The Journal of Supercomputing (2024) 80:23848–23866
https://doi.org/10.1007/s11227-024-06305-4

Comprehensive analysis on the existence and uniqueness of solutions for fractional q-integro-differential equations

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Accepted: 13 June 2024 / Published online: 24 July 2024

Abstract
In this work, we study the coupled system of fractional integro-differential equations, which includes the fractional derivatives of the Riemann–Liouville type and the fractional q-integral of the Riemann–Liouville type. We focus on the utilization of two significant fixed-point theorems, namely the Schauder fixed theorem and the Banach contraction principle. These mathematical tools play a crucial role in investigating the existence and uniqueness of a solution for a coupled system of fractional q-integro-differential equations. Our analysis specifically incorporates the fractional derivative and integral of the Riemann–Liouville type. To illustrate the implications of our findings, we present two examples that demonstrate the practical applications of our results. These examples serve as tangible scenarios where the aforementioned theorems can effectively address real-world problems and elucidate the underlying mathematical principles. By leveraging the power of the Schauder fixed theorem and the Banach contraction principle, our work contributes to a deeper understanding of the solutions to coupled systems of fractional q-integro-differential equations. Furthermore, it highlights the potential practical significance of these mathematical tools in various fields where such equations arise, offering a valuable framework for addressing complex problems.

Keywords Fractional derivative · q-Integro-differential equation · Existence and uniqueness of solution · Applications

Mathematics Subject Classification 35-XX · 65-XX

1 Introduction

The topics of q-calculus and fractional calculus have received a lot of interest in view of their role in describing some real-world problems in numerous fields. It is worth noting that both fractional calculus and q-calculus are generalizations

Extended author information available on the last page of the article of classical calculus for any order. Note that the field of fractional calculus has a large number of fractional operators, but in our paper, we are interested in studying the equations that contain the fractional derivative of the Riemann–Liouville type. Mathematics modeling is used to convert a range of applied problems into a set of fractional differential and integral equation [1–3]. The study of theoretical aspects of q-calculus and fractional calculus has been the focus of many studies, and it is now considered a significant area of research. There are numerous papers on the solvability of nonlinear fractional differential equations. At the same time, the study of coupled systems of nonlinear fractional differential equations is also important due to their numerous applications. In Ahmad and Nieto [4] studied the existence of result for the following coupled system:

$$\mathcal{D}^{\alpha}\mathfrak{U}(Ł) = \mathfrak{f}(Ł, w(Ł), \mathcal{D}^{\lambda}w(Ł)), \qquad \mathcal{D}^{\beta}w(Ł) = \mathfrak{g}(Ł, \mathfrak{U}(Ł), \mathcal{D}^{\gamma}w(Ł)), \qquad Ł \in (0,1),$$
$$\mathfrak{U}(0) = 0, \quad \mathfrak{U}(1) = \varsigma\mathfrak{U}(\zeta), \quad w(0) = 0, \quad w(1) = \varsigma w(\zeta),$$

where $\alpha, \lambda, \beta, \zeta, \gamma, and\varsigma$ satisfy certain conditions. In [5], Zhang et al. applied a variety of fixed-point theorems to the following coupled system of nonlinear fractional differential equations to investigate the existence and uniqueness of solutions:

$$\begin{aligned}
&\mathcal{D}^{\rho_1}\mathfrak{U}(Ł) = \mathfrak{g}_1\Big(Ł, \mathfrak{U}(Ł), w(Ł), \mathcal{D}^{\alpha_1}\mathfrak{U}(Ł), \mathcal{D}^{\alpha_2}w(Ł)\Big), \quad Ł \in (0,1),\\
&\mathcal{D}^{\rho_2}w(Ł) = \mathfrak{g}_2\Big(Ł, \mathfrak{U}(Ł), w(Ł), \mathcal{D}^{\alpha_1}\mathfrak{U}(Ł), \mathcal{D}^{\alpha_2}w(Ł)\Big), \quad Ł \in (0,1),\\
&\mathfrak{U}(0) = \mathfrak{U}'(0) = 0, w(0) = w'(0) = 0,\\
&\mathfrak{U}(1) = \gamma_1 I^{\beta_1}\mathfrak{U}(\mu_1), w(1) = \gamma_2 I^{\beta_2} w(\mu_2),
\end{aligned}$$

where $\mathcal{D}^{\rho_i}$ and $\mathcal{D}^{\alpha_i}$ represent the standard Riemann–Liouville fractional derivative, $2 < \rho_i \le 3, 0 < \alpha_i \le 1, 0 < \mu_i \le 1, \gamma_i, \beta_i > 0, i = 1, 2$. For more details, see [6–19].

In contrast, several studies have been written about the existence and uniqueness of solutions for fractional q-integro-differential equations; for further information, see [20–24]. In [25, 26], the authors discussed the numerical and analytical solutions of the Fredholm and Fredholm–Volterra integro-differential equations of the first and second orders, respectively. In addition, they study the numerical solution using a merge of finite difference with Simpson's and finite difference with trapezoidal methods. In [27], they discussed the existence and uniqueness of a solution for the nonlocal fractional q-integro-differential equation:

$$\mathfrak{U}''(Ł) = \mathfrak{g}\Big(Ł, \mathfrak{U}(Ł), {}^{CF}\mathcal{D}^{\beta}\mathfrak{U}(Ł), I_q^{\alpha}\mathfrak{f}(Ł, \mathfrak{U}'(Ł))\Big), \quad Ł \in (0,1],$$

under the q-nonlocal condition:

$$(1-q)\nu\sum_{l=0}^{n} q^l\mathfrak{U}(q^i\nu) = \rho_0, \quad \mathfrak{U}'(0) = \alpha_0, \quad \nu \in (0,1).$$

where ${}^{CF}\mathcal{D}^{\beta}\mathfrak{U}(Ł)$ is the fractional derivative of Caputo–Fabrizio type, I_q^{α} is the Riemann–Liouville fractional q-integral, and ρ_0, α_0 are constants, $q, \beta \in (0, 1)$. In addition, they solved it numerically by using two methods: the finite-trapezoidal method and the cubic-trapezoidal method.

Inspired by the aforementioned results, we examine in this work the following coupled system of a nonlocal fractional q-integro-differential equation:

$$\begin{aligned} \mathcal{D}^{\beta_1}\mathfrak{U}(Ł) &= \mathcal{F}_1\left(Ł, \mathfrak{U}(Ł), w(Ł), \mathcal{D}^{\alpha_1}\mathfrak{U}(Ł), \mathcal{D}^{\alpha_2}w(Ł), I_{q_1}^{\delta_1}\mathfrak{U}(Ł), I_{q_2}^{\delta_2}w(Ł)\right), \quad Ł \in (0,1), \\ \mathcal{D}^{\beta_2}w(Ł) &= \mathcal{F}_2\left(Ł, \mathfrak{U}(Ł), w(Ł), \mathcal{D}^{\alpha_1}\mathfrak{U}(Ł), \mathcal{D}^{\alpha_2}w(Ł), I_{q_1}^{\delta_1}\mathfrak{U}(Ł), I_{q_2}^{\delta_2}w(Ł)\right), \quad Ł \in (0,1), \end{aligned} \tag{1}$$

considering the q-nonlocal conditions:

$$\begin{aligned} &(1-q_1)\chi_1 \sum_{\kappa=0}^{\nu_1} q_1^{\kappa}\mathfrak{U}(q_1^{\kappa}\chi_1) = \rho_1, \quad \mathfrak{U}(0) = \mathfrak{U}'(0) = 0, \quad \chi_1 \in (0,1), \\ &(1-q_2)\chi_2 \sum_{\kappa=0}^{\nu_2} q_2^{\kappa}w(q_2^{\kappa}\chi_2) = \rho_2, \quad w(0) = w'(0) = 0, \quad \chi_2 \in (0,1), \end{aligned} \tag{2}$$

where $\mathcal{D}^{\beta_i}$ and $\mathcal{D}^{\alpha_i}$ represent the standard fractional derivative of the Riemann–Liouville type, $I_{q_i}^{\delta_i}$ represent the Riemann–Liouville fractional q-integrals of the order $\delta_i > 0, 2 < \beta_i \leq 3, 0 < \alpha_i \leq 1, \rho_i$ are constants, and $q_i, \alpha_i \in (0, 1), i = 1, 2$.

This is how the essay is organized: We list several lemmas and definitions that are indeed in this work in Sect. 2. The existence and uniqueness of the solution to the nonlocal coupled system of a fractional q-integro-differential equation (1)–(2) were examined in Sect. 3. Section 4 contains applications. In Sect. 5, the conclusion is presented.

2 Preliminaries

We now go through some fundamental ideas in q-calculus and fractional calculus, as well as some lemmas that will be applied in this article.

Definition 2.1 [20, 21] Let z be a function that is defined on the interval [0, 1]. The Riemann–Liouville fractional q-integral of order $\varrho > 0$ can be defined as

$$(I_q^{\varrho}z)(Ł) = \begin{cases} z(Ł), & \varrho = 0, \\ \frac{1}{\Gamma_q(\varrho)}\int_0^{Ł}(Ł - qs)^{(\varrho-1)}z(s)\mathrm{d}_q s, \end{cases} \tag{3}$$

where

$$(Ł - qs)^{(0)} = 1, \quad (Ł - qs)^{(\phi)} = \prod_{j=0}^{\phi-1}(Ł - q^{j+1}s), \phi \in \mathbb{N},$$

$$(Ł - qs)^{(\Omega)} = \prod_{j=0}^{\infty} \frac{(Ł - q^{j+1}s)}{(Ł - q^{j+\Omega+1}s)}, \quad \Omega \in \mathbb{R}.$$

Lemma 2.2 [20] *As a result of q-integration by parts, we obtain*

$$(I_q^{\varrho}1)(Ł) = \frac{Ł^{(\varrho)}}{\Gamma_q(\varrho+1)}, \qquad \varrho > 0. \tag{4}$$

Definition 2.3 [28] The Riemann–Liouville fractional derivative of order $\beta > 0$ of a function $\mathcal{F} : (0, \infty) \to \mathbb{R}$ can be defined as

$$\mathcal{D}^{\beta}\mathcal{F}(Ł) = \frac{1}{\Gamma(\nu-\beta)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\nu} \int_0^{Ł} (Ł - \theta)^{\nu-\beta-1}\mathcal{F}(\theta)\mathrm{d}\theta.$$

where $\nu = [\beta] + 1, [\beta]$ represents the integer part of number β, provided that the right-hand side is pointwise defined on $(0, \infty)$.

