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Reliability analysis of the augmented cubes in terms of the *h*-extra *r*-component edge-connectivity

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Abstract

In order to meet ever-increasing demands for reliable parallel and distributed systems, it is crucial to evaluate the reliability and fault tolerance of their underlying interconnection networks. Such an interconnection network is usually modeled as a connected graph G, where the vertex set and edge set represent the processors and links between processors in the network, respectively. In this paper, we combine Fàbrega's idea about h-extra edge-connectivity and Sampathkumar's concept about r-component edge-connectivity to introduce a more refined parameter for characterizing fault tolerance of interconnection networks, named as h-extra r-component edge-connectivity. Given a connected graph G, for two integers $h \geq 1$ and $r \geq 2$, the h-extra r-component edge-connectivity of G, denoted as $c\lambda_r^h(G)$, is the minimum cardinality among all edge subsets $F \subset E(G)$, if any, such that $G - F$ has at least r components, and each component has at least h vertices. As an enhancement on hypercube, the n-dimensional augmented cube AQ_n, introduced by Choudum and Sunitha in 2002, reserves several excellent topological properties. As $|V(\mathrm{AQ}_n)| = 2^n$, the h-extra three-component edge-connectivity of AQ_n is well-defined for each integer h with $1 \leq h \leq \lfloor 2^n/3 \rfloor$. In this paper, a generalization of Xu et al.'s conclusion is obtained that finds an upper bound for the exact value of general h-extra three-component edge-connectivity of AQ_n and shows that it is sharp for $1 \leq h \leq 2^{\lfloor \frac{n}{2} \rfloor - 1}$ and $h = 2^c$ where $1 \leq c \leq n-2$. Let $h = \sum_{i=0}^{s} 2^{t_i}$ be a positive integer with $t_0 > t_1 > \cdots > t_s \geq 0$. Let $\delta = 0$ if h is even and $\delta = 1$ if h is odd. Specifically, $c\lambda_3^h(\mathrm{AQ}_n) = (4n-4)h - 2\sum_{i=0}^{s}(2t_i - 1)2^{t_i} - 2\sum_{i=0}^{s} 4i \cdot 2^{t_i} - \delta$ for $n \geq 4, h \leq 2^{\lfloor \frac{n}{2} \rfloor - 1}$, and $c\lambda_3^{2^c}(\mathrm{AQ}_n) = (2n - 2c - 1)2^{c+1}$ for $n \geq 4$ and $1 \leq c \leq n-2$.

Keywords Interconnection network · Reliability and fault tolerance · h-Extra r-component edge-connectivity · Augmented cube

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1 Introduction

An interconnection network is a network composed of switching elements in a certain topology and control mode to achieve interconnection between multiple processors or functional components within a computer system. As the brain of interconnection networks, data centers have developed vigorously in recent years. With the increase in the number of processors in the interconnection networks, there will exist several almost inevitable errors that may result in communication interruption between some processors in the interconnection networks, and then lead to the communication delay of the whole network or even network paralysis. As a faulty processor will lose communication with other processors, these faulty links that disconnect the interconnection network are modeled as an edge-cut in the corresponding graph. Given a connected graph G, an edge subset $F \subset E(G)$ is called an *edge-cut* of G if its deletion disconnects G. We call the numbers of vertices and edges in G as the order and size of G, respectively. The classical Menger's *edge-connectivity* is the minimum cardinality of all edge-cuts of G, denoted as $\lambda(G)$ [17]. In other words, edge-connectivity is the minimum number of faulty links that disconnect the network.

In a specific interconnection network, the processors and links that do not fail are called fault-free vertices and fault-free edges of the corresponding graph, respectively, which are collectively referred to as the fault-free set. Due to the different demands of fault-free sets in distinct connected graph G such as the number of components and the order of each component, we need to evaluate the reliability and fault tolerance of large-scale parallel and distributed systems using multiple parameters. Since the classical edge-connectivity does not exert any restriction on its surviving components, Harary proposed *conditional edge-connectivity* as a generalization of the classical edge-connectivity in 1983, denoted as $\lambda(G, \mathcal{P})$, where $\mathcal{P}$ is the given properties of fault-free set in graph G [12]. There are two typical examples of conditional edge-connectivity, one is *h-extra edge-connectivity* and the other is *r-component edge-connectivity*. An edge subset of G, if any, is called an *h-extra edge-cut* if its deletion disconnects G, and each remaining component has at least h vertices. In 1996, Fàbrega and Fiol introduced h-extra edge-connectivity of the connected graph G which denoted as $\lambda_h(G)$, is the minimum cardinality of any h-extra edge-cut of G [8]. Another well-known conditional edge-connectivity was introduced by Sampathkumar [19] in 1984 called r-component edge-connectivity. For a given positive integer r, an *r-component edge-cut* of a connected graph G, if any, is defined as an edge subset F of graph G, whose deletion yields a disconnected graph with at least r components. The r-component edge-connectivity of a connected graph G, denoted by $c\lambda_r(G)$, is the minimum cardinality taken over all r-component edge-cuts of G. Some researches have obtained the r-component edge-connectivity of many special graphs with small r [4, 5]. In addition, let F be a minimum r-component edge-cut of connected graph G, then the extremal structure of $G - F$ is usually composed of $r - 1$ isolated vertices and a giant component [11]. For some other researches on a variety of networks about conditional edge-connectivity, see [3, 7, 9, 10, 20, 21, 25, 27–30].

For an interconnection network with some faulty edges, in order to restrict the number of connected components and ensure the scale of normal working processors in each component, more recently, Li et al. [15] gave the definition of *h-extra r-component connectivity* by combining h-extra connectivity and r-component connectivity in 2021. In details, the h-extra r-component connectivity of connected graph G is the minimum cardinality of any vertex subset of G, whose removal disconnects G and then results in at least r components, and each component contains at least h vertices, denoted as $c\kappa_r^h(G)$ [15]. In addition, they determined the h-extra r-component connectivity of n-dimensional hypercube Q_n that $c\kappa_r^2(Q_n) = 2(r-1)(n-r+1)$ for $r \in \{2, 3, 4\}$.

Motivated by the ideas of Fàbrega and Sampathkumar, as a generalization of [15], we consider the edge version of h-extra r-component connectivity to characterize the fault tolerance of interconnection networks and give the definition of *h-extra r-component edge-connectivity* as follows:

Definition 1 Given a connected graph $G = (V, E)$, for two integers $h \geq 1$ and $r \geq 2$, a subset $F \subset E$ is called an h-extra r-component edge-cut of G, if any, if there are at least r components in $G - F$, and each component has at least h vertices. The h-extra r-component edge-connectivity of G, denoted as $c\lambda_r^h(G)$, is the minimum cardinality of any h-extra r-component edge-cut of G.

