary n-cube. Section 3 constructs a g-good-neighbor $(r+1)$-component edge-cut. Section 4 determines g-good-neighbor $(r+1)$-component edge-connectivity of 3-ary n-cube. Section 5 presents some examples. Section 6 concludes the work.

2 Preliminaries

First of all, we assume that all parameters are nonnegative integers. Let $G(V(G), E(G))$ be a network or graph, where $|V(G)|$, $|E(G)|$ denote the order and size of G, respectively. For $W \subseteq V(G)(\text{resp.}, E(G))$, $G[W]$ denotes the subgraph of G induced by W, and $G-W$ denotes the subgraph of G induced by $V(G) \setminus W$ (resp., $E(G) \setminus W$). For vertex subsets $T_1, T_2, \dots, T_r \subset V(G)$, we denote by $E(T_1, T_2, \dots, T_r)$ the edge set of G with one vertex in T_i, and the other in T_j, where $T_i \cap T_j = \emptyset, 1 \leq i < j \leq r$. The degree of a vertex v, denoted by $d_G(v)$, is the number of vertices incident to v. We denote by $\delta(G)$ the minimum degree of graph G. A graph G is k-regular if $d_G(v) = k$ for any vertex $v \in V(G)$. The component of G is the maximal connected subgraph of G. The graph G_1 is isomorphic to G_2 which is indicated by the notation $G_1 \cong G_2$. Let S_r be the set $\{0, 1, 2, \dots, r-1\}$, L_l^n be the set $\{t_i^n\}_{i=0}^{l-1}$, where t_i^n is the n-ternary string conversed by the decimal number i. Let $D(u, v)$ be the number of different positions in the n-ternary strings u and v. For graph definitions and notations not defined here, we follow [4].

The definition of 3-ary n-cube is recalled as follows.

Definition 1 [10] The 3-ary n-cube, denoted by Q_n^3, has 3^n vertices, where each vertex has form $\{z_n z_{n-1} \dots z_1 \mid z_i \in S_3, 1 \leq i \leq n\}$. For any two vertices u, v, $(u, v) \in E(Q_n^3)$ if and only if $D(u, v) = 1$.

The 3-ary n-cube Q_n^3 is a $2n$-regular $2n$-connected graph [6]. Q_n^3 can be decomposed into three vertex and edge-disjoint 3-ary $(n-1)$-cubes, denoted by $Q_n^3[0]$, $Q_n^3[1]$, and $Q_n^3[2]$, which are induced by the vertices of Q_n^3 with the *ith* coordinate 0, 1, and 2, respectively. Clearly, $Q_n^3[i]$ and $Q_n^3[j]$ are joined by one perfect matching, so $|E(V(Q_n^3[i]), V(Q_n^3[j]))| = 3^{n-1}$ for $i \neq j \in S_3$. For convenience, we denote Q_n^3 as $Q_n^3[0] \bigoplus Q_n^3[1] \bigoplus Q_n^3[2]$. We denote by $t_l^{n-m} Z^m$ the vertex set

$$\{x_n x_{n-1} \dots x_{m+1} z_m \dots z_2 z_1 \mid z_i \in S_3, 1 \leq i \leq t\},$$

where $x_n x_{n-1} \dots x_{m+1}$ is $(n-m)$-ternary string conversed by the decimal number l, Z represents variable in S_3. By the definition of the Q_n^3, the subgraph $Q_n^3[t_l^{n-m} Z^m] \cong Q_m^3$. The 3-ary 3-cube Q_3^3 is illustrated in Fig. 1.

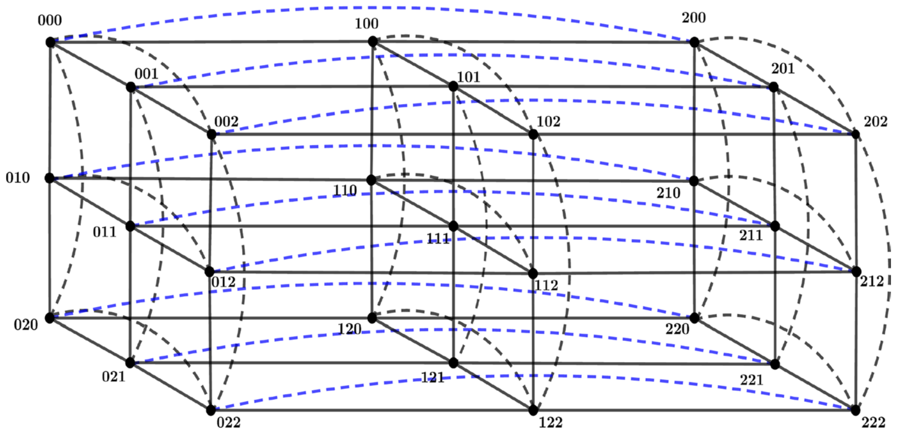


Fig. 1 3-ary 3-cube Q_3^3

Denote by $\frac{ex_l(Q_n^3)}{2}$ the maximum size (the number of edges) of the subgraph induced by a vertex set with a given size l in Q_n^3, where $ex_l(Q_n^3)$ represents the maximum degree sum of the subgraph induced by l vertices of Q_n^3, i.e.,

$$ex_l(Q_n^3) = \max\{2|E(Q_n^3[W])| \mid W \subseteq V(Q_n^3) \text{ and } |W| = l\}.$$

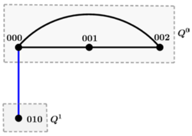
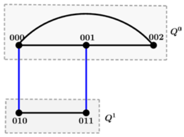
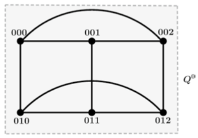
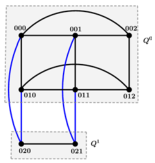
Fan et al. [7] showed that $\frac{ex_l(Q_n^3)}{2}$ can be achieved by the induced subgraph $Q_n^3[L_l^n]$, i.e., $E(Q_n^3[L_l^n]) = \frac{ex_l(Q_n^3)}{2}$. Whereafter, Zhang et al. [32] characterized structural features of subgraph $Q_n^3[L_l^n]$ in accordance with the ternary decomposition of l. Let $l = \sum_{i=0}^{s} a_i 3^{k_i}$ be the ternary decomposition of l such that $k_0 = [\log_3 l]$, $a_0 = [l - 2 \cdot 3^{k_0}]^+ + 1$ and $k_i = [\log_3(l - \sum_{j=0}^{i-1} a_j 3^{k_j})]$, $a_i = [l - \sum_{j=0}^{i-1} a_j 3^{k_j} - 2 \cdot 3^{k_i}]^+ + 1$ for $1 \le i \le s$, where

$$[x]^+ = \begin{cases} 1, & \text{if } x \ge 0; \\ 0, & \text{if } x < 0. \end{cases}$$

In [32], L_l^n can be expressed as $V(Q^0 \cup Q^1 \cup \cdots \cup Q^s)$, where each Q^i is a subgraph induced by vertex set with $a_i 3^{k_i}$ vertices, that is, each Q^i is either 3-ary k_i-subcube or disjoint union of two 3-ary k_i-subcubes connected by a perfect matching. In particular, the single vertex is deemed to 3-ary 0-subcube. In addition, Q^i is taken from a 3-ary n-cube which is obtained from Q^{i-1} by changing the 0 of $(t_i + 1)th$ coordinate of Q^{i-1} to 1 or the 0 and 1 of $(t_i + 1)th$ coordinate of Q^{i-1} to 2 for $1 \le i \le s$. So there exists at least one edge between the vertices in different Q^is. We present some examples in the Table 2 to illustrate this structure.