Definition 2.4 [28] The Riemann–Liouville fractional integral of order $\beta > 0$ of a function $\mathcal{F} : (0, \infty) \to \mathbb{R}$ is given by

$$I^{\beta}\mathcal{F}(Ł) = \frac{1}{\Gamma(\beta)} \int_0^{Ł} (Ł - \theta)^{\beta-1}\mathcal{F}(\theta)\mathrm{d}\theta.$$

Lemma 2.5 [28] *Suppose that* $\beta > 0, \nu - 1 < \beta < \nu, \nu \in N$. *Then*

1. *For any* $z \in L^1(c, d), D^{\beta}(I^{\beta}z) = z$.
2. *If* $I^{\nu-\beta}z \in AC^{\nu}[c, d]$, *then*

$$I^{\beta}\mathcal{D}^{\beta}z(Ł) = z(Ł) + k_1Ł^{\beta-1} + k_2Ł^{\beta-2} + \cdots + k_{\nu}Ł^{\beta-\nu},$$

where $k_i \in \mathbb{R}(i = 1, 2, \ldots, \nu), \nu$ is the lowest integer smaller than or equal to β.

Lemma 2.6 [28]

1. *If* $\mathcal{F} \in L^1(c, d), \sigma > \varphi > 0$, *then*

$$I^{\sigma}I^{\varphi}\mathcal{F}(Ł) = I^{\sigma+\varphi}\mathcal{F}(Ł), \quad \mathcal{D}^{\varphi}I^{\sigma}\mathcal{F}(Ł) = I^{\sigma-\varphi}\mathcal{F}(Ł), \quad \mathcal{D}^{\sigma}I^{\sigma}\mathcal{F}(Ł) = \mathcal{F}(Ł).$$

2. *If* $\sigma > \varphi > 0$, *then*

$$I^{\sigma}Ł^{\varphi} = \frac{\Gamma(\varphi+1)}{\Gamma(\varphi+1+\sigma)}Ł^{\sigma+\varphi}, \quad \mathcal{D}^{\varphi}Ł^{\sigma} = \frac{\Gamma(\sigma+1)}{\Gamma(\sigma+1-\varphi)}Ł^{\sigma-\varphi}, \quad \mathcal{D}^{\sigma}Ł^{\varphi} = 0.$$

Lemma 2.7 [29] *Suppose that* $\sigma > 0, \mathfrak{U} \in L^1([c, d], \mathbb{R})$. *Then, we have*

$$I^{\nu+1}\mathfrak{U}(Ł) \leq \|I^{\nu}\mathfrak{U}(Ł)\|_{L^1}, Ł \in [c,d].$$

For the convenience, we set

$$\Lambda_i = \frac{1}{(1-q_i)\chi_i \sum_{\kappa=0}^{\nu_i} q_i^{\kappa}(q_i^{\kappa}\chi_i)^{(\beta_i-1)}}, \quad i = 1,2.$$

Lemma 2.8 *Suppose that* $\mathfrak{U} \in AC^3[0,1]$, *and* $2 < \beta_1 \leq 3$. *Therefore, we can get a unique solution to the following nonlocal problem*:

$$\begin{cases} \mathcal{D}^{\beta_1}\mathfrak{U}(Ł) = \mathcal{F}_1\left(Ł, \mathfrak{U}(Ł), w(Ł), \mathcal{D}^{\alpha_1}\mathfrak{U}(Ł), \mathcal{D}^{\alpha_2}w(Ł), I_{q_1}^{\delta_1}\mathfrak{U}(Ł), I_{q_2}^{\delta_2}w(Ł)\right), \quad Ł \in (0,1), \\ (1-q_1)\chi_1 \sum_{\kappa=0}^{\nu_1} q_1^{\kappa}\mathfrak{U}(q_1^{\kappa}\chi_1) = \rho_1, \quad \mathfrak{U}(0) = \mathfrak{U}'(0) = 0, \quad \chi_1 \in (0,1), \end{cases} \tag{5}$$

as

$$\begin{aligned} \mathfrak{U}(Ł) = \mathfrak{U}(Ł) = I^{\beta_1}\mathcal{F}_1\left(Ł, \mathfrak{U}(Ł), w(Ł), \mathcal{D}^{\alpha_1}\mathfrak{U}(Ł), \mathcal{D}^{\alpha_2}w(Ł), I_{q_1}^{\delta_1}\mathfrak{U}(Ł), I_{q_2}^{\delta_2}w(Ł)\right) \\ + \Lambda_1\Bigg[\rho_1 - (1-q_1)\chi_1 \sum_{\kappa=0}^{\nu_1} q_1^{\kappa} I^{\beta_1}\mathcal{F}_1\Big(q_1^{\kappa}\chi_1, \mathfrak{U}(q_1^{\kappa}\chi_1), w(q_1^{\kappa}\chi_1), \mathcal{D}^{\alpha_1}\mathfrak{U}(q_1^{\kappa}\chi_1), \\ \mathcal{D}^{\alpha_2}w(q_1^{\kappa}\chi_1), I_{q_1}^{\delta_1}\mathfrak{U}(q_1^{\kappa}\chi_1), I_{q_2}^{\delta_2}w(q_1^{\kappa}\chi_1)\Big)\Bigg] Ł^{\beta_1-1}. \end{aligned} \tag{6}$$

Proof Considering the lemma (), we can get the solution of (5) as follows:

$$\begin{aligned} \mathfrak{U}(Ł) = I^{\beta_1}\mathcal{F}_1\left(Ł, \mathfrak{U}(Ł), w(Ł), \mathcal{D}^{\alpha_1}\mathfrak{U}(Ł), \mathcal{D}^{\alpha_2}w(Ł), I_{q_1}^{\delta_1}\mathfrak{U}(Ł), I_{q_2}^{\delta_2}w(Ł)\right) \\ + k_1 Ł^{\beta_1-1} + k_2 Ł^{\beta_1-2} + k_3 Ł^{\beta_1-3}. \end{aligned}$$

Using the condition $\mathfrak{U}(0) = \mathfrak{U}'(0) = 0$, we get $k_2 = k_3 = 0$. Therefore,

$$\mathfrak{U}(Ł) = I^{\beta_1}\mathcal{F}_1\left(Ł, \mathfrak{U}(Ł), w(Ł), \mathcal{D}^{\alpha_1}\mathfrak{U}(Ł), \mathcal{D}^{\alpha_2}w(Ł), I_{q_1}^{\delta_1}\mathfrak{U}(Ł), I_{q_2}^{\delta_2}w(Ł)\right) + k_1 Ł^{\beta_1-1}. \tag{7}$$

Using the q-nonlocal condition $(1-q_1)\chi_1 \sum_{\kappa=0}^{\nu_1} q_1^{\kappa}\mathfrak{U}(q_1^{\kappa}\chi_1) = \rho_1$, we obtain

$$\begin{aligned} (1-q_1)\chi_1 \sum_{\kappa=0}^{\nu_1} q_1^{\kappa}\mathfrak{U}(q_1^{\kappa}\chi_1) = (1-q_1)\chi_1 \sum_{\kappa=0}^{\nu_1} q_1^{\kappa} I^{\beta_1}\mathcal{F}_1\Big(q_1^{\kappa}\chi_1, \mathfrak{U}(q_1^{\kappa}\chi_1), w(q_1^{\kappa}\chi_1), \mathcal{D}^{\alpha_1}\mathfrak{U}(q_1^{\kappa}\chi_1), \\ \mathcal{D}^{\alpha_2}w(q_1^{\kappa}\chi_1), I_{q_1}^{\delta_1}\mathfrak{U}(q_1^{\kappa}\chi_1), I_{q_2}^{\delta_2}w(q_1^{\kappa}\chi_1)\Big) + k_1(1-q_1)\chi_1 \sum_{\kappa=0}^{\nu_1} q_1^{\kappa}(q_1^{\kappa}\chi_1)^{\beta_1-1}. \end{aligned}$$

Therefore,

$$k_1 = \Lambda_1 \Bigg[\rho_1 - (1-q_1)\chi_1 \sum_{\kappa=0}^{\nu_1} q_1^{\kappa} I^{\beta_1} \mathcal{F}_1 \Bigg(q_1^{\kappa}\chi_1, \mathfrak{U}(q_1^{\kappa}\chi_1), w(q_1^{\kappa}\chi_1), \mathcal{D}^{\alpha_1}\mathfrak{U}(q_1^{\kappa}\chi_1),$$
$$\mathcal{D}^{\alpha_2} w(q_1^{\kappa}\chi_1), I_{q_1}^{\delta_1}\mathfrak{U}(q_1^{\kappa}\chi_1), I_{q_2}^{\delta_2} w(q_1^{\kappa}\chi_1) \Bigg) \Bigg].$$

Substituting k_1 into (7), we obtain

$$\mathfrak{U}(Ł) = I^{\beta_1}\mathcal{F}_1\Big(Ł, \mathfrak{U}(Ł), w(Ł), \mathcal{D}^{\alpha_1}\mathfrak{U}(Ł), \mathcal{D}^{\alpha_2} w(Ł), I_{q_1}^{\delta_1}\mathfrak{U}(Ł), I_{q_2}^{\delta_2} w(Ł) \Big)$$
$$+ \Lambda_1 \Bigg[\rho_1 - (1-q_1)\chi_1 \sum_{\kappa=0}^{\nu_1} q_1^{\kappa} I^{\beta_1} \mathcal{F}_1 \Bigg(q_1^{\kappa}\chi_1, \mathfrak{U}(q_1^{\kappa}\chi_1), w(q_1^{\kappa}\chi_1), \mathcal{D}^{\alpha_1}\mathfrak{U}(q_1^{\kappa}\chi_1),$$
$$\mathcal{D}^{\alpha_2} w(q_1^{\kappa}\chi_1), I_{q_1}^{\delta_1}\mathfrak{U}(q_1^{\kappa}\chi_1), I_{q_2}^{\delta_2} w(q_1^{\kappa}\chi_1) \Bigg) \Bigg] Ł^{\beta_1 - 1}.$$

