



Reliability evaluation of generalized exchanged X -cubes under the R_g -conditional restriction

Wanling Lin^{1,2} · Hongbin Zhuang^{1,2} · Xiao-Yan Li^{1,2} · Yufang Zhang^{1,2}

Accepted: 19 December 2023 / Published online: 26 January 2024

© The Author(s), under exclusive licence to Springer Science+Business Media, LLC, part of Springer Nature 2024

Abstract

In the realm of multiprocessor systems, the evaluation of interconnection network reliability holds utmost significance, both in terms of design and maintenance. The intricate nature of these systems calls for a systematic assessment of reliability metrics, among which, two metrics emerge as vital: connectivity and diagnosability. The R_g -conditional connectivity is the minimum number of processors whose deletion will disconnect the multiprocessor system and every processor has at least g fault-free neighbors. The R_g -conditional diagnosability is a novel generalized conditional diagnosability, which is the maximum number of faulty processors that can be identified under the condition that every processor has no less than g fault-free neighbors. In this paper, we first investigate the R_g -conditional connectivity of generalized exchanged X -cubes $GEX(s, t)$ and present the lower (upper) bounds of the R_g -conditional diagnosability of $GEX(s, t)$ under the PMC model. Applying our results, the R_g -conditional connectivity and the lower (upper) bounds of R_g -conditional diagnosability of generalized exchanged hypercubes, generalized exchanged crossed cubes, and locally generalized exchanged twisted cubes under the PMC model are determined. Our comparative analysis highlights the superiority of R_g -conditional diagnosability, showcasing its effectiveness in guiding reliability studies across a diverse set of networks.

Keywords Reliability · R_g -conditional restriction · PMC model · Fault tolerance · Generalized exchanged X -cubes

1 Introduction

Increasingly, the complex functionalities of emerging real-time applications, such as in automotive, industrial automation, and robotics domains, require *multiprocessor systems* (MPS for short) to be implemented. Compared to the uniprocessor, multiprocessor systems provide a multitude of advantages, such as superior performance

and reliability, enhanced reconfigurability, and scalability. With advancements in *very large-scale integration* (VLSI for short) and software technologies, multiprocessor systems may incorporate tens of thousands of processors. However, as multiprocessor systems scale up, the likelihood of processor failures doubles or even grows geometrically. Once the processor fails, it may significantly compromise the reliability of the system, such as data transmission failure, packet loss, and increased latency. It is important to quantify the effect of the faults so that fault-tolerant designs can be pursued. One fundamental criterion in the design of MPS is reliability [1]. However, the accurate identification and replacement of faulty processors remain a key challenge in maintaining the system's reliability.

As we all know, *connectivity* refers to the ability of the interconnection network to establish and maintain communication between processors when failures occur, which is an important metric to assess the fault tolerance of MPS. In general, connectivity is denoted by $\kappa(G)$, which is the minimum number of processors that make the system disconnected. However, the traditional connectivity in actual network applications has been subject to significant limitations. Specifically, it overlooks the practical possibility of all adjacent processors to a processor malfunctioning simultaneously [2]. To overcome this drawback, Esfahanian and Hakimi [3] introduced a novel measure known as the *restricted connectivity*, which restricts the simultaneous failure of adjacent processors of arbitrary one processor. Subsequently, Latifi et al. [4] proposed a generalized restricted connectivity notion, the *g-restricted connectivity* (i.e., R_g -conditional connectivity), which mandates that each processor should have at least g fault-free neighbors in the system. The R_g -conditional connectivity offers a more refined and comprehensive means of evaluating the resilience and robustness of a multiprocessor system against failures.

In addition, *system-level diagnostics* is a crucial technique to diagnose a system, as it enables the identification of all faulty processors within the system. Such a fault location method, based on test outcomes, possesses the potential for real-time diagnostics and can be automated. The *diagnosability* of a system is a pivotal metric in assessing its reliability, which refers to the maximum number of faulty processors that the system can self-identify. A system is said to be *t-diagnosable* when all faulty processors can be accurately detected, provided that the number of faulty processors does not exceed t [5]. The *PMC model*, proposed by Preparata et al. [5], is utilized to identify the faulty processors. To execute fault diagnosis in a system, this model assumes that a processor, acting as a *tester* u , sends a test message to its neighbor, acting as a *testee* v , where the ordered pair $\langle u, v \rangle$ represents the test. When u is fault-free, the outcome of $\langle u, v \rangle$ can be used to deduce the state of v . Specifically, if the outcome of $\langle u, v \rangle$ is 1 (resp. 0), then v is faulty (resp. fault-free). However, if u is faulty, then the outcome of $\langle u, v \rangle$ becomes unreliable, rendering the state of v unreliable as well. Meanwhile, Maeng and Malek [6] proposed a comparison-based model, the *MM model*, for diagnosing a system. In this model, a processor w called *comparator*, sends the same test to its neighbors u, v and then compares their responses. Assume that such a comparison is denoted by $(u, v)_w$. Specifically, $(u, v)_w = 0$ (resp. $(u, v)_w = 1$) indicates that the test outcomes for u and v are identical (resp. distinct). For a fault-free comparator w , $(u, v)_w = 0$ implies that both u

and v are fault-free, whereas $(u, v)_w = 1$ reveals that there exists at least one faulty processor between u and v . If the comparator w is faulty, then the result of $(u, v)_w$ is unreliable. Based on the MM model, Sengupta et al. [7] introduced the *MM* model* as a special case of the MM model, in which each processor must test any pair of adjacent processors. In this paper, we focus on the PMC model for diagnosing faulty processors in a system.

The distribution pattern of faulty processors has a significant impact on the diagnosability of a system. However, the classical diagnosability of a system is often limited due to the lack of restrictions on the distribution pattern of faulty processors. To address this limitation, Lai et al. [8] introduced the concept of *conditional diagnosability*, which imposes a restriction on the system such that all the neighbors of any processor cannot be faulty at the same time. Peng et al. [9] further extended this concept by proposing the *g -good-neighbor conditional diagnosability*, which aims to identify the maximum number of faulty processors under the condition that every fault-free processor has at least g fault-free neighbors. Over the years, researchers have conducted extensive studies on the conditional diagnosability and g -good-neighbor conditional diagnosability of various interconnection networks, such as star graph, alternating group networks, BC networks, dual cubes, and so on [4, 10–17]. In particular, Guo et al. [18] proposed a novel generalized system-level diagnosis measure called *R_g -conditional diagnosability*, which assumes that every processor has at least g good neighbors. This measure builds upon the concept of g -good-neighbor conditional diagnosability and extends it to cover a broader range of scenarios. Guo et al. also conducted a study of the R_g -conditional diagnosability of hypercubes under the PMC model. That is, the R_g -conditional diagnosability of n -dimensional hypercubes under the PMC model is $s^{2g}(n - 2g) + 2^{2g-1} - 1$. Later, Wang et al. [19] proposed that the lower bound of R_g -conditional diagnosability of n -dimensional hypercubes is $2^{2g-2}(2n - 2^{2g} + 1) + (n - g)2^{g-1} - 1$ under the PMC model when $g \geq 1$ and $2^{2g} \leq n - 1$. Yuan et al. [20] studied the relationships between the R_g -conditional connectivity $\kappa^g(G)$ and the R_g -conditional diagnosability $t_{R_g}(G)$ under the PMC model and MM* model and showed that $t_{R_g}(G) = \kappa^{2g+1}(G) + g$ under some reasonable conditions, except $t_{R_1}(G)$ under the MM* model. They also investigated the R_g -conditional diagnosability of star graphs and bubble-sort graphs under the PMC model and MM* model. In addition, Yuan et al. [21] also investigated the R_g -conditional diagnosability of general networks, such as hypercubes and exchanged hypercubes under the PMC model, and presented the lower and upper bounds of R_g -conditional diagnosability of networks under some reasonable conditions.

The study of fault tolerance in interconnection networks is of great importance for improving the reliability and stability of computer systems. Recently, Li et al. [22] proposed a new framework, called the *generalized exchanged X -cubes GEX(s, t)*, which enables the construction of network architectures using various connecting rules. The g -good-neighbor conditional diagnosability of the generalized exchanged X -cubes framework was also determined under the PMC model and MM* model. Generalized exchanged X -cubes possess several excellent properties, including small diameter, fewer edges, low cost, and low latency. These properties indicate

that the framework offers a well-balanced consideration of the network's hardware and communication costs. In addition, the generalized exchanged X -cube $GEX(s, t)$ is designed around the BC network, comprising three principal interconnection network types: *generalized exchanged hypercubes* [23], *generalized exchanged crossed cubes* and *locally generalized exchanged twisted cubes* (the detailed definitions are provided in Sect. 5). As shown in Fig. 1, the dual-cube-like network is a special case of generalized exchanged hypercubes, which has attracted several researchers to study its properties [24]. Also, the locally exchanged twisted cube [25] is a novel interconnection that scales upward with lower edge costs than the locally twisted cube and provides more interconnection flexibility. Observably, the generalized exchanged X -cube framework integrates numerous established, contemporary, and extensively embraced interconnection networks. This emphasizes the need for a thorough investigation into the reliability of $GEX(s, t)$, which stands as a universal approach for analyzing reliability across diverse networks.

To further enhance the reliability index of generalized exchanged X -cubes, this paper investigates the R_g -conditional connectivity and establishes the lower and upper bounds of R_g -conditional diagnosability under the PMC model. Overall, our study demonstrates the usefulness and applicability of the R_g -conditional diagnosability measure for evaluating the reliability of various interconnection networks. The major contributions of our work are as follows.

- We investigate the R_g -conditional connectivity of BC networks and generalized exchanged X -cubes $GEX(s, t)$.
- We establish the lower and upper bounds of R_g -conditional diagnosability of $GEX(s, t)$ under the PMC model.
- As applications, we determine the R_g -conditional connectivity, and the lower and upper bounds of R_g -conditional diagnosability of generalized exchanged hypercubes, generalized exchanged crossed cubes, and locally generalized exchanged twisted cubes under the PMC model.
- We compare the R_g -conditional connectivity (resp. diagnosability) with the traditional connectivity (resp. diagnosability) and g -good-neighbor conditional con-

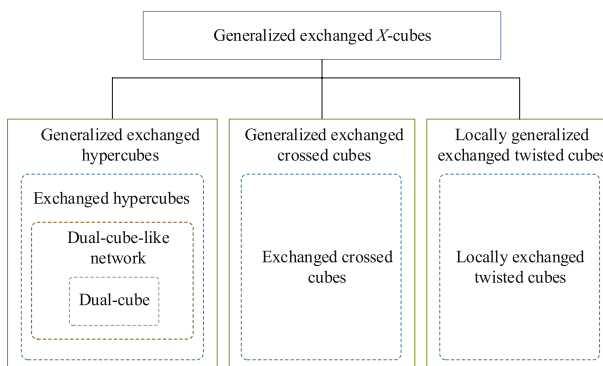


Fig. 1 The generalized exchanged X -cubes structure contains various popular networks

nectivity (resp. diagnosability) in $GEX(s, t)$. Moreover, for generalized exchanged hypercubes, the most typical case of $GEX(s, t)$, we conduct comprehensive comparisons involving R_g -conditional diagnosability, g -component diagnosability, and g -extra diagnosability in $GEH(s, t)$, along with evaluating R_g -conditional connectivity against a spectrum of novel connectivity.

- The comparative findings highlight the notable reliability of $GEX(s, t)$ under the R_g restriction, surpassing the performance of a majority of established connectivity and diagnosability.

Organization. The rest of this paper is organized as follows. Basic notations and definitions for generalized exchanged X -cubes, R_g -conditional diagnosability, and other terms are provided in Sect. 2. Section 3 demonstrates the R_g -conditional connectivity of generalized exchanged X -cubes. Section 4 shows the lower and upper bounds of R_g -conditional diagnosability of generalized exchanged X -cubes. Section 5 gives some applications based on the results regarding the R_g -conditional connectivity and the R_g -conditional diagnosability. Moreover, we draw some comparisons in Sect. 6. Finally, we offer some concluding remarks in the last section.