Motivated by the idea of Fan et al. [7], Xu et al. [22] determined the exact expression of $ex_l(Q_n^3)$ as follows.

Table 2 The relationship between ternary decomposition of l and structure of $Q_3^3[L_l^3]$

Value of l	4	5	6	8
Ternary decomposition of l	$3^1 + 3^0$	$3^1 + 2 \cdot 3^0$	$2 \cdot 3^1$	$2 \cdot 3^1 + 2 \cdot 3^0$
Structure of $Q_3^3[L_l^3]$				

Lemma 1 [22] *Let* $1 \leq l \leq 3^n$ *and* $l = \sum_{i=0}^{s} a_i 3^{k_i}$ *be the ternary decomposition of l. Then, we have* $\frac{1}{2}ex_l(Q_n^3) = |E(Q_n^3[L_l^n])| = \sum_{i=0}^{s}(a_ik_i + a_i - 1)3^{k_i} + \sum_{i=1}^{s}\left(\sum_{j=0}^{i-1} a_j\right) a_i 3^{k_i}$.

3 Construction of *g*-good-neighbor $(r+1)$-component edge-cut

In this section, we will construct a g-good-neighbor $(r+1)$-component edge-cut in the 3-ary n-cube.

Lemma 2 [32] *For two positive integers n, l, let* $\overline{L_l^n} = V(Q_n^3) \setminus L_l^n$. *Both* $Q_n^3[L_l^n]$ *and* $Q_n^3[\overline{L_l^n}]$ *are connected.*

3.1 $g = 0 \pmod 2$

First of all, we define some notations. Let n, g, r be three integers such that $(r+1) \cdot 3^{\frac{g}{2}} \leq 3^n$. For any $i \in S_r$, let $T_i = t_i^{n-\frac{g}{2}} Z^{\frac{g}{2}}$ and $G_i = Q_n^3[T_i]$. Let $T_r = V(Q_n^3) - \bigcup_{i \in S_r} T_i$ and $G_r = Q_n^3[T_r]$. Obviously, $G_i \cong Q_{\frac{g}{2}}^3$ for any $i \in S_r$ and $T_i \cap T_j = \emptyset$ for $i \neq j \in S_r$. We contract each T_i into a vertex t_i for $i \in S_r$ and delete the multiple edges between T_i and T_j when $E(T_i, T_j) \neq \emptyset$ for $i \neq j \in S_r$. Then the graph G^* induced by $\{t_0, t_1, \ldots, t_{r-1}\}$ is isomorphic to $Q_n^3[L_r^n]$. Note that $(t_i, t_j) \in E(G^*)$ if and only if $|E(T_i, T_j)| = 3^{\frac{g}{2}}$, hence, $|E(T_0, T_1, \ldots, T_{r-1})| = |E(G^*)| \cdot 3^{\frac{g}{2}} = \frac{1}{2}ex_r(Q_n^3) \cdot 3^{\frac{g}{2}}$.

Lemma 3 *If* $g = 0 \pmod 2$, *then*

$$ex_{r \cdot 3^{\frac{g}{2}}}(Q_n^3) = r \cdot ex_{3^{\frac{g}{2}}}(Q_n^3) + ex_r(Q_n^3) \cdot 3^{\frac{g}{2}}.$$

Proof Let T_i be defined as above for $i \in S_r$. In light of definition of T_i, we have $\bigcup_{i\in S_r} T_i = L^n_{r\cdot 3^{\frac{g}{2}}}$. By Lemma 1, $2|E(Q_n^3[\bigcup_{i\in S_r} T_i])| = ex_{r\cdot 3^{\frac{g}{2}}}(Q_n^3)$. Therefore,

$$\begin{aligned} ex_{r\cdot 3^{\frac{g}{2}}}(Q_n^3) &= 2|E(Q_n^3[\bigcup_{i\in S_r} T_i])| \\ &= 2\sum_{i=0}^{r-1} |E(Q_n^3[T_i])| + 2|E(T_0, T_1, \dots, T_{r-1})| \\ &= r \cdot ex_{3^{\frac{g}{2}}}(Q_n^3) + ex_r(Q_n^3) \cdot 3^{\frac{g}{2}}. \end{aligned}$$

□

Lemma 4 *For any $i \in S_{r+1}$, the subgraph G_i is connected and $\delta(G_i) \geq g$.*

Proof For any $i \in S_r$, $G_i \cong Q^3_{\frac{g}{2}}$, thereby, G_i is connected and $\delta(G_i) \geq g$. In light of definition of T_i, we have $\bigcup_{i\in S_r} T_i = L^n_{r\cdot 3^{\frac{g}{2}}}$ and $T_r = \overline{L^n_{r\cdot 3^{\frac{g}{2}}}}$. By Lemma 2, G_r is connected. Let $r = \sum_{j=0}^{s} a_j 3^{k_j}$ be the ternary decomposition of r, $k_j + \frac{g}{2} = l_j$. Then,

$$r' = 3^n - r \cdot 3^{\frac{g}{2}} = 3^n - \sum_{j=0}^{s} a_j 3^{l_j} = b_0 3^{n-1} + b_1 3^{n-2} + \dots + b_h 3^{n-1-h},$$

where $n-1-h = l_s, b_h = 3 - a_s$ and

$$b_i = \begin{cases} 2 - a_j, & \text{if } l_j = n-1-i; \\ 2, & \text{otherwise,} \end{cases}$$

for $0 \leq i \leq h-1$. According to the above expression of r', $b_i = 0$ for some i's. Therefore, r' can be rephrased as $r' = \sum_{i=0}^{h'} b'_i 3^{q_i}$, where $q_0 > q_1 > \dots > q_{h'} = l_s$ and $b'_{h'} = 3 - a_s$. According to the construction of G_r, each vertex in G_r falls in some $Q^3_{q_i}$. In view of $q_i \geq l_s = k_s + \frac{g}{2}$ for any $i \in S_{h'+1}$, $\delta(G_r) \geq g$. □

Lemma 5 *Let n, g, r be three integers such that $(r+1)\cdot 3^{\frac{g}{2}} \leq 3^n$, where $g = 0 \pmod 2$. Then*

$$\lambda_{g,r+1}(Q_n^3) \leq (2n-g)r \cdot 3^{\frac{g}{2}} - \frac{1}{2} ex_r(Q_n^3) \cdot 3^{\frac{g}{2}}.$$

Proof We prove this lemma by constructing a g-good-neighbor $(r+1)$-component edge-cut with size $(2n-g)r \cdot 3^{\frac{g}{2}} - \frac{1}{2} ex_r(Q_n^3) \cdot 3^{\frac{g}{2}}$. Suppose that G_i and T_i for any $i \in S_{r+1}$ are defined as above. Then, by Lemma 4, $Q_n^3 - E(T_0, T_1, \dots, T_r)$ is disconnected and has exactly $r+1$ components $G_0, G_1, \dots, G_r$ and $\delta(G_i) \geq g$ for any $i \in S_{r+1}$. Moreover, $|E(T_0, T_1, \dots, T_{r-1})| = |E(G^*)| \cdot 3^{\frac{g}{2}} = \frac{1}{2} ex_r(Q_n^3) \cdot 3^{\frac{g}{2}}$. Therefore, we have

$$\begin{aligned}|E(T_0, T_1, \dots, T_r)| &= \sum_{i=0}^{r-1} |E(T_i, \overline{T_i})| - |E(T_0, T_1, \dots, T_{r-1})| \\ &= r(2n-g)3^{\frac{g}{2}} - \frac{1}{2} ex_r(Q_n^3) \cdot 3^{\frac{g}{2}}.\end{aligned}$$