The proof is finished. □

In the same way, the solution of

$$\begin{cases} \mathcal{D}^{\beta_2} w(Ł) = \mathcal{F}_2\Big(Ł, \mathfrak{U}(Ł), w(Ł), \mathcal{D}^{\alpha_1}\mathfrak{U}(Ł), \mathcal{D}^{\alpha_2} w(Ł), I_{q_1}^{\delta_1}\mathfrak{U}(Ł), I_{q_2}^{\delta_2} w(Ł) \Big), \quad Ł \in (0,1), \\ (1-q_2)\chi_2 \sum_{j=0}^{\nu_2} q_2^j w(q_2^j \chi_2) = \rho_2, \quad w(0) = w'(0) = 0, \quad \chi_2 \in (0,1), \end{cases}$$

is

$$w(Ł) = I^{\beta_2}\mathcal{F}_2\Big(Ł, \mathfrak{U}(Ł), w(Ł), \mathcal{D}^{\alpha_1}\mathfrak{U}(Ł), \mathcal{D}^{\alpha_2} w(Ł), I_{q_1}^{\delta_1}\mathfrak{U}(Ł), I_{q_2}^{\delta_2} w(Ł) \Big)$$
$$+ \Lambda_2 \Bigg[\rho_2 - (1-q_2)\chi_2 \sum_{\kappa=0}^{\nu_2} q_2^{\kappa} I^{\beta_2} \mathcal{F}_2 \Bigg(q_1^{\kappa}\chi_1, \mathfrak{U}(q_1^{\kappa}\chi_1), w(q_1^{\kappa}\chi_1), \mathcal{D}^{\alpha_1}\mathfrak{U}(q_1^{\kappa}\chi_1),$$
$$\mathcal{D}^{\alpha_2} w(q_1^{\kappa}\chi_1), I_{q_1}^{\delta_1}\mathfrak{U}(q_1^{\kappa}\chi_1), I_{q_2}^{\delta_2} w(q_1^{\kappa}\chi_1) \Bigg) \Bigg] Ł^{\beta_2 - 1}.$$

Denote that $L^1([0,1], \mathbb{R})$ is the Banach space of Lebesgue integrable functions from $[0,1] \to \mathbb{R}$ with the norm $\|\mathfrak{U}\| = \int_0^1 |\mathfrak{U}(Ł)| dŁ$.

3 Main results

The spaces $\mathcal{X}$ and $\mathcal{Y}$ should now be introduced as follows: $\mathcal{X} = \{\mathfrak{U} | \mathfrak{U} \in \mathcal{C}[0,1], \mathcal{D}^{\beta_1}\mathfrak{U} \in \mathcal{C}[0,1]$ and $I_{q_1}^{\delta_1}\mathfrak{U} \in \mathcal{C}[0,1]\}$ equipped with the norm $\|\mathfrak{U}\| = \max_{Ł \in [0,1]} |\mathfrak{U}(Ł)| + \max_{Ł \in [0,1]} |\mathcal{D}^{\beta_1}\mathfrak{U}(Ł)| + \max_{Ł \in [0,1]} |I_{q_1}^{\delta_1}\mathfrak{U}(Ł)|$, and also $\mathcal{Y}=\{w|w\in\mathcal{C}[0,1],\mathcal{D}^{\beta_2}w\in\mathcal{C}[0,1]$ and $I_{q_2}^{\delta_2}w\in\mathcal{C}[0,1]\}$ equipped with the norm $\|w\|=\max_{Ł\in[0,1]}|w(Ł)|+\max_{Ł\in[0,1]}|\mathcal{D}^{\beta_2}w(Ł)|+\max_{Ł\in[0,1]}|I_{q_2}^{\delta_2}w(Ł)|$. Evidently, $(\mathcal{X},\|.\|)$ and $(\mathcal{Y},\|.\|)$ are Banach spaces. Therefore, the product space $(\mathcal{X}\times\mathcal{Y},\|(\mathfrak{U},w)\|_{\mathcal{X}\times\mathcal{Y}})$ is also a Banach space equipped with the norm $\|(\mathfrak{U},w)\|_{\mathcal{X}\times\mathcal{Y}}=\|\mathfrak{U}\|_{\mathcal{X}}+\|w\|_{\mathcal{Y}}$. Now, we define the operator $G:\mathcal{X}\times\mathcal{Y}\to\mathcal{X}\times\mathcal{Y}$ by

$$G(\mathfrak{U},w)(Ł)=\Big(G_1(\mathfrak{U},w)(Ł),G_2(\mathfrak{U},w)(Ł)\Big), \tag{8}$$

where

$$G_1(\mathfrak{U},w)(Ł)=I^{\beta_1}\vartheta(Ł)+\Lambda_1\left[\rho_1-(1-q_1)\chi_1\sum_{\kappa=0}^{\nu_1}q_1^{\kappa}I^{\beta_1}\vartheta(q_1^{\kappa}\chi_1)\right]Ł^{\beta_1-1},$$

and

$$G_2(\mathfrak{U},w)(Ł)=I^{\beta_2}z(Ł)+\Lambda_2\left[\rho_2-(1-q_2)\chi_2\sum_{\kappa=0}^{\nu_2}q_2^{\kappa}I^{\beta_2}z(q_2^{\kappa}\chi_2)\right]Ł^{\beta_2-1}.$$

Conveniently, we set

1. $$\mathcal{A}_{ij}=\left(1+|\Lambda_i|(1-q_i)\chi_i\sum_{\kappa=0}^{\nu_i}q_i^{\kappa}\right)\|I^{\beta_i-1}b_{ij}\|_{L^1},i=1,2,\quad j=1,\dots,7.$$
2. $$\mathcal{B}_{ij}=\|I^{\beta_i-\alpha_i-1}b_{ij}\|_{L^1}+\frac{|\Lambda_i|\Gamma(\beta_i)}{\Gamma(\beta_i-\alpha_i)}(1-q_i)\chi_i\sum_{\kappa=0}^{\nu_i}q_i^{\kappa}\|I^{\beta_i-1}b_{ij}\|_{L^1},$$
$$i=1,2,\quad j=1,\dots,7.$$
3. $$C_{ij}=\left(1+\frac{1}{\Gamma_{q_i}(\delta_i+1)}\right)\mathcal{A}_{ij}+\mathcal{B}_{ij},i=1,2,\quad j=1,\dots,7.$$
4. $$E_i=\left(1+\frac{1}{\Gamma_{q_i}(\delta_i+1)}+\frac{\Gamma(\beta_i)}{\Gamma(\beta_i-\alpha_i)}\right)|\rho_i\Lambda_i|,\qquad i=1,2.$$
5. $$D_i=\max\{C_{i1},C_{i2},C_{i3},C_{i4},C_{i5},C_{i6}\},\qquad i=1,2.$$

Theorem 3.1 *Suppose that $\mathcal{F}_i$ are continuous for almost every $Ł\in(0,1)$ and measurable in $Ł$ for any $w_1,w_2,w_3,w_4,w_5,w_6\in\mathbb{R}$. There exist nonnegative functions $b_{ij}(Ł)\in L^1([0,1],\mathbb{R}_+),i=1,2,j=1,2,\dots,7,$ such that*

(a)
$$|\mathcal{F}_i(Ł, w_1, w_2, w_3, w_4, w_5, w_6)| \le \sum_{j=1}^{6} b_{ij}(Ł)|w_j|^{\tau_{ij}} + b_{i7}(Ł), \quad \tau_{ij} \in (0,1).$$

Consequently, there is at least one solution for the nonlocal coupled system (1)-(2).

Proof To demonstrate that the operator G defined by (8) has a fixed point, we shall employ the Schauder fixed-point theorem.

First, let $Q_{\mathfrak{r}} \subset \mathcal{X} \times \mathcal{Y} = \{(\mathfrak{U}, w)(Ł) \in \mathbb{R}^2 : \|(\mathfrak{U}, w)\|_{\mathcal{X}\times\mathcal{Y}} \le \mathfrak{r}\}$, where

$$\begin{aligned}\mathfrak{r} \ge \max\{&(16C_{i1})^{\frac{1}{1-\tau_{i1}}}, (16C_{i2})^{\frac{1}{1-\tau_{i2}}}, (16C_{i3})^{\frac{1}{1-\tau_{i3}}}, (16C_{i4})^{\frac{1}{1-\tau_{i4}}},\\ &(16C_{i5})^{\frac{1}{1-\tau_{i5}}}, (16C_{i6})^{\frac{1}{1-\tau_{i6}}}, 16C_{i7}, i = 1,2\}.\end{aligned}$$