2 Preliminaries

2.1 Terminologies and notations

In this subsection, we provide Table 1 that describes some of the important notations used in this paper. It is a standard method to regard the multiprocessor system as a simple undirected graph $G = (V(G), E(G))$. For any node $u \in V(G)$, we define the *neighborhood* $N_G(u)$ of u in G to be the set of nodes adjacent to u . Also,

Table 1 Notations

Symbol	Meaning
$[n, m]$	The set of integers $\{n, n + 1, \dots, m\}$ where $n < m$
$V(G)$	The node set of a graph G
$E(G)$	The edge set of a graph G
(u, v)	An edge with two ends u and v
$d_G(v)$	The degree of the node v in G (or simply $d(v)$)
$G - U$	The graph obtained from G by removing $U \subseteq V(G)$
$G \cong H$	Two graphs G and H are isomorphic
$G[U]$	The subgraph of G induced by a subset $U \subseteq V(G)$
$\delta(G)$	The minimum degree of G
$\Delta(G)$	The maximum degree of G
K_n	The complete graph with n nodes
$F_1 \Delta F_2$	The symmetric difference of F_1 and F_2

we set $N_G(R) = \{v \in V(G) \setminus R | (u, v) \in E(G) \text{ and } u \in R\} = \bigcup_{u \in R} N_G(u) \setminus R$ and $N_G[R] = N_G(R) \cup R$ with $R \subseteq V(G)$. For neighborhoods, we always omit the subscript for the graph when no confusion arises. A graph G is called n -regular if $deg_G(u) = n$ for any node $u \in V(G)$. $G - R$ is denoted as $G[V(G) \setminus R]$, where R is called a *node cut* if $G - R$ is disconnected. Two binary strings $u = u_1u_0$ and $v = v_1v_0$ are *pair related*, denoted by $u \sim v$, if and only if $u, v \in \{(00, 00), (01, 11), (10, 10), (11, 01)\}$. The case that u and v are not pair related is denoted by $u \not\sim v$. A *component* is defined as a maximally connected subgraph of a graph. A *matching* M in G is a set of pairwise nonadjacent edges. M is a *perfect matching* of G if each node in $V(G)$ is incident with an edge in M . Given two sets $F_1, F_2 \subset V(G)$, the *symmetric difference* of F_1 and F_2 is denoted by $F_1 \triangle F_2 = (F_1 \setminus F_2) \cup (F_2 \setminus F_1)$.

2.2 Generalized exchanged X -cubes

In this subsection, we review the definition and properties of *generalized exchanged X -cubes*. Since generalized exchanged X -cubes are derived by *bijective connection networks* (BC networks for short), we first review the definition and properties of the BC networks. Specifically, BC networks are a class of cube-based graphs including several well-known interconnection graphs such as hypercubes [9], Möbius cubes [26], crossed cubes [27], and locally twisted cubes [28]. An n -dimensional BC network, denoted by X_n , is an n -regular graph with 2^n nodes and $n \times 2^{n-1}$ edges. The set of all the n -dimensional BC networks is called the *family of the n -dimensional BC networks*, denoted by \mathbb{L}_n . X_n and \mathbb{L}_n can be recursively defined as follows.

Definition 1 (See [29]). The one-dimensional BC network X_1 contains only two nodes that form an edge. The family of the one-dimensional BC network is defined as $\mathbb{L}_1 = \{X_1\}$. A graph G belongs to the family of n -dimensional BC networks \mathbb{L}_n if and only if there exist $V_0, V_1 \subset V(G)$ such that the following two conditions hold:

- (1) $V(G) = V_0 \cup V_1, V_0 \neq \emptyset, V_1 \neq \emptyset, V_0 \cap V_1 = \emptyset$, and $G[V_0], G[V_1] \in \mathbb{L}_{n-1}$; and
- (2) $E(V_0, V_1) = \{(u, v) | u \in V_0, v \in V_1, (u, v) \in E(G)\}$ is a perfect matching M of G .

We illustrate two three-dimensional BC networks in Fig. 2. In addition, for any $X_n \in \mathbb{L}_n$, there exist V_0, V_1 and M satisfying the above two conditions by Definition 1. It is clear that $G[V_0]$ and $G[V_1]$ are both $(n - 1)$ -dimensional BC networks, as well as $E(G[V_0]), E(G[V_1])$ and M are a decomposition of $E(X_n)$. Thus, we use X_{n-1}^0 (resp. X_{n-1}^1) to denote the induced subgraph $G[V_0]$ (resp. $G[V_1]$) and define the decomposition as $X_n = G(X_{n-1}^0, X_{n-1}^1; M)$.

Lemma 1 (See [30]). *Suppose that $0 \leq g \leq n$ and $Y \subset V(X_n)$. If $\delta(X_n[Y]) \geq g$, then $|Y| \geq 2^g$.*

Lemma 2 (See [31]). *If S is a subgraph of X_n with $|V(S)| = g + 1$ for $0 \leq g \leq n$, then $|N_{X_n}(V(S))| \geq (g + 1)n - \frac{g(g-1)}{2} - 2g$.*

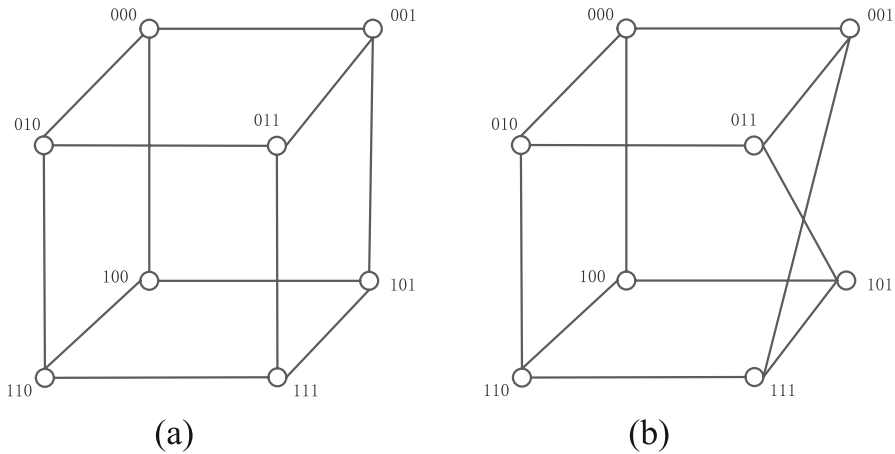


Fig. 2 Two three-dimensional BC networks

Lemma 3 (See [32]). *Suppose that $0 \leq g \leq n$, $n \geq 1$ and $Y \subset V(X_n)$. If $\delta(X_n[Y]) = g$, then $|N_{X_n}[Y]| \geq (n - g + 1)2^g$.*

Next, we introduce the definition and properties of generalized exchanged X-cubes.

Definition 2 (See [22]). The (s, t) -dimensional generalized exchanged X-cube is defined as a graph $GEX(s, t) = (V(GEX(s, t)), E(GEX(s, t)))$ with $s, t \geq 1$. $GEX(s, t)$ consists of two disjoint subgraphs \tilde{L} and \tilde{R} , where \tilde{L} consists of 2^t subgraphs, denoted by \tilde{L}_i for $i \in [1, 2^t]$. Similarly, \tilde{R} consists of 2^s subgraphs, denoted by \tilde{R}_j for $j \in [1, 2^s]$. Moreover, $GEX(s, t)$ satisfies the following conditions:

- (1) For any $1 \leq i \leq 2^t$ and $1 \leq j \leq 2^s$, $\tilde{L}_i \cong X_s$ and $\tilde{R}_j \cong X_t$. Further, $|V(\tilde{L}_i)| = 2^s$ and $|V(\tilde{R}_j)| = 2^t$;
- (2) Each node in $V(\tilde{L})$ has a unique neighbor in $V(\tilde{R})$ and vice versa. In addition, for distinct nodes in each \tilde{L}_i , their neighbors locate in different \tilde{R}_j ;
- (3) For any two different subgraphs \tilde{L}_i and $\tilde{L}_{i'}$ with $i \neq i'$, there exists no edge between them. Similar for \tilde{R}_j and $\tilde{R}_{j'}$ with $j \neq j'$.

According to Definition 2, we can deduce that $|V(GEX(s, t))| = 2^{s+t+1}$. Let each \tilde{L}_i and \tilde{R}_j be a cluster of $GEX(s, t)$. Obviously, $GEX(s, t)$ consists of $2^t + 2^s$ clusters. If we contract each cluster into a node, then $GEX(s, t)$ can be abstracted as a complete bipartite graph $K_{2^t, 2^s}$ (see Fig. 3). The edges that connect different clusters are called *cross edges*. In the following discussion, we consider $s \leq t$. Thus, $\delta(GEX(s, t)) = s + 1$ and $\Delta(GEX(s, t)) = t + 1$.

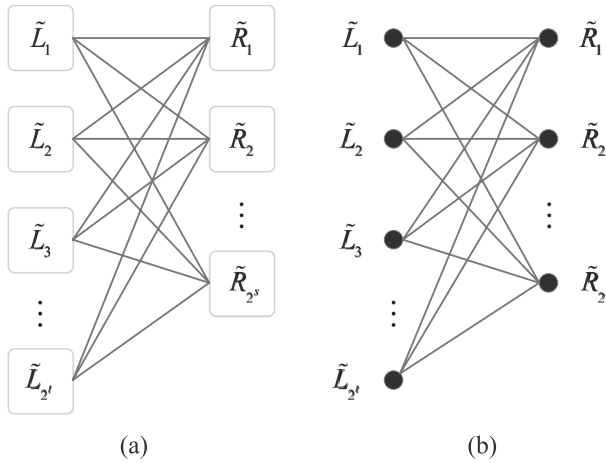


Fig. 3 **a** The partition of $GEX(s, t)$, where the rectangle boxes indicate the clusters; **b** The contraction of $GEX(s, t)$, where each cluster is represented as a node

Lemma 4 (See [22]). *For any integers $3 \leq s \leq t$ and $1 \leq g \leq s - 2$, $t_g(GEX(s, t)) = (s - g + 2)2^g - 1$ under the PMC model.*

2.3 The R_g -conditional diagnosability

Guo et al. [18] proposed a novel conditional diagnosability named R_g -conditional diagnosability, which requires every processor to have at least g good neighbors. In what follows, some concepts and results on R_g -conditional diagnosability of a system are listed.

Suppose that G is an n -regular graph and a set of faulty nodes is called a faulty set. Given a node $u \in V(G)$, \mathcal{A}_u is called a *forbidden faulty set* of u if and only if $\mathcal{A}_u \subset N(u)$ and $|\mathcal{A}_u| \geq n - g$ when $0 < g \leq n$ [33]. Let $R_g = \{Y \subset V(G) | \mathcal{A}_u \not\subset Y, \mathcal{A}_u \text{ is a forbidden faulty set of } u, \text{ for any } u \in V(G)\}$. Thus, if $F \in R_g$, then every node, including fault-free and faulty nodes, has at least g good neighbors (i.e., fault-free nodes). A faulty set F is called an R_g -conditional faulty set if $F \in R_g$. A node cut R is called an R_g -cut of G if $R \in R_g$.

Definition 3 (See [4]). A system G is R_g -connected if G has an R_g -cut. The R_g -conditional connectivity of G , denoted by $\kappa^g(G)$, is the minimum cardinality of all R_g -cuts in G .

Definition 4 (See [18]). A system G is R_g -conditionally t -diagnosable if an arbitrary pair of distinct R_g -conditional faulty sets (F_1, F_2) is distinguishable with

$|F_1|, |F_2| \leq t$. The R_g -conditional diagnosability of G , denoted by $t_{R_g}(G)$, is the maximum t such that G is R_g -conditionally t -diagnosable.

Lemma 5 (See [18]). *For any two distinct indistinguishable R_g -conditional faulty sets F_1 and F_2 of $G = (V(G), E(G))$, the following conditions are satisfied:*

- (1) *Each component U of $F_1 \triangle F_2$ has $\delta(U) \geq 2g$; and*
- (2) *Each component C of $V(G) \setminus (F_1 \cup F_2)$ has $\delta(C) \geq g$.*

In addition, we give a sufficient and necessary condition for two distinct subsets \hat{F}_1 and \hat{F}_2 to be a distinguishable pair under the PMC model.

Lemma 6 (See [34]). *For any two distinct subsets \hat{F}_1 and \hat{F}_2 in a system $G = (V(G), E(G))$, the sets \hat{F}_1 and \hat{F}_2 are distinguishable under the PMC model if and only if there exists at least one edge from $V(G) \setminus (\hat{F}_1 \cup \hat{F}_2)$ to $\hat{F}_1 \triangle \hat{F}_2$.*

3 The R_g -conditional connectivity of $GEX(s, t)$

In this section, we will investigate the R_g -conditional connectivity of $GEX(s, t)$. Before that, we first discuss the R_g -conditional connectivity of BC networks X_n and propose Lemma 7 as follows.

Lemma 7 *For BC networks X_n with $n \geq 3$ and $0 \leq g \leq \lfloor \frac{n}{2} \rfloor$, the R_g -conditional connectivity of X_n is $\kappa^g(X_n) = (n - g)2^g$.*

Proof At first, we prove that $\kappa^g(X_n) \leq (n - g)2^g$. Let F be a set of all neighbors of $V(X_g)$, that is $F = \{N_{X_n}(u) \setminus V(X_g) \text{ for } u \in V(X_g)\}$. Since X_n is an n -regular graph with 2^n nodes, we can deduce that $X_n - F$ is disconnected and $|F| = (n - g)2^g$. Then, we now verify that $F \in R_g$. Obviously, every node in X_g has at least g good neighbors. According to the structural properties of X_n , F is composed of $n - g$ disjoint subgraphs X_g . Hence, each node in F has at most g neighbors u with $u \in V(F)$ and has at least $n - g$ neighbors in $X_n - F$. Further, since $g \leq \lfloor \frac{n}{2} \rfloor$ and $n - g \geq g$, it is clear that every node in F has at least g good neighbors. Also, each node in $X_n - F - X_g$ has at most $n - g$ neighbors in F and $n - g \geq g$, thus any node in $X_n - F - X_g$ has at least g neighbors in $X_n - F$. This proves that such an F belongs to R_g .