Thus, the edge set $E(T_0, T_1, \dots, T_r)$ is a g-good-neighbor $(r+1)$-component edge-cut with size $(2n-g)r \cdot 3^{\frac{g}{2}} - \frac{1}{2} ex_r(Q_n^3) \cdot 3^{\frac{g}{2}}$. By the definition of $\lambda_{g,r+1}(Q_n^3)$, we have $\lambda_{g,r+1}(Q_n^3) \le (2n-g)r \cdot 3^{\frac{g}{2}} - \frac{1}{2} ex_r(Q_n^3) \cdot 3^{\frac{g}{2}}$. □

Algorithm 1 Find one perfect matching in $Q_n^3[L_{2r}^n]$

```
Input: r, L^n_{2r} = {t^n_j}_{j=0}^{2r-1}.
Output: a perfect matching M.
Let M = ∅, a = [log_3 2r], S = S_{2r};
if r = 1 then
  M = M ∪ (t^n_0, t^n_1);
else
  M = M ∪ (t^n_0, t^n_{3^a});
  S = S − {0, 3^a};
  While S ≠ ∅ do
    Choose the minimum number w in S;
    Let w = a_0 3^{k_0} + a_1 3^{k_1} + ··· + a_s 3^{k_s} be the ternary decomposition of w;
    Select the maximum number l in {k_s, k_s − 1, ..., 0} such that w + 3^l ∈ S;
    M = M ∪ (t^n_w, t^n_{w+3^l});
    S = S − {w, w + 3^l};
  end While
end if
return M.
```

3.2 $g = 1 \pmod 2$

Similarly, we define some notations. Let n, g, r be three integers such that $2(r+1) \cdot 3^{\lfloor \frac{g}{2} \rfloor} \le 3^n$. For any $i \in S_{2r}$, let $T_i = t_i^{n-\lfloor \frac{g}{2} \rfloor} Z^{\lfloor \frac{g}{2} \rfloor}$. Obviously, $T_i \cap T_j = \emptyset$ for $i \neq j \in S_{2r}$. We contract each T_i into a vertex t_i for $i \in S_{2r}$ and delete the multiple edges between T_i and T_j when $E(T_i, T_j) \neq \emptyset$ for $i \neq j \in S_{2r}$. Then the graph G^* induced by $\{t_0, t_1, \dots, t_{2r-1}\}$ is isomorphic to $Q_n^3[L_{2r}^n]$. Note that $(t_i, t_j) \in E(G^*)$ if and only if $|E(T_i, T_j)| = 3^{\lfloor \frac{g}{2} \rfloor}$, so a perfect matching in $Q_n^3[L_{2r}^n]$ corresponds to a paired-partitiom of $\{T_0, T_1, \dots, T_{2r-1}\}$ such that each pairwise paired induced subgraph is isomorphic to $Q^3_{\lfloor \frac{g}{2} \rfloor+1}[0] \oplus Q^3_{\lfloor \frac{g}{2} \rfloor+1}[1]$.

Theorem 1 *Algorithm* 1 *outputs one perfect matching M in* $Q^3_n[L^n_{2r}]$.

Proof By Algorithm 1, it is easy to see that M covers all vertices in L^n_{2r}. Next, it suffices to show that for each $(x, y) \in M$, $(x, y) \in E(Q^3_n[L^n_{2r}])$. Obviously, $D(t^n_0, t^n_1) = 1$ and $D(t^n_0, t^n_{3^a}) = 1$. By the definitions of Q^3_n and t^n_j, if $r = 1$, then $(t^n_0, t^n_1) \in E(Q^3_n[L^n_{2r}])$, while if $r > 1$, $(t^n_0, t^n_{3^a}) \in E(Q^3_n[L^n_{2r}])$. According to the lines 7 to 13 in Algorithm 1 and choice of w, if $l = k_s$, then $a_s \neq 2$. Thus, $D(t^n_w, t^n_{w+3^l}) = 1$, which implies that $(t^n_w, t^n_{w+3^l}) \in E(Q^3_n[L^n_{2r}])$. □

For example, let $n = 3$ and $r = 8$, then by Algorithm 1, we can find one perfect matchinig in $Q^3_3[L^3_{16}]$, that is $\{(000, 100), (001, 002), (010, 020), (011, 012), (021, 022), (101, 102), (110, 120), (111, 112)\}$ (see Fig. 2).

By Theorem 1, there are r pairs of subgraphs $G_0, G_1, \dots, G_{r-1}$ such that $G_j \cong Q^3_{\lfloor \frac{g}{2} \rfloor+1}[0] \oplus Q^3_{\lfloor \frac{g}{2} \rfloor+1}[1]$, for any $j \in S_r$. Let $T_{2r} = V(Q^3_n) - \bigcup_{i \in S_{2r}} T_i$ and $G_r = Q^3_n[T_{2r}]$.

Lemma 6 *If* $g = 1 \pmod 2$, *then*

$$ex_{2r\cdot 3^{\lfloor \frac{g}{2} \rfloor}}(Q^3_n) = r \cdot ex_{2\cdot 3^{\lfloor \frac{g}{2} \rfloor}}(Q^3_n) + (\frac{1}{2} ex_{2r}(Q^3_n) - r) \cdot 2 \cdot 3^{\lfloor \frac{g}{2} \rfloor}.$$

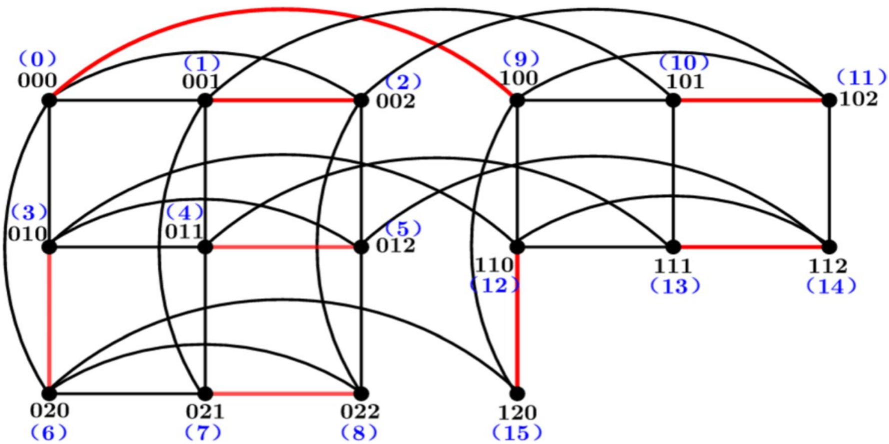


Fig. 2 perfect matching in $Q^3_3[L^3_{16}]$

Proof Let T_i and G_j be defined as above for any $i \in S_{2r}$ and $j \in S_r$, respectively. In light of definition of T_i, we have $\bigcup_{i\in S_{2r}} T_i = L^n_{2r\cdot 3^{\lfloor \frac{g}{2} \rfloor}}$. By Lemma 1, $2|E(Q_n^3[\bigcup_{i\in S_{2r}} T_i])| = ex_{2r\cdot 3^{\lfloor \frac{g}{2} \rfloor}}(Q_n^3)$. Therefore,

$$\begin{aligned} ex_{2r\cdot 3^{\lfloor \frac{g}{2} \rfloor}}(Q_n^3) &= 2|E(Q_n^3[\bigcup_{i\in S_{2r}} T_i])| \\ &= 2\sum_{j=0}^{r-1} |E(G_j)| + 2|E(V(G_0), V(G_1), \dots, V(G_{r-1}))| \\ &= r \cdot ex_{2\cdot 3^{\lfloor \frac{g}{2} \rfloor}}(Q_n^3) + (\frac{1}{2}ex_{2r}(Q_n^3) - r) \cdot 2 \cdot 3^{\lfloor \frac{g}{2} \rfloor}. \end{aligned}$$

