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A unified approach to reliability and edge fault tolerance of cube-based interconnection networks under three hypotheses

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Abstract

The topological structures of the interconnection networks of some parallel and distributed systems are designed as n-dimensional hypercube Q_n or n-dimensional folded hypercube FQ_n with $N = 2^n$ processors. For integers $0 \leq k \leq n-1$, let $\mathcal{P}_1^k$, $\mathcal{P}_2^k$ and $\mathcal{P}_3^k$ be the property of having at least k neighbors for each processor, containing at least 2^k processors and admitting average neighbors at least k, respectively. $\mathcal{P}$-conditional edge-connectivity of G, $\lambda(\mathcal{P}, G)$, is the minimum cardinality of faulty edge-cut, whose malfunction divides this network into several components, with each component satisfying the property of $\mathcal{P}$. For each integer $0 \leq k \leq n-1$, and $1 \leq i \leq 3$, this paper offers a unified method to investigate the $\mathcal{P}_i^k$-conditional edge-connectivity of Q_n and FQ_n. Exact value of $\mathcal{P}_i^k$-conditional edge-connectivity of Q_n, $\lambda(\mathcal{P}_i^k, Q_n)$, is $(n-k)2^k$, and that of $\mathcal{P}_i^k$-conditional edge-connectivity of FQ_n, $\lambda(\mathcal{P}_i^k, FQ_n)$, is $(n-k+1)2^k$. Our method generalizes the result of Guo and Guo in [The Journal of Supercomputing, 2014, 68:1235-1240] and the previous other results.

Keywords Interconnection network · Reliability and fault tolerability · Parallel and distributed systems · $\mathcal{P}$-conditional edge-connectivity · Hypercube and folded hypercube

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1 Introduction

With the onset of digital age, big data based on massively parallel distribute processing systems greatly facilitates our lives. To deal with the problem of processing and storage of ever-increasing volume of massive data, various topological structures of interconnection networks of parallel and distributed system are designed and extensively investigated. The performance of a parallel and distributed system is not only related to properties of each single processor, but also highly relied on the topological structures and their parameters of the interconnection networks. The topological interconnection network of parallel and distributed system is usually modeled as a graph $G(V, E)$, where the vertices (nodes) and edges (links) represent processors and communicative links between them. If no confusion caused, the terms graphs and networks are replaceable each other.

Both hypercube and folded hypercube (one of the most famous variants, which was first proposed by El-Amawy and Latifi in 1991 [6]) share some attractive topological properties, such as regularity, highly symmetric property, vertex transitivity, lower diameter, easy scalability and high reliability [3, 6, 24]. They are applied to underlying topological structures of most potential interconnection networks of parallel and distributed system, such as ATM switches [19, 20], 3D Fold-Noc network [8] and PM2I networks [15].

Definition 1 An n-dimensional hypercube Q_n is an undirected graph with $|V(Q_n)| = 2^n$ and $|E(Q_n)| = n2^{n-1}$. Each vertex can be represented by an n-bit binary string in $\{x_n x_{n-1} \cdots x_2 x_1 : x_i \in \{0, 1\}, 1 \leq i \leq n\}$. There is an edge between the vertex $x = x_n x_{n-1} \cdots x_2 x_1$ and $y = y_n y_{n-1} \cdots y_2 y_1$ if and only if their binary string representations differ in only one bit position.

Definition 2 [6] An edge between vertex $x = x_n x_{n-1} \cdots x_2 x_1$ and $y = y_n y_{n-1} \cdots y_2 y_1$ of Q_n, (x, y), is a complementary edge if and only if the bits of x and y are complements of each other, that is, $y_i = \overline{x_i}$ for each $i = 1, 2, \ldots, n$. The n-dimensional folded hypercube, denoted by FQ_n, is an undirected graph obtained from Q_n by adding all complementary edges. M_n is edge set containing all complementary edges of FQ_n.