Next, we prove $\kappa^g(X_n) \geq (n - g)2^g$ by induction on g . Assume that S is a minimum R_g -cut of X_n , Y is the node set of a minimum connected component of $X_n - S$, and the node set $Z = V(X_n - S - Y)$. Let $S_i = S \cap V(X_{n-1}^i)$, $Y_i = Y \cap V(X_{n-1}^i)$ and $Z_i = Y \cap V(X_{n-1}^i)$ for $i \in \{0, 1\}$. It is obvious that the result is true for $g = 0$. We

assume that the result is true for $g - 1$ with $g \geq 1$. The following two cases are considered.

Case 1: $Y_0 = \emptyset$ or $Y_1 = \emptyset$.

Without loss of generality, suppose that $Y_1 = \emptyset$, that is $Y \subseteq V(X_{n-1}^0)$. By Lemma 3 and $X_n = G(X_{n-1}^0, X_{n-1}^1; M)$, we can deduce that $|N_{X_n}(Y)| = |N_{X_{n-1}^0}(Y)| + |N_{X_{n-1}^1}(Y)| = |N_{X_{n-1}^0}(Y)| \geq ((n - 1) - g + 1)2^g = (n - g)2^g$. Since S is an R_g -cut, it shows that $N_{X_n}(Y) \subseteq S$. Thus, $|S| \geq |N_{X_n}(Y)| \geq (n - g)2^g$.

Case 2: $Y_0 \neq \emptyset$ and $Y_1 \neq \emptyset$.

Subcase 2.1: $Z_0 \neq \emptyset$ and $Z_1 \neq \emptyset$.

As S is an R_g -cut of X_n , for each $i \in \{0, 1\}$, every node $v \in Y_i \cup Z_i$ has at least g neighbors in $X_n - S$, as well as every node $u \in S_i$ has at least g neighbors in $X_n - S$. If a node v has at most one neighbor in $X_{n-1}^0 - S$ (resp. $X_{n-1}^1 - S$), then v has at least $g - 1$ neighbors in $X_{n-1}^1 - S$ (resp. $X_{n-1}^0 - S$). Moreover, a node $u \in S$ has at least $g - 1$ neighbors in $X_{n-1}^0 - S$ (resp. $X_{n-1}^1 - S$), which implies that S_0 (resp. S_1) is an R_{g-1} -cut of X_{n-1}^0 (resp. X_{n-1}^1). By the induction hypothesis, $|S_0| \geq [(n - 1) - (g - 1)]2^{g-1} = (n - g)2^{g-1}$ and $|S_1| \geq 2^{g-1}(n - g)$. Hence, $|S| = |S_0 \cup S_1| \geq (n - g)2^g$.

Subcase 2.2: $Z_0 = \emptyset$ or $Z_1 = \emptyset$.

Without loss of generality, suppose that $Z_0 \neq \emptyset$ and $Z_1 = \emptyset$. As discussed in Subcase 2.1, S_0 is an R_{g-1} -cut of X_{n-1}^1 and $|S_0| \geq (n - g)2^{g-1}$. Then, $|S_0| = |X_{n-1}^0| - |Y_0| - |Z_0|$ and $|S_1| = |X_{n-1}^1| - |Y_1|$. If $|S_1| < |S_0|$, then $|Z| < |Y_0| + |Z| < |Y_1|$, which is a contradiction. Therefore, $|S_1| \geq |S_0|$ and $|S| \geq 2|S_0| \geq (n - g)2^g$.

By the above discussion, we can conclude the R_g -conditional connectivity of X_n is $\kappa^g(X_n) = (n - g)2^g$. □

Lemma 8 For $GEX(s, t)$ with $3 \leq s \leq t$ and $1 \leq g \leq \lfloor \frac{s}{2} \rfloor$, the R_g -conditional connectivity of $GEX(s, t)$ is $\kappa^g(GEX(s, t)) \leq (s - g + 1)2^g$.

Proof By Definition 2, it shows that $GEX(s, t)$ is composed of two disjoint subgraphs \tilde{L} and \tilde{R} , where \tilde{L} (resp. \tilde{R}) can be partitioned into 2^i (resp. 2^s) subgraphs (clusters). Without loss of generality, suppose that there exists $A \subseteq V(\tilde{L}_1)$ such that $GEX(s, t)[A] \cong X_g$. By Definitions 1 and 2, we can get that $|A| = 2^g$, $\tilde{L}_1 \cong X_s$ and $|N_{\tilde{L}_1}(A)| = (s - g)2^g$. Moreover, each node in $V(\tilde{L})$ has an unique neighbor in $V(\tilde{R})$ by Definition 2. For distinct nodes in each \tilde{L}_i , their neighbors of \tilde{R} locate in different \tilde{R}_j , where $i \in [1, 2^i]$ and $j \in [1, 2^s]$. In addition, for any two different subgraphs \tilde{L}_i and $\tilde{L}_{i'}$ with $i \neq i'$, the edge between them does not exist, so each node in \tilde{L}_1 has only one neighbor in $GEX(s, t) - \tilde{L}_1$. Then, we can deduce that $|N_{GEX(s, t) - \tilde{L}_1}(A)| = |A| = 2^g$. Thus, we have

$$\begin{aligned} |N_{GEX(s, t)}(A)| &= |N_{\tilde{L}_1}(A)| + |N_{GEX(s, t) - \tilde{L}_1}(A)| \\ &= (s - g)2^g + 2^g \\ &= (s - g + 1)2^g. \end{aligned}$$

Since $|N_{GEX(s,t)}[A]| = (s - g + 1)2^g + 2^g = (s - g + 2)2^g$ and $|V(GEX(s, t))| = 2^{s+t+1} > (s - g + 2)2^g$, we can infer that $GEX(s, t) - N_{GEX(s,t)}(A)$ is disconnected and $N_{GEX(s,t)}(A)$ is a node cut of $GEX(s, t)$. It shows that $GEX(s, t)[A] \cong X_g$, and thus $\delta(GEX(s, t)[A]) = g$. By Lemma 7, $N_{\tilde{L}_1}(A)$ is an R_g -cut of \tilde{L}_1 with $1 \leq g \leq \lfloor \frac{s}{2} \rfloor$, thus $\delta(\tilde{L}_1 - N_{\tilde{L}_1}[A]) \geq g$. From $\delta(GEX(s, t)) = s + 1$ ($s \leq t$) and Definition 2, we can deduce that $\delta(GEX(s, t) - \tilde{L}_1 - N_{GEX(s,t)-\tilde{L}_1}(A)) \geq s + 1 - 1 > g$ for any node in $GEX(s, t) - \tilde{L}_1 - N_{GEX(s,t)-\tilde{L}_1}(A)$. Based on the above discussion, we have $\delta(GEX(s, t) - N_{GEX(s,t)}(A)) \geq g$. By Definition 2, for distinct nodes in each \tilde{L}_i , their neighbors of \tilde{R} locate in different \tilde{R}_j . For any two different subgraphs \tilde{R}_j and $\tilde{R}_{j'}$ with $j \neq j'$, there exists no edge between them. Hence, it shows that any node in $N_{GEX(s,t)-\tilde{L}_1}(A)$ does not have neighbors in $N_{GEX(s,t)-\tilde{L}_1}(A)$, but has at least s neighbors in $GEX(s, t) - N_{GEX(s,t)}(A)$. Since $N_{\tilde{L}_1}(A)$ is an R_g -cut of \tilde{L}_1 , each node in $N_{\tilde{L}_1}(A)$ has at least g neighbors in $\tilde{L}_1 - N_{\tilde{L}_1}(A)$ and $N_{GEX(s,t)}(A)$ is an R_g -cut of $GEX(s, t)$. Hence, $\kappa^g(GEX(s, t)) \leq (s - g + 1)2^g$ with $1 \leq g \leq \lfloor \frac{s}{2} \rfloor$ and $3 \leq s \leq t$, the lemma holds. \square

Lemma 9 For $GEX(s, t)$ with $3 \leq s \leq t$ and $1 \leq g \leq \lfloor \frac{s}{2} \rfloor$, the R_g -conditional connectivity of $GEX(s, t)$ is $\kappa^g(GEX(s, t)) \geq (s - g + 1)2^g$.

Proof Assume that I is a minimum R_g -cut of $GEX(s, t)$. Let $I_{\tilde{L}_i} = I \cap V(\tilde{L}_i)$ and $I_{\tilde{R}_j} = I \cap V(\tilde{R}_j)$ for each $1 \leq i \leq 2^t$ and $1 \leq j \leq 2^s$. Then, we will prove that $\kappa^g(GEX(s, t)) = |I| \geq (s - g + 1)2^g$ with $1 \leq g \leq \lfloor \frac{s}{2} \rfloor$ and $s \geq 3$ by the following three cases.

Case 1: $\tilde{L}_i - I_{\tilde{L}_i}$ and $\tilde{R}_j - I_{\tilde{R}_j}$ are connected for each i and j .

We prove this case by contradiction. Suppose that $|I| \leq (s - g + 1)2^g - 1$. Since I is a minimum R_g -cut of $GEX(s, t)$, it is no doubt that $GEX(s, t) - I$ is disconnected. Clearly, a component C necessarily traverses r clusters (i.e., subgraph \tilde{L}_i or \tilde{R}_j), $r \geq 1$, say B_1, B_2, \dots, B_r . Let $I_\alpha = B_\alpha \cap I$ with $\alpha \in [1, r]$. Accordingly, $C = \bigcup_{\alpha=1}^r (B_\alpha - I_\alpha)$. By Definition 2(2) and (3), we can deduce that at most $r - 1$ cross edges between $B_\alpha - I_\alpha$ and $B_{\beta \setminus \{\alpha\}}$ exist, where $\beta = \{1, 2, \dots, r\}$. Moreover, there are at least $2^s - |I_\alpha| - (r - 1)$ cross edges between $B_\alpha - I_\alpha$ and \mathcal{W} , where $\mathcal{W} = GEX(s, t) - \bigcup_{\alpha=1}^r B_\alpha$ and $s \leq t$. Let $\mathcal{M} = I \setminus \bigcup_{\alpha=1}^r I_\alpha$. Since there is no edge between $B_\alpha - I_\alpha$ and $\mathcal{W} - \mathcal{M}$, $|\mathcal{M}| \geq \sum_{\alpha=1}^r [2^s - |I_\alpha| - (r - 1)]$. Then, we have

$$\begin{aligned} |I| &= |I_1| + |I_2| + \dots + |I_r| + |\mathcal{M}| \\ &\geq |I_1| + |I_2| + \dots + |I_r| + \sum_{\alpha=1}^r [2^s - |I_\alpha| - (r - 1)] \\ &= r(2^s - r + 1). \end{aligned}$$

Let the function $f(r) = r(2^s - r + 1)$ with $r \geq 1$. We can obtain that $\frac{df(r)}{dr} = 2^s - 2r + 1 > 0$ and $f(r)$ is an increasing function. Thus, $f(r) \geq f(1) = 2^s$ and $|I| \geq f(r) \geq 2^s$. Similarly, we suppose that $f(g) = 2^s - [(s - g + 1)2^g - 1]$ with

$1 \leq g \leq \lfloor \frac{s}{2} \rfloor$. We can calculate that $\frac{\partial f(g)}{\partial g} = 2^g [g \ln 2 + 1 - (s + 1) \ln 2] < 0$, which implies that $f(g)$ is a decreasing function and $f(g) \geq f(\lfloor \frac{s}{2} \rfloor) > 0$. Clearly, $|I| \geq 2^s > (s - g + 1)2^g - 1$, which violates the assumption that $|I| \leq (s - g + 1)2^g - 1$.

Case 2: Only one of $\tilde{L}_i - I_{\tilde{L}_i}$ and $\tilde{R}_j - I_{\tilde{R}_j}$ is disconnected.

Similarly, we also prove this case by contradiction. Suppose that $|I| \leq (s - g + 1)2^g - 1$. Without loss of generality, we assume that $\tilde{L}_1 - I_{\tilde{L}_1}$ is disconnected. Since I is an R_g -cut of $GEX(s, t)$, $\delta(\tilde{L}_1 - I_{\tilde{L}_1}) \geq g - 1$ and $I_{\tilde{L}_1}$ is an R_{g-1} -cut of \tilde{L}_1 . By Lemma 7, it shows that $|I_{\tilde{L}_1}| \geq [s - (g - 1)]2^{g-1} = (s - g + 1)2^{g-1}$. Let $I_1 = I_{\tilde{L}_1}$ and $\mathcal{P} = I \setminus I_1$. Then, we can deduce that $|\mathcal{P}| \leq (s - g + 1)2^{g-1} - 1$.