□

Lemma 7 *For any $i \in S_{r+1}$, the subgraph G_i is connected and $\delta(G_i) \geq g$.*

Proof For any $i \in S_r$, $G_i \cong Q^3_{\lfloor \frac{g}{2} \rfloor+1}[0] \oplus Q^3_{\lfloor \frac{g}{2} \rfloor+1}$ [1], thereby, G_i is connected and $\delta(G_i) \geq g$. In light of definition of T_i, we have $\bigcup_{i\in S_{2r}} T_i = L^n_{2r\cdot 3^{\lfloor \frac{g}{2} \rfloor}}$ and $T_{2r} = \overline{L^n_{2r\cdot 3^{\lfloor \frac{g}{2} \rfloor}}}$. By Lemma 2, G_r is connected. Let $2r = \sum_{j=0}^{s} a_j 3^{k_j}$ be the ternary decomposition of $2r$, $k_j + \left\lfloor \frac{g}{2} \right\rfloor = l_j$. Then,

$$r' = 3^n - 2r \cdot 3^{\lfloor \frac{g}{2} \rfloor} = 3^n - \sum_{j=0}^{s} a_j 3^{l_j} = b_0 3^{n-1} + b_1 3^{n-2} + \dots + b_h 3^{n-1-h},$$

where $n - 1 - h = l_s, b_h = 3 - a_s$ and

$$b_i = \begin{cases} 2 - a_j, & \text{if } l_j = n - 1 - i; \\ 2, & \text{otherwise,} \end{cases}$$

for $0 \leq i \leq h - 1$. According to the above expression of r', $b_i = 0$ for some i's. Therefore, r' can be rephrased as $r' = \sum_{i=0}^{h'} b_i' 3^{q_i}$, where $q_0 > q_1 > \dots > q_{h'} = l_s$ and $b'_{h'} = 3 - a_s$. According to the construction of G_r, each vertex in G_r falls in some $Q^3_{q_i}$. Because of $2(r + 1) \cdot 3^{\lfloor \frac{g}{2} \rfloor} \leq 3^n$, $|G_r| \geq 2 \cdot 3^{\lfloor \frac{g}{2} \rfloor}$. If $q_{h'} = l_s = \left\lfloor \frac{g}{2} \right\rfloor$, then each vertex in $Q^3_{q_{h'}}$ has at least one neighbor outside $Q^3_{q_{h'}}$, and so $\delta(G_r) \geq g$. If $q_{h'} = l_s > \left\lfloor \frac{g}{2} \right\rfloor$, then $\delta(G_r) \geq g$ in view of $q_i \geq l_s > \left\lfloor \frac{g}{2} \right\rfloor$ for any $i \in S_{h'+1}$. □

Lemma 8 *Let n, g, r be three integers such that $2(r + 1) \cdot 3^{\lfloor \frac{g}{2} \rfloor} \leq 3^n$, where $g = 1 \pmod 2$. Then*

$$\lambda_{g,r+1}(Q_n^3) \leq 2(2n - g)r \cdot 3^{\lfloor \frac{g}{2} \rfloor} - \left(\frac{1}{2}ex_{2r}(Q_n^3) - r\right) \cdot 3^{\lfloor \frac{g}{2} \rfloor}.$$

Proof We prove this lemma by constructing a g-good-neighbor $(r+1)$-component edge-cut with size $2(2n-g)r \cdot 3^{\lfloor \frac{g}{2} \rfloor} - (\frac{1}{2}ex_{2r}(Q_n^3) - r) \cdot 3^{\lfloor \frac{g}{2} \rfloor}$. Suppose that G_i is defined as above for any $i \in S_{r+1}$. Then, by Lemma 7, $Q_n^3 - E(V(G_0), V(G_1), \dots, V(G_r))$ is disconnected and has exactly $r+1$ components $G_0, G_1, \dots, G_r$ and $\delta(G_i) \geq g$ for any $i \in S_{r+1}$. Moreover, $|E(V(G_0), V(G_1), \dots, V(G_{r-1}))| = (\frac{1}{2}ex_{2r}(Q_n^3) - r) \cdot 3^{\lfloor \frac{g}{2} \rfloor}$. Therefore, we have

$$
\begin{aligned}
|E(V(G_0), V(G_1), \dots, V(G_r))| &= \sum_{i=0}^{r-1} |E(V(G_i), \overline{V(G_i)})| \\
&\quad - |E(V(G_0), V(G_1), \dots, V(G_{r-1}))| \\
&= 2r(2n-g)3^{\lfloor \frac{g}{2} \rfloor} - \left(\frac{1}{2}ex_{2r}(Q_n^3) - r\right) \cdot 3^{\lfloor \frac{g}{2} \rfloor}.
\end{aligned}
$$

Thus, the edge set $E(V(G_0), V(G_1), \dots, V(G_r))$ is a g-good-neighbor $(r+1)$-component edge-cut with size $2(2n-g)r \cdot 3^{\lfloor \frac{g}{2} \rfloor} - (\frac{1}{2}ex_{2r}(Q_n^3) - r) \cdot 3^{\lfloor \frac{g}{2} \rfloor}$. By the definition of $\lambda_{g,r+1}(Q_n^3)$, we have $\lambda_{g,r+1}(Q_n^3) \leq 2(2n-g)r \cdot 3^{\lfloor \frac{g}{2} \rfloor} - (\frac{1}{2}ex_{2r}(Q_n^3) - r) \cdot 3^{\lfloor \frac{g}{2} \rfloor}$. □

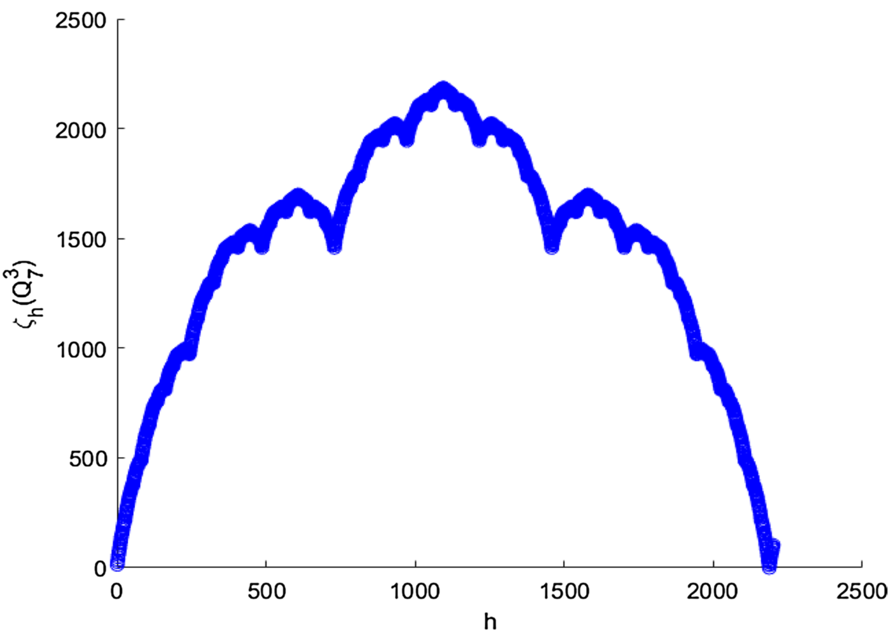


Fig. 3 Image of $\zeta_h(Q_7^3)$

4 *g*-good-neighbor $(r+1)$-component edge-connectivity

Define

$$\zeta_h(Q_n^3) = \min\{|[W,\overline{W}]| \mid W \subset V(G), |W| = h, \text{ both } Q_n^3[W] \text{ and } Q_n^3[\overline{W}] \text{ are connected}\}.$$