Because of coagulation and robustness of local networks, all processors and links incident to the same processor cannot malfunction simultaneously. Although the classical Menger's theorem about connectivity or edge-connectivity (defined as the minimum number of vertices or edges whose removal from G results in a disconnected graph) lays a cornerstone for evaluating the reliability and fault tolerance of a static interconnection networks, once coagulation and robustness are enhanced dynamically, that is, when the size of faulty-free sets varies, Menger's theorem no longer offers a more accurate parameter for the reliability of a large-scale parallel and distributed processing systems [3, 7, 9, 10, 12, 18, 23–25]. In 1983, Harary generalized the Menger's theorem of connectivity in both vertex and edge version by introducing the notion of conditional connectivity and theoretically enriched the theory of connectedness of networks [12]. Instead of focusing on the vertex version

[4, 14, 16, 22], this paper mainly studies the conditional edge-connectivity of a connected graph G, which is defined as follows.

Definition 3 Let $\mathcal{P}$ be a property of a graph $G = (V, E)$. The edge subset $F \subset E(G)$ is defined as a $\mathcal{P}$-connected edge-cut of G, if any, G–F is disconnected, and each component of G–F satisfies the condition $\mathcal{P}$. $\lambda(\mathcal{P}, G)$, $\mathcal{P}$-conditional edge-connectivity of G, is defined as the minimum cardinality $\mathcal{P}$-conditional edge-cuts F of G.

Based on the different properties and the faulty-free set, various kinds of $\mathcal{P}$-conditional edge-connectivity are studied. Let
$\mathcal{P}_1^k$ = {having the minimum degree $\delta \geq k$},
$\mathcal{P}_2^k$ = {containing at least 2^k vertices},
and $\mathcal{P}_3^k$ = {satisfying the average degree at least k}.

Then $\lambda(\mathcal{P}_1^k, G)$, $\lambda(\mathcal{P}_2^k, G)$ and $\lambda(\mathcal{P}_3^k, G)$ are the k-super edge-connectivity $\lambda_\delta^k(G)$ [1, 11], 2^k-extra edge-connectivity $\lambda_{2^k}(G)$) [2, 9], and k-average degree edge-connectivity $\overline{\lambda}^k(G)$, respectively. As a k-super edge-connectivity of G, $\lambda(\mathcal{P}_1^k, G) = \lambda_\delta^k(G)$ is the minimum cardinality of all the k-super edge-cut of G [10, 18]. While given a positive integer k, if $\mathcal{P} = \mathcal{P}_2^k$, it gives the 2^k-extra edge-connectivity of G, $\lambda_{2^k}(G)$, which was first introduced by Fàbrega and Foil in [9]. If F is a $\mathcal{P}_2^k$-conditional edge-cut, F is also called a 2^k-extra edge-cut. $\lambda(\mathcal{P}_2^k, G) = \lambda_{2^k}(G)$ is the minimum cardinality of the 2^k-extra edge-cuts of G. Similarly, $\lambda(\mathcal{P}_3^k, G) = \overline{\lambda}^k(G)$ is the minimum cardinality of tht k-average degree edge-cuts of G.

In recent years, the exact values of $\lambda(\mathcal{P}_i^k, Q_n)$ and $\lambda(\mathcal{P}_i^k, FQ_n)$ of cube-based interconnection networks were widely investigated. The focus of these results is either on Q_n and FQ_n for some special $k \leq 4$ under some $\mathcal{P}_i^k$-conditional constrain for a fixed i [3, 11, 25] or on some special cube-based graphs for some fixed i and general $0 \leq k \leq n-1$ [17]. For above special cases, the related results are summarized in

Table 1 Previous known results and our results on $\lambda(\mathcal{P}_i^k, G)$ of the cube-based G