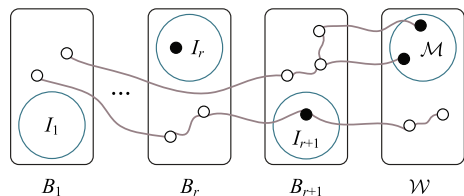
Moreover, we assume that $GEX(s, t) - \tilde{L}_1 - \mathcal{P}$ is disconnected. At this time, a component C must traverse r clusters with $r \geq 1$, that is, B_2, B_3, \dots, B_{r+1} . Let $I_\alpha = B_\alpha \cap I$ with $\alpha \in [2, r + 1]$. Accordingly, $C = \bigcup_{\alpha=2}^{r+1} (B_\alpha - I_\alpha)$. By Definition 2(2) and (3), we can deduce that at most r cross edges between $B_\alpha - I_\alpha$ and $B_{\beta \setminus \{\alpha\}}$ exist, where $\beta = \{1, 2, \dots, r + 1\}$. Meanwhile, there are at least $2^s - |I_\alpha| - r$ cross edges between $B_\alpha - I_\alpha$ and \mathcal{W} , where $\mathcal{W} = GEX(s, t) - \bigcup_{\alpha=1}^{r+1} B_\alpha$ and $B_1 \cong \tilde{L}_1$. Let $\mathcal{M} = I \setminus \bigcup_{\alpha=1}^{r+1} I_\alpha$. Since there is no edge between $B_\alpha - I_\alpha$ and $\mathcal{W} - \mathcal{M}$, $|\mathcal{M}| \geq \sum_{\alpha=2}^{r+1} [2^s - |I_\alpha| - r]$. Figure 4 shows an illustration of this case. Then, we have

$$\begin{aligned} |\mathcal{P}| &= |I_2| + \dots + |I_{r+1}| + |\mathcal{M}| \\ &\geq |I_2| + \dots + |I_{r+1}| + \sum_{\alpha=2}^{r+1} [2^s - |I_\alpha| - r] \\ &= r(2^s - r). \end{aligned}$$

Let $f(r) = r(2^s - r)$ with $r \geq 1$. We can obtain that $\frac{\partial f(r)}{\partial r} = 2^s - 2r > 0$ and $f(r)$ is an increasing function. Hence, $f(r) \geq f(1) = 2^s - 1$ and $|\mathcal{P}| \geq f(r) \geq 2^s - 1$. Similarly, we define the function $f(g) = 2^s - 1 - [(s - g + 1)2^{g-1} - 1]$ with $1 \leq g \leq \lfloor \frac{s}{2} \rfloor$ and $s \geq 3$. It shows that $\frac{\partial f(g)}{\partial g} = 2^{g-1} [g \ln 2 + 1 - (s + 1) \ln 2] < 0$. Thus, $f(g)$ is a decreasing function and $f(g) \geq f(\lfloor \frac{s}{2} \rfloor) > 0$. Then, $|\mathcal{P}| \geq 2^s - 1 > (s - g + 1)2^{g-1} - 1$, which results in a contradiction with $|\mathcal{P}| \leq (s - g + 1)2^{g-1} - 1$. Therefore, $GEX(s, t) - \tilde{L}_1 - \mathcal{P}$ is connected.

Since $\tilde{L}_1 - I_1$ is disconnected, there must be a component U in $\tilde{L}_1 - I_1$ such that no edge between U and $GEX(s, t) - \tilde{L}_1 - \mathcal{P}$ exists. Then, $|\mathcal{P}| \geq |V(U)|$. Obviously,

Fig. 4 An illustration of the proof of Case 2 in Lemma 9



$N_{\tilde{L}_1}(U) \subseteq I_1$, which means that $|I| = |I_1| + |\mathcal{P}| \geq |N_{\tilde{L}_1}(U)| + |V(U)| = |N_{\tilde{L}_1}[U]|$. Considering that I is an R_g -cut of $GEX(s, t)$, $\delta(U) \geq g$. By Lemma 3 and $\tilde{L}_1 \cong X_s$, we deduce that $|I| = |N_{\tilde{L}_1}[U]| \geq (s - g + 1)2^g$, which contradicts with $|I| \leq (s - g + 1)2^g - 1$.

Case 3: At least two of $\tilde{L}_i - I_{\tilde{L}_i}$ and $\tilde{R}_j - I_{\tilde{R}_j}$ are disconnected.

Without loss of generality, suppose that $\tilde{L}_1 - I_{\tilde{L}_1}$ and $\tilde{L}_2 - I_{\tilde{L}_2}$ are disconnected. Since I is an R_g -cut of $GEX(s, t)$, we have that $\delta(\tilde{L}_1 - I_{\tilde{L}_1}) \geq g - 1$ and $\delta(\tilde{L}_2 - I_{\tilde{L}_2}) \geq g - 1$ by Definition 2. In addition, it is clear that every node in \tilde{L}_1 (resp. \tilde{L}_2) has at least $g - 1$ neighbors in $\tilde{L}_1 - I_{\tilde{L}_1}$ (resp. $\tilde{L}_2 - I_{\tilde{L}_2}$). Thus, $I_{\tilde{L}_1}$ (resp. $I_{\tilde{L}_2}$) is an R_{g-1} -cut of \tilde{L}_1 (resp. \tilde{L}_2). By Lemma 7, $|I_{\tilde{L}_1}| \geq (s - g + 1)2^{g-1}$ and $|I_{\tilde{L}_2}| \geq (s - g + 1)2^{g-1}$. Then,

$$\begin{aligned} |I| &\geq |I_{\tilde{L}_1}| + |I_{\tilde{L}_2}| \\ &\geq (s - g + 1)2^{g-1} + (s - g + 1)2^{g-1} \\ &= (s - g + 1)2^g. \end{aligned}$$

As a result, based on the above discussion, we can conclude that $\kappa^g(GEX(s, t)) = |I| \geq (s - g + 1)2^g$. □

Thus, we combine Lemmas 8 and 9, and the following theorem holds.

Theorem 10 For $GEX(s, t)$ with $3 \leq s \leq t$ and $1 \leq g \leq \lfloor \frac{s}{2} \rfloor$, the R_g -conditional connectivity of $GEX(s, t)$ is $\kappa^g(GEX(s, t)) = (s - g + 1)2^g$.

4 The R_g -conditional diagnosability of $GEX(s, t)$ under the PMC model

In this section, we will establish the upper and lower bounds of R_g -conditional diagnosability of $GEX(s, t)$.

Lemma 11 If H is a subgraph of $GEX(s, t)$ with $|V(H)| = g + 1$ for $0 \leq g \leq s - 1$, $s \geq 2$, then $|N_{GEX(s, t)}(V(H))| \geq (g + 1)s - \frac{g(g+1)}{2} + 1$.

Proof We shall prove the lemma by induction on $|V(H)|$. By Definition 2, it is clear that the result holds for $|V(H)| = 1$. Suppose that the conclusion holds for all H with $|V(H)| \leq g$, where $g \geq 1$. Then, we show the results are true for H with $|V(H)| = g + 1 \geq 2$ by the following two cases.

Case 1: $V(H)$ is distributed in exactly one cluster.

Without loss of generality, we let $H \subset B_\alpha$, where $B_\alpha \cong X_s$ and a certain $\alpha \in [1, 2^s + 2^t]$. By Lemma 2, it is easy to deduce that $|N_{B_\alpha}(V(H))| \geq (g + 1)s - \frac{g(g-1)}{2} - 2g$. Since each node in B_α has only one neighbor in $V(GEX(s, t) - B_\alpha)$, $|N_{GEX(s, t)}(V(H))| \geq (g + 1)s - \frac{g(g-1)}{2} - 2g + (g + 1) = (g + 1)s - \frac{g(g+1)}{2} + 1$.

Case 2: $V(H)$ is distributed in at least two different clusters.

Suppose that $H_0 = H \cap B_\alpha$ and $H_1 = H \cap (\text{GEX}(s, t) - B_\alpha)$, where $|V(H_0)| = x$, $|V(H_1)| = g + 1 - x$ and $1 \leq x \leq g$. By Lemma 2 and the induction hypothesis, we have

$$\begin{aligned} |N_{B_\alpha}(V(H_0))| &\geq xs - \frac{(x-1)(x-2)}{2} - 2(x-1) \\ &= xs - \frac{x(x+1)}{2} + 1, \end{aligned}$$

and

$$|N_{\text{GEX}(s,t)}(V(H_1))| \geq (g + 1 - x)s - \frac{(g+1-x)(g-x)}{2} + 1.$$

Since each node in $V(H_1)$ has at most one neighbor in B_α , $|N_{B_\alpha}(V(H_1))| \leq g + 1 - x$. Then,

$$\begin{aligned} |N_{\text{GEX}(s,t)-B_\alpha}(V(H_1))| &= |N_{\text{GEX}(s,t)}(V(H_1))| - |N_{B_\alpha}(V(H_1))| \\ &\geq [(g + 1 - x)s - \frac{(g+1-x)(g-x)}{2} + 1] - (g + 1 - x). \end{aligned}$$

Hence, we have

$$\begin{aligned} |N_{\text{GEX}(s,t)}(V(H))| &\geq |N_{B_\alpha}(V(H_0))| + |N_{\text{GEX}(s,t)-B_\alpha}(V(H_1))| \\ &\geq xs - \frac{x(x+1)}{2} + 1 + [(g + 1 - x)s - \frac{(g+1-x)(g-x)}{2} + 1] \\ &\quad - (g + 1 - x) \\ &= (g + 1)s - \frac{x(x+1)}{2} - \frac{(g-x)(g-x+3)}{2} + 1. \end{aligned}$$

Further, we deduce

$$\begin{aligned} |N_{\text{GEX}(s,t)}(V(H))| - [(g + 1)s - \frac{g(g+1)}{2} + 1] &\geq (g + 1)s - \frac{x(x+1)}{2} - \frac{(g-x)(g-x+3)}{2} \\ &\quad + 1 - [(g + 1)s - \frac{g(g+1)}{2} + 1] \\ &= (g - x)(x - 1) \\ &\geq 0. \end{aligned}$$

Therefore, $|N_{\text{GEX}(s,t)}(V(H))| \geq (g + 1)s - \frac{g(g+1)}{2} + 1$. □

Lemma 12 Suppose that H is a subgraph of $\text{GEX}(s, t)$ with $\delta(H) \geq g$ for $1 \leq s \leq t$ and $1 \leq g \leq s$, it shows that $|V(H)| \geq 2^g$.

Proof We prove this lemma through the following two cases.

Case 1: $V(H)$ is distributed in exactly one cluster.

At this time, it is easy to deduce that $|V(H)| \geq 2^g$ by Lemma 1.

Case 2: $V(H)$ is distributed at least two different clusters.

We suppose that each B_α represents a cluster of $\text{GEX}(s, t)$ for $\alpha \in [1, 2^s + 2^t]$. Let $\mathcal{H} = \{H_1, H_2, \dots, H_m\}$ with $m \geq 2$, where $H_d = H \cap B_\alpha$ for $d \in [1, m]$ and some $\alpha \in [1, 2^s + 2^t]$. By Definition 2, we can deduce that $\delta(H_d) \geq g - 1$.

Thus, it is obvious that $|V(H_d)| \geq 2^{g-1}$ by Lemma 1. Then, we can conclude that $|V(H)| = \bigcup_{d=1}^m |V(H_d)| \geq 2^g$ because $m \geq 2$. Hence, this lemma holds. \square

Theorem 13 For $g \geq 1$ and $2^{2g} \leq s \leq t$, $t_{R_g}(GEX(s, t)) \geq (2s - 2^{2g} + 3)2^{2g-2} + (s - g + 1)2^{g-1}$ under the PMC model.

Proof At first, we define the function $f(g) = (2s - 2^{2g} + 3)2^{2g-2} + (s - g + 1)2^{g-1}$. Let F_1 and F_2 be two distinct R_g -conditional faulty sets with $\max\{|F_1|, |F_2|\} \leq f(g)$. We prove this theorem by contradiction. Suppose that F_1 and F_2 are indistinguishable under the PMC model. We can deduce that $|V(GEX(s, t))| = 2^{s+t+1} > 2f(g) \geq |F_1 \cup F_2|$, thus $|V(GEX(s, t)) \setminus F_1 \cup F_2| \geq 1$. Since F_1 and F_2 are indistinguishable, and $F_1 \neq F_2$, $F_1 \cap F_2$ is an R_g -conditional faulty node cut set and $N_{GEX(s,t)}(F_1 \triangle F_2) \subseteq F_1 \cap F_2$. By Theorem 10, $|F_1 \cap F_2| \geq \kappa^g(GEX(s, t)) = (s - g + 1)2^g$. By Lemma 5 and Lemma 12, we have $|F_1 \triangle F_2| \geq 2^{2g}$.