As Q_n^3 is $2n$-regular, by the definition of $ex_h(Q_n^3)$, we have

$$\zeta_h(Q_n^3) = 2nh - ex_h(Q_n^3).$$

In this section, we will endeavor to establish the lower bound of g-good-neighbor $(r+1)$-component edge-connectivity in Q_n^3. In this process, we observe that the value of the lower bound is closely related to the function $\zeta_h(Q_n^3)$. More specifically, it relies heavily on the monotonicity of $\zeta_h(Q_n^3)$. Figure 3 depicts the image of the function $\zeta_h(Q_7^3)$, it possesses a fractal structure and symmetry. In order to determine the exact value of $\lambda_{g,r+1}(Q_n^3)$, based on a series of results by Xu et al. [22] and Zhang et al. [32] concerning the properties of the function $\zeta_h(Q_n^3)$, we consider only two special cases that $1 \le (2-d)r \cdot 3^{\lfloor \frac{g}{2} \rfloor} \le (2-c) \cdot 3^{\lceil \frac{n}{2} \rceil - 1}$ and $r = 3^{k_0}, 0 \le k_0 \le \left\lfloor \frac{n}{2} \right\rfloor - 1,\ 0 \le g \le 2(n - 2k_0 - 1)$. In what follows, we review some properties of the function $\zeta_h(Q_n^3)$.

We define

$$c = \begin{cases} 1, & \text{if } n = 1 \ (\text{mod} 2); \\ 0, & \text{if } n = 0 \ (\text{mod} 2), \end{cases}$$

and

$$d = \begin{cases} 1, & \text{if } g = 0 \ (\text{mod} 2); \\ 0, & \text{if } g = 1 \ (\text{mod} 2). \end{cases}$$

Lemma 9 [22] *$\zeta_h(Q_n^3)$ is increasing with respect to h in the interval* $[1, (2-c) \cdot 3^{\lceil \frac{n}{2} \rceil} - 2]$.

Lemma 10 [22] *If $(2-c) \cdot 3^{\lceil \frac{n}{2} \rceil} - 3 \le h \le 3^{\lceil \frac{n+2-c}{2} \rceil}$ for $n \ge 3$, then $\zeta_h(Q_n^3) \ge \zeta_{(2-c)\cdot 3^{\lceil \frac{n}{2} \rceil}}(Q_n^3)$. In particular, $\zeta_{(2-c)\cdot 3^{\lceil \frac{n}{2} \rceil}}(Q_n^3) = \zeta_{(2-c)\cdot 3^{\lceil \frac{n}{2} \rceil} - 3}(Q_n^3)$.*

Lemma 11 [22, 32] *If $3^k \le h \le 2 \cdot 3^{n-1}$ for $0 \le k \le n-1$ and $n \ge 3$, then $\zeta_h(Q_n^3) \ge \zeta_{3^k}(Q_n^3)$. If $2 \cdot 3^k \le h \le 3^{n-1}$ for $0 \le k \le n-2$ and $n \ge 3$, then $\zeta_h(Q_n^3) \ge \zeta_{2\cdot 3^k}(Q_n^3)$.*

Lemma 12 [27] *For $n \ge 3$ and $0 \le g \le 2n$, if H is a connected subgraph in Q_n^3 with $\delta(H) \ge g$, then $|V(H)| \ge (2-d)3^{\lfloor \frac{g}{2} \rfloor}$.*

Lemma 13 *For* $n \geq 3$, *if* $1 \leq (2-d)r \cdot 3^{\lfloor \frac{g}{2} \rfloor} \leq (2-c) \cdot 3^{\lceil \frac{n}{2} \rceil - 1}$ *or* $r = 3^{k_0}, 0 \leq k_0 \leq \lceil \frac{n}{2} \rceil - 1$ *and* $0 \leq g \leq 2(n - 2k_0 - 1)$, *then*

$$\lambda_{g,r+1}(Q_n^3) \geq (2-d)(2n-g)r \cdot 3^{\lfloor \frac{g}{2} \rfloor} - [\frac{1}{2} ex_{(2-d)r}(Q_n^3) - (1-d)r] \cdot 3^{\lfloor \frac{g}{2} \rfloor}.$$

Proof Let F be minimum g-good-neighbor $(r+1)$-component edge-cut of Q_n^3. By the minimality of F, $Q_n^3 - F$ has exactly $r+1$ components, denoted by $H_0, H_1, \ldots, H_r$, satisfying that the minimum degree of each component is at least g. Let $|V(H_i)| = p_i$ for $i \in S_{r+1}$. By Lemma 12, we have $p_i \geq (2-d) \cdot 3^{\lfloor \frac{g}{2} \rfloor}$. Without loss of generality, suppose that $p_0 \geq p_1 \geq \ldots \geq p_r$. Note that $\lceil \frac{3^n}{r+1} \rceil \leq p_0 \leq 3^n - r \cdot p_r$, we have

$$(2-d)r \cdot 3^{\lfloor \frac{g}{2} \rfloor} \leq r \cdot p_r \leq \sum_{i=1}^{r} p_i \leq 3^n - \left\lceil \frac{3^n}{r+1} \right\rceil = \left\lfloor \frac{r \cdot 3^n}{r+1} \right\rfloor.$$

Therefore,

$$\begin{aligned}
|F| &= \sum_{i=1}^{r} |E(V(H_i), \overline{V(H_i)})| - |E(V(H_1), V(H_2), \ldots, V(H_r))| \\
&= 2n \sum_{i=1}^{r} p_i - 2 \sum_{i=1}^{r} |E(H_i)| - \left(\left| E\left(\bigcup_{i=1}^{r} H_i \right) \right| - \sum_{i=1}^{r} |E(H_i)| \right) \\
&= 2n \sum_{i=1}^{r} p_i - \sum_{i=1}^{r} |E(H_i)| - \left| E\left(\bigcup_{i=1}^{r} H_i \right) \right| \\
&\geq 2n \sum_{i=1}^{r} p_i - \frac{1}{2} \sum_{i=1}^{r} ex_{p_i}(Q_n^3) - \frac{1}{2} ex_{\sum_{i=1}^{r} p_i}(Q_n^3) \\
&= \frac{1}{2} \sum_{i=1}^{r} \zeta_{p_i}(Q_n^3) + \frac{1}{2} \zeta_{\sum_{i=1}^{r} p_i}(Q_n^3) \\
&\geq \frac{1}{2} \sum_{i=1}^{r} \zeta_{(2-d) \cdot 3^{\lfloor \frac{g}{2} \rfloor}}(Q_n^3) + \frac{1}{2} \zeta_{\sum_{i=1}^{r} p_i}(Q_n^3) \text{ (by Lemma 11)} \\
&= \frac{1}{2}(2-d)(2n-g)r \cdot 3^{\lfloor \frac{g}{2} \rfloor} + \frac{1}{2} \zeta_{\sum_{i=1}^{r} p_i}(Q_n^3).
\end{aligned}$$

Case 1. $(2-d)r \cdot 3^{\lfloor \frac{g}{2} \rfloor} \leq \sum_{i=1}^{r} p_i \leq 2 \cdot 3^{n-1}$.