Lemma 1 *If F is a minimum h-extra r-component edge-cut of G, then $G - F$ has exactly r components.*

Proof Suppose to the contrary that $G - F$ has exactly p components as $G_1, G_2, \dots, G_p$ with $p > r$ and $|V(G_i)| \geq h$, $1 \leq i \leq p$. Since G is connected, then there exists an edge xy that $x \in V(G_i)$ and $y \in V(G_j)$ for some $i, j \in \{1, 2, \dots, p\}$ and $i \neq j$. Let $F_1 = F \backslash \{xy\}$. Note that $G[V(G_i) \cup V(G_j)]$ is connected with at least $2h > h$ vertices, then $G - F_1$ has $p - 1 \geq r$ components, and each component has at least h vertices. In other words, F_1 is an h-extra r-component edge-cut of G with $|F_1| < |F|$, which contradicts the minimality of F. Hence, $G - F$ has exactly r components. □

For the given connected graph G, let $V_1, V_2, \dots, V_t$ be a partition of $V(G)$. That is, $V_i \subset V(G)$ for $1 \leq i \leq t$, $\cup_{k=1}^t V_k = V(G)$, and $V_i \cap V_j = \emptyset$ for $1 \leq i < j \leq t$. Let $[V_i, V_j]$ be the edges with one end-vertex in V_i and the other in V_j and $[V_1, V_2, \dots, V_k] = \cup_{1 \leq i < j \leq k} [V_i, V_j]$. Define the function $\xi_m(G)$ be the minimum cardinality of any edge-cut F of G such that $G - F$ has one component with exactly m vertices [23, 24]. In other words, $\xi_m(G) = \min\{|[V_0, \overline{V_0}]| : V_0 \subset V(G), G[V_0]$ is connected, $|V_0| = m\}$. In addition, we say that G is λ_h-optimal if $\lambda_h(G) = \xi_h(G)$. As a generalization of $\xi_m(G)$ denotes the minimum cardinality of any edge-cut F of connected graph G such that $G - F$ has exactly $r + 1$ components with r components which have exactly m vertices as $\xi_{m,r+1}(G)$. Let $V_1, V_2, \dots, V_{r+1}$ be a partition of $V(G)$. In detail, $\xi_{m,r+1}(G) = \min\{|[V_1, V_2, \dots, V_{r+1}]| : |V_i| = m \leq \lfloor |V(G)|/(r+1) \rfloor$ for $1 \leq i \leq r$, each $G[V_j]$ is connected for $1 \leq j \leq r+1\}$. Therefore,

$\lambda_h(G) = \min\{\xi_{m,2}(G) : h \le m \le \lfloor |V(G)|/2 \rfloor\}$ by the definition of $\lambda_h(G)$. Furthermore, if $c\lambda_r^h(G) = \xi_{h,r}(G)$, we say that G is $c\lambda_r^h$-optimal; otherwise, G is not $c\lambda_r^h$-optimal. Motivated by the idea of introducing $\xi_{m,2}(G)$ to solve $\lambda_h(G)$, $\xi_{m,r}(G)$ can be used to solve $c\lambda_r^h(G)$, and whether G is $c\lambda_r^h$-optimal or not similarly.

From the definition, it can be immediately obtained that the 1-extra 2-component edge-connectivity of G equals to the edge-connectivity of G as $c\lambda_2^1(G) = \lambda(G)$, the 1-extra r-component edge-connectivity of G equals to the r-component edge-connectivity of G as $c\lambda_r^1(G) = c\lambda_r(G)$, and the h-extra 2-component edge-connectivity of G equals to the h-extra edge-connectivity of G as $c\lambda_2^h(G) = \lambda_h(G)$. In addition, let m be a positive integer, and $ex_m(G) = \max\{d(G[X]) : X \subset V(G), |X| = m\}$ be the maximum sum of the degrees of the subgraph induced by a vertex set with the given cardinality m in G, i.e., $ex_m(G)/2$ is the maximum possible sizes of the subgraph induced by m vertices in G. If G is d-regular, then $\xi_{m,2}(G) = dm - ex_m(G)$.

As an enhancement on hypercube, the augmented cube, introduced by Choudum and Sunitha in 2002 [6], not only reserves several of the advantages of the hypercube such as strong connectivity, small diameter, symmetry, recursive construction, relatively small degree, and regularity [1, 14], but also carries some embedding properties that the hypercube does not have [13, 18]. Due to its excellent topological properties, the augmented cube is often used for the underlying topological structure of parallel and distributed systems [26].

Definition 2 ([6]) Let $n \ge 1$ be an integer. The n-dimensional augmented cube, denoted by AQ_n, is a vertex transitive and $(2n-1)$-regular graph with 2^n vertices, each labeled by an n-bit binary string $x_n x_{n-1} \cdots x_2 x_1$ where $x_i \in \{0,1\}, 1 \le i \le n$. Write $V(\mathrm{AQ}_n)$ as $X_n X_{n-1} \cdots X_2 X_1 = \{x_n x_{n-1} \cdots x_2 x_1 : x_i \in \{0,1\}, 1 \le i \le n\}$. Define AQ_1 be the complete graph K_2 with two vertices labeled by 0 and 1, respectively. As for $n \ge 2$, AQ_n has recursive structure. That is, AQ_n consists of two copies of $(n-1)$-dimensional augmented cubes, denoted by $0\mathrm{AQ}_{n-1}$ and $1\mathrm{AQ}_{n-1}$ that $V(0\mathrm{AQ}_{n-1}) = 0X_{n-1} \cdots X_2 X_1$ and $V\left(1\mathrm{AQ}_{n-1}\right) = 1X_{n-1} \cdots X_2 X_1$, and adding 2^n edges (two perfect matchings of AQ_n) between $0\mathrm{AQ}_{n-1}$ and $1\mathrm{AQ}_{n-1}$. The vertex $a = 0a_{n-1} \cdots a_2 a_1 \in V(0\mathrm{AQ}_{n-1})$ is joined to the vertex $b = 1b_{n-1} \cdots b_2 b_1 \in V(1\mathrm{AQ}_{n-1})$ if and only if,

(i) $a_i = b_i$ for $1 \le i \le n-1$; or
(ii) $a_i = 1 - b_i$ for $1 \le i \le n-1$.

From the definition, each vertex in $V(0\mathrm{AQ}_{n-1})$ has two neighbors in $V(1\mathrm{AQ}_{n-1})$ and vice versa. Hence, AQ_n can be written as $0\mathrm{AQ}_{n-1} \oplus 1\mathrm{AQ}_{n-1}$ and $E(\mathrm{AQ}_n)$ can be partitioned into three disjoint edge subsets of AQ_n for $n \ge 2$. Let $u = u_n u_{n-1} \cdots u_2 u_1$ and $v = v_n v_{n-1} \cdots v_2 v_1$ be any two adjacent vertices in AQ_n. If $uv \in E(0\mathrm{AQ}_{n-1})$ or $uv \in E(1\mathrm{AQ}_{n-1})$, then uv is called an original edge (O-edge for short). Otherwise, uv is called a hypercube edge (H-edge for short) or a complement edge (C-edge for short) if uv satisfies the case (i) or the case (ii) in Definition 2, respectively. In detail,