Let Q be a component of $F_1 \triangle F_2$ with $|V(Q)| = 2^{2g}$, and H be a subset of $|F_1 \cap F_2|$ with $|H| = (s - g + 1)2^g$. By Lemma 11, $|N_{GEX(s,t)}(V(Q))| \geq 2^{2g}s - \frac{2^{2g}(2^{2g}-1)}{2} + 1$. Thus, $|V(Q) \cup N_{GEX(s,t)}(V(Q))| \geq (s + 1)2^{2g} - \frac{2^{2g}(2^{2g}-1)}{2} + 1$. Since $V(Q) \cup N_{GEX(s,t)} \subseteq F_1 \cup F_2$ and $H \subseteq F_1 \cap F_2$, we have

$$\begin{aligned} |F_1 \cup F_2 \setminus H| &= |F_1 \cup F_2| - |H| \\ &\geq |V(Q) \cup N_{GEX(s,t)}(V(Q))| - |H| \\ &\geq (s + 1)2^{2g} - \frac{2^{2g}(2^{2g}-1)}{2} + 1 - (s - g + 1)2^g \\ &= (s + 1)2^{2g} - \frac{2^{2g}(2^{2g}-1)}{2} - (s - g + 1)2^g + 1 \\ &= (2s - 2^{2g} + 3)2^{2g-1} - 2^g(s - g + 1) + 1. \end{aligned}$$

Hence,

$$\begin{aligned} \max\{|F_1|, |F_2|\} &\geq \frac{|F_1 \cup F_2| + |F_1 \cap F_2|}{2} \\ &\geq \frac{|F_1 \cup F_2| + |H|}{2} \\ &= \frac{|F_1 \cup F_2 \setminus H|}{2} + |H| \\ &\geq (2s - 2^{2g} + 3)2^{2g-2} - (s - g + 1)2^{g-1} + \frac{1}{2} + (s - g + 1)2^g \\ &= (2s - 2^{2g} + 3)2^{2g-2} + (s - g + 1)2^{g-1} + \frac{1}{2} \\ &= f(g) + \frac{1}{2} \\ &> f(g). \end{aligned}$$

This is a contradiction, which implies that F_1 and F_2 are distinguishable. Thus, $t_{R_g}(GEX(s, t)) \geq (2s - 2^{2g} + 3)2^{2g-2} + (s - g + 1)2^{g-1}$. \square

Theorem 14 For $4 \leq s \leq t$ and $1 \leq g \leq \lfloor \frac{s}{3} \rfloor$, $t_{R_g}(GEX(s, t)) \leq 2^{2g-1} + 2^{2g}(s + t + 1 - 2g) - 1$ under the PMC model.

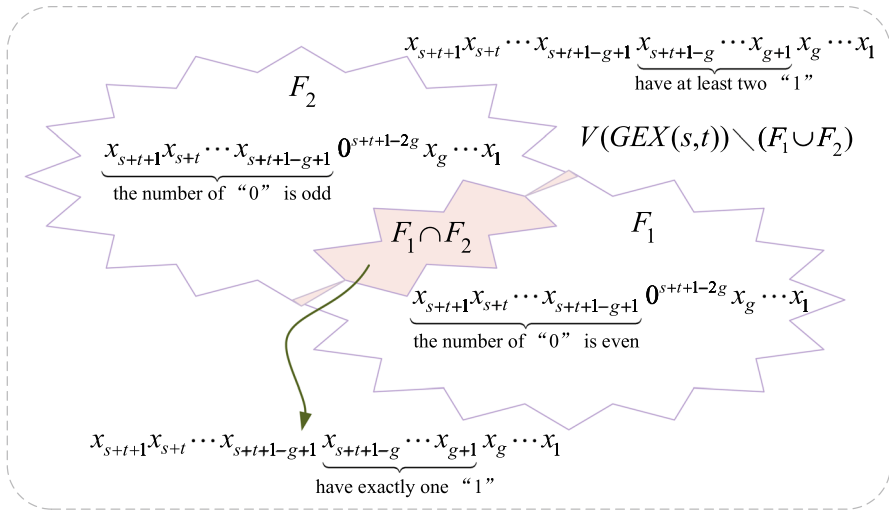


Fig. 5 An illustration of F_1 and F_2

Proof By Definition 4, if there are two distinct and indistinguishable R_g -conditional faulty sets F_1 and F_2 , where $|F_1|, |F_2| \leq 2^{2g-1} + 2^{2g}(s+t+1-2g)$, then $GEX(s, t)$ is not R_g -conditionally $2^{2g-1} + 2^{2g}(s+t+1-2g)$ -diagnosable with $t_{R_g}(GEX(s, t)) \leq 2^{2g-1} + 2^{2g}(s+t+1-2g) - 1$. As shown in Fig. 5, we have

$$F_1 \setminus F_2 = \{x_{s+t+1}x_{s+t} \cdots x_{s+t+1-g+1} \underbrace{0 \cdots 0}_{s+t+1-2g} x_g \cdots x_1 \mid x_{s+t+1}x_{s+t} \cdots x_{s+t+1-g+1} \text{ have an even number of "0"s, } x_i \in \{0, 1\}\},$$

$$F_2 \setminus F_1 = \{x_{s+t+1}x_{s+t} \cdots x_{s+t+1-g+1} \underbrace{0 \cdots 0}_{s+t+1-2g} x_g \cdots x_1 \mid x_{s+t+1}x_{s+t} \cdots x_{s+t+1-g+1} \text{ have an odd number of "0"s, } x_i \in \{0, 1\}\},$$

$$F_1 \cap F_2 = \{x_{s+t+1}x_{s+t} \cdots x_{s+t+1-g+1}x_{s+t+1-g} \cdots x_{g+1} \cdots x_1 \mid x_{s+t+1-g} \cdots x_{g+1} \text{ have exactly one "1", } x_i \in \{0, 1\}\}, \text{ and}$$

$$V(GEX(s, t)) \setminus (F_1 \cup F_2) = \{x_{s+t+1}x_{s+t} \cdots x_{s+t+1-g+1}x_{s+t+1-g} \cdots x_{g+1}x_g \cdots x_1 \mid x_{s+t+1-g} \cdots x_{g+1} \text{ have at least two "1"s, } x_i \in \{0, 1\}\}.$$

Each node in $F_1 \triangle F_2$ has g neighbors in $F_1 \setminus F_2$ and $F_2 \setminus F_1$, respectively. In addition, each node in $F_1 \cap F_2$ has at least $s - 2g$ neighbors in $V(GEX(s, t)) \setminus (F_1 \cup F_2)$. Since $1 \leq g \leq \lfloor \frac{s}{3} \rfloor$, we have $s - 2g \geq g$. Each node in $V(GEX(s, t)) \setminus (F_1 \cup F_2)$ has at least $2g$ neighbors in $V(GEX(s, t)) \setminus (F_1 \cup F_2)$. At this time, there is no edge between $F_1 \triangle F_2$ and $V(GEX(s, t)) \setminus (F_1 \cup F_2)$. Hence, F_1 and F_2 are two distinct and indistinguishable R_g -conditional faulty sets.

Since $|F_2 \setminus F_1| = |F_1 \setminus F_2| = 2^{2g-1}$ and $|F_1 \cap F_2| = 2^{2g}(s+t+1-2g)$, we have that $|F_1| = |F_1 \setminus F_2| + |F_1 \cap F_2| = 2^{2g-1} + 2^{2g}(s+t+1-2g)$ and $|F_2| = |F_2 \setminus F_1| + |F_2 \cap F_1| = 2^{2g-1} + 2^{2g}(s+t+1-2g)$. Thus, F_1 and F_2 are two distinct and indistinguishable R_g -conditional faulty sets of $GEX(s, t)$, where $|F_1| \leq 2^{2g-1} + 2^{2g}(s+t+1-2g)$ and $|F_2| \leq 2^{2g-1} + 2^{2g}(s+t+1-2g)$. By Definition 4, $GEX(s, t)$ is not R_g -conditionally $2^{2g-1} + 2^{2g}(s+t+1-2g)$ -diagnosable. Therefore, the upper bound of $t_{R_g}(GEX(s, t))$ is $2^{2g-1} + 2^{2g}(s+t+1-2g) - 1$. \square

5 Applications to a family of famous networks

Based on Sects. 3 and 4, we obtain the R_g -conditional connectivity of $GEX(s, t)$ and determine the lower and upper bounds of R_g -conditional diagnosability of $GEX(s, t)$ under the PMC model. Applying the theorems of Sects. 3 and 4, we can directly establish the R_g -conditional connectivity and lower and upper bounds of R_g -conditional diagnosability of some generalized exchanged X -cubes, including generalized exchanged hypercubes, generalized exchanged crossed cubes, and locally generalized exchanged twisted cubes under the PMC model. In this section, we will give the applications to these networks.

5.1 The generalized exchanged hypercube

The *exchanged hypercube* $EH(s, t)$ was proposed by Loh et al. [35], which is a variant of hypercube obtained by removing some edges from a hypercube Q_{s+t+1} . We denote $\mathbb{I}_n = \{1, 2, \dots, n\}$, where n is a given position integer. The definition of exchanged hypercubes is presented as follows.

Definition 5 (See [35]). For $s, t \geq 1$, the exchanged hypercube $EH(s, t) = (V, E)$ is an undirected graph with the node set $V = \{u_{s+t} \dots u_{t+1} u_t \dots u_1 u_0 \mid u_i \in \{0, 1\} \text{ for } i \in \{0\} \cup \mathbb{I}_{s+t}\}$, two nodes $u = u_{s+t} \dots u_{t+1} u_t \dots u_1 u_0$ and $v = v_{s+t} \dots v_{t+1} v_t \dots v_1 v_0$ are linked by an edge, called an *r-dimensional edge*, if and only if the following conditions are satisfied:

- (1) u and v differ exactly in the r -th bit for $r \in [0, s+t]$,
- (2) if $r \in \mathbb{I}_t$, then $u_0 = v_0 = 1$,
- (3) if $r \in \mathbb{I}_{s+t} \setminus \mathbb{I}_t$, then $u_0 = v_0 = 0$.

Later, Cheng et al. [23] further introduced the *generalized exchange hypercube*. The following is the formal definition of the generalized exchange hypercube $GEH(s, t, f)$.

Definition 6 (See [23]). For $s, t \geq 1$, $GEH(s, t, f)$ consists of two disjoint subgraphs L' and R' , where L' contains 2^t subgraphs, denoted by the Class-0 clusters $L'_i (i \in [1, 2^t])$

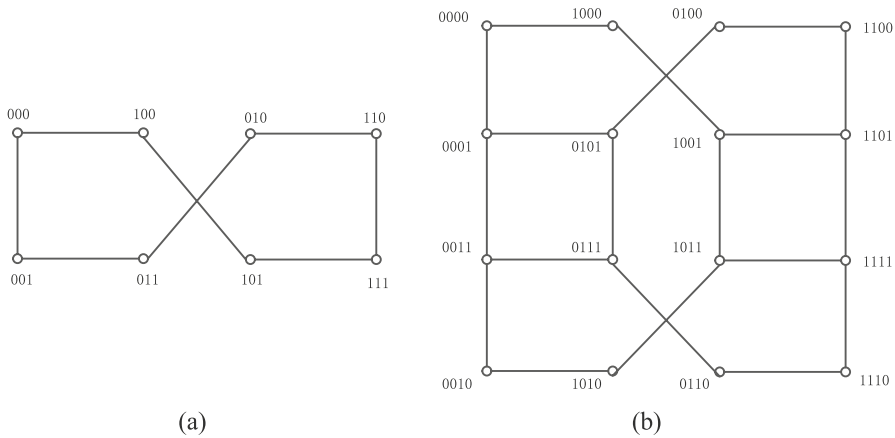


Fig. 6 a $GEH(1, 1)$; b $GEH(1, 2)$

and R' contains 2^s subgraphs, denoted by the Class-1 clusters $R'_j (j \in [1, 2^s])$. Class-0 and Class-1 clusters will be referred to as clusters of opposite classes of each other, same class otherwise, and collectively as clusters. In addition, $GEH(s, t, f)$ fulfills the following conditions:

- (1) for any $i \in [1, 2^t]$ and $j \in [1, 2^s]$, $L'_i \cong Q_s$ and $R'_j \cong Q_t$;
- (2) the function f is a bijection between nodes of Class-0 clusters and those of Class-1 clusters. If two nodes u and v are in the same cluster, then $f(u)$ and $f(v)$ belong to two different clusters, where the edge $(u, f(u))$ is called a *cross edge*.

Furthermore, the bijection function f ensures the existence of perfect matching between nodes of Class-0 clusters and those in the Class-1 clusters but ignores the specifics of the perfect matching. For convenient, we simplify $GEH(s, t, f)$ as $GEH(s, t)$. According to the definition of generalized exchanged X -cubes $GEX(s, t)$ and the properties of generalized exchanged hypercube $GEH(s, t)$, we can deduce that $GEH(s, t)$ is a member of generalized exchanged X -cubes, where the X -cube is a hypercube. Figure 6 illustrates $GEH(1, 1)$ and $GEH(1, 2)$.