In this scenario, by iteratively using Lemmas 9, 10 and 11, we have

$$\zeta_{\sum_{i=1}^{r} p_i}(Q_n^3) \geq \zeta_{(2-d)r \cdot 3^{\lfloor \frac{g}{2} \rfloor}}(Q_n^3) = 2n(2-d)r \cdot 3^{\lfloor \frac{g}{2} \rfloor} - ex_{(2-d)r \cdot 3^{\lfloor \frac{g}{2} \rfloor}}(Q_n^3).$$

Therefore,

$$|F| \geq \frac{1}{2}(2-d)(2n-g)r \cdot 3^{\lfloor \frac{g}{2} \rfloor} + \frac{1}{2}[2n(2-d)r \cdot 3^{\lfloor \frac{g}{2} \rfloor} - ex_{(2-d)r \cdot 3^{\lfloor \frac{g}{2} \rfloor}}(Q_n^3)]$$
$$= (2-d)(2n-g)r \cdot 3^{\lfloor \frac{g}{2} \rfloor} - [\frac{1}{2}ex_{(2-d)r}(Q_n^3) - (1-d)r] \cdot 3^{\lfloor \frac{g}{2} \rfloor}.$$

Case 2. $2 \cdot 3^{n-1} < \sum_{i=1}^{r} p_i \leq \left\lfloor \frac{r \cdot 3^n}{r+1} \right\rfloor$.

Case 2.1. $1 \leq (2-d)r \cdot 3^{\lfloor \frac{g}{2} \rfloor} \leq (2-c) \cdot 3^{\lceil \frac{n}{2} \rceil - 1}$.

In this scenario, we have

$$(2-c) \cdot 3^{\lceil \frac{n}{2} \rceil - 1} \leq \left\lceil \frac{3^n}{\frac{1}{(2-d)}(2-c) \cdot 3^{\lceil \frac{n}{2} \rceil - 1 - \lfloor \frac{g}{2} \rfloor} + 1} \right\rceil \leq \left\lceil \frac{3^n}{r+1} \right\rceil \leq 3^n - \sum_{i=1}^{r} p_i \leq 3^{n-1}.$$

By the symmetry of edge-cut, we have $\zeta_{\sum_{i=1}^{r} p_i}(Q_n^3) = \zeta_{3^n - \sum_{i=1}^{r} p_i}(Q_n^3)$. By Lemmas 9 and 11,

$$\zeta_{3^n - \sum_{i=1}^{r} p_i}(Q_n^3) \geq \zeta_{(2-d)r \cdot 3^{\lfloor \frac{g}{2} \rfloor}}(Q_n^3) = 2n(2-d)r \cdot 3^{\lfloor \frac{g}{2} \rfloor} - ex_{(2-d)r \cdot 3^{\lfloor \frac{g}{2} \rfloor}}(Q_n^3)$$

holds. Therefore, by Lemmas 3 and 6, we have

$$|F| \geq (2-d)(2n-g)r \cdot 3^{\lfloor \frac{g}{2} \rfloor} - [\frac{1}{2}ex_{(2-d)r}(Q_n^3) - (1-d)r] \cdot 3^{\lfloor \frac{g}{2} \rfloor}.$$

Case 2.2. $r = 3^{k_0}$ and $0 \leq g \leq 2(n - 2k_0 - 1)$, where $0 \leq k_0 \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$.

In this scenario, we have

$$(2-d)r \cdot 3^{\lfloor \frac{g}{2} \rfloor} \leq 2 \cdot 3^{n-k_0-1} \leq \left\lceil \frac{3^n}{3^{k_0}+1} \right\rceil \leq \left\lceil \frac{3^n}{r+1} \right\rceil \leq 3^n - \sum_{i=1}^{r} p_i \leq 3^{n-1}.$$

By the symmetry of edge-cut, we have $\zeta_{\sum_{i=1}^{r} p_i}(Q_n^3) = \zeta_{3^n - \sum_{i=1}^{r} p_i}(Q_n^3)$. By Lemma 11,

$$\zeta_{3^n - \sum_{i=1}^{r} p_i}(Q_n^3) \geq \zeta_{(2-d)r \cdot 3^{\lfloor \frac{g}{2} \rfloor}}(Q_n^3) = 2n(2-d)r \cdot 3^{\lfloor \frac{g}{2} \rfloor} - ex_{(2-d)r \cdot 3^{\lfloor \frac{g}{2} \rfloor}}(Q_n^3)$$

holds. Therefore, by Lemmas 3 and 6, we have

$$|F| \geq (2-d)(2n-g)r \cdot 3^{\lfloor \frac{g}{2} \rfloor} - [\frac{1}{2}ex_{(2-d)r}(Q_n^3) - (1-d)r] \cdot 3^{\lfloor \frac{g}{2} \rfloor}.$$

Summing up above, we have

$$\lambda_{g,r+1}(Q_n^3) \geq (2-d)(2n-g)r \cdot 3^{\lfloor \frac{g}{2} \rfloor} - \left[\frac{1}{2}ex_{(2-d)r}(Q_n^3) - (1-d)r\right] \cdot 3^{\lfloor \frac{g}{2} \rfloor}.$$

□

Combining Lemma 13 with Lemmas 5 and 8, the following result is immediately obtained.

Theorem 2 *Let n, g, r be three integers such that $n \geq 3$, $1 \leq (2-d)r \cdot 3^{\lfloor \frac{g}{2} \rfloor} \leq (2-c) \cdot 3^{\lceil \frac{n}{2} \rceil - 1}$ or $r = 3^{k_0}, 0 \leq k_0 \leq \lfloor \frac{n}{2} \rfloor - 1$ and $0 \leq g \leq 2(n - 2k_0 - 1)$. Then,*

$$\lambda_{g,r+1}(Q_n^3) = \begin{cases} (2n-g)r \cdot 3^{\frac{g}{2}} - \frac{1}{2}ex_r(Q_n^3) \cdot 3^{\frac{g}{2}}, & \text{if } g = 0 \text{ (mod2)}; \\ 2(2n-g)r \cdot 3^{\lfloor \frac{g}{2} \rfloor} - (\frac{1}{2}ex_{2r}(Q_n^3) - r) \cdot 3^{\lfloor \frac{g}{2} \rfloor}, & \text{if } g = 1 \text{ (mod2)}, \end{cases}$$

Corollary 1 [15, 22] (1) *For $n \geq 2$ and $g \leq n$, $\lambda^g(Q_n^3) = (2-d)(2n-g)3^{\lfloor \frac{g}{2} \rfloor}$.*

(2) *For $n \geq 6$ and $r \leq 3^{\lceil \frac{n}{2} \rceil}$, $c\lambda_{r+1}(Q_n^3) = 2nr - \frac{ex_r(Q_n^3)}{2}$.*