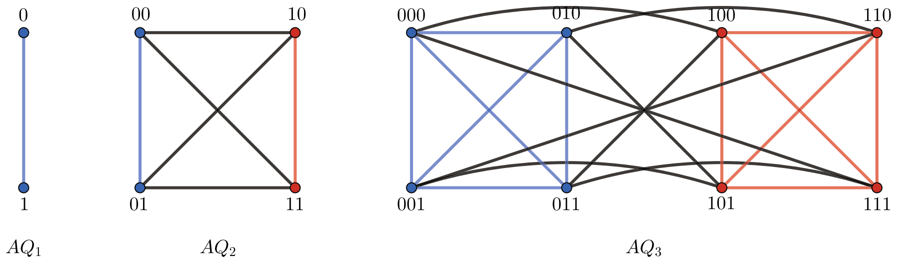

Fig. 1 Illustration of AQ_n

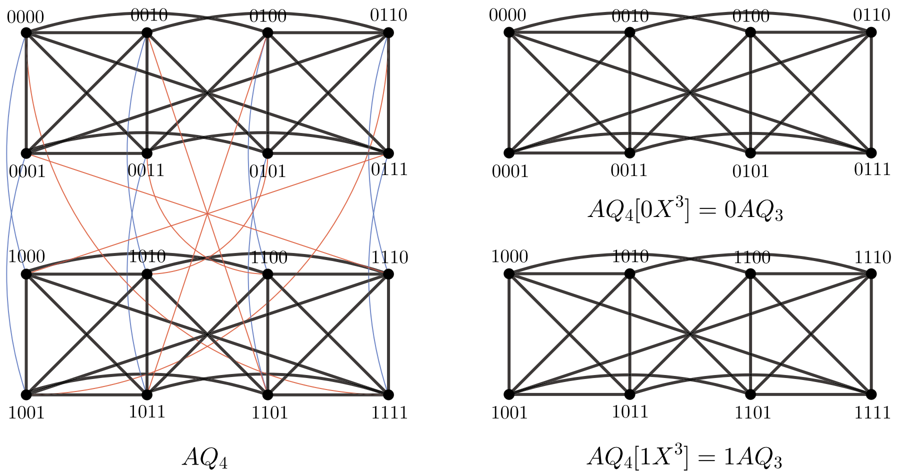

Fig. 2 $AQ_4[0X^3] = 0AQ_3$ and $AQ_4[1X^3] = 1AQ_3$

uv is an O-edge in AQ_n if and only if there exists an integer k with $1 \leq k < n$ such that,

(*i*) $u_i = v_i$ for $k+1 \leq i \leq n$ and $u_j = 1 - v_j$ for $1 \leq j \leq k$; or
(*ii*) $u_i = v_i$ for $i \neq k$ and $u_k = 1 - v_k$.

In other case, uv is an H-edge in AQ_n if and only if $u_n = 1 - v_n$ and $u_i = v_i$ for $1 \leq i \leq n-1$. Furthermore, uv is a C-edge in AQ_n if and only if $u_i = 1 - v_i$ for $1 \leq i \leq n$. The n-dimensional augmented cubes for $n = 1, 2, 3$ are illustrated in Fig. 1. In addition, for $n = 2, 3$, the O-edges, H-edges, and C-edges in AQ_n are marked in black, blue (dark gray in print), and red (light gray in print), respectively.

Let X^n and x^n denote $X_n X_{n-1} \cdots X_2 X_1$ and $x_n x_{n-1} \cdots x_2 x_1$, respectively. Denote the vertex set $\{z_n z_{n-1} \cdots z_{k+1} x_k x_{k-1} \cdots x_1 : x_i \in \{0, 1\}, 1 \leq i \leq k, z_j$ is fixed, $k+1 \leq j \leq n\}$ as $z_n z_{n-1} \cdots z_{k+1} X^k$. It is obvious that $AQ_n[z_n z_{n-1} \cdots z_{k+1} X^k]$ is a k-dimensional augmented subcube in AQ_n. By this way, $0AQ_{n-1} = AQ_n[0X^{n-1}]$ and

$1AQ_{n-1} = AQ_n[1X^{n-1}]$. We use $z_n z_{n-1} \cdots z_{k+1} X^k$ to represent $AQ_n[z_n z_{n-1} \cdots z_{k+1} X^k]$, if no confusion arises (Fig. 2).

Let m and S_m be a positive integer with $m \le 2^n$ and the set $\{0, 1, 2, \ldots, m-1\}$, respectively. Denote the corresponding set of S_m that is represented by n-binary strings as L_m^n. Let $m = \sum_{i=0}^{s} 2^{t_i}$ be the decomposition of m where $t_0 = \lfloor \log_2 m \rfloor, t_i = \lfloor \log_2(m - \sum_{k=0}^{i-1} 2^{t_k}) \rfloor$ for $i \ge 1$. In 2014, Chien *et al.* showed that $ex_m(AQ_n) = \sum_{i=0}^{s}(2t_i - 1)2^{t_i} + \sum_{i=0}^{s} 4i \cdot 2^{t_i}$ [2], but this result is not true for m is odd. In 2021, Zhang *et al.* fixed it and obtained the value of $ex_m(AQ_n)$ that $ex_m(AQ_n) = \sum_{i=0}^{s} (2t_i - 1)2^{t_i} + \sum_{i=0}^{s} 4i \cdot 2^{t_i} + \delta$ where if m is even, then $\delta = 0$; if m is odd, then $\delta = 1$ [26]. It is noteworthy that they gave a lower bound of $ex_m(AQ_n)$ by showing that the vertex subset L_m^n in $V(AQ_n)$ satisfies $|L_m^n| = m$ and $|E(AQ_n[L_m^n])| = \frac{1}{2}(\sum_{i=0}^{s} (2t_i - 1)2^{t_i} + \sum_{i=0}^{s} 4i \cdot 2^{t_i} + \delta)$.

For the given $m = \sum_{i=0}^{s} 2^{t_i}$, take $s+1$ t_i-dimensional augmented subcubes in an n-dimensional augmented cube for $i = 0, 1, \ldots, s$ as follows:

$A_m^0 : 0 \ldots 0 \underbrace{X_{t_0} \ldots X_1}_{t_0}$

(t_0-dimensional augmented cube)

$A_m^1 : 0 \ldots 0 \underbrace{1\, 0 \ldots 0 \underbrace{X_{t_1} \ldots X_1}_{t_1}}_{t_0}$

(take a t_1-dimensional augmented cube from $0 \ldots 01X_{t_0} \ldots X_1$)

$A_m^2 : 0 \ldots 010 \ldots 0 \underbrace{1\, 0 \ldots 0 \underbrace{X_{t_2} \ldots X_1}_{t_2}}_{t_1}$

(take a t_2-dimensional augmented cube from $0 \ldots 010 \ldots 01X_{t_1} \ldots X_1$)

…

$A_m^s : 0 \ldots 010 \ldots \ldots 0 \underbrace{1\, 0 \ldots 0 \underbrace{X_{t_s} \ldots X_1}_{t_s}}_{t_{s-1}}$

(take a t_s-dimensional augmented cube from $0 \ldots 010 \ldots \ldots 01X_{t_{s-1}} \ldots X_1$)

Note that $L_m^n = V(A_m^0) \cup \cdots \cup V(A_m^s)$ and A_m^0 is fixed, A_m^i is taken from a t_{i-1}-dimensional augmented subcube which is obtained from A_m^{i-1} by changing the 0 of $(t_{i-1}+1)$th-coordinate of A_m^{i-1} to 1 for $i = 1, \ldots, s$. Hence, $V(A_m^i) \cap V(A_m^j) = \emptyset$ for $i \ne j$, $i, j \in \{0, \ldots, s\}$ and $|V(A_m^0) \cup \cdots \cup V(A_m^s)| = \sum_{i=0}^{s} 2^{t_i} = |L_m^n| = m$. In [26], $AQ_n - AQ_n[L_m^n]$ is connected and $|E(AQ_n[L_m^n])| = \sum_{i=0}^{s-1} (2t_i - 1)2^{t_i - 1} + \sum_{i=0}^{s} 2i \cdot 2^{t_i}$ when $t_s > 0$; $|E(AQ_n[L_m^n])| = \sum_{i=0}^{s} (2t_i - 1)2^{t_i - 1} + \sum_{i=0}^{s} 2i \cdot 2^{t_i}$ when $t_s = 0$ thus $|E(AQ_n[L_m^n])| = \frac{1}{2}(\sum_{i=0}^{s}(2t_i - 1)2^{t_i} + \sum_{i=0}^{s} 4i \cdot 2^{t_i} + \delta)$. The $AQ_4[L_7^4]$ and $AQ_4[L_{14}^4]$ are illustrated in Fig. 3.