G	k	i	$\lambda(\mathcal{P}_i^k, G)$	Authors and Reference
Q_n	0	1,2,3	n	Zhu and Xu [25]
FQ_n	0	1,2,3	$n+1$	El-Amawy and Latifi [6]
Q_n	1	2	$2n-2$	Zhu and Xu [25]
FQ_n	1	1,2	$2n$	Guo and Guo [11]
FQ_n	1	1,2	$2n, n \geq 2$	Zhu and Xu [25]
FCQ_n	1	2	$2n, n \geq 4$	Cai and Vumar [2]
Q_n	2	2	$4n-8, n \geq 3$	Chang and Hsieh [3] Guo and Guo [11]
FQ_n	2	2	$4n-4, n \geq 4$	Chang and Hsieh [3] Guo and Guo [11]
Q_n	3	1	$8n-24, n \geq 6$	Guo and Guo [11]
FQ_n	3	1	$8n-16, n \geq 16$	Guo and Guo [11]
Q_n	$k, 0 \leq k \leq n-1$	1	$2^k(n-k)$	Li and Xu [17]
$\boldsymbol{Q_n}$	$\boldsymbol{k, 0 \leq k \leq n-1}$	**1, 2, 3**	$\boldsymbol{2^k(n-k)}$	**Current Authors**
$\boldsymbol{FQ_n}$	$\boldsymbol{k, 0 \leq k \leq n-1}$	**1, 2, 3**	$\boldsymbol{2^k(n-k+1)}$	**Current Authors**

Table 1. However, seldom do the researchers pay their attentions to investigating $\lambda(\mathcal{P}_i^k, Q_n)$ and $\lambda(\mathcal{P}_i^k, FQ_n)$ for each nonnegative integer $0 \le k \le n-1$ and $1 \le i \le n$.

For each nonnegative integer $0 \le k \le n-1$, $n \ge 1$ and $1 \le i \le 3$, this paper mainly finds a unified approach to explore the $\mathcal{P}_i^k$-conditional edge-connectivity of Q_n and FQ_n, or rather, the minimum cardinality of faulty edge-cut of Q_n and FQ_n, whose malfunction divides network into several components, with each resulting component satisfying the property of $\mathcal{P}_i^k$, respectively. The exact values of them are equal to the minimum number of edges to delete such that the remaining one connected subgraph is induced by 2^k vertices and the other is also connected, respectively. Our main results are reflected in the last two lines of Table 1, and marked in bold. Before giving the proof of our main results, we show some preliminaries.

2 Preliminaries

By the definitions of Q_n and FQ_n, the vertex set $V(Q_n)$ and $V(FQ_n)$ is denoted by $X_nX_{n-1}\cdots X_2X_1 = \{x_nx_{n-1}\cdots x_2x_1 : x_i \in \{0,1\}, i = 1,2,\ldots,n\}$. For further simplification, let X^n denote $X_nX_{n-1}\cdots X_2X_1$, x^n denote $x_nx_{n-1}\cdots x_2x_1$, and the same is true for the following. Let $0X^{n-1}$ and $1X^{n-1}$ be the vertex subsets $\{0x^{n-1} : x_i \in \{0,1\}, i = 1,2,\ldots,n-1\}$ and $\{1x^{n-1} : x_i \in \{0,1\}, i = 1,2,\ldots,n-1\}$. Let D_0 and D_1 be induced subgraphs $Q_n[0X^{n-1}]$ and $Q_n[1X^{n-1}]$. They are both $(n-1)$-dimensional hypercube and $E(D_0)$, $E(D_1)$ and M_n is a decomposition of $E(Q_n)$. we can express Q_n as $D_0 \bigoplus D_1$. Similarly, we write $z^{n-k}X^k$ for the vertex $\{z^{n-k}X^k : X_i \in \{0,1\}, i = 1,2,\ldots,k, z_j$ is fixed, for $j = k+1, k+2,\ldots,n\}$. Inductively, the subgraph $Q_n[z^{n-k}X^k]$ is a k-dimensional subcube in Q_n induced by $z^{n-k}X^k$. Since FQ_n can be obtained from Q_n by adding complementary edges $\overline{M_n}$, $E(D_0)$, $E(D_1)$, M_n and $\overline{M_n}$ be a decomposition of $E(FQ_n)$. We express FQ_n as $D_0 \bigoplus D_1$. A vertex in $0X^{n-1}$ has exactly one neighbor in $1X^{n-1}$ in Q_n, but have two neighbors in FQ_n.