5 Application and comparisons

This work concentrates on the g-good-neighbor $(r+1)$-component edge-connectivity under the conditions $1 \leq (2-d)r \cdot 3^{\lfloor \frac{g}{2} \rfloor} \leq (2-c) \cdot 3^{\lceil \frac{n}{2} \rceil - 1}$ or $r = 3^{k_0}, 0 \leq k_0 \leq \lfloor \frac{n}{2} \rfloor - 1$ and $0 \leq g \leq 2(n - 2k_0 - 1)$. In what follows, we present an example in each of these two cases. For instance, let $r = 2$ and $n = 10$. In this case, $r = 2$ satisfies the former condition, hence, we have $0 \leq g \leq 8$. Based on the formulas of Theorem 2 and Corollary 1, we derive the corresponding values of $\lambda_{g,3}(Q_{10}^3)$, $\lambda^g(Q_{10}^3)$ and $c\lambda_{g+1}(Q_{10}^3)$ with respect to $0 \leq g \leq 8$ (see Table 3). Another example is $r = 3$ and $n = 9$. In this case, $r = 3$ satisfies the latter condition, hence, we deduce that $0 \leq g \leq 12$. Subsequently, we obtain the corresponding values of $\lambda_{g,4}(Q_9^3)$, $\lambda^g(Q_9^3)$ and $c\lambda_{g+1}(Q_9^3)$ with respect to $0 \leq g \leq 12$ in the same manner (see Table 3). As can be seen from Table 3, the value of $\lambda_{g,3}(Q_{10}^3)$ is almost twice as large

Table 3 The value of $\lambda_{g,r+1}(Q_n^3)$, $\lambda^g(Q_n^3)$ and $c\lambda_{g+1}(Q_n^3)$

g	$\lambda_{g,3}(Q_{10}^3)$	$\lambda^g(Q_{10}^3)$	$c\lambda_{g+1}(Q_{10}^3)$	$\lambda_{g,4}(Q_9^3)$	$\lambda^g(Q_9^3)$	$c\lambda_{g+1}(Q_9^3)$
0	39	20	0	51	18	0
1	74	38	20	96	34	18
2	105	54	39	135	48	35
3	198	102	57	252	90	51
4	279	144	76	351	126	68
5	522	270	94	648	234	84
6	729	378	111	891	324	99
7	1350	702	129	1620	594	115
8	1863	972	146	2187	810	130
9	–	–	–	3888	1458	144
10	–	–	–	5103	1944	161
11	–	–	–	8748	3402	177
12	–	–	–	10935	4374	192

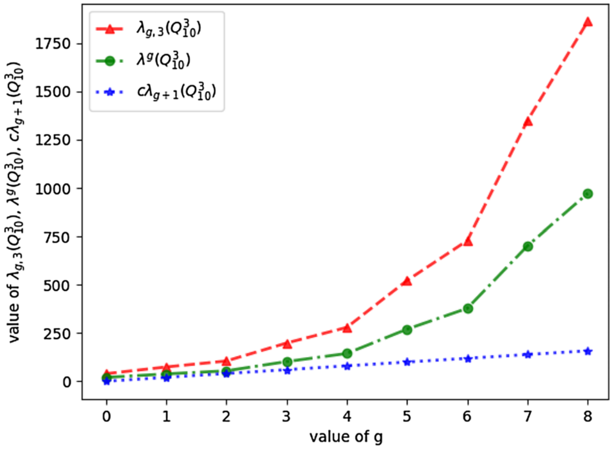


Fig. 4 Illustration that $r = 2$, $n = 10$ and $0 \leq g \leq 8$

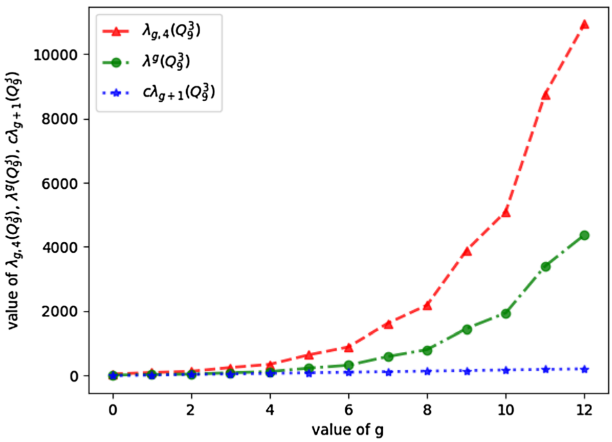


Fig. 5 Illustration that $r = 3$, $n = 9$ and $0 \leq g \leq 12$

as $\lambda^g(Q^3_{10})$ and much larger than the value of $c\lambda_{g+1}(Q^3_{10})$ regardless of the value of g. A similar conclusion can be drawn in Q^3_9, the value of $\lambda_{g,4}(Q^3_9)$ is almost three times the value of $\lambda^g(Q^3_9)$ and much larger than the value of $c\lambda_{g+1}(Q^3_9)$.

The quantitative relationship among $\lambda_{g,r+1}(Q^3_n)$, $\lambda^g(Q^3_n)$ and $c\lambda_{g+1}(Q^3_n)$ can be clearly observed from Table 3. Next, we plot Figs. 4 and 5 to better present the growth rates of these three indicators as g increases. Consistent results can be observed that when r and n are fixed, $\lambda_{g,r+1}(Q^3_n)$ grows at a rapid rate as g increases, whereas $c\lambda_{g+1}(Q^3_n)$ is relatively flat. This implies that only a few failed links can disrupt this network to generate multiple components, but to allow each processor to communicate with more processors would require large-scale link failures. In

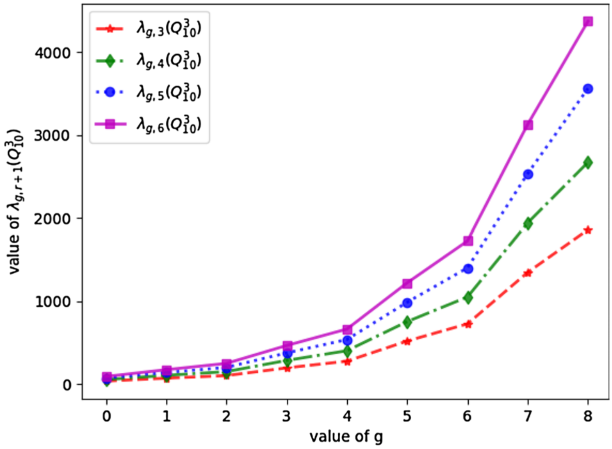


Fig. 6 Illustration that $2 \leq r \leq 5$, $n = 10$ and $0 \leq g \leq 8$

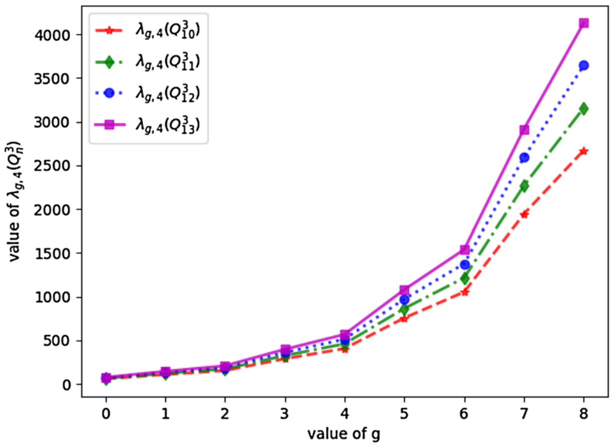


Fig. 7 Illustration that $r = 3, 10 \leq n \leq 13$ and $0 \leq g \leq 8$

addition, Figs. 6 and 7 depict the variation of $\lambda_{g,r+1}(Q_n^3)$ with the growth of g for fixed n and r, respectively.