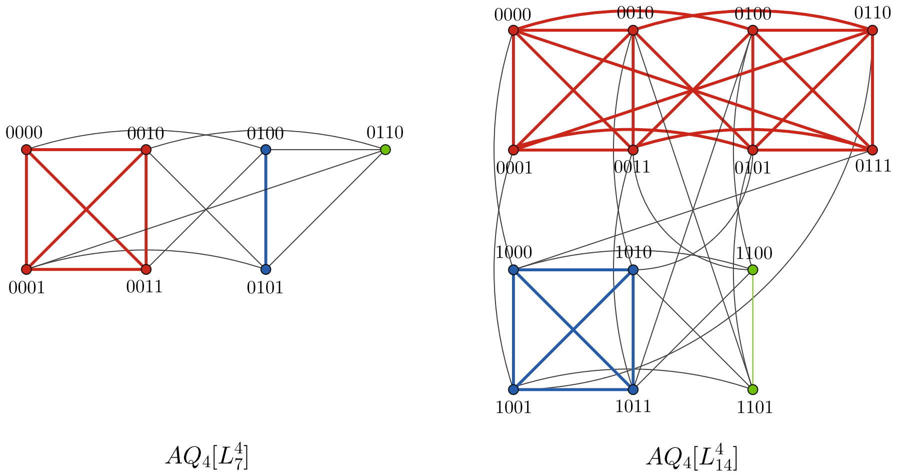

Fig. 3 The induced graphs $\mathrm{AQ}_4[L_7^4]$ and $\mathrm{AQ}_4[L_{14}^4]$

In 2021, Zhang et al. [26] showed that AQ_n is λ_h-optimal for $n \geq 2$ and $h \leq 2^{\lfloor \frac{n}{2} \rfloor}$. In this paper, we determine the exact value of h-extra 3-component edge-connectivity of AQ_n and show that AQ_n is $c\lambda_3^h$-optimal for $n \geq 4, h \leq 2^{\lfloor \frac{n}{2} \rfloor - 1}$ and $c\lambda_3^{2^c}(\mathrm{AQ}_n) = \xi_{2^c,3}(\mathrm{AQ}_n) = (2n-2c-1)2^{c+1}$ for $n \geq 4$ and $1 \leq c \leq n-2$ as the following theorems.

Theorem 1 *For* $n \geq 4$ *and* $1 \leq h \leq 2^{\lfloor \frac{n}{2} \rfloor - 1}$, $c\lambda_3^h(\mathrm{AQ}_n) = \xi_{h,3}(\mathrm{AQ}_n) = (4n-4)h - 2ex_h(\mathrm{AQ}_n) + \delta$ *where* $\delta = 0$ *if* h *is even, and* $\delta = 1$ *if* h *is odd.*

Theorem 2 *Given a positive integer* $n \geq 4$, *then* $c\lambda_3^{2^c}(\mathrm{AQ}_n) = \xi_{2^c,3}(\mathrm{AQ}_n) = (2n-2c-1)2^{c+1}$ *for* $1 \leq c \leq n-2$.

The rest of this paper is organized as follows. In Sect. 2, we introduce some useful properties of AQ_n. In Sect. 3, the proofs of Theorem 1 and Theorem 2 will be provided. In Sect. 4, we conclude this paper and propose some prospects.

2 Some properties and lemmas about AQ_n

As AQ_n is a $(2n-1)$-regular connected graph, then $\xi_{m,2}(\mathrm{AQ}_n) = (2n-1)m - ex_m(\mathrm{AQ}_n)$ and $\lambda_h(\mathrm{AQ}_n) = \min\{\xi_{m,2}(\mathrm{AQ}_n) : h \leq m \leq 2^{n-1}\}$ by the definition of $\lambda_h(\mathrm{AQ}_n)$. Basis on this, Zhang et al. [26] determined the exact value of $\lambda_h(\mathrm{AQ}_n)$ by showing that $\lambda_h(AQ_n) = \xi_{h,2}(\mathrm{AQ}_n)$ for $n \geq 2$ and $h \leq 2^{\lfloor \frac{n}{2} \rfloor}$ in 2021. Motivated by the above, we can use $\xi_{h,3}(\mathrm{AQ}_n)$ to determine the exact value of $c\lambda_3^h(\mathrm{AQ}_n)$, and whether AQ_n is $c\lambda_3^h$-optimal or not. Let F be a minimum h-extra 3-component edge-cut of AQ_n. Actually, $|F| = c\lambda_3^h(\mathrm{AQ}_n)$ and $\mathrm{AQ}_n - F$ have exactly three components. In Sect. 3, we will prove that two of the three components have exactly

h vertices for $n \geq 4$ and $1 \leq h \leq 2^{\lfloor \frac{n}{2} \rfloor -1}$ by the following lemmas. In other words, $c\lambda_3^h(\mathrm{AQ}_n) = \xi_{h,3}(\mathrm{AQ}_n)$, i.e., AQ_n is $c\lambda_3^h$-optimal for $n \geq 4$ and $1 \leq h \leq 2^{\lfloor \frac{n}{2} \rfloor -1}$.

Lemma 2 ([26]) *For two integers m_1, m_2 with $m_1 \leq m_2$ and $m_1 + m_2 > 2$, $ex_{m_1+m_2}(\mathrm{AQ}_n) \geq ex_{m_1}(\mathrm{AQ}_n) + ex_{m_2}(\mathrm{AQ}_n) + 4m_1$.*

Lemma 3 ([26]) *For $n \geq 4$ and $1 \leq h \leq 2^{\lfloor \frac{n}{2} \rfloor} - 1$, $\lambda_h(\mathrm{AQ}_n) = \xi_{h,2}(\mathrm{AQ}_n) = (2n-1)h - ex_h(\mathrm{AQ}_n)$ and $\xi_{h,2}(\mathrm{AQ}_n) \leq \xi_{h+1,2}(\mathrm{AQ}_n)$.*

Lemma 4 ([26]) *For any three positive integers m, n, and b with $1 \leq 2^b \leq m \leq 2^{n-1}$, $\xi_{m,2}(\mathrm{AQ}_n) \geq \xi_{2^b,2}(\mathrm{AQ}_n)$.*