After deeply mining the properties of the three definitions of the above $\mathcal{P}_i^k$-conditional edge-connectivity of G, one can see that the optimal $\mathcal{P}_i^k$-conditional edge-cut is reached if and only if there are just two components left after removing such edge-cuts from G. Thus, it is necessary to introduce the following two important functions $\xi_m(G)$ and $ex_m(G)$.

Given a vertex set $X \subset V(G)$, we denote $G[X]$ the subgraph of G induced by X. For two vertex sets X and $\overline{X}$, we denote $[X, \overline{X}]$ the edge subset of G with one end in X and the other end in $\overline{X}$. Let $\xi_m(G) = min\{|[X, \overline{X}]| : X \subset V(G), |X| = m \le \lfloor |V(G)|/2 \rfloor$, both $G[X]$ and $G[\overline{X}]$ are connected$\}$, where $|[X, \overline{X}]|$ is the number of elements of $[X, \overline{X}]$. In other words, $\xi_m(G)$ is the minimum number of edges to delete such that the remaining one connected subgraph is induced by m vertices and the other is also connected, where $m \le \lfloor |V(G)|/2 \rfloor$. For a d-regular graph, it follows that $\xi_m(G) = dm - ex_m(G)$, where $ex_m(G)$ is the twice of the maximum number edges of the subgraph of G induced by m vertices.

For convenience, the vertex x^n of n-dimensional hypercube and n-dimensional folded hypercube also can be represented by decimal number $\sum_{i=1}^{n} x_i 2^{i-1}$

in this paper. For instance, $x^5 = x_5x_4x_3x_2x_1 = 01011$ can be disassembled into $1 \times 2^0 + 1 \times 2^1 + 0 \times 2^2 + 1 \times 2^3 + 0 \times 2^4 = 1 + 2 + 8 = 11$. A positive integer m can be decomposed into $\sum_{i=0}^{s} 2^{t_i}$, where $t_0 = \lfloor log_2 m \rfloor$, $t_i = \lfloor log_2(m - \sum_{i=0}^{i-1} 2^{t_i}) \rfloor$ for $i \geq 1$ and $t_i > t_{i+1} \geq 0$. Let S_m be the set $\{0, 1, 2, \dots, m-1\}$ under decimal representation and L_m^n the corresponding set represented by their n-binary strings. Let $\overline{L_m^n}$ be the complement set of $V(Q_n) \setminus L_m^n$. Obviously, by the definition of Q_n, both of L_m^n and $\overline{L_m^n}$ are the subset of $V(Q_n)$. Both $Q_n[L_m^n]$ and $Q_n[\overline{L_m^n}]$ are the subgraphs induced by L_m^n and $\overline{L_m^n}$, respectively. Since Q_n is n-regular, in [13] and [18], exact values of $\xi_m(Q_n)$ and $ex_m(Q_n)$ had been given.

Lemma 1 [13, 18] *For a positive integer m, $m = \sum_{i=0}^{s} 2^{t_i} \leq 2^n$, $\xi_m(Q_n) = nm - ex_m(Q_n)$, where $ex_m(Q_n) = 2|E(Q_n[L_m^n])| = \sum_{i=0}^{s} t_i 2^{t_i} + \sum_{i=0}^{s} 2i2^{t_i}$.*