6 Concluding Remarks

This work suggests a novel indicator, namely g-good-neighbor r-component edge-connectivity, for measuring network reliability and fault tolerability of 3-ary n-cube, which is proposed by Liu et al. [17]. This concept breaks down the inherent idea to emphasize the double constraints on the resulting network in the presence of the large-scale faulty links, which not only limits the number of component in the resulting network, but also requires that each vertex is adjacent to at least g fault-free links. More importantly, g-good-neighbor $(r+1)$-component edge-connectivity is almost r times the size of g-good-neighbor edge-connectivity and much larger than $(r+1)$-component edge-connectivity in 3-ary n-cube. As a consequence, this metric offers a more refined quantitative indicator of the robustness of a multiprocessor system in the presence of link disruption.

Acknowledgements This work was partly supported by the National Natural Science Foundation of China (Nos. 61977016, 61572010, and 62277010), Natural Science Foundation of Fujian Province (Nos. 2020J01164, 2017J01738). This work was also partly supported by China Scholarship Council (CSC No. 202108350054).

Author contributions Qifan Zhang wrote the main manuscript text and prepared all figures. Shuming Zhou and Lulu Yang reviewed and edited the manuscript. All authors reviewed the manuscript.

Data availability No datasets were generated or analyzed during the current study.

Declarations

Conflict of interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

1. Adiga NR, Blumrich MA, Chen D, Coteus P, Gara A, Giampapa ME, Hei-delberger P, Singh S, Steinmacher-Burow BD, Takken T, Tsao M, Vranas P (2005) Blue Gene/L torus interconnection network. IBM J Res Dev 49(2–3):265–276
2. Anderson E, Brooks J, Grassl C, Scott S (1997) Performance of the CRAY T3E multiprocessor. In: Proceedings of the ACM/IEEE Conference on Supercomputing (SC'97), pp 1–17
3. Bhuyan L, Yang Q, Agrawal D (1989) Performance of multiprocessor intercommectionnetworks. Computer 22(2):25–37
4. Bondy JA, Murty USR (2008) Graph theory and its application. Academic Press, London
5. Chen L, Li X-J, Ma M (2022) Reliability assessment for modified bubble-sort networks. Discret Appl Math 307:88–94
6. Day K, Al-Ayyoub AE (1997) Fault diameter of k-ary n-cube networks. IEEE Trans Parallel Distrib Syst 8(9):903–907
7. Fan W, Fan J, Lin C, Wang Y, Han Y, Wang R (2019) Optimally embedding 3-ary n-cubes into grids. J Comput Sci Tech 34:372–387
8. Făbrega J, Fiol MA (1996) On the extraconnectivity of graphs. Discret Math 155:49–57
9. Fonseca P, Mota ES (2017) A survey on fault management in software-defined networks. IEEE Commun Surv Tutor 19(4):2284–2321
10. Hsieh S-Y, Lin T-J, Huang H-L (2007) Panconnectivity and edgepancyclicity of 3-ary n-cube. J Supercomput 42:225–233
11. Hsieh S-Y, Chang Y-H (2012) Extraconnectivity of k-ary n-cube networks. Theor Comput Sci 443:63–69
12. Kessler RE, Schwarzmeier JL Cray T3D: A new dimension for Cray research. In: 38th Annual IEEE Computer Society International Computer Conference-COMPCON SPRING '93(Publ by IEEE, Piscataway, NJ, United States, San Fran-cisco, CA, USA, 1993), pp 176–182
13. Latifi S, Hegde M, Naraghi-Pour M (1994) Conditional connectivity measures for large miltiprocessor systems. IEEE Trans Comput 43:218–221
14. Li X-J, Xu J-M (2013) Edge-fault tolerance of hypercube-like networks. Inf Process Lett 113(19–21):760–763
15. Li S-Y, Li X-J, Ma M (2024) Reliability assessment for k-ary n-cubes with faulty edges. J Parallel Distrib Comput 27:104886

16. Liu D, Li P, Zhang B (2022) Component edge connectivity of hypercube-like networks. Theor Comput Sci 911:19–25
17. Liu H, Zhang M, Hsieh SY (2024) A novel links fault tolerant analysis: g-good r-component edge-connectivity of interconnection networks with applications to hypercube, IEEE Trans Reliab. https://doi.org/10.1109/TR.2024.3410526
18. Noakes MD, Wallach DA, Dally WJ (1993) The J-machine multicomputer: anarchitectural evaluation. Comput Arch News 21(2):224–235
19. Qiao H, Zhang M, Ma W, Yang X (2023) An $O(\log(N))$ algorithm view: reliability evaluation of folded-crossed hypercube in terms of h-extra edge-connectivity. Parallel Process Lett 33:2350003
20. Sampathkumar E (1984) Connectivity of a graph-a generalization. J Comb Inf Syst Sci 9:71–78
21. Shang H, Sabir E, Meng J (2019) Conditional edge connectivity of the locally twisted cubes. J Oper Res Soc China 7(3):501–509
22. Xu L, Zhou S, Hsieh S-Y (2022) An $O(\log_3 N)$ algorithm for reliability assessment of 3-ary n-cubes based on h-extra edge connectivity. IEEE Trans Reliab 71(3):1230–1240
23. Xu L, Zhou S (2022) An $O(\log_2 N)$ algorithm for reliability assessment of augmented cubes based on h-extra edge-connectivity. J Supercomput 78(5):6739–6751
24. Xu L (2024) Fault tolerability evaluation for the component edge-connectivity of folded Petersen networks. Appl Math Comput 473:128673
25. Yang Y, Zhang M, Meng J (2023) Fault tolerance analysis for hamming graphs with large-scale faulty links based on r-component edge-connectivity. J Parallel Distrib Comput 173:107–114
26. Yang Y, Zhang M, Meng J (2024) Link fault tolerance of BC networks and folded hypercubes on h -extra r-component edge-connectivity. Appl Math Comput 462:128343
27. Yuan J, Liu A-X, Qin X, Zhang J-F, Li J (2016) g-Good-neighbor conditional diagnosability measures for 3-ary n-cube networks. Theor Comput Sci 626:144–162
28. Zhang M, Meng J, Yang W, Tian Y (2014) Reliability analysis of bijective connection networks in terms of the extra edge-connectivity. Inf Sci 279:374–382
29. Zhang M, Zhang L, Feng X, Lai H-J (2018) An $O(\log_2 N)$ algorithm for reliability evaluation of h -extra edge-connectivity of folded hypercubes. IEEE Trans Reliab 67(1):297–307
30. Zhang M, Yang X, He X, Qin Z, Ma Y (2022) Reliability evaluation of augmented cubes on degree. J Interconnect Netw 22(1):2142019
31. Zhang Q, Xu L, Yang W (2021) Reliability analysis of the augmented cubes in terms of the extra edge-connectivity and the component edge-connectivity. J Parallel Distrib Comput 147:124–131
32. Zhang Q, Zhou S, Zhang H (2023) Reliability analysis of 3-ary n-cube in terms of average degree edge-connectivity. Discret Appl Math 341:31–39
33. Zhao S, Yang W, Zhang S, Xu L (2018) Component edge connectivity of hypercubes. Int J Found Comput Sci 29(6):995–1001