Lemma 5 $\mathrm{AQ}_n[L^n_{2m}]$ *contains two disjoint subgraphs that are both isomorphic to* $\mathrm{AQ}_n[L^n_m]$ *for $n \geq 2$ and $1 \leq m \leq 2^{n-1}$.*

Proof Note that $m \leq 2^{n-1}$, then any vertex $u \in L^n_m$ can be written as follows: $u = 0u_{n-1}u_{n-2} \cdots u_1$. Define two bijections θ_1: $0u_{n-1}u_{n-2} \cdots u_1 \rightarrow u_{n-1}u_{n-2} \cdots u_1 u_1$ and θ_2: $0u_{n-1}u_{n-2} \cdots u_1 \rightarrow u_{n-1}u_{n-2} \cdots u_1\overline{u_1}$ where $\overline{u_1} = 1 - u_1$. For any $t \in S_m = \{0, 1, \cdots, m-1\}$, let $0t_{n-1}t_{n-2} \cdots t_1 \in L^n_m$ be the n-binary string corresponding to t. Denote $\theta_1(t)$ and $\theta_2(t)$ be the decimal representation of $t_{n-1}t_{n-2} \cdots t_1 t_1$ and $t_{n-1}t_{n-2} \cdots t_1\overline{t_1}$, respectively. If $t_1 = 0$, then $\theta_1(t) = 2t$, $\theta_2(t) = 2t + 1$, and if $t_1 = 1$, then $\theta_1(t) = 2t + 1$, $\theta_2(t) = 2t$. Hence, $\theta_i(t) \leq 2m - 1$ and $\theta_i(0t_{n-1}t_{n-2} \cdots t_1) \in L^n_{2m}$, $i = 1, 2$. Denote $T^n_m = \{\theta_1(u) : u \in L^n_m\}$ and $H^n_m = \{\theta_2(u) : u \in L^n_m\}$. Therefore, $T^n_m \subset L^n_{2m}$ and $H^n_m \subset L^n_{2m}$. Due to $|T^n_m| = |H^n_m| = m$ and $T^n_m \cap H^n_m = \emptyset$, L^n_{2m} can be partitioned into T^n_m and H^n_m.

For any two vertices $p = p_n p_{n-1} \cdots p_2 p_1$ and $q = q_n q_{n-1} \cdots q_2 q_1$ with $p_n = q_n = 0$ in L^n_m, we prove that $\mathrm{AQ}_n[T^n_m]$ is isomorphic to $\mathrm{AQ}_n[L^n_m]$ by showing that $pq \in E(\mathrm{AQ}_n[L^n_m])$ if and only if $\theta_1(p)\theta_1(q) \in E(\mathrm{AQ}_n[T^n_m])$. Suppose that $pq \in E(\mathrm{AQ}_n[L^n_m])$, then pq is an O-edge in AQ_n. Therefore, there exists an integer k with $1 \leq k < n$ satisfying one of the following two cases.

Case 1. $p_i = q_i$ for $k+1 \leq i \leq n$ and $p_j = 1 - q_j$ for $1 \leq j \leq k$.

In this case, if $k = n-1$, then $p_i = 1 - q_i$ for $1 \leq i \leq n-1$. Note that $\theta_1(p) = p_{n-1}p_{n-2} \cdots p_1 p_1$ and $\theta_1(q) = q_{n-1}q_{n-2} \cdots q_1 q_1 = \overline{p_{n-1}}\,\overline{p_{n-2}} \cdots \overline{p_1}\,\overline{p_1}$, $\theta_1(p)\theta_1(q)$ is a C-edge in AQ_n. Otherwise, $1 \leq k \leq n-2$. Considering $p_i = q_i$ for $i = n-1, n-2, \cdots, k+1$ and $p_j = 1 - p_j$ for $j = k, k-1, \cdots, 1$, $\theta_1(p)\theta_1(q)$ is an O-edge in AQ_n.

Case 2. $p_i = q_i$ for $i \neq k$ and $p_k = 1 - q_k$.

In this case, if $k = n-1$, then $p_{n-1} = 1 - q_{n-1}$ and $p_i = q_i$ for $1 \leq i \leq n-2$. Hence, $\theta_1(p)\theta_1(q)$ is an H-edge in AQ_n. Otherwise, $1 \leq k \leq n-2$. Considering $p_i = q_i$ for $i = n-1, n-2, \cdots, k+1, k-1, \cdots, 1$ and $p_k = 1 - q_k$, $\theta_1(p)\theta_1(q)$ is an O-edge in AQ_n.

In a nutshell, if $pq \in E(\mathrm{AQ}_n[L^n_m])$, then $\theta_1(p)\theta_1(q) \in E(\mathrm{AQ}_n)$ and $\theta_1(p)\theta_1(q) \in E(\mathrm{AQ}_n[T^n_m])$ naturally. Conversely, let $\theta_1(p)\theta_1(q) \in E(\mathrm{AQ}_n[T^n_m])$. If $\theta_1(p)\theta_1(q)$ is an O-edge in AQ_n, then there exists an integer k with $1 \leq k < n-1$

satisfying $p_i = q_i$ for $i = n-1, n-2, \cdots, k+1$, $p_j = 1 - q_j$ for $j = k, k-1, \cdots, 2, 1$, or $p_i = q_i$ for $i = n-1, n-2, \cdots, k+1, k-1, \cdots, 1$, $p_k = q_k$, respectively. Note that $p = p_n p_{n-1} \cdots p_2 p_1$ and $q = q_n q_{n-1} \cdots q_2 q_1$ with $p_n = q_n = 0$, pq is an O-edge in AQ_n. In other case, if $\theta_1(p)\theta_1(q)$ is an H-edge in AQ_n, then $p_{n-1} = 1 - q_{n-1}$ and $p_i = q_i$ for $1 \le i \le n-2$. Let $k_1 = n-1$. Since $p_i = q_i$ for $i \ne k_1$ and $p_{k_1} = 1 - q_{k_1}$, pq is an O-edge in AQ_n. In addition, if $\theta_1(p)\theta_1(q)$ is a C-edge in AQ_n, then $p_i = 1 - q_i$ for $1 \le i \le n-1$. Let $k_2 = n-1$. Since $p_i = q_i$ for $k_2 + 1 \le i \le n$ and $p_j = 1 - q_j$ for $1 \le j \le k_2$, pq is an O-edge in AQ_n. Briefly, if $\theta_1(p)\theta_1(q) \in E(\mathrm{AQ}_n[T_m^n])$, then $pq \in E(\mathrm{AQ}_n)$ and $pq \in E(\mathrm{AQ}_n[L_m^n])$ naturally. Summarize the above in one sentence, $pq \in E(\mathrm{AQ}_n[L_m^n])$ if and only if $\theta_1(p)\theta_1(q) \in E(\mathrm{AQ}_n[T_m^n])$. Hence, $\mathrm{AQ}_n[T_m^n]$ is isomorphic to $\mathrm{AQ}_n[L_m^n]$. Similarly, we can prove that $AQ_n[H_m^n]$ is isomorphic to $\mathrm{AQ}_n[L_m^n]$ by showing that $pq \in E(\mathrm{AQ}_n[L_m^n])$ if and only if $\theta_2(p)\theta_2(q) \in E(\mathrm{AQ}_n[H_m^n])$. Thus, this lemma holds. □

Incidentally, since $ex_{2m}(\mathrm{AQ}_n) = 2ex_m(\mathrm{AQ}_n) + 4m - 2\delta$, there are $2m - \delta$ edges between $\mathrm{AQ}_n[T_m^n]$ and $\mathrm{AQ}_n[H_m^n]$ where $\delta = 0$ if m is even, $\delta = 1$ if m is odd. In Fig. 4, $\mathrm{AQ}_4[L_{14}^4]$ contains two disjoint connected subgraphs $\mathrm{AQ}_4[T_7^4]$ (marked in red, light gray in print) and $\mathrm{AQ}_n[H_7^4]$ (marked in blue, dark gray in print).