In particular, the *dual-cube* D_n is a special case of the exchanged hypercube $EH(s, t)$ when $s = t$, that is, $EH(n, n) \cong D_n$, which was proposed by Li and Peng [36]. The dual-cube-like network DC_n is a generalization of dual cubes and $DC_n \cong EH(n - 1, n - 1)$ [24]. Clearly, DC_n is a special case of $GEH(n - 1, n - 1)$. Then, the following theorems hold obviously.

Theorem 15 *The R_g -conditional connectivity of $GEH(s, t)$ and DC_n are as follows:*

- (1) For any integers $3 \leq s \leq t$ and $1 \leq g \leq \lfloor \frac{s}{2} \rfloor$, $\kappa^g(GEH(s, t)) = (s - g + 1)2^g$;

(2) For any integers $n \geq 4$ and $1 \leq g \leq \lfloor \frac{n}{2} \rfloor$, $\kappa^g(DC_n) = (n - g)2^g$.

Theorem 16 The lower and upper bounds of R_g -conditional diagnosability of $GEH(s, t)$ under the PMC model are as follows:

(1) For any integers $g \geq 1$ and $2^{2g} \leq s \leq t$, $t_{R_g}(GEH(s, t)) \geq (2s - 2^{2g} + 3)2^{2g-2} + (s - g + 1)2^{g-1}$;

(2) For any integers $4 \leq s \leq t$ and $1 \leq g \leq \lfloor \frac{s}{3} \rfloor$, $t_{R_g}(GEH(s, t)) \leq 2^{2g-1} + 2^{2g}(s + t + 1 - 2g) - 1$.

Theorem 17 The lower and upper bounds of R_g -conditional diagnosability of DC_n under the PMC model are as follows:

(1) For any integers $g \geq 1$ and $2^{2g} \leq n - 1$, $t_{R_g}(DC_n) \geq (2n - 2^{2g} + 1)2^{2g-2} + (n - g)2^{g-1}$;

(2) For any integers $n \geq 5$ and $1 \leq g \leq \lfloor \frac{n-1}{3} \rfloor$, $t_{R_g}(DC_n) \leq 2^{2g-1} + 2^{2g}(2n - 2g + 1) - 1$.

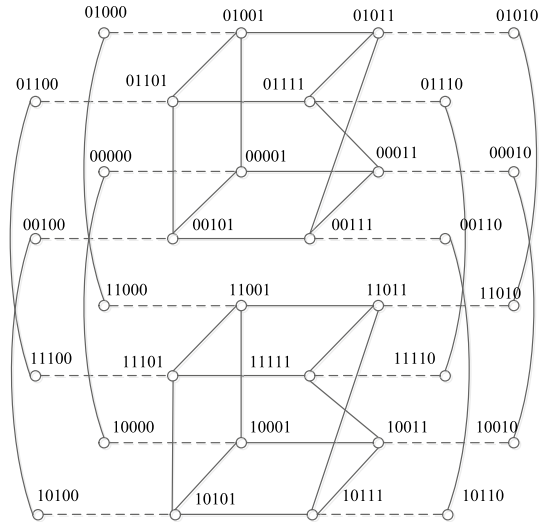
5.2 The generalized exchanged crossed cube

Li et al. [37] proposed the *exchanged crossed cube* $ECQ(s, t)$, which is obtained by removing edges from a crossed cube CQ_{s+t+1} . In what follows, we review the definition of exchanged crossed cubes.

Definition 7 (See [37]). The (s, t) -dimensional exchanged crossed cube is defined as a graph $ECQ(s, t) = G(V, E)$ with $s, t \geq 1$, where the node set $V = \{a_{s-1} \dots a_0 b_{t-1} \dots b_0 c \mid a_i, b_j, c \in \{0, 1\}, i \in [0, s - 1], j \in [0, t - 1]\}$ and the edge set $E = \{(u, v) \mid (u, v) \in V \times V\}$. Especially, E is composed of three disjoint sets E_1, E_2 , and E_3 , as shown below:

- (1) $E_1 : u[0] \neq v[0], u \oplus v = 1$, where \oplus is the exclusive-OR operator;
- (2) $E_2 : u[0] = v[0] = 1, u[s + t : t + 1] = v[s + t : t + 1]$, where $\lambda[x : y]$ denotes the bit pattern of the node λ between dimensions y and x inclusive. Thus, $u[t : 1]$ is denoted by $b = b_{t-1} \dots b_0$ and $v[t : 1]$ is denoted by $b' = b'_{t-1} \dots b'_0$, where u and v are adjacent by the following rule: for any $t \geq 1$, if and only if there is an ℓ ($1 \leq \ell \leq t$) such that $b_{t-1} \dots b_\ell = b'_{t-1} \dots b'_\ell$ and $b_{\ell-1} \neq b'_{\ell-1}, b_{\ell-2} = b'_{\ell-2}$ if ℓ is even, as well as $b_{2i+1}b_{2i} \sim b'_{2i+1}b'_{2i}$ with $0 \leq i < \lfloor (\ell - 1)/2 \rfloor$;
- (3) $E_3 : u[0] = v[0] = 0, u[t : 1] = v[t : 1]$, and $u[s + t : t + 1]$ and $v[s + t : t + 1]$ are denoted by $a = a_{s-1} \dots a_0$ and $a' = a'_{t-1} \dots a'_0$, respectively. Then, u and v are adjacent by the following rule: for all $s \geq 1$, if and only if there exists an ℓ

Fig. 7 An exchanged crossed cube $GECQ(1, 3)$



($1 \leq \ell \leq s$) such that $a_{s-1} \cdots a_\ell = a'_{s-1} \cdots a'_\ell$ and $a_{\ell-1} \neq a'_{\ell-1}, a_{\ell-2} = a'_{\ell-2}$ if ℓ is even, as well as $a_{2i+1}a_{2i} \sim a'_{2i+1}a'_{2i}$ for $0 \leq i < \lfloor (\ell - 1)/2 \rfloor$.

The *generalized exchanged crossed cube*, denoted as $GECQ(s, t, f)$ with $s, t \geq 1$, comprises two distinct classes of crossed cubes, known as the Class-0 clusters and the Class-1 clusters, respectively. The former contains $2^t CQ_s$'s and the latter encompasses $2^s CQ_t$'s. Class-0 clusters and Class-1 clusters are referred to as clusters of opposite classes of each other. The function f establishes a bijective relationship between the nodes of Class-0 clusters and Class-1 clusters. This bijection ensures that for any two nodes u and v within the same cluster, their corresponding values $f(u)$ and $f(v)$ reside in distinct clusters, with the edge $(u, f(u))$ (resp. $(v, f(v))$) representing a cross edge. The purpose of the bijection f is to ensure the existence of perfect matching between two nodes in different clusters, although the specific arrangement of this matching is not specified.

For the sake of convenience, the notation $GECQ(s, t, f)$ is often abbreviated as $GECQ(s, t)$. Based on the definition of generalized exchanged X -cubes and the properties of generalized exchanged crossed cube, it can be inferred that the generalized exchanged crossed cube $GECQ(s, t)$ belongs to the family of generalized exchanged X -cubes, where the X -cube specifically represents a crossed cube (see $GECQ(1, 3)$ in Fig. 7). Then, the following theorems hold obviously.

Theorem 18 For any integers $s \geq 3$ and $1 \leq g \leq \lfloor \frac{s}{2} \rfloor$, $\kappa^g(GECQ(s, t)) = (s - g + 1)2^g$.

Theorem 19 The lower and upper bounds of R_g -conditional diagnosability of $GECQ(s, t)$ under the PMC model are as follows:

- (1) For any integers $g \geq 1$ and $2^{2g} \leq s \leq t$, $t_{R_g}(GECQ(s, t)) \geq (2s - 2^{2g} + 3)2^{2g-2} + (s - g + 1)2^{g-1}$;

(2) For any integers $4 \leq s \leq t$ and $1 \leq g \leq \lfloor \frac{s}{3} \rfloor$, $t_{R_g}(GECQ(s, t)) \leq 2^{2g-1} + 2^{2g}(s + t + 1 - 2g) - 1$.

5.3 The locally generalized exchanged twisted cube

The *locally exchanged twisted cube* $LETQ(s, t)$ was proposed by Chang et al. [25], which is obtained by removing edges from a locally twisted cube LTC_{s+t+1} . The definition of the locally exchanged twisted cube is introduced as follows.

Definition 8 (See [25]). The (s, t) -dimensional locally exchanged twisted cube $LETQ(s, t) = (V, E)$ with $s, t \geq 1$, where the node set $V = \{x_{t+s} \cdots x_{t+1}x_t \cdots x_1 x_0 | x_i \in \{0, 1\}, i \in [0, s+t]\}$ and E is the edge set consisting of the following three types of disjoint sets E_1, E_2 , and E_3 .

- (1) $E_1 = \{(x, y) \in V \times V : x \oplus y = 2^0\}$,
- (2) $E_2 = \{(x, y) \in V \times V : x_0 = y_0 = 1, x_1 = y_1 = 0 \text{ and } x \oplus y = 2^k \text{ for } k \in [3, t]\} \cup \{(x, y) \in V \times V : x_0 = y_0 = x_1 = y_1 = 1 \text{ and } x \oplus y = 2^k + 2^{k-1} \text{ for } k \in [3, t]\} \cup \{(x, y) \in V \times V : x_0 = y_0 = 1 \text{ and } x \oplus y \in \{2^1, 2^2\}\}$,
- (3) $E_3 = \{(x, y) \in V \times V : x_0 = y_0 = x_{t+1} = y_{t+1} = 0 \text{ and } x \oplus y = 2^k \text{ for } k \in [t+3, t+s]\} \cup \{(x, y) \in V \times V : x_0 = y_0 = 0, x_{t+1} = y_{t+1} = 1 \text{ and } x \oplus y = 2^k + 2^{k-1} \text{ for } k \in [t+3, t+s]\} \cup \{(x, y) \in V \times V : x_0 = y_0 = 0 \text{ and } x \oplus y \in \{2^{t+1}, 2^{t+2}\}\}$.

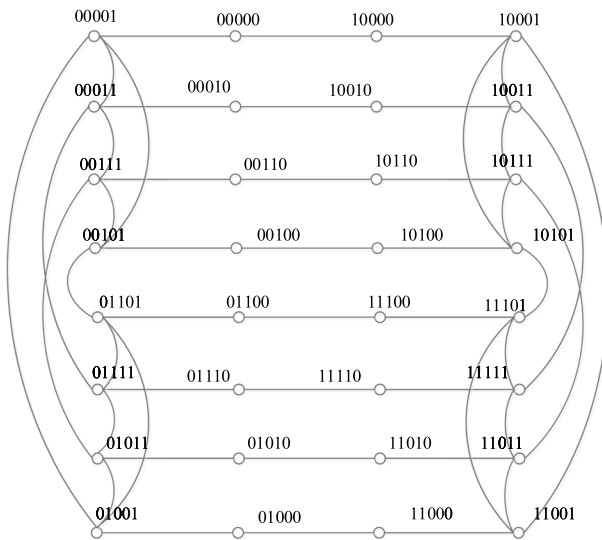


Fig. 8 A locally exchanged twisted cube $LGETQ(1, 3)$

Let $s, t \geq 1$, there are two classes of locally twisted cubes in the *locally generalized exchanged twisted cube* $LGETQ(s, t, f)$: one class, referred to as the Class-0 clusters, contains 2^t LTQ_s 's; and the other, referred to as the Class-1 clusters, contains 2^s LTQ_t 's. They will be referred to as clusters of opposite classes of each other, same class otherwise. There exists a bijection function f between nodes of Class-0 clusters and those of Class-1 clusters. For two nodes u, v in the same cluster, $f(u)$ and $f(v)$ belong to two different ones, and the edge $(u, f(u))$ is a cross edge. The bijection f ensures the existence of perfect matching between nodes of Class-0 clusters and those in the Class-1 clusters, but the specifics of the perfect matching can be ignored. For convenient, $GEH(s, t, f)$ is denoted by $GEH(s, t)$. According to the definition of generalized exchanged X -cubes and the properties of locally generalized exchanged twisted cube, we can deduce that the locally generalized exchanged twisted cube $LGETQ(s, t)$ is a member of generalized exchanged X -cubes, where the X -cube is a locally twisted cube (see $LGETQ(1, 3)$ in Fig. 8). Then, the following theorems hold obviously.

Theorem 20 For any integers $s \geq 3$ and $1 \leq g \leq \lfloor \frac{s}{2} \rfloor$, $\kappa^g(LGETQ(s, t)) = (s - g + 1)2^g$.