It is obvious that $Q_{\mathfrak{r}}$ is nonempty, bounded, closed and convex subset of $\mathcal{C}[0,1]$. We demonstrate $G : Q_{\mathfrak{r}} \to Q_{\mathfrak{r}}$. For any $(\mathfrak{U}, w)(Ł) \in Q_{\mathfrak{r}}$, we use Lemma () and condition (a), to obtain

$$\begin{aligned}|G_1(\mathfrak{U}, w)(Ł)| \le& I^{\beta_1}|\mathcal{F}_1(Ł, \mathfrak{U}(Ł), w(Ł), \mathcal{D}^{\alpha_1}\mathfrak{U}(Ł), \mathcal{D}^{\alpha_2}w(Ł), I_{q_1}^{\delta_1}\mathfrak{U}(Ł), I_{q_2}^{\delta_2}w(Ł))| + |\rho_1\Lambda_1|\\
&+ |\Lambda_1|(1-q_1)\chi_1 \sum_{\kappa=0}^{\nu_1} q_1^{\kappa} I^{\beta_1}|\mathcal{F}_1(q_1^{\kappa}\chi_1, \mathfrak{U}(q_1^{\kappa}\chi_1), w(q_1^{\kappa}\chi_1), \mathcal{D}^{\alpha_1}\mathfrak{U}(q_1^{\kappa}\chi_1),\\
&\mathcal{D}^{\alpha_2}w(q_1^{\kappa}\chi_1), I_{q_1}^{\delta_1}\mathfrak{U}(q_1^{\kappa}\chi_1), I_{q_2}^{\delta_2}w(q_1^{\kappa}\chi_1))|\\
&\le I^{\beta_1}\left(\sum_{j=1}^{6} b_{1j}(Ł)R^{\tau_{1j}} + b_{17}(Ł)\right) + |\rho_1\Lambda_1|\\
&+ |\Lambda_1|(1-q_1)\chi_1 \sum_{\kappa=0}^{\nu_1} q_1^{\kappa} I^{\beta_1}\left(\sum_{j=1}^{6} b_{1j}(q_1^{\kappa}\chi_1)R^{\tau_{1j}} + b_{17}(q_1^{\kappa}\chi_1)\right)\\
&\le \sum_{j=1}^{6}\left[\left(1 + |\Lambda_1|(1-q_1)\chi_1 \sum_{\kappa=0}^{\nu_1} q_1^{\kappa}\right)\|I^{\beta_1-1}b_{1j}\|_{L^1}\right]R^{\tau_{1j}} + |\rho_1\Lambda_1|\\
&+ \left(1 + |\Lambda_1|(1-q_1)\chi_1 \sum_{\kappa=0}^{\nu_1} q_1^{\kappa}\right)\|I^{\beta_1-1}b_{17}\|_{L^1}\\
&= \sum_{j=1}^{6} \mathcal{A}_{1j}R^{\tau_{1j}} + \mathcal{A}_{17} + |\rho_1\Lambda_1|.\end{aligned}$$

Similarly, by using Lemma (), we get

$$
\begin{aligned}
|\mathcal{D}^{\alpha_1} G_1(\mathfrak{U}, w)(Ł)| = & \Big| I^{\beta_1-\alpha_1}\mathcal{F}_1(Ł, \mathfrak{U}(Ł), w(Ł), \mathcal{D}^{\alpha_1}\mathfrak{U}(Ł), \mathcal{D}^{\alpha_2} w(Ł), I_{q_1}^{\delta_1}\mathfrak{U}(Ł), I_{q_2}^{\delta_2} w(Ł)) \\
& + \frac{\Lambda_1 \Gamma(\beta_1)}{\Gamma(\beta_1-\alpha_1)} Ł^{\beta_1-\alpha_1-1}\Big[\rho_1 - (1-q_1)\chi_1 \sum_{\kappa=0}^{\nu_1} q_1^{\kappa} I^{\beta_1}\mathcal{F}_1(q_1^{\kappa}\chi_1, \mathfrak{U}(q_1^{\kappa}\chi_1), w(q_1^{\kappa}\chi_1), \\
& \mathcal{D}^{\alpha_1}\mathfrak{U}(q_1^{\kappa}\chi_1), \mathcal{D}^{\alpha_2} w(q_1^{\kappa}\chi_1), I_{q_1}^{\delta_1}\mathfrak{U}(q_1^{\kappa}\chi_1), I_{q_2}^{\delta_2} w(q_1^{\kappa}\chi_1))\Big]\Big| \\
& \leq I^{\beta_1-\alpha_1}\Big|\mathcal{F}_1(Ł, \mathfrak{U}(Ł), w(Ł), \mathcal{D}^{\alpha_1}\mathfrak{U}(Ł), \mathcal{D}^{\alpha_2} w(Ł), I_{q_1}^{\delta_1}\mathfrak{U}(Ł), I_{q_2}^{\delta_2} w(Ł))\Big| \\
& + \frac{|\rho_1 \Lambda_1| \Gamma(\beta_1)}{\Gamma(\beta_1-\alpha_1)} + \frac{|\Lambda_1| \Gamma(\beta_1)}{\Gamma(\beta_1-\alpha_1)}(1-q_1)\chi_1 \sum_{\kappa=0}^{\nu_1} q_1^{\kappa} I^{\beta_1}\Big|\mathcal{F}_1(q_1^{\kappa}\chi_1, \mathfrak{U}(q_1^{\kappa}\chi_1), w(q_1^{\kappa}\chi_1), \\
& \mathcal{D}^{\alpha_1} u(q_1^{\kappa}\chi_1), \mathcal{D}^{\alpha_2} w(q_1^{\kappa}\chi_1), I_{q_1}^{\delta_1}\mathfrak{U}(q_1^{\kappa}\chi_1), I_{q_2}^{\delta_2} w(q_1^{\kappa}\chi_1))\Big| \\
& \leq I^{\beta_1-\alpha_1}\Big(\sum_{j=1}^{6} b_{1j}(Ł) R^{\tau_{1j}} + b_{17}(Ł)\Big) + \frac{|\rho_1 \Lambda_1| \Gamma(\beta_1)}{\Gamma(\beta_1-\alpha_1)} + \frac{|\Lambda_1| \Gamma(\beta_1)}{\Gamma(\beta_1-\alpha_1)} \\
& (1-q_1)\chi_1 \sum_{\kappa=0}^{\nu_1} q_1^{\kappa} I^{\beta_1}\Big(\sum_{j=1}^{6} b_{1j}(q_1^{\kappa}\chi_1) R^{\tau_{1j}} + b_{17}(q_1^{\kappa}\chi_1)\Big) \\
& \leq \sum_{j=1}^{6}\Big[\|I^{\beta_1-\alpha_1-1} b_{1j}\|_{L_1} + \frac{|\Lambda_1| \Gamma(\beta_1)}{\Gamma(\beta_1-\alpha_1)}(1-q_1)\chi_1 \sum_{\kappa=0}^{\nu_1} q_1^{\kappa}\|I^{\beta_1-1} b_{1j}\|_{L^1}\Big] R^{\tau_{1j}} \\
& + \Big[\|I^{\beta_1-\alpha_1-1} b_{17}\|_{L^1} + \frac{|\Lambda_1| \Gamma(\beta_1)}{\Gamma(\beta_1-\alpha_1)}(1-q_1)\chi_1 \sum_{\kappa=0}^{\nu_1} q_1^{\kappa}\|I^{\beta_1-1} b_{17}\|_{L^1}\Big] \\
& + \frac{|\rho_1 \Lambda_1| \Gamma(\beta_1)}{\Gamma(\beta_1-\alpha_1)} \\
& = \sum_{j=1}^{6} \mathcal{B}_{1j} R^{\tau_{1j}} + \mathcal{B}_{17} + \frac{|\rho_1 \Lambda_1| \Gamma(\beta_1)}{\Gamma(\beta_1-\alpha_1)}.
\end{aligned}
\tag{9}
$$

Analogously, we get

$$
\begin{aligned}
|I_{q_1}^{\delta_1} G_1(\mathfrak{U}, w)(Ł)| \leq & I_{q_1}^{\delta_1} I^{\beta_1}|\mathcal{F}_1(Ł, \mathfrak{U}(Ł), w(Ł), \mathcal{D}^{\alpha_1}\mathfrak{U}(Ł), \\
& \mathcal{D}^{\alpha_2} w(Ł), I_{q_1}^{\delta_1}\mathfrak{U}(Ł), I_{q_2}^{\delta_2} w(Ł))| + I_{q_1}^{\delta_1}|\rho_1 \Lambda_1| \\
& + I_{q_1}^{\delta_1}|\Lambda_1|(1-q_1)\chi_1 \sum_{\kappa=0}^{\nu_1} q_1^{\kappa} I^{\beta_1}|\mathcal{F}_1(q_1^{\kappa}\chi_1, \mathfrak{U}(q_1^{\kappa}\chi_1), w(q_1^{\kappa}\chi_1), \mathcal{D}^{\alpha_1}\mathfrak{U}(q_1^{\kappa}\chi_1), \\
& \mathcal{D}^{\alpha_2} w(q_1^{\kappa}\chi_1), I_{q_1}^{\delta_1}\mathfrak{U}(q_1^{\kappa}\chi_1), I_{q_2}^{\delta_2} w(q_1^{\kappa}\chi_1))| \\
& \leq I_{q_1}^{\delta_1}\Big(\sum_{j=1}^{6} \mathcal{A}_{1j} R^{\tau_{1j}} + \mathcal{A}_{17} + |\rho_1 \Lambda_1|\Big).
\end{aligned}
$$

Using lemma (), we get

$$
|I_{q_1}^{\delta_1} G_1(\mathfrak{U}, w)(Ł)| = \frac{1}{\Gamma_{q_1}(\delta_1+1)}\Big(\sum_{j=1}^{6} \mathcal{A}_{1j} R^{\tau_{1j}} + \mathcal{A}_{17} + |\rho_1 \Lambda_1|\Big).
$$

Therefore,

$$\|G_1(\mathfrak{U}, w)\|_{\mathcal{X}} \leq \sum_{j=1}^{6} C_{1j}R^{\tau_{1j}} + C_{17} + E_1 \leq \frac{1}{16}\mathfrak{r} + \frac{1}{16}\mathfrak{r} + \frac{1}{16}\mathfrak{r} + \frac{1}{16}\mathfrak{r} + \frac{1}{16}\mathfrak{r} + \frac{1}{16}\mathfrak{r} + \frac{1}{16}\mathfrak{r} + \frac{1}{16}\mathfrak{r} = \frac{\mathfrak{r}}{2}.$$

Similarly, we have

$$\|G_2(\mathfrak{U}, w)\|_{\mathcal{X}} \leq \sum_{j=1}^{6} C_{2j}R^{\tau_{2j}} + C_{27} + E_2 \leq \frac{1}{16}\mathfrak{r} + \frac{1}{16}\mathfrak{r} + \frac{1}{16}\mathfrak{r} + \frac{1}{16}\mathfrak{r} + \frac{1}{16}\mathfrak{r} + \frac{1}{16}\mathfrak{r} + \frac{1}{16}\mathfrak{r} + \frac{1}{16}\mathfrak{r} = \frac{\mathfrak{r}}{2}.$$

Thus,

$$\|G(\mathfrak{U}, w)\|_{\mathcal{X}\times\mathcal{Y}} = \|G_1(\mathfrak{U}, w)\|_{\mathcal{X}} + \|G_2(\mathfrak{U}, w)\|_{\mathcal{Y}} \leq \mathfrak{r}.$$