Lemma 6 *For* $n \ge 2$ *and* $1 \le m < 2^{n-1}$, $\xi_{m,3}(\mathrm{AQ}_n) = (4n-4)m - 2ex_m(\mathrm{AQ}_n) + \delta$ *where* $\delta = 0$ *if m is even, and* $\delta = 1$ *if m is odd.*

Proof We prove this lemma by giving an edge-cut F of AQ_n of size $(4n-4)m - 2ex_m(\mathrm{AQ}_n) + \delta$ such that $G - F$ has exactly three components with two components which have exactly m vertices and showing that $(4n-4)m - 2ex_m(\mathrm{AQ}_n) + \delta$ is the lower bound of $\xi_{m,3}(\mathrm{AQ}_n)$.

As for $\xi_{m,3}(\mathrm{AQ}_n) \le (4n-4)m - 2ex_m(\mathrm{AQ}_n) + \delta$, given $n \ge 2$ and $1 \le m < 2^{n-1}$, by Lemma 5, $\mathrm{AQ}_n[L_{2m}^n]$ contains two disjoint connected subgraphs with order m and size $\frac{1}{2}ex_m(\mathrm{AQ}_n)$ as $M_1 = \mathrm{AQ}_n[T_m^n]$ and $M_2 = \mathrm{AQ}_n[H_m^n]$. Let $M_3 = \mathrm{AQ}_n[\overline{L_{2m}^n}]$. Note that M_1, M_2, and M_3 are all connected and $|T_m^n| = |H_m^n| = m$, then

$$\xi_{m,3}(\mathrm{AQ}_n) \le |[T_m^n, H_m^n, \overline{L_{2m}^n}]| = |[T_m^n, H_m^n]| + |[L_{2m}^n, \overline{L_{2m}^n}]| = 2m - \delta + \xi_{2m,2}(\mathrm{AQ}_n)$$
$$= 2m(2n-1) - (2ex_m(\mathrm{AQ}_n) + 4m - 2\delta) + 2m - \delta = (4n-4)m - 2ex_m(\mathrm{AQ}_n) + \delta.$$

Let F be any edge-cut of AQ_n that $\mathrm{AQ}_n - F$ has exactly three connected components A_1, A_2, and A_3 with $|V(A_1)| = |V(A_2)| = m$. Note that

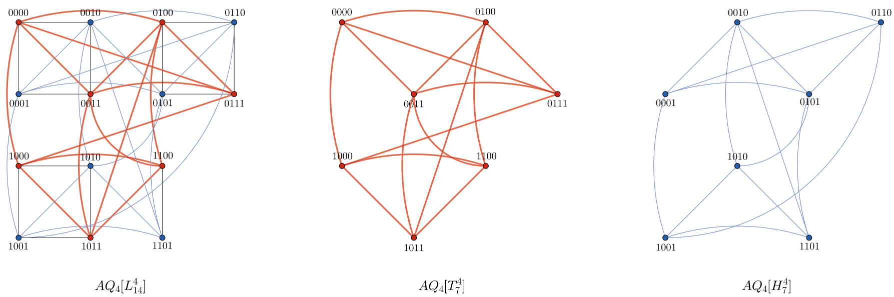

Fig. 4 $\mathrm{AQ}_4[T_7^4]$ and $\mathrm{AQ}_4[H_7^4]$ are both isomorphic to $\mathrm{AQ}_4[L_7^4]$ and $|[T_7^4, H_7^4]| = 2 \times 7 - 1$

$$\begin{aligned}2|F| &= 2|[V(A_1), V(A_2), V(A_3)]| \\ &= 2(|[V(A_1), V(A_2)]| + |[V(A_2), V(A_3)]| + |[V(A_1), V(A_3)]|) \\ &\geq \xi_{m,2}(AQ_n) + \xi_{m,2}(AQ_n) + \xi_{2^n-2m,2}(AQ_n) \\ &= 2\xi_{m,2}(AQ_n) + \xi_{2m,2}(AQ_n) \\ &= 2(2n-1)m - 2ex_m(AQ_n) + (2n-1)2m - ex_{2m}(AQ_n).\end{aligned}$$

Since $ex_{2m}(\mathrm{AQ}_n) = 2ex_m(\mathrm{AQ}_n) + 4m - 2\delta$, we have $|F| \geq (4n-4)m - 2ex_m(\mathrm{AQ}_n) + \delta$. By the generality of F, $\xi_{m,3}(\mathrm{AQ}_n) \geq (4n-4)m - 2ex_m(\mathrm{AQ}_n) + \delta$, this lemma holds. □

3 The main proof of our results about *h*-extra *r*-component edge-connectivity of AQ$_n$

The proof of Theorem 1 for $c\lambda_3^h(\mathrm{AQ}_n) = \xi_{h,3}(\mathrm{AQ}_n)$ for $n \geq 4$ and $1 \leq h \leq 2^{\lfloor \frac{n}{2} \rfloor - 1}$

Proof For $n \geq 4$ and $1 \leq h \leq 2^{\lfloor \frac{n}{2} \rfloor - 1} < \lfloor \frac{2^n}{3} \rfloor$, considering an upper bound for the exact value of general h-extra 3-component edge-connectivity of AQ$_n$ is offered by Lemma 6 that $c\lambda_3^h(\mathrm{AQ}_n) \leq \xi_{h,3} = (4n-4)h - 2ex_h(\mathrm{AQ}_n) + \delta$, we prove Theorem 1 by showing that $(4n-4)h - 2ex_h(\mathrm{AQ}_n) + \delta$ is the lower bound of $c\lambda_3^h(\mathrm{AQ}_n)$.

Let F be a minimum h-extra 3-component edge-cut of AQ$_n$. By Lemma 1, $\mathrm{AQ}_n - F$ has exactly three components, denoted as C_1, C_2, C_3 with $|V(C_1)| \leq |V(C_2)| \leq |V(C_3)|$. As a general rule, two of the three components have small scales, and the other is a giant component as Fig. 5.