For example, assume that $m = 5$ and $n = 4$. We have $S_5 = \{0, 1, 2, 3, 4\}$, then $L_5^4 = \{0000, 0001, 0010, 0011, 0100\}$, and hence $[L_5^4, \overline{L_5^4}]$ is the minimum edge-cut of Q_4 when deleting many edges and resulting in one connected subgraph having five vertices and the other being also connected. Since $5 = 2^2 + 2^0$, we have $t_0 = 2$, $t_1 = 0$ and $s = 1$. By Lemma 1, we can obtain that

$$\begin{aligned} ex_5(Q_4) &= 2\left|E\left(Q_4[L_5^4]\right)\right| \\ &= \sum_{i=0}^{1} t_i 2^{t_i} + \sum_{i=0}^{1} 2i2^{t_i} \\ &= t_0 \times 2^{t_0} + t_1 \times 2^{t_0} + 2 \times 0 \times 2^{t_0} + 2 \times 1 \times 2^{t_1} \\ &= 2 \times 2^2 + 0 \times 2^0 + 2 \times 0 \times 2^2 + 2 \times 1 \times 2^0 \\ &= 10. \end{aligned}$$

By the definition of $\xi_5(Q_4) = min\{|[X, \overline{X}]| : X \subset V(Q_4), |X| = 5 \leq \lfloor |V(Q_4)|/2 \rfloor$, both $Q_4[X]$ and $Q_4[\overline{X}]$ are connected$\}$, we have

$$\begin{aligned} \xi_5(Q_4) &= \left|[L_5^4, \overline{L_5^4}]\right| \\ &= nm - ex_m(Q_n) \\ &= 4 \times 5 - 10 \\ &= 10. \end{aligned}$$

Furthermore, if $S' = \{0000, \underline{0001}, 0010, \underline{0100}, 1000\}$ and $S'' = \{00\underline{00}, 0010, 0011, 0100, \underline{01}01\}$, then $[S', \overline{S'}]$ and $[S'', \overline{S''}]$ are edge-cuts of Q_n with $|[S', \overline{S'}]| = 12 > 10$, $|[S'', \overline{S''}]| = 12 > 10$. But $[S', \overline{S'}]$ and $[S'', \overline{S''}]$ are not minimum number of edges to delete when the remaining one connected subgraph is induced by five vertices and the other is also connected. $[L_5^4, \overline{L_5^4}]$ in Q_4, $[S', \overline{S'}]$ and $[S'', \overline{S''}]$ are represented in imaginary lines in Fig. 1.

Obviously, by the definition of FQ_n, both L_m^n and $\overline{L_m^n}$ are also the subset of $V(FQ_n)$. Both $FQ_n[L_m^n]$ and $FQ_n[\overline{L_m^n}]$ are the subgraphs induced by L_m^n and $\overline{L_m^n}$, respectively. From the following lemma, one can obtain that $ex_m(Q_n) = ex_m(FQ_n)$ for each $m \leq 2^{n-1}$.

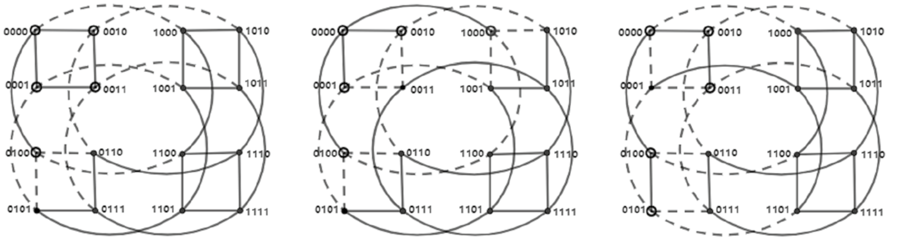

Fig. 1 $[L_5^4, \overline{L_5^4}]$ in Q_4 (imaginary lines in left one), $[S', \overline{S'}]$ and $[S'', \overline{S''}]$ (imaginary lines in right two)