Then, $G : Q_{\mathfrak{r}} \to Q_{\mathfrak{r}}$, and also, the class of functions $\{G(\mathfrak{U}, w)(Ł)\}$ is uniformly bounded in $Q_{\mathfrak{r}}$. Observe that $G_1(\mathfrak{U}, w)(Ł), G_2(\mathfrak{U}, w)(Ł), \mathcal{D}^{\alpha_1}G_1(\mathfrak{U}, w)(Ł)$, $\mathcal{D}^{\alpha_2}G_2(\mathfrak{U}, w)(Ł), I_{q_1}^{\delta_1}G_1(\mathfrak{U}, w)(Ł), I_{q_2}^{\delta_2}G_2(\mathfrak{U}, w)(Ł)$ are continuous on [0, 1]. Clearly, G is also continuous. Next, we demonstrate that G isequicontinuous. Let

$$N_i = \max_{Ł\in[0,1]} \left\{ |\mathcal{F}_i(Ł, \mathfrak{U}(Ł), w(Ł), \mathcal{D}^{\alpha_1}\mathfrak{U}(Ł), \mathcal{D}^{\alpha_2}w(Ł), I_{q_1}^{\delta_1}U(Ł), I_{q_2}^{\delta_2}w(Ł))| \right\}, \forall(\mathfrak{U}, w) \in Q_{\mathfrak{r}}, i = 1, 2.$$

For $Ł_1, Ł_2 \in [0, 1], Ł_1 < Ł_2$, we obtain

$$\begin{aligned}
&|G_1(\mathfrak{U}, w)(Ł_2) - G_1(\mathfrak{U}, w)(Ł_1)| \leq I^{\beta_1}\Big|\mathcal{F}_1\Big(Ł_2, \mathfrak{U}(Ł_2), w(Ł_2), \mathcal{D}^{\alpha_1}\mathfrak{U}(Ł_2), \mathcal{D}^{\alpha_2}w(Ł_2),\\
&I_{q_1}^{\delta_1}\mathfrak{U}(Ł_2), I_{q_2}^{\delta_2}w(Ł_2)\Big) - \mathcal{F}_1\Big(Ł_1, \mathfrak{U}(Ł_1), w(Ł_1), \mathcal{D}^{\alpha_1}\mathfrak{U}(Ł_1), \mathcal{D}^{\alpha_2}w(Ł_1), I_{q_1}^{\delta_1}\mathfrak{U}(Ł_1), I_{q_2}^{\delta_2}w(Ł_1)\Big)\Big|\\
&+ \Big[|\Lambda_1\rho_1| + |\Lambda_1|(1-q_1)\chi_1\sum_{\kappa=0}^{\nu_1} q_1^{\kappa} I^{\beta_1}\Big|\mathcal{F}_1(q_1^{\kappa}\chi_1, \mathfrak{U}(q_1^{\kappa}\chi_1), w(q_1^{\kappa}\chi_1), \mathcal{D}^{\alpha_1}\mathfrak{U}(q_1^{\kappa}\chi_1),\\
&\mathcal{D}^{\alpha_2}w(q_1^{\kappa}\chi_1), I_{q_1}^{\delta_1}\mathfrak{U}(q_1^{\kappa}\chi_1), I_{q_2}^{\delta_2}w(q_1^{\kappa}\chi_1))\Big|\Big](Ł_2^{\beta_1-1} - Ł_1^{\beta_1-1})\\
&\leq N_1\left[\int_0^{Ł_1} \frac{(Ł_2 - s)^{\beta_1-1} - (Ł_1 - s)^{\beta_1-1}}{\Gamma(\beta_1)}ds + \int_{Ł_1}^{Ł_2} \frac{(Ł_2 - s)^{\beta_1-1}}{\Gamma(\beta_1)}ds\right]\\
&+ \left[|\Lambda_1\rho_1| + N_1|\Lambda_1|(1-q_1)\chi_1\sum_{\kappa=0}^{\nu_1} q_1^{\kappa}\int_0^{q_1^{\kappa}\chi_1} \frac{(q_1^{\kappa}\chi_1 - s)^{\beta_1-1}}{\Gamma(\beta_1)}ds\right](Ł_2^{\beta_1-1} - Ł_1^{\beta_1-1})\\
&\leq \frac{N_1}{\Gamma(\beta_1+1)}(Ł_2^{\beta_1} - Ł_1^{\beta_1}) + \left[|\rho_1\Lambda_1| + N_1|\Lambda_1|(1-q_1)\chi_1\sum_{\kappa=0}^{\nu_1} q_1^{\kappa}\frac{(q_1^{\kappa}\chi_1)^{\beta_1}}{\Gamma(\beta_1+1)}\right]\\
&(Ł_2^{\beta_1-1} - Ł_1^{\beta_1-1}).
\end{aligned}$$

In contrast, we obtain

$$|\mathcal{D}^{\alpha_1} G_1(\mathfrak{U}, w)(Ł_2) - \mathcal{D}^{\alpha_1} G_1(\mathfrak{U}, w)(Ł_1)| \leq \Bigg| I^{\beta_1-\alpha_1} \Bigg[\mathcal{F}_1 \Bigg(Ł_2, \mathfrak{U}(Ł_2), w(Ł_2), \mathcal{D}^{\alpha_1} \mathfrak{U}(Ł_2), \mathcal{D}^{\alpha_2} w(Ł_2),$$
$$I_{q_1}^{\delta_1} \mathfrak{U}(Ł_2), I_{q_2}^{\delta_2} w(Ł_2) \Bigg) - \mathcal{F}_1 \Bigg(Ł_1, \mathfrak{U}(Ł_1), w(Ł_1), \mathcal{D}^{\alpha_1} \mathfrak{U}(Ł_1), \mathcal{D}^{\alpha_2} w(Ł_1), I_{q_1}^{\delta_1} \mathfrak{U}(Ł_1), I_{q_2}^{\delta_2} w(Ł_1) \Bigg) \Bigg] \Bigg|$$
$$+ \left[\frac{|\rho_1 \Lambda_1| \Gamma(\beta_1)}{\Gamma(\beta_1 - \alpha_1)} + |\Lambda_1|(1 - q_1)\chi_1 \sum_{\kappa=0}^{\nu_1} q_1^{\kappa} I^{\beta_1} \right.$$
$$\left. \Big| \mathcal{F}_1(q_1^{\kappa}\chi_1, \mathfrak{U}(q_1^{\kappa}\chi_1), w(q_1^{\kappa}\chi_1), \mathcal{D}^{\alpha_1} \mathfrak{U}(q_1^{\kappa}\chi_1), \mathcal{D}^{\alpha_2} w(q_1^{\kappa}\chi_1)) \Big| \right] (Ł_2^{\beta_1-\alpha_1-1} - Ł_1^{\beta_1-\alpha_1-1})$$
$$\leq N_1 \left[\int_0^{Ł_1} \frac{(Ł_2 - s)^{\beta_1-\alpha_1-1} - (Ł_1 - s)^{\beta_1-\alpha_1-1}}{\Gamma(\beta_1 - \alpha_1)} ds + \int_{Ł_1}^{Ł_2} \frac{(Ł_2 - s)^{\beta_1-\alpha_1-1}}{\Gamma(\beta_1 - \alpha_1)} ds \right]$$
$$+ \left[\frac{|\rho_1 \Lambda_1| \Gamma(\beta_1)}{\Gamma(\beta_1 - \alpha_1)} + N_1 |\Lambda_1|(1 - q_1)\chi_1 \sum_{\kappa=0}^{\nu_1} q_1^{\kappa} \int_0^{q_1^{\kappa}\chi_1} \frac{(q_1^{\kappa}\chi_1 - s)^{\beta_1-1}}{\Gamma(\beta_1)} \right]$$
$$(Ł_2^{\beta_1-\alpha_1-1} - Ł_1^{\beta_1-\alpha_1-1})$$
$$\leq \frac{N_1}{\Gamma(\beta_1 - \alpha_1 + 1)} (Ł_2^{\beta_1-\alpha_1} - Ł_1^{\beta_1-\alpha_1}) + \left[\frac{|\rho_1 \Lambda_1| \Gamma(\beta_1)}{\Gamma(\beta_1 - \alpha_1)} + N_1 |\Lambda_1|(1 - q_1)\chi_1 \right.$$
$$\left. \sum_{\kappa=0}^{\nu_1} q_1^{\kappa} \frac{(q_1^{\kappa}\chi_1)^{\beta_1}}{\Gamma(\beta_1 + 1)} \right] (Ł_2^{\beta_1-\alpha_1-1} - Ł_1^{\beta_1-\alpha_1-1}),$$

and also,

$$|I_{q_1}^{\delta_1} G_1(\mathfrak{U}, w)(Ł_2) - I_{q_1}^{\delta_1} G_1(\mathfrak{U}, w)(Ł_1)| \leq I_{q_1}^{\delta_1} \frac{N_1}{\Gamma(\beta_1 + 1)} (Ł_2^{\beta_1} - Ł_1^{\beta_1}) + I_{q_1}^{\delta_1} \Bigg[|\rho_1 \Lambda_1|$$
$$+ N_1 |\Lambda_1|(1 - q_1)\chi_1 \sum_{\kappa=0}^{\nu_1} q_1^{\kappa} \frac{(q_1^{\kappa}\chi_1)^{\beta_1}}{\Gamma(\beta_1 + 1)} \Bigg] (Ł_2^{\beta_1-1} - Ł_1^{\beta_1-1})$$
$$\leq \frac{N_1}{\Gamma(\beta_1 + 1)} (Ł_2^{\beta_1} - Ł_1^{\beta_1}) \frac{Ł^{\delta_1}}{\Gamma_{q_1}(\delta_1 + 1)} + \frac{Ł^{\delta_1}}{\Gamma_{q_1}(\delta_1 + 1)} \Bigg[|\rho_1 \Lambda_1|$$
$$+ N_1 |\Lambda_1|(1 - q_1)\chi_1 \sum_{\kappa=0}^{\nu_1} q_1^{\kappa} \frac{(q_1^{\kappa}\chi_1)^{\beta_1}}{\Gamma(\beta_1 + 1)} \Bigg] (Ł_2^{\beta_1-1} - Ł_1^{\beta_1-1})$$
$$\leq \frac{N_1}{\Gamma(\beta_1 + 1)\Gamma_{q_1}(\delta_1 + 1)} (Ł_2^{\beta_1} - Ł_1^{\beta_1}) + \frac{1}{\Gamma_{q_1}(\delta_1 + 1)} \Bigg[|\rho_1 \Lambda_1|$$
$$+ N_1 |\Lambda_1|(1 - q_1)\chi_1 \sum_{\kappa=0}^{\nu_1} q_1^{\kappa} \frac{(q_1^{\kappa}\chi_1)^{\beta_1}}{\Gamma(\beta_1 + 1)} \Bigg] (Ł_2^{\beta_1-1} - Ł_1^{\beta_1-1}).$$