Let $h_i = |V(C_i)|$ for $i \in \{1, 2, 3\}$. Note that $h \leq h_1 \leq \lfloor \frac{2^n}{3} \rfloor$ and $\left\lceil \frac{2^n - h_1}{2} \right\rceil \leq h_3 \leq 2^n - 2h_1$, then $2h \leq 2h_1 \leq h_1 + h_2 \leq 2^n - \left\lceil \frac{2^n - h_1}{2} \right\rceil$. For any $1 \leq i \leq 3$, we have $|[V(C_i), \overline{V(C_i)}]| = (2n-1)h_i - 2|E(C_i)| \geq \xi_{h_i,2}(\mathrm{AQ}_n) = (2n-1)h_i - ex_{h_i}(\mathrm{AQ}_n)$. By Lemma 2, it follows that

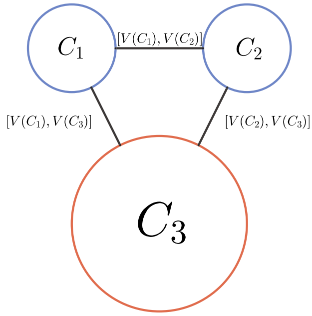

Fig. 5 Illustration of $\mathrm{AQ}_n - F$

$$\begin{aligned}|F| &= |[V(C_1),\overline{V(C_1)}]| + |[V(C_2),\overline{V(C_2)}]| - |[V(C_1),V(C_2)]|\\ &= (2n-1)(h_1+h_2) - 2|E(C_1)| - 2|E(C_2)| - (|E(C_1\cup C_2)| - |E(C_1)| - |E(C_2)|)\\ &\geq (2n-1)\big(h_1+h_2\big) - \frac{1}{2}ex_{h_1}(\mathrm{AQ}_n) - \frac{1}{2}ex_{h_2}(\mathrm{AQ}_n) - \frac{1}{2}ex_{h_1+h_2}(\mathrm{AQ}_n)\\ &\geq (2n-1)\big(h_1+h_2\big) - ex_{h_1+h_2}\big(\mathrm{AQ}_n\big) + 2h_1.\end{aligned}$$

Case 1. $2h \leq h_1 + h_2 \leq 2^{\lfloor \frac{n}{2} \rfloor}$.

By Lemma 3, we have $(2n-1)\big(h_1+h_2\big) - ex_{h_1+h_2}\big(\mathrm{AQ}_n\big) + 2h_1 = \xi_{h_1+h_2,2}\big(\mathrm{AQ}_n\big)$ $+2h_1 \geq \xi_{2h,2}\big(\mathrm{AQ}_n\big) + 2h_1 = (4n-2)h - ex_{2h}(\mathrm{AQ}_n) + 2h_1$. Therefore, $|F| \geq (4n-2)h - ex_{2h}\big(\mathrm{AQ}_n\big) + 2h$ for $1 \leq h \leq 2^{\lfloor \frac{n}{2} \rfloor - 1}$ and $n \geq 4$.

Case 2. $2^{\lfloor \frac{n}{2} \rfloor} \leq h_1 + h_2 \leq 2^{n-1}$.

By Lemma 4, we have $(2n-1)\big(h_1+h_2\big) - ex_{h_1+h_2}(\mathrm{AQ}_n) + 2h_1 = \xi_{h_1+h_2,2}\big(\mathrm{AQ}_n\big)$ $+2h_1 \geq \xi_{2^{\lfloor \frac{n}{2} \rfloor},2}(\mathrm{AQ}_n) + 2h_1$. By Lemma 3, $|F| \geq \xi_{2^{\lfloor \frac{n}{2} \rfloor},2}(\mathrm{AQ}_n) + 2h_1 \geq \xi_{2h,2}\big(\mathrm{AQ}_n\big)$ $+2h_1 = (4n-2)h - ex_{2h}\big(\mathrm{AQ}_n\big) + 2h_1 \geq (4n-2)h - ex_{2h}\big(\mathrm{AQ}_n\big) + 2h$ for $1 \leq h \leq 2^{\lfloor \frac{n}{2} \rfloor - 1}$ and $n \geq 4$.

Case 3. $2^{n-1} \leq h_1 + h_2 \leq 2^n - \left\lceil \frac{2^n - h_1}{2} \right\rceil$.

In this case, $\left\lceil \frac{2^n - h_1}{2} \right\rceil \leq 2^n - \big(h_1+h_2\big) \leq 2^{n-1}$. Note that $\xi_{2^n-(h_1+h_2),2}\big(\mathrm{AQ}_n\big) = \xi_{h_1+h_2,2}$ $\big(\mathrm{AQ}_n\big) = (2n-1)\big(h_1+h_2\big) - ex_{h_1+h_2}\big(\mathrm{AQ}_n\big)$ and $\left\lceil \frac{2^n - h_1}{2} \right\rceil > 2^{\lfloor \frac{n}{2} \rfloor}$ for any $1 \leq h \leq 2^{\lfloor \frac{n}{2} \rfloor - 1}$ and $h \leq h_1 \leq \left\lfloor \frac{2^n}{3} \right\rfloor$. By Lemma 3 and Lemma 4, $|F| \geq (2n-1)\big(h_1+h_2\big) - ex_{h_1+h_2}\big(\mathrm{AQ}_n\big) + 2h_1 = \xi_{h_1+h_2,2}\big(\mathrm{AQ}_n\big) + 2h_1 >$ $\xi_{2^{\lfloor \frac{n}{2} \rfloor},2}\big(\mathrm{AQ}_n\big) + 2h_1 \geq \xi_{2h,2}\big(\mathrm{AQ}_n\big) + 2h_1 \geq 2nh - ex_{2h}\big(\mathrm{AQ}_n\big) + 2h$ for $1 \leq h \leq 2^{\lfloor \frac{n}{2} \rfloor - 1}$ and $n \geq 4$.

Thus, we have $c\lambda_3^h(\mathrm{AQ}_n) \geq (4n-2)h - ex_{2h}\big(\mathrm{AQ}_n\big) + 2h = (4n-2)h - (2ex_h(\mathrm{AQ}_n)$ $+4h - 2\delta) + 2h \geq (4n-4)h - 2ex_h(\mathrm{AQ}_n) + \delta$. Combining with $c\lambda_3^h(\mathrm{AQ}_n) \leq \xi_{h,3}(\mathrm{AQ}_n) = (4n-4)h - 2ex_h(\mathrm{AQ}_n) + \delta$, we can obtain that $c\lambda_3^h(\mathrm{AQ}_n) = \xi_{h,3}(\mathrm{AQ}_n) = (4n-4)h - 2ex_h(\mathrm{AQ}_n) + \delta$ for $n \geq 4$ and $1 \leq h \leq 2^{\lfloor \frac{n}{2} \rfloor - 1}$. □

The proof of Theorem 2 for $c\lambda_3^{2^c}(\mathrm{AQ}_n) = \xi_{2^c,3}(\mathrm{AQ}_n)$ for $n \geq 4$ and $1 \leq c \leq n-2$

Proof It is sufficient to show that $c\lambda_3^{2^c}(\mathrm{AQ}_n) \leq (2n-2c-1)2^{c+1}$ by constructing a 2^c-extra 3-component edge-cut of cardinality $(2n-2c-1)2^{c+1}$. For any $1 \leq c \leq n-2, n \geq 4$, note that $\mathrm{AQ}_n[L^n_{2^{c+1}}]$ can be divided into two c-dimensional augmented cubes as $N_1 = \mathrm{AQ}_n[T^n_{2^c}]$ and $N_2 = \mathrm{AQ}_n[H^n_{2^c}]$. Let $N_3 = \mathrm{AQ}_n - \mathrm{AQ}_n[L^n_{2^{c+1}}]$. Since N_1, N_2, and N_3 are both connected and $|V(N_3)| = 2^n - 2^{c+1} \geq 2^c$, $F = [V(N_1), V(N_2), V(N_3)]$ is a 2^c-extra 3-component edge-cut of AQ_n.Hence, $c\lambda_3^{2^c}(\mathrm{AQ}_n) \leq |F| = 2\xi_{2^c,2}(\mathrm{AQ}_n) - 2^{c+1} = (4n-2)2^c - (4c-2)2^c - 2^{c+1} = (2n-2c-1)2^{c+1}$.