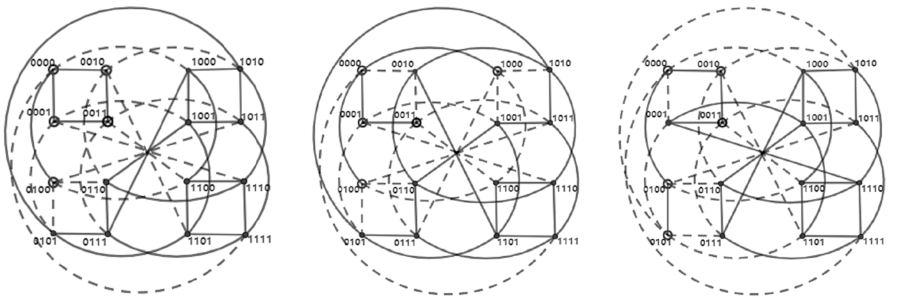

Fig. 2 $[L_5^4, \overline{L_5^4}]$ in FQ_4 (imaginary lines in left one), $[S''', \overline{S'''}]$ and $[S'''', \overline{S''''}]$ with one component 5 vertices (imaginary lines in right two)

Lemma 2 [21, 24] *For a positive integer* $m = \sum_{i=0}^{s} 2^{t_i} \leq 2^n$, $\xi_m(FQ_n) = (n+1)m - ex_m(FQ_n)$, *where*

$$ex_m(FQ_n) = 2|E\big(FQ_n[L_m^n]\big)| = \begin{cases} \sum_{i=0}^{s} t_i 2^{t_i} + \sum_{i=0}^{s} 2i2^{t_i} & \textit{if } 1 \leq m \leq 2^{n-1} \\ \sum_{i=0}^{s} t_i 2^{t_i} + \sum_{i=0}^{s} 2(i+1)2^{t_i} - 2^n & \textit{if } m > 2^{n-1}. \end{cases}$$

Thus, for $m = 5$ and $n = 4$, $ex_5(FQ_4) = ex_5(Q_4) = 10$ and $\xi_5(FQ_4) = (4+1) \times 5 - 10 = 15$. If $S''' = \{0000, 0001, 0011, 0100, 1000\}$ and $S'''' = \{0000, 0010, 0011, 0100, 0101\}$, then $[S''', \overline{S'''}]$ and $[S'''', \overline{S''''}]$ are edge-cuts of FQ_4 with $|[S''', \overline{S'''}]| = 17 > 15$, $|[S'''', \overline{S''''}]| = 16 > 15$. But $[S''', \overline{S'''}]$ and $[S'''', \overline{S''''}]$ are not the minimum number of edges to delete when deleting some edges of FQ_4 and resulting in one connected subgraph having 5 vertices and the other being also connected. $[L_5^4, \overline{L_5^4}]$ in FQ_4, $[S''', \overline{S'''}]$ and $[S'''', \overline{S''''}]$ are represented by imaginary lines in Fig. 2.

Note that for each $m \leq 2^{n-1}$, the subgraph $Q_n[L_m^n]$ is isomorphic to $FQ_n[L_m^n]$. The subgraphs $Q_n[L_m^n]$, $Q_n[\overline{L_m^n}]$, $FQ_n[L_m^n]$ and $FQ_n[\overline{L_m^n}]$ are connected. They are the components which can be separated from Q_n and FQ_n by deleting the edge-cuts $[L_m^n, \overline{L_m^n}]$ in Q_n and FQ_n for each $m \leq 2^n$.

3 The bounds of $\mathcal{P}_i^k$-conditional edge-connectivity of FQ_n and Q_n

A unified method for the bounds of $\mathcal{P}_i^k$-conditional edge-connectivity of FQ_n and Q_n under three hypotheses can be illustrated. First, we show the upper bounds.