Similar to that, we can demonstrate

$$|G_2(\mathfrak{U}, w)(Ł_2) - G_2(\mathfrak{U}, w)(Ł_1)| \leq \frac{N_2}{\Gamma(\beta_2 + 1)} (Ł_2^{\beta_2} - Ł_1^{\beta_2})$$
$$+ \left[|\rho_2 \Lambda_2| + N_2 |\Lambda_2|(1 - q_2)\chi_2 \sum_{\kappa=0}^{\nu_2} q_2^{\kappa} \frac{(q_2^{\kappa}\chi_2)^{\beta_2}}{\Gamma(\beta_2 + 1)} \right] (Ł_2^{\beta_2-1} - Ł_1^{\beta_2-1}),$$

$$|\mathcal{D}^{\alpha_2}G_2(\mathfrak{U},w)(Ł_2)-\mathcal{D}^{\alpha_2}G_2(\mathfrak{U},w)(Ł_1)|\le\frac{N_2}{\Gamma(\beta_2-\alpha_2+1)}(Ł_2^{\beta_2-\alpha_2}-Ł_1^{\beta_2-\alpha_2})$$
$$+\left[|\rho_2\Lambda_2|\frac{\Gamma(\beta_2)}{\Gamma(\beta_2-\alpha_2)}+N_2|\Lambda_2|(1-q_2)\chi_2\sum_{\kappa=0}^{\nu_2}q_2^{\kappa}\frac{(q_2^{\kappa}\chi_2)^{\beta_2}}{\Gamma(\beta_2+1)}\right](Ł_2^{\beta_2-\alpha_2-1}-Ł_1^{\beta_2-\alpha_2-1}),$$

$$|I_{q_2}^{\delta_2}G_2(\mathfrak{U},w)(Ł_2)-I_{q_2}^{\delta_2}G_2(\mathfrak{U},w)(Ł_1)|\le\frac{N_2}{\Gamma(\beta_2+1)\Gamma_{q_2}(\delta_2+1)}(Ł_2^{\beta_2}-Ł_1^{\beta_2})+\frac{1}{\Gamma_{q_2}(\delta_2+1)}\Bigg[|\rho_2\Lambda_2|$$
$$+N_2|\Lambda_2|(1-q_2)\chi_2\sum_{\kappa=0}^{\nu_2}q_2^{\kappa}\frac{(q_2^{\kappa}\chi_2)^{\beta_2}}{\Gamma(\beta_2+1)}\Bigg](Ł_2^{\beta_2-1}-Ł_1^{\beta_2-1}).$$

Letting $Ł_1\to Ł_2$, then

$$\begin{aligned}&|G_1(\mathfrak{U},w)(Ł_2)-G_1(\mathfrak{U},w)(Ł_1)|\to0,\quad |\mathcal{D}^{\alpha_1}G_1(\mathfrak{U},w)(Ł_2)-\mathcal{D}^{\alpha_1}G_1(\mathfrak{U},w)(Ł_1)|\to0,\\&|I_{q_1}^{\delta_1}G_1(\mathfrak{U},w)(Ł_2)-I_{q_1}^{\delta_1}G_1(\mathfrak{U},w)(Ł_1)|\to0,\\&|G_2(\mathfrak{U},w)(Ł_2)-G_2(\mathfrak{U},w)(Ł_1)|\to0,\quad |\mathcal{D}^{\alpha_2}G_2(\mathfrak{U},w)(Ł_2)-\mathcal{D}^{\alpha_2}G_2(\mathfrak{U},w)(Ł_1)|\to0,\\&|I_{q_2}^{\delta_2}G_2(\mathfrak{U},w)(Ł_2)-I_{q_2}^{\delta_2}G_2(\mathfrak{U},w)(Ł_1)|\to0.\end{aligned}$$

Therefore,

$$\|G_1(\mathfrak{U},w)(Ł_2)-G_1(\mathfrak{U},w)(Ł_1)\|_{\mathcal{X}}\to0,\quad \|G_2(\mathfrak{U},w)(Ł_2)-G_2(\mathfrak{U},w)(Ł_1)\|_{\mathcal{Y}}\to0.$$

That is, as $Ł_1\to Ł_2$,

$$\|G(\mathfrak{U},w)(Ł_2)-G(\mathfrak{U},w)(Ł_1)\|_{\mathcal{X}\times\mathcal{Y}}\to0.$$

As a result, we establish that the operator G is equicontinuous. Then, using the Arzela–Ascoli theorem, we conclude that G is a completely continuous operator. So, it follows from the Schauder fixed theorem that (1)–(2) possesses at least one solution $(\mathfrak{U},w)\in Q_{\mathfrak{r}}$. This proof has been established. □

Now, we demonstrate the uniqueness of solutions using the Banach contraction principle.

Theorem 3.2 *Suppose that $\mathcal{F}_i$ are continuous for almost all $Ł\in(0,1)$ and measurable in t for any $w_1,w_2,w_3,w_4,w_5,w_6\in\mathbb{R}$. There exist nonnegative functions $b_{ij}(Ł)\in L^1([0,1],\mathbb{R}_+), i=1,2, j=1,2,\dots,7$, such that the following requirements are fulfilled*:

(H_1) $|\mathcal{F}_i(Ł,w_1,w_2,w_3,w_4,w_5,w_6)-\mathcal{F}_i(Ł,z_1,z_2,z_3,z_4,z_5,z_6)|\le\sum_{j=1}^{6}b_{ij}(Ł)|w_j-z_j|, i=1,2.$
(H_1) $3D_1+3D_2<1.$

Following that, the coupled system (1)–(2) possesses a unique solution.
Proof Let $\sup_{Ł\in[0,1]}\mathcal{F}_i(Ł,0,0,0,0,0,0)=\mu_i<\infty, i=1,2$ and take

$$R \geq \frac{\mu_1' + \mu_2'}{1 - 3D_1 - 3D_2},$$

where $\mu_i' = |\rho_i \Lambda_i| + \frac{|\rho_i \Lambda_i| \Gamma(\beta_i)}{\Gamma(\beta_i - \alpha_i)} + \frac{|\rho_i \Lambda_i|}{\Gamma_{q_i}(\delta_i+1)} + \Big[\frac{1}{\Gamma(\beta_i-\alpha_i+1)} + \frac{(1+|\Lambda_i|(1-q_i)\chi_i \sum_{\kappa=0}^{\nu_i} q_i^{\kappa})}{\Gamma(\beta_i+1)}(1 + \frac{1}{\Gamma_{q_i}(\delta_i+1)}) + \frac{|\Lambda_i|(1-q_i)\chi_i \sum_{\kappa=0}^{\nu_i} q_i^{\kappa}}{\Gamma(\beta_i-\alpha_i)\Gamma(\beta_i+1)}\Big]\mu_i$. First, we prove that $G(Q_R) \subset Q_R$, where $Q_R = \{(\mathfrak{U}, w) | (\mathfrak{U}, w) \in \mathcal{X} \times \mathcal{Y} : \|(\mathfrak{U}, w)\|_{\mathcal{X}\times\mathcal{Y}} \leq R\}$. For any $(\mathfrak{U}, w)(Ł) \in Q_R$, we can obtain

$$\begin{aligned}
|G_1(\mathfrak{U}, w)(Ł)| \leq & I^{\beta_1}\Big[\Big|\mathcal{F}_1\Big(Ł, \mathfrak{U}(Ł), w(Ł), \mathcal{D}^{\alpha_1}\mathfrak{U}(Ł), \mathcal{D}^{\alpha_2} w(Ł), I_{q_1}^{\delta_1}\mathfrak{U}(Ł), I_{q_2}^{\delta_2} w(Ł)\Big) \\
& - \mathcal{F}_1\Big(Ł, 0, 0, 0, 0, 0, 0\Big)\Big| + |\mathcal{F}_1(Ł, 0, 0, 0, 0, 0, 0)|\Big] + |\rho_1 \Lambda_1| \\
& + |\Lambda_1|(1-q_1)\chi_1 \sum_{\kappa=0}^{\nu_1} q_1^{\kappa} I^{\beta_1}\Big|\mathcal{F}_1\Big(q_1^{\kappa}\chi_1, \mathfrak{U}(q_1^{\kappa}\chi_1), w(q_1^{\kappa}\chi_1), \mathcal{D}^{\alpha_1}\mathfrak{U}(q_1^{\kappa}\chi_1), \mathcal{D}^{\alpha_2} w(q_1^{\kappa}\chi_1), \\
& I_{q_1}^{\delta_1}\mathfrak{U}(q_1^{\kappa}\chi_1), I_{q_2}^{\delta_2} w(q_1^{\kappa}\chi_1)\Big) - \mathcal{F}_1\Big(q_1^{\kappa}\chi_1, 0, 0, 0, 0, 0, 0\Big)\Big| + |\mathcal{F}_1(q_1^{\kappa}\chi_1, 0, 0, 0, 0, 0, 0)|\Big] \\
\leq & I^{\beta_1}\Big[(b_{11}(Ł) + b_{13}(Ł) + b_{15}(Ł))\|\mathfrak{U}\|_{\mathcal{X}} + (b_{12}(Ł) + b_{14}(Ł) + b_{16}(Ł))\|w\|_{\mathcal{Y}} + \mu_1\Big] \\
& + |\rho_1 \Lambda_1| + |\Lambda_1|(1-q_1)\chi_1 \sum_{\kappa=0}^{\nu_1} q_1^{\kappa} I^{\beta_1}\Big[(b_{11}(Ł) + b_{13}(Ł) + b_{15}(Ł))\|\mathfrak{U}\|_{\mathcal{X}} \\
& + (b_{12}(Ł) + b_{14}(Ł) + b_{16}(Ł))\|w\|_{\mathcal{Y}} + \mu_1\Big] \\
\leq & (\mathcal{A}_{11} + \mathcal{A}_{13} + \mathcal{A}_{15})\|\mathfrak{U}\|_{\mathcal{X}} + (\mathcal{A}_{12} + \mathcal{A}_{14} + \mathcal{A}_{16})\|w\|_{\mathcal{Y}} + |\rho_1 \Lambda_1| \\
& + \frac{(1 + |\Lambda_1|(1-q_1)\chi_1 \sum_{\kappa=0}^{\nu_1} q_1^{\kappa})\mu_1}{\Gamma(\beta_1 + 1)}.
\end{aligned}$$