As for $c\lambda_3^{2^c}(\mathrm{AQ}_n) \geq (2n-2c-1)2^{c+1}$, let F be a minimum 2^c -extra 3-component edge-cut of AQ_n.By Lemma 1, $\mathrm{AQ}_n - F$ has exactly three components, denoted as W_1, W_2, W_3. Let $2^c \leq |V(W_1)| \leq |V(W_2)| \leq |V(W_3)|$ and $|V(W_i)| = h_i$ for $1 \leq i \leq 3$. Note that $\xi_{2^c,2}(\mathrm{AQ}_n) = (2n-1)2^c - ex_{2^c}(\mathrm{AQ}_n) = (2n-1)2^c - (2c-1)2^c = (2n-2c)2^c$, then

Case 1. $1 \leq c \leq \lfloor \frac{n}{2} \rfloor - 1$.

By Theorem 1, we have $|F| \geq (4n-2)2^c - 2ex_{2^c}(\mathrm{AQ}_n) - 2^{c+1} = (4n-2)2^c - (2c-1)2^{c+1} - 2^{c+1} = (2n-2c-1)2^{c+1}$.

Case 2. $\lfloor \frac{n}{2} \rfloor \leq c \leq n-3$.

If $2^c \leq h_1 \leq h_2 \leq h_3 \leq 2^{c+1} \leq 2^{n-2}$, then $h_1 + h_2 + h_3 \leq 3 \cdot 2^{n-2} < 2^n$, a contradiction. Hence, $h_3 > 2^{c+1}$. Since $2|F| = |[V(W_1), \overline{V(W_1)}]| + |[V(W_2), \overline{V(W_2)}]| + |[V(W_3), \overline{V(W_3)}]| \geq \xi_{h_1,2}(\mathrm{AQ}_n) + \xi_{h_2,2}(\mathrm{AQ}_n) + \xi_{h_3,2}(\mathrm{AQ}_n)$, by Lemma 4, there is

Subcase 1. $2^{c+1} < h_3 \leq 2^{n-1}$.

$$\begin{aligned} 2|F| &\geq \xi_{h_1,2}(\mathrm{AQ}_n) + \xi_{h_2,2}(\mathrm{AQ}_n) + \xi_{h_3,2}(\mathrm{AQ}_n) \\ &\geq \xi_{2^c,2}(\mathrm{AQ}_n) + \xi_{2^c,2}(\mathrm{AQ}_n) + \xi_{2^{c+1},2}(\mathrm{AQ}_n) \\ &= 2 \cdot (2n-2c) \cdot 2^c + (2n-2c-2) \cdot 2^{c+1} \\ &= (4n-4c-2) \cdot 2^{c+1}. \end{aligned}$$

Subcase 2. $h_3 > 2^{n-1}$.

As $2^{c+1} \leq h_1 + h_2 \leq 2^{n-1}$, we have

$$\begin{aligned} 2|F| &\geq \xi_{h_1,2}(\mathrm{AQ}_n) + \xi_{h_2,2}(\mathrm{AQ}_n) + \xi_{h_3,2}(\mathrm{AQ}_n) \\ &= \xi_{h_1,2}(\mathrm{AQ}_n) + \xi_{h_2,2}(\mathrm{AQ}_n) + \xi_{h_1+h_2,2}(\mathrm{AQ}_n) \\ &\geq \xi_{2^c,2}(\mathrm{AQ}_n) + \xi_{2^c,2}(\mathrm{AQ}_n) + \xi_{2^{c+1},2}(\mathrm{AQ}_n) \\ &= 2 \cdot (2n-2c) \cdot 2^c + (2n-2c-2) \cdot 2^{c+1} \\ &= (4n-4c-2) \cdot 2^{c+1}. \end{aligned}$$

Case 3. $c = n-2$.

In this case, there is $2^{n-2} = 2^c \leq h_1 \leq h_2 \leq h_3$ and then $h_1 + h_2 \geq 2^{n-1}$, $h_3 \leq 2^{n-1}$,

$$\begin{aligned} 2|F| &\geq \xi_{h_1,2}(\mathrm{AQ}_n) + \xi_{h_2,2}(\mathrm{AQ}_n) + \xi_{h_3,2}(\mathrm{AQ}_n) \\ &\geq 3 \cdot \xi_{2^c,2}(\mathrm{AQ}_n) \\ &= 3 \cdot (2n-2c) \cdot 2^c \\ &= 3 \cdot (n-c) \cdot 2^{c+1} \\ &= (4n-4c-2) \cdot 2^{c+1}. \end{aligned}$$

According to the discussion above, we have $|F| \geq (2n-2c-1) \cdot 2^{c+1}$. Similar to the proof of Theorem 1, it can be obtained that $c\lambda_3^{2^c}(\mathrm{AQ}_n) = \xi_{2^c,3}(\mathrm{AQ}_n) = (2n-2c-1)2^{c+1}$ for $1 \leq c \leq n-2$. The proof is completed. □

4 Conclusions

In this paper, we combine Fàbrega and Sampathkumar's concepts about the parameters of network fault tolerance as h-extra edge-connectivity and r-component edge-connectivity to introduce a more refined parameter for characterizing fault tolerance of interconnection networks as h-extra r-component edge-connectivity. Inspired by introducing $\xi_{m,2}(G)$ to solve the exact value of $\lambda_h(G)$ and whether G is λ_h-optimal or not, we introduce the concept of $c\lambda_r^h$-optimal and define the function $\xi_{m,r}(G)$ to solve the exact value of $c\lambda_r^h(G)$ and whether G is $c\lambda_r^h$-optimal or not. Basis on this, we determine the h-extra 3-component edge-connectivity of AQ_n and show that AQ_n is $c\lambda_3^h$-optimal for $n \geq 4$ and $1 \leq h \leq 2^{\lfloor \frac{n}{2} \rfloor -1}$, that is, $c\lambda_3^h(AQ_n) = \xi_{h,3}(AQ_n) = (4n-4)h - 2ex_h(AQ_n) + \delta$. In addition, AQ_n is $c\lambda_3^{2^c}$-optimal for $n \geq 4$ and $1 \leq c \leq n-2$, that is, $c\lambda_3^{2^c}(AQ_n) = \xi_{2^c,3}(AQ_n) = (2n-2c-1)2^{c+1}$. In the future work, we would like to consider a more general case and get more results about $c\lambda_r^h(AQ_n)$ and $\xi_{h,r}(AQ_n)$ to discuss the $c\lambda_r^h$-optimality of AQ_n for $r = 3, 2^{\lfloor \frac{n}{2} \rfloor -1} < h \leq \lfloor 2^n/3 \rfloor$ and $r \geq 4$, respectively.

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Data availability Not applicable.

Declarations

Conflict of interest We declare that the authors have no competing interests as defined by Springer, or other interests that might be perceived to influence the results and/or discussion reported in this paper.

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