Lemma 3 *For each integer $0 \le k \le n-1$, and $1 \le i \le 3$, $n \ge 1$, $\lambda(\mathcal{P}_i^k, Q_n) \le \xi_{2^k}(Q_n) = (n-k)2^k$; $\lambda(\mathcal{P}_i^k, FQ_n) \le \xi_{2^k}(FQ_n) = (n-k+1)2^k$.*

Proof For Q_n, it is sufficient to show that there exists a $\mathcal{P}_i^k$-conditional edge-cut of the size $2^k(n-k)$ in Q_n for each $0 \le k \le n-1$ and $1 \le i \le 3$. On the one hand, the size of the edge-cut $[L_{2^k}^n, \overline{L_{2^k}^n}]$ is $\xi_{2^k}(Q_n) = (n-k)2^k$ for each $0 \le k \le n-1$. $Q_n[L_{2^k}^n] = Q_n[0^{n-k-1}0X^k]$ is isomorphic to k-dimensional subcube Q_k, and Q_k is k-regular and $|\overline{L_{2^k}^n}| = 2^n - 2^k = 2^{n-1} + 2^{n-2} + \cdots + 2^{k+1} + 2^k$. $Q_n[\overline{L_{2^k}^n}] = Q_n[0^{n-k-1}1X^k \cup 0^{n-k-2}1X^{k+1} \cup \cdots \cup 01X^{n-2} \cup 1X^{n-1}]$. $Q_n[L_{2^k}^n]$ is k-regular, with $|L_{2^k}^n| = 2^k$. So both minimum-degree and average-degree of $Q_n[L_{2^k}^n]$ are at least k. As $Q_n[\overline{L_{2^k}^n}]$ has minimum degree k, its average degree is also at least k. This is because that for each subgraph $Q_n[0^{n-s-1}1X^s]$ is vertex disjoint for each $k \le s \le n-1$. It is isomorphic to Q_s and is s-regular for each $k \le s \le n-1$. There exists at least one edge between the vertex in different $Q_n[0^{n-s-1}1X^s]$. So $Q_n[\overline{L_{2^k}^n}]$ is connected. $|L_{2^k}^n| = 2^k \ge 2^k$, $|\overline{L_{2^k}^n}| = 2^n - 2^k \ge 2^k$. Based on the above facts, removing the edge-cut $[L_{2^k}, \overline{L_{2^k}^n}]$ from Q_n results in exactly two components $Q_n[L_{2^k}^n]$ and $Q_n[\overline{L_{2^k}^n}]$. Both $Q_n[L_{2^k}^n]$ and $Q_n[\overline{L_{2^k}^n}]$ satisfy the properties $\mathcal{P}_1^k$, $\mathcal{P}_2^k$ and $\mathcal{P}_3^k$. Thus, in Q_n, $[\overline{L_{2^k}^n}, L_{2^k}^n]$ is a $\mathcal{P}_i^k$-conditional edge-cut in Q_n for each positive $0 \le k \le n-1$ and $1 \le i \le 3$.

Similarly, because $Q_n[L_{2^k}^n] = FQ_n[L_{2^k}^n]$ and $E(Q_n[\overline{L_{2^k}^n}]) \subset E(FQ_n[\overline{L_{2^k}^n}])$ for each $k \le n-1$, in FQ_n, $[L_{2^k}^n, \overline{L_{2^k}^n}]$ is also a $\mathcal{P}_i^k$-conditional edge-cut. As $2|E(Q_n[L_{2^k}^n])| = k2^k$ and FQ_n is $(n+1)$-regular, $|[L_{2^k}^n, \overline{L_{2^k}^n}]| = (n+1)2^k - k2^k = \xi_{2^k}(FQ_n)$ for each positive $0 \le k \le n-1$ and $1 \le i \le 3$.

All in all, by the minimality of $\lambda(\mathcal{P}_i^k, Q_n)$ and $\lambda(\mathcal{P}_i^k, FQ_n)$, the results $\lambda(\mathcal{P}_i^k, Q_n) \le (n-k)2^k = \xi_{2^k}(Q_n)$ and $\lambda(\mathcal{P}_i^k, Q_n) \le (n+1-k)2^k = (n-k+1)2^k = \xi_{2^k}(FQ_n)$ hold. □

The lower bounds of $\mathcal{P}_i^k$-conditional edge-connectivity of FQ_n and Q_n under three hypotheses can be shown as follows.