As an alternative, using (9), we may get

$$\begin{aligned}
|\mathcal{D}^{\alpha_1} G_1(\mathfrak{U}, w)(Ł)| \leq & (\mathcal{B}_{11} + \mathcal{B}_{13} + \mathcal{B}_{15})\|\mathfrak{U}\|_{\mathcal{X}} + (\mathcal{B}_{12} + \mathcal{B}_{14} + \mathcal{B}_{16})\|w\|_{\mathcal{Y}} \\
& + \frac{\mu_1}{\Gamma(\beta_1 - \alpha_1 + 1)} + \frac{|\rho_1 \Lambda_1| \Gamma(\beta_1)}{\Gamma(\beta_1 - \alpha_1)} + \frac{|\Lambda_1|(1-q_1)\chi_1 \sum_{\kappa=0}^{\nu_1} q_1^{\kappa}}{\Gamma(\beta_1 - \alpha_1)\Gamma(\beta_1 + 1)}\mu_1.
\end{aligned}$$

Also,

$$\begin{aligned}
|I_{q_1}^{\delta_1} G_1(\mathfrak{U}, w)(Ł)| \leq & \frac{1}{\Gamma_{q_1}(\delta_1 + 1)}(\mathcal{A}_{11} + \mathcal{A}_{13} + \mathcal{A}_{15})\|\mathfrak{U}\|_{\mathcal{X}} + \frac{1}{\Gamma_{q_1}(\delta_1 + 1)} \\
& (\mathcal{A}_{12} + \mathcal{A}_{14} + \mathcal{A}_{16})\|w\|_{\mathcal{Y}} + \frac{|\rho_1 \Lambda_1|}{\Gamma_{q_1}(\delta_1 + 1)} + \frac{(1 + |\Lambda_1|(1-q_1)\chi_1 \sum_{\kappa=0}^{\nu_1} q_1^{\kappa})\mu_1}{\Gamma(\beta_1 + 1)\Gamma_{q_1}(\delta_1 + 1)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}\|G_1(\mathfrak{U},w)(Ł)\| \leq & (C_{11}+C_{13}+C_{15})\|\mathfrak{U}\|_{\mathcal{X}}+(C_{12}+C_{14}+C_{16})\|w\|_{\mathcal{Y}}+\mu_1' \\ & \leq 3D_1R+\mu_1'.\end{aligned}$$

Similarly, we can get

$$\begin{aligned}\|G_2(\mathfrak{U},w)(Ł)\| \leq & (C_{21}+C_{23}+C_{25})\|\mathfrak{U}\|_{\mathcal{X}}+(C_{22}+C_{24}+C_{26})\|w\|_{\mathcal{Y}}+\mu_2' \\ & \leq 3D_2R+\mu_2'.\end{aligned}$$

Thus,

$$\|G(\mathfrak{U},w)\|_{X\times Y}=\|G_1(\mathfrak{U},w)\|_{\mathcal{X}}+\|G_2(\mathfrak{U},w)\|_{\mathcal{Y}} \leq 3(D_1+D_2)R+\mu_1'+\mu_2' \leq R.$$

Second, for any $(\mathfrak{U}_1,w_1)(Ł),(\mathfrak{U}_2,w_2)(Ł)\in Q_R$, we have

$$\begin{aligned}&|G_1(\mathfrak{U}_2,w_2)(Ł)-G_1(\mathfrak{U}_1,w_1)(Ł)| \leq (\mathcal{A}_{11}+\mathcal{A}_{13}+\mathcal{A}_{15}) \\ &\|\mathfrak{U}_2-\mathfrak{U}_1\|_{\mathcal{X}}+(\mathcal{A}_{12}+\mathcal{A}_{14}+\mathcal{A}_{16})\|w_2-w_1\|_{\mathcal{Y}},\end{aligned}$$

and

$$\begin{aligned}&\|\mathcal{D}^{\alpha_1}G_1(\mathfrak{U}_2,w_2)(Ł)-\mathcal{D}^{\alpha_1}G_1(\mathfrak{U}_1,w_1)(Ł)\| \\ &\leq (\mathcal{B}_{11}+\mathcal{B}_{13}+\mathcal{B}_{15})\|\mathfrak{U}_2-\mathfrak{U}_1\|_{\mathcal{X}}+(\mathcal{B}_{12}+\mathcal{B}_{14}+\mathcal{B}_{16})\|w_2-w_1\|_{\mathcal{Y}},\end{aligned}$$

also,

$$\begin{aligned}|I_{q_1}^{\delta_1}G_1(\mathfrak{U}_2,w_2)(Ł)-I_{q_1}^{\delta_1}G_1(\mathfrak{U}_1,w_1)(Ł)| \leq & \frac{1}{\Gamma_{q_1}(\delta_1+1)}(\mathcal{A}_{11}+\mathcal{A}_{13}+\mathcal{A}_{15})\|\mathfrak{U}_2-\mathfrak{U}_1\|_{\mathcal{X}} \\ & +\frac{1}{\Gamma_{q_1}(\delta_1+1)}(\mathcal{A}_{12}+\mathcal{A}_{14}+\mathcal{A}_{16})\|w_2-w_1\|_{\mathcal{Y}}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|G_1(\mathfrak{U}_2,w_2)-G_1(\mathfrak{U}_1,w_1)\| \leq & (C_{11}+C_{13}+C_{15})\|\mathfrak{U}_2-\mathfrak{U}_1\|_{\mathcal{X}}+(C_{12}+C_{14}+C_{16})\|w_2-w_1\|_{\mathcal{Y}} \\ & \leq 3D_1\|\mathfrak{U}_2-\mathfrak{U}_1\|_{\mathcal{X}}+3D_1\|w_2-w_1\|_{\mathcal{Y}}.\end{aligned}$$

Similarly, we can get

$$\|G_2(\mathfrak{U}_2,w_2)-G_2(\mathfrak{U}_1,w_1)\| \leq 3D_2\|\mathfrak{U}_2-\mathfrak{U}_1\|_{\mathcal{X}}+3D_2\|w_2-w_1\|_{\mathcal{Y}}.$$

Thus,

$$\|G(\mathfrak{U}_2,w_2)-G(\mathfrak{U}_1,w_1)\|_{\mathcal{X}\times\mathcal{Y}} \leq (3D_1+3D_2)\Big[\|\mathfrak{U}_2-\mathfrak{U}_1\|_{\mathcal{X}}+\|w_2-w_1\|_{\mathcal{Y}}\Big].$$

Since $3D_1+3D_2<1$, G is a contraction operator. This implies that G has a unique fixed point, and thus, (1)–(2) has a unique solution. □

4 Applications

We now give the two examples below to demonstrate our findings.

Test problem 1: Consider the following coupled system:

$$\begin{aligned}
\mathcal{D}^{\frac{9}{4}}\mathfrak{U}(Ł) &= b_{11}(Ł)(\mathfrak{U}(Ł))^{\tau_{11}} + b_{12}(Ł)(w(Ł))^{\tau_{12}} + b_{13}(Ł)(\mathcal{D}^{\frac{1}{2}}\mathfrak{U}(Ł))^{\tau_{13}} + b_{14}(Ł)(\mathcal{D}^{\frac{1}{5}}w(Ł))^{\tau_{14}} \\
&\quad + b_{15}(Ł)(I^{\frac{1}{4}}_{\frac{1}{7}}\mathfrak{U}(Ł))^{\tau_{15}} + b_{16}(Ł)(I^{\frac{1}{2}}_{\frac{1}{3}}w(Ł))^{\tau_{16}} + b_{17}(Ł), Ł \in (0,1), \\
\mathcal{D}^{\frac{5}{2}}w(Ł) &= b_{21}(Ł)(\mathfrak{U}(Ł))^{\tau_{21}} + b_{22}(Ł)(w(Ł))^{\tau_{22}} + b_{23}(Ł)(\mathcal{D}^{\frac{1}{2}}\mathfrak{U}(Ł))^{\tau_{23}} + b_{24}(Ł)(\mathcal{D}^{\frac{1}{5}}w(Ł))^{\tau_{24}} \\
&\quad + b_{25}(Ł)(I^{\frac{1}{4}}_{\frac{1}{7}}\mathfrak{U}(Ł))^{\tau_{25}} + b_{26}(Ł)(I^{\frac{1}{2}}_{\frac{1}{3}}w(Ł))^{\tau_{26}} + b_{27}(Ł), Ł \in (0,1), \\
&(1-q_1)\chi_1 \sum_{\kappa=0}^{\nu_1} q_1^{\kappa}\mathfrak{U}(q_1^{\kappa}\chi_1) = \rho_1, \quad \mathfrak{U}(0) = \mathfrak{U}'(0) = 0, \quad \chi_1 \in (0,1), \\
&(1-q_2)\chi_2 \sum_{\kappa=0}^{\nu_2} q_2^{\kappa}w(q_2^{\kappa}\chi_2) = \rho_2, \quad w(0) = w'(0) = 0, \quad \chi_2 \in (0,1),
\end{aligned}$$

where $\beta_1 = \frac{9}{4}, \quad \beta_2 = \frac{5}{2}, \quad \alpha_1 = \frac{1}{2}, \quad \alpha_2 = \frac{1}{5}, \quad q_1 = \frac{1}{7}, \quad q_2 = \frac{1}{3}, \quad \delta_1 = \frac{1}{4}, \quad \delta_2 = \frac{1}{2},$ $\chi_1 = 0.5, \quad \chi_2 = 0.4, \quad \nu_1 = 1, \quad \nu_2 = 2, \quad 0 < \tau_{ij} < 1 (j = 1, 2, \ldots, 6, \quad i = 1, 2)$ and $b_{ij}(Ł)(j = 1, 2, \ldots, 7, \quad i = 1, 2)$ are nonnegative functions. The test problem 1 must have at least one solution, according to Theorem ().