Theorem 21 The lower and upper bounds of R_g -conditional diagnosability of $LGETQ(s, t)$ under the PMC model are as follows:

(1) For any integers $g \geq 1$ and $2^{2g} \leq s \leq t$, $t_{R_g}(LGETQ(s, t)) \geq (2s - 2^{2g} + 3)2^{2g-2} + (s - g + 1)2^{g-1}$;

(2) For any integers $4 \leq s \leq t$ and $1 \leq g \leq \lfloor \frac{s}{3} \rfloor$, $t_{R_g}(LGETQ(s, t)) \leq 2^{2g-1} + 2^{2g}(s + t + 1 - 2g) - 1$.

6 Comparison results

In this section, we will demonstrate the superiority of our method by providing some comparative analysis with existing results. First, we focus on conducting comparisons and analyses pertaining to connectivity in Sect. 6.1. Following this, our emphasis will transition to conducting diagnosability analyses under the PMC and MM* models in Sect. 6.2.

6.1 Connectivity

In the following, we compare our results to the traditional connectivity, g -good-neighbor conditional connectivity, component connectivity, extra connectivity, and cyclic connectivity. We provide the definitions of these indicators as follows.

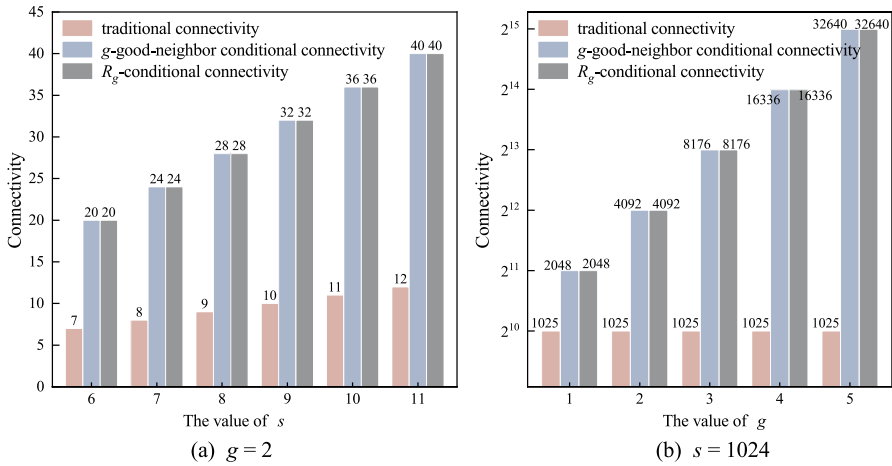


Fig. 9 The comparisons among traditional connectivity, g -good-neighbor conditional connectivity and R_g -conditional connectivity in $GEH(s, t)$

Table 2 A brief summary of known results related to connectivity for $GEH(s, t)$

	$\kappa^g(GEH(s, t))$	$\bar{\kappa}_g(GEH(s, t))$	$\tilde{\kappa}_g(GEH(s, t))$	$\kappa'(GEH(s, t))$
Reference	Theorem 15	[38]	[39]	[38]
Value	$(s - g + 1)2^g$	$s(g - 1) - \frac{(g-1)(g-2)}{2} + 1$	$\frac{g(2s-g+1)}{2} + 1$	$4s - 4$

- The *traditional connectivity* of a graph G , denoted by $\kappa(G)$, is defined as the cardinality of a minimum node set F whose removal results in a disconnected or trivial graph.
- A node set F of a graph G is a *g -good-neighbor node cut* if and only if $G - F$ is disconnected and each remaining component has minimum degree at least g . The *g -good-neighbor conditional connectivity* of G , denoted by $\hat{\kappa}_g(G)$, is defined as the cardinality of a minimum g -good-neighbor node cut of G . Clearly, $\hat{\kappa}_0(G) = \kappa(G)$.
- A node set F of a graph G is a *g -component node cut* if and only if $G - F$ is disconnected and has at least g components. The *g -component connectivity* of G , denoted by $\bar{\kappa}_g(G)$, is defined as the cardinality of a minimum g -component node cut of G . Clearly, $\bar{\kappa}_2(G) = \kappa(G)$.
- A node set F of a graph G is a *g -extra node cut* if and only if $G - F$ is disconnected and every component of $G - F$ has at least g nodes. The *g -extra connectivity* of G , denoted by $\tilde{\kappa}_g(G)$, is defined as the cardinality of a minimum g -extra node cut of G . Obviously, $\tilde{\kappa}_1(G) = \kappa(G)$.

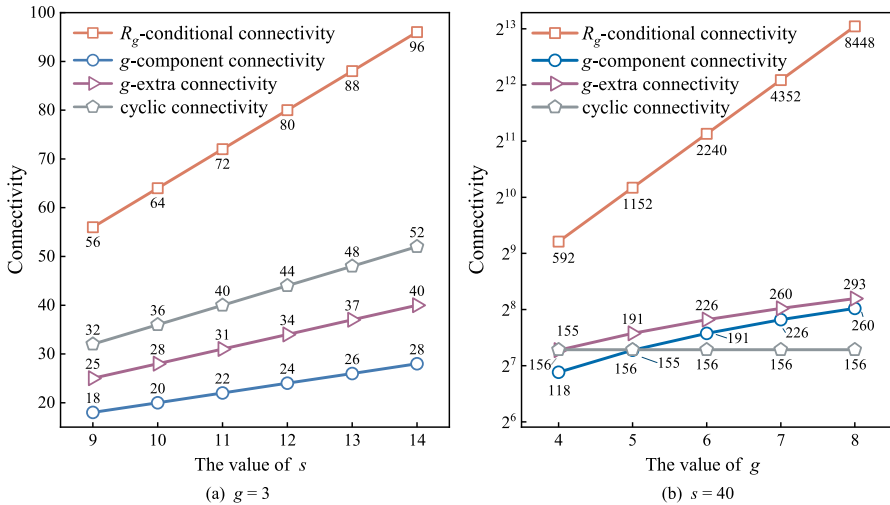


Fig. 10 The comparisons among R_g -conditional connectivity, g -component connectivity, g -extra connectivity and cyclic connectivity in $GEH(s, t)$

- A node set F of a graph G is a *cyclic node cut* if and only if $G - F$ is disconnected and at least two components in $G - F$ contain a cycle. The *cyclic connectivity* of G , denoted by $\kappa'(G)$, is defined as the cardinality of a minimum cyclic node cut of G .

From Theorem 10, we have that the R_g -conditional connectivity of $GEX(s, t)$ is $\kappa^g = (s - g + 1)2^g$. It is clear that $\kappa(GEX(s, t)) = s + 1$ and $\hat{\kappa}_g(GEX(s, t)) = (s - g + 1)2^g$ [22]. As shown in Fig. 9, we can observe that the traditional connectivity significantly lags behind both R_g -conditional connectivity and g -good-neighbor conditional connectivity. Notably, the fault tolerance values for R_g -conditional connectivity and g -good-neighbor conditional connectivity remain identical. This uniformity stems from the sole distinction between the two being the constraints imposed on the faulty processor, a factor that does not impact the determination of connectivity.

As a subclass of $GEX(s, t)$, the generalized exchanged hypercube $GEH(s, t)$ has been the subject of extensive research, with Table 2 presenting a detailed compilation of the current results on the connectivity of $GEH(s, t)$.

For $GEH(s, t)$, we compare our method with g -component connectivity, g -extra connectivity, and cyclic connectivity in Fig. 10. It is evident that each of these connectivity is monotonically increasing on s . Moreover, with a fixed value of s , the R_g -conditional connectivity asserts its dominance, exhibiting a significantly higher level than the other three connectivity. Figure 10b further illustrates that the parameter g exerts a more substantial influence on the R_g -conditional connectivity, in contrast to its negligible impact on the cyclic connectivity.

Table 3 A brief summary of known results related to diagnosability for $GEX(s, t)$ and $GEH(s, t)$

	$t(GEX(s, t))$	$\hat{t}_g(GEX(s, t))$	$t_{R_g}(GEX(s, t)) \geq$
Reference	[22]	[22]	Theorem 13
PMC model	$s + 1$	$(s - g + 2)2^g - 1$	$(2s - 2^{2g} + 3)2^{2g-2} + (s - g + 1)2^{g-1}$
MM* model	$s + 1$	$(s - g + 2)2^g - 1$	—
	$t_{R_g}(GEH(s, t)) \geq$	$\hat{t}_g(GEH(s, t))$	$\tilde{t}_g(GEH(s, t))$
Reference	Theorem 16	[40]	[39]
PMC model	$(2s - 2^{2g} + 3)2^{2g-2} + (s - g + 1)2^{g-1}$	$gs - \frac{g(g-1)}{2} + 1$	$\frac{g(2(s+1)-g+1)}{2}$
MM* model	—	$gs - \frac{g(g-1)}{2} + 1$	$\frac{g(2(s+1)-g+1)}{2}$

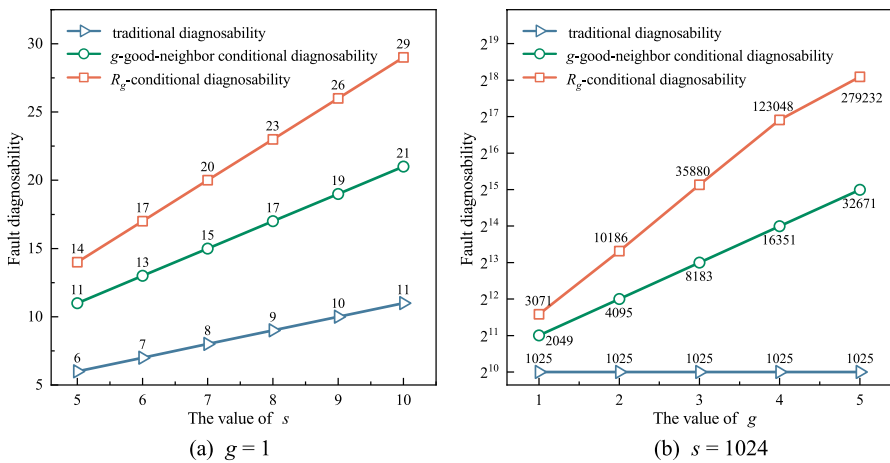


Fig. 11 The comparisons among traditional diagnosability, g -good-neighbor diagnosability and R_g -conditional diagnosability in $GEX(s, t)$ under the PMC model

6.2 Diagnosability

In this subsection, we compare our results to the traditional diagnosability, g -good-neighbor conditional diagnosability, component diagnosability, and extra diagnosability under the PMC model or MM* model. Then, the definitions and results of these indicators are listed as follows.

- The *traditional diagnosability* of a graph G , denoted by $t(G)$, is the maximum number of faulty processors that can be accurately identified.
- The *g -good-neighbor conditional diagnosability* of a graph G , denoted by $\hat{t}_g(G)$, is the maximum number of faulty processors that can be identified under the condition that every fault-free processor has at least g fault-free neighbors. In particular, $\hat{t}_0(G) = t(G)$.

- The g -component diagnosability of a graph G , denoted by $\bar{t}_g(G)$, is the maximum number of faulty processors that can be identified under the condition that the number of components in $G - F$ is at least g . Clearly, $\bar{t}_2(G) = t(G)$.
- The g -extra diagnosability of a graph G , denoted by $\tilde{t}_g(G)$, is the maximum number of faulty processors that can be identified under the condition that each component in $G - F$ has at least g nodes. Obviously, $\tilde{t}_1(G) = t(G)$.

By Lemma 4, the g -good-neighbor conditional diagnosability of $GEX(s, t)$ is $\hat{t}_g(GEX(s, t)) = (s - g + 2)2^g - 1$ under the PMC model with $s \geq 3$ and $1 \leq g \leq s - 2$ (resp. under the MM* model with $s \geq 4$ and $1 \leq g \leq s - 2$). Thus, we can deduce that $t(GEX(s, t)) = s + 1$ under the PMC model and MM* model. Furthermore, the lower (resp. upper) bound of R_g -conditional diagnosability of $GEX(s, t)$ is $(2s - 2^{2g} + 3)2^{2g-2} + (s - g + 1)2^{g-1}$ (resp. $2^{2g-1} + 2^{2g}(s + t + 1 - 2g) - 1$). It is evident that the upper bound of R_g -conditional diagnosability of $GEX(s, t)$ is larger than its lower bound by $(2s + 4t + 4 - 8g + 2^{2g-1})2^{2g-2} + (s - g + 1)2^{g-1} - 1$ through calculation. Therefore, we compare the lower bound of R_g -conditional diagnosability of $GEX(s, t)$ with these other diagnosabilities. In Table 3, we show some outcomes based on these indicators and our methods for $GEX(s, t)$ and $GEH(s, t)$. We can observe that the traditional diagnosability (or g -good-neighbor conditional diagnosability) remains consistent, whether under the PMC model or the MM* model.