Test problem 2: Consider the following coupled system:

$$\begin{aligned}
\mathcal{D}^{\frac{5}{2}}\mathfrak{U}(Ł) &= \frac{Ł^5}{30}\mathfrak{U}(Ł) + \frac{Ł^4}{60}w(Ł) + \frac{Ł^4}{40}\mathcal{D}^{\frac{1}{2}}\mathfrak{U}(Ł) + \frac{Ł^6}{25}\mathcal{D}^{\frac{1}{5}}w(Ł) \\
&\quad + \frac{Ł^7}{45}I^{\frac{1}{4}}_{0.2}\mathfrak{U}(Ł) + \frac{Ł^5}{50}I^{\frac{1}{3}}_{0.5}w(Ł) + \frac{1}{50}(1-Ł)^5, Ł \in (0,1), \\
\mathcal{D}^{\frac{7}{3}}w(Ł) &= \frac{(1-Ł)^3}{150}\mathfrak{U}(Ł) + \frac{(1-Ł)^5}{100}w(Ł) + \frac{Ł^7}{100}\mathcal{D}^{\frac{1}{2}}\mathfrak{U}(Ł) + \frac{Ł^5}{80}\mathcal{D}^{\frac{1}{5}}w(Ł) \\
&\quad + \frac{Ł^8}{50}I^{\frac{1}{4}}_{0.2}\mathfrak{U}(Ł) + \frac{Ł^8}{100}I^{\frac{1}{3}}_{0.5}w(Ł) + \frac{1}{100}(1-Ł)^8, Ł \in (0,1), \\
&(1-q_1)\chi_1 \sum_{\kappa=0}^{\nu_1} q_1^{\kappa}\mathfrak{U}(q_1^{\kappa}\chi_1) = \rho_1, \quad \mathfrak{U}(0) = \mathfrak{U}'(0) = 0, \quad \chi_1 \in (0,1), \\
&(1-q_2)\chi_2 \sum_{\kappa=0}^{\nu_2} q_2^{\kappa}w(q_2^{\kappa}\chi_2) = \rho_2, \quad w(0) = w'(0) = 0, \quad \chi_2 \in (0,1),
\end{aligned}$$

where$\beta_1 = \frac{5}{2}, \quad \beta_2 = \frac{7}{3}, \quad \alpha_1 = \frac{1}{2}, \quad \alpha_2 = \frac{1}{5}, \quad q_1 = 0.2, \quad q_2 = 0.5, \quad \delta_1 = \frac{1}{4}, \quad \delta_2 = \frac{1}{3},$ $\chi_1 = 0.5, \quad \chi_2 = 0.4, \quad \nu_1 = \nu_2 = 4.$

Then, we have

$$|\mathcal{F}_1(Ł, w_1, w_2, w_3, w_4, w_5, w_6)| = \frac{Ł^5}{30}|w_1| + \frac{Ł^4}{60}|w_2| + \frac{Ł^4}{40}|w_3| + \frac{Ł^6}{25}|w_4| + \frac{Ł^7}{45}|w_5| + \frac{Ł^5}{50}|w_6| + \frac{1}{50}(1-Ł)^5,$$

$$|\mathcal{F}_2(Ł, w_1, w_2, w_3, w_4, w_5, w_6)| \leq \frac{(1-Ł)^3}{150}|w_1| + \frac{(1-Ł)^5}{100}|w_2| + \frac{Ł^7}{100}|w_3| + \frac{Ł^5}{80}|w_4| + \frac{Ł^8}{50}|w_5| + \frac{Ł^8}{100}|w_6| + \frac{1}{100}(1-Ł)^8,$$

and

$$|\mathcal{F}_1(Ł, w_1, w_2, w_3, w_4, w_5, w_6) - \mathcal{F}_1(Ł, z_1, z_2, z_3, z_4, z_5, z_6)| \leq \frac{Ł^5}{30}|w_1 - z_1| + \frac{Ł^4}{60}|w_2 - z_2| + \frac{Ł^4}{40}|w_3 - z_3| + \frac{Ł^6}{25}|w_4 - z_4| + \frac{Ł^7}{45}|w_5 - z_5| + \frac{Ł^5}{50}|w_6 - z_6| + \frac{1}{50}(1-Ł)^5,$$

$$|\mathcal{F}_2(Ł, w_1, w_2, w_3, w_4, w_5, w_6) - \mathcal{F}_2(Ł, z_1, z_2, z_3, z_4, z_5, z_6)| \leq \frac{(1-Ł)^3}{150}|w_1 - z_1| + \frac{(1-Ł)^5}{100}|w_2 - z_2| + \frac{Ł^7}{100}|w_3 - z_3| + \frac{Ł^5}{80}|w_4 - z_4| + \frac{Ł^8}{50}|w_5 - z_5| + \frac{Ł^8}{100}|w_6 - z_6| + \frac{1}{100}(1-Ł)^8,$$

where $b_{11} = \frac{Ł^5}{30}$, $b_{12} = \frac{Ł^4}{60}$, $b_{13} = \frac{Ł^4}{40}$, $b_{14} = \frac{Ł^6}{25}$, $b_{15} = \frac{Ł^7}{45}$, $b_{16} = \frac{Ł^5}{50}$, $b_{17} = \frac{(1-Ł)^5}{50}$, $b_{21} = \frac{(1-Ł)^3}{150}$, $b_{22} = \frac{(1-Ł)^5}{100}$, $b_{23} = \frac{Ł^7}{100}$, $b_{24} = \frac{Ł^5}{80}$, $b_{25} = \frac{Ł^8}{50}$, $b_{26} = \frac{Ł^8}{50}$, $b_{27} = \frac{(1-Ł)^8}{100}$. By direct calculation, we get $\mathcal{A}_{11} = 0.0095576$, $\mathcal{A}_{12} = 0.0062124$, $\mathcal{A}_{13} = 0.0093187$, $\mathcal{A}_{14} = 0.0091753$, $\mathcal{A}_{15} = 0.00419785$, $\mathcal{A}_{16} = 0.0057346$, $\mathcal{A}_{17} = 0.0155236$, $\mathcal{A}_{21} = 0.0108042$, $\mathcal{A}_{22} = 0.0110885$, $\mathcal{A}_{23} = 0.00381566$, $\mathcal{A}_{24} = 0.0069399$, $\mathcal{A}_{25} = 0.0065411$, $\mathcal{A}_{26} = 0.0032706$, $\mathcal{A}_{27} = 0.0075244$, $\mathcal{B}_{11} = 0.0154193$, $\mathcal{B}_{12} = 0.00974474$, $\mathcal{B}_{13} = 0.0146171$, $\mathcal{B}_{14} = 0.0151834$, $B_{15} = 0.00711007$, $\mathcal{B}_{16} = 0.00925156$, $\mathcal{B}_{17} = 0.01935415$, $\mathcal{B}_{21} = 0.0118813$, $\mathcal{B}_{22} = 0.0121674$, $\mathcal{B}_{23} = 0.00452789$, $\mathcal{B}_{24} = 0.00814884$, $\mathcal{B}_{25} = 0.00779734$, $\mathcal{B}_{26} = 0.0038987$, $\mathcal{B}_{27} = 0.008244$, $C_{11} = 0.0195134$, $C_{12} = 0.0126837$, $C_{13} = 0.0190256$, $C_{14} = 0.0339163$, $C_{15} = 0.0156807$, $C_{16} = 0.0209596$, $C_{17} = 0.0510482, C_{21} = 0.0343762$, $C_{22} = 0.0352543$, $C_{23} = 0.0124723$, $C_{24} = 0.022598$, $C_{25} = 0.0214163$, $C_{26} = 0.0107081$, $C_{27} = 0.0239103$. Thus, $D_1 = 0.051048$, $D_2 = 0.03525$. Therefore, $3D_1 + 3D_2 = 0.258907 < 1$. The coupled system (1)–(2) must have at least one solution, according to Theorem ().

5 Conclusion

In the present work, we have discussed the existence of solutions for a coupled system of Riemann–Liouville fractional q-integro-differential equations. We have established the conditions under which these solutions exist. Furthermore, we have demonstrated the uniqueness of the solution by utilizing the contraction principle. By employing the powerful tools of fixed-point theorems, specifically the Schauder fixed theorem and the Banach contraction principle, we have provided a rigorous analysis of the coupled system. These theorems have allowed us to establish the existence and uniqueness of solutions, showcasing their effectiveness in addressing complex mathematical problems. Additionally, we have presented two illustrative examples that highlight the practical applications of our results. These examples serve to demonstrate how the findings of our work can be applied to real-world scenarios, further emphasizing the relevance and significance of our research. In conclusion, this work contributes to the understanding of coupled systems of Riemann–Liouville fractional q-integro-differential equations by discussing the existence and uniqueness of solutions. Our findings provide a solid foundation for future research in this area and offer practical insights for solving similar problems in various fields of study.

Acknowledgements The authors extend their appreciation to the Deanship of Research and Graduate Studies at King Khalid University for funding this work through the Large Research Project under grant number RGP2/222/45.

Author contributions The first author developed the idea, the second and third authors developed theories and appropriate analysis, and the fourth author formulated these ideas as theories and prepared their proofs. Everyone participated in writing the paper and preparing it for its final form.

Data availability No underlying data were collected or produced in this study.

Declarations

Conflict of interest The writers state unequivocally that they do not have any Conflict of interest.

Ethics approval and consent to participate There are no issues that need to be resolved between the authors and any other party.

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