As shown in Fig. 11a, we can observe that each diagnosability of these increases with the value of s when g is fixed. This is not difficult to understand because when s increases, the size of $GEX(s, t)$ increases as well. Obviously, the R_g -conditional diagnosability of $GEX(s, t)$ is greater than its g -good-neighbor conditional diagnosability and traditional diagnosability under the PMC model. At the same time, Fig. 11b shows that if s is the same, as g increases, the value of the traditional diagnosability

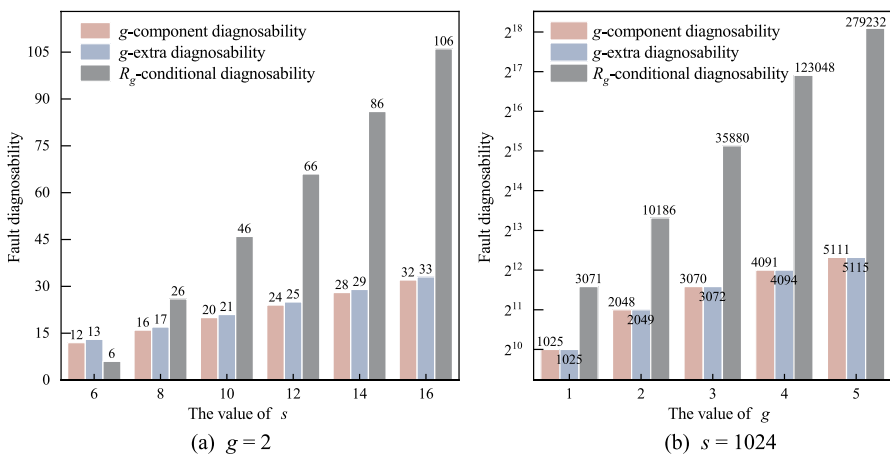


Fig. 12 The comparisons among R_g -conditional diagnosability, g -component diagnosability and g -extra diagnosability in $GEH(s, t)$ under the PMC model

remains unchanged, but the rate of increase of the R_g -conditional diagnosability of $GEX(s, t)$ is much higher than that g -good-neighbor conditional diagnosability of $GEX(s, t)$ under the PMC model. Thus, R_g -conditional diagnosability can better improve the reliability of the interconnection network.

Meanwhile, we delve into diagnosability results for $GEH(s, t)$ under the PMC model and MM^* model, as detailed in Table 3. It is evident that for $GEH(s, t)$, g -component diagnosability (resp. g -extra diagnosability) remains identical whether under the PMC model or the MM^* model. Additionally, Fig. 12 reveals that, under the PMC model, the disparities between the values of g -component diagnosability and g -extra diagnosability are minimal. However, the reliability of $GEH(s, t)$ under the R_g constraint consistently surpasses g -component diagnosability and g -extra diagnosability. Interestingly, when $s = 6$, the R_g -conditional diagnosability is lower than the corresponding g -component diagnosability and g -extra diagnosability, highlighting the efficacy of R_g -conditional diagnosability in large-scale networks.

7 Conclusion

The concepts of R_g -conditional connectivity and diagnosability hold immense importance in enhancing the reliability of multiprocessor systems. These properties help ensure seamless communication and connectivity within the system, minimizing potential failures and enhancing performance. Generalized exchanged X -cube is a promising solution for improving the performance of multiprocessor systems. These structures not only leverage the benefits of traditional X -cubes but also offer a reduction in interconnection complexity. This reduction allows for simpler and more efficient communication between nodes, thereby increasing the overall speed and reliability of the system.

In this paper, we first determine the R_g -conditional connectivity of generalized exchanged X -cubes. Further, we study the lower and upper bounds of R_g -conditional diagnosability of generalized exchanged X -cubes under the PMC model. As applications, the lower and upper bounds of R_g -conditional diagnosability of generalized exchanged hypercubes, generalized exchanged crossed cubes, and locally exchanged twisted cubes are established directly under the PMC model. The research results indicate that compared to the other existing results in terms of connectivity or diagnosability, $GEX(s, t)$ has showcased remarkable reliability under the R_g -conditional restriction. This significantly advances the exploration of network reliability, such as generalized exchanged hypercubes, generalized exchanged crossed cubes, and locally generalized exchanged twisted cubes.

Finally, we conclude this paper by proposing some directions worth exploring in future research:

1. We will attempt to study the R_g -conditional diagnosability of $GEX(s, t)$ under the MM^* model.
2. We may extend the R_g -conditional restriction to edge faults, and study R_g -conditional edge connectivity/diagnosability.

3. We will try to apply R_g -conditional connectivity/diagnosability to real-world systems (e.g., data center networks and autonomous robots systems [41]) such that the systems can tolerate a range of node failures to keep operation effective.

Author Contributions WL was involved in validation, supervision, writing. HZ helped in investigation, visualization, conceptualization. X-Y L contributed to resources, funding acquisition, project administration, validation. YZ assisted in investigation, formal analysis, methodology.

Funding This work was supported by the National Natural Science Foundation of China under Grant (62002062) and the Natural Science Foundation of Fujian Province under Grant (2022J05029).

Data availability All of the material is owned by the authors and/or no permissions are required.

Declarations

Conflict of interest I declare that the authors have no competing interests as defined by Springer, or other interests that might be perceived to influence the results and/or discussion reported in this paper.

Ethical approval Not applicable.

References

1. Zhang Q, Xu L, Zhou S, Yang W (2020) Reliability analysis of subsystem in dual cubes. *Theor Comput Sci* 816:249–259
2. Gu MM, Hao R-X, Chang J-M (2021) Reliability analysis of alternating group graphs and split-stars. *Comput J* 64(9):1425–1436
3. Esfahanian A-H, Hakimi S (1988) On computing a conditional edge-connectivity of a graph. *Inform Process Lett* 27:195–199
4. Latifi S, Hedge M, Naraghi-Pour M (1994) Conditional connectivity measures for large multiprocessor systems. *IEEE Trans Comput* 43(2):218–222
5. Preparata FP, Metze G, Chien RT (1967) On the connection assignment problem of diagnosable systems. *IEEE Trans Comput* 16(6):848–854
6. Maeng J, Malek M (1980) A comparison connection assignment for diagnosis of multiprocessor systems. In: *Proceedings of 7th annual symposium on Computer Architecture (ISCA 1980)*, La Baule USA, 6–8 May, pp 31–36
7. Sengupta A, Dahbura A (1992) On self-diagnosable multiprocessor systems: diagnosis by the comparison approach. *IEEE Trans Comput* 41(11):1386–1396
8. Lai P-L, Tan JJ-M, Chang C-P, Hsu L-H (2005) Conditional diagnosability measure for large multiprocessors systems. *IEEE Trans Comput* 54(2):165–175
9. Peng S-L, Lin C-K, Tan JJ-M, Hsu L-H (2012) The g -good-neighbor conditional diagnosability of hypercube under PMC model. *Appl Math Comput* 218(21):10406–10412
10. Chang N-W, Hsieh S-Y (2020) Conditional diagnosability of alternating group networks under the PMC model. *IEEE ACM Trans Netw* 28(5):196–1980
11. Li X, Lu L, Zhou S (2014) Conditional diagnosability of twisted-cube connected networks. In: *Proceedings of International Conference on Computer Science and Information Technology (CSAIT 2013)*, Kunming, China, 21–23 Sept, pp 359–370
12. Xu X, Li X, Zhou S, Hao R-X, Gu MM (2017) The g -good-neighbor diagnosability of (n, k) -star graphs. *Theor Comput Sci* 659:53–63
13. Hao R-X, Feng Y-Q, Zhou J-X (2012) Conditional diagnosability of alternating group graphs. *IEEE Trans Comput* 62(4):827–831

14. Lin L, Xu L, Wang D, Zhou S (2018) The g -good-neighbor conditional diagnosability of arrangement graphs. *IEEE Trans Depend Secur Comput* 15(3):542–548
15. Hsieh S-Y, Kao C-Y (2012) The conditional diagnosability of k -ary n -cubes under the comparison diagnosis model. *IEEE Trans Comput* 62(4):839–843
16. Chang J-M, Pai K-J, Wu R-Y, Yang J-S (2019) The 4-component connectivity of alternating group networks. *Theor Comput Sci* 766:38–45
17. Hao R-X, Tian Z-X, Xu J-M (2016) Relationship between conditional diagnosability and 2-extra connectivity of symmetric graphs. *Theor Comput Sci* 627:36–53
18. Guo C, Xiao Z, Liu Z, Peng S (2020) R_g -conditional diagnosability: a novel generalized measure of system-level diagnosis. *Theor Comput Sci* 814:19–27
19. Wang Y, Lin C-K, Zhou Q (2020) Note on R_g -conditional diagnosability of hypercube. *Theor Comput Sci* 849:197–201
20. Yuan J, Qiao H, Liu A (2022) The R_g -conditional diagnosability of international networks. *Theor Comput Sci* 898:30–43
21. Yuan J, Qiao H, Liu A (2022) The upper and lower bounds of R_g -conditional diagnosability of networks. *Inf Process Lett* 176:106248
22. Li X, Zhuang H, Zhou S, Cheng H, Lin C-K, Guo W (2020) Reliability evaluation of generalized exchanged X -Cubes based on the condition of g -good-neighbor. *Wirel Commun Mob Comput* 9793082:1–16
23. Cheng E, Qiu K, Shen Z (2017) Structural properties of generalized exchanged hypercubes. *Emergent Comput* 24:215–232
24. Angjeli A, Cheng E, Lipták L (2013) Linearly many faults in dual-cube-like networks. *Theor Comput Sci* 472:1–8
25. Chang J-M, Chen X-R, Yang J-S, Wu R-Y (2016) Locally exchanged twisted cubes: connectivity and super connectivity. *Inf Process Lett* 116(7):460–466
26. Fan J (1998) Diagnosability of the Möbius cubes. *IEEE Trans Parallel Distrib Syst* 9(9):923–928
27. Wang D (2008) On embedding Hamiltonian cycles in crossed cubes. *IEEE Trans Parallel Distrib Syst* 19(3):334–346
28. Yang X, Evans DJ, Megson GM (2005) The locally twisted cubes. *Int J Comput Math* 82(4):401–413
29. Fan J, Lin X (2005) The t/k -diagnosability of the BC graphs. *IEEE Trans Comput* 54(2):176–184
30. Li X, Xu J (2013) Edge-fault tolerance of hypercube-like networks. *Inf Process Lett* 113:760–763
31. Yang W, Lin H (2014) Reliability evaluation of BC networks in terms of the extra vertex- and edge-connectivity. *IEEE Trans Comput* 63(10):2540–2548
32. Ye L, Liang J (2016) On conditional h -vertex connectivity of some networks. *Chin J Electron* 25:556–560
33. Wu J, Guo G (1998) Fault tolerance measures for m -ary n -dimensional hypercubes based on forbidden faulty sets. *IEEE Trans Comput* 47(8):888–893
34. Dahbura AT, Masson GM (1984) An $O(n^{2.5})$ fault identification algorithm for diagnosable systems. *IEEE Trans Comput* 33(6):486–492
35. Loh PKK, Hsu WJ, Pan Y (2005) The exchanged hypercube. *IEEE Trans Parallel Distrib Syst* 16(9):866–874
36. Li Y, Peng S (2000) Dual-cubes: a new interconnection network for high-performance computer clusters. In: *Proceedings of the 2000 international computer symposium, workshop on computer architecture*, pp 51–57
37. Li K, Mu Y, Min G (2013) Exchanged crossed cube: a novel interconnection network for parallel computation. *IEEE Trans Parallel Distrib Syst* 24(11):2211–2219
38. Cheng E, Qiu K, Shen Z (2017) Structural properties of generalized exchanged hypercubes. *Emerg Comput* 24:215–232
39. Cheng E, Qiu K, Shen Z (2020) The g -extra diagnosability of the generalized exchanged hypercube. *Int J Comput Math Comput Syst Theory* 5(2):112–123
40. Zhuang H, Guo W, Li X, Liu X, Lin C-K (2022) The component diagnosability of general networks. *Int J Found Comput Sci* 33(1):67–89
41. Zhang H, Hao R-X, Qin X-W, Lin C-K, Hsieh S-Y (2023) The high faulty tolerant capability of the alternating group graphs. *IEEE Trans Parallel Distrib Syst* 34(1):225–233
42. Zhu Q (2008) On conditional diagnosability and reliability of the BC networks. *J Supercomput* 45(2):173–184

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.

Authors and Affiliations

Wanling Lin^{1,2} · Hongbin Zhuang^{1,2} · Xiao-Yan Li^{1,2} · Yufang Zhang^{1,2}

✉ Xiao-Yan Li
xyli@fzu.edu.cn

Wanling Lin
linwl595@163.com

Hongbin Zhuang
hbzhuang476@gmail.com

Yufang Zhang
yoguptzyf@gmail.com

¹ College of Computer and Data Science, Fuzhou University, Fuzhou 350108, China

² Fujian Provincial Key Laboratory of Information Security of Network Systems, Fuzhou University, Fuzhou 350108, China