Lemma 4 *For each $0 \le c \le n-2$, $2^c \le h \le 2^{n-1}$, then $\xi_h(Q_n) \ge \xi_{2^c}(Q_n) = (n-c)2^c$; $\xi_h(FQ_n) \ge \xi_{2^c}(FQ_n) = (n-c+1)2^c$.*

Proof By Lemmas 1 and 2, $ex_{2^c}(Q_n) = ex_{2^c}(FQ_n) = c2^c$. For any $d = c, c+1, \cdots, n-2$,

$$\begin{aligned} & ex_{2^{d+1}}(Q_n) - ex_{2^d}(Q_n) \\ & \quad = ex_{2^{d+1}}(FQ_n) - ex_{2^d}(FQ_n) \\ & \quad = (d+1)2^{d+1} - d2^d \\ & \quad = (d+2)2^d. \end{aligned}$$

So,

$$\begin{aligned} & \xi_{2^{d+1}}(Q_n) - \xi_{2^d}(Q_n) \\ & \quad = n2^{d+1} - ex_{2^{d+1}}(Q_n) - [n2^d - ex_{2^d}(Q_n)] \\ & \quad = n2^d - (d+2)2^d \\ & \quad = (n-d-2)2^d \\ & \quad \geq 0. \end{aligned}$$

The equality of the last inequality above holds if and only if $n = d + 2$.
Similarly,

$$\begin{aligned} & \xi_{2^{d+1}}(FQ_n) - \xi_{2^d}(FQ_n) \\ & \quad = (n+1)2^{d+1} - ex_{2^{d+1}}(FQ_n) - [(n+1)2^d - ex_{2^d}(FQ_n)] \\ & \quad = (n+1)2^d - (d+2)2^d \\ & \quad = (n-d-1)2^d \\ & \quad \geq 2^d \\ & \quad > 0. \end{aligned}$$

On the other hand, $h = 2^d + m_0 < 2^{d+1}$, let $h = \sum_{i=0}^{s} 2^{t_i}$, $t_0 = 2^d$, $m_0 = h - 2^d$, then $m_0 = \sum_{i=1}^{s} 2^{t_i} = \sum_{i=0}^{s-1} 2^{t_{i+1}} < 2^k \leq 2^{n-2}$. Since $h \leq 2^{n-1}$, $ex_h(Q_n) = ex_h(Q_{n-2})$,

$$\begin{aligned} & \xi_h(Q_n) - \xi_{2^d}(Q_n) \\ & \quad = nh - ex_m(Q_n) - [n2^d - ex_{2^d}(Q_n)] \\ & \quad = n\sum_{i=0}^{s} 2^{t_i} - \sum_{i=0}^{s} t_i 2^{t_i} - \sum_{i=0}^{s} 2i2^{t_i} - n2^d \\ & \quad = n\sum_{i=0}^{s-1} 2^{t_{i+1}} - \sum_{i=0}^{s-1} t_{i+1}2^{t_{i+1}} - \sum_{i=0}^{s-1} 2(i+1)2^{t_{i+1}} \\ & \quad = (n-2)(h-2^d) - \sum_{i=0}^{s-1} t_{i+1}2^{t_{i+1}} - \sum_{i=0}^{s-1} 2i2^{t_{i+1}} \\ & \quad = (n-2)m_0 - ex_{m_0}(Q_n) \\ & \quad = (n-2)m_0 - ex_{m_0}(Q_{n-2}) \\ & \quad = \xi_{m_0}(Q_{n-2}) \\ & \quad \geq 0. \end{aligned}$$

Similarly, $h = 2^d + m_0 < 2^{d+1}$, in FQ_n,

$$
\begin{aligned}
&\xi_h(FQ_n) - \xi_{2^d}(FQ_n) \\
&\quad = (n+1)h - ex_{m_0}(FQ_n) - (n+1)2^d - ex_{2^d}(FQ_n) \\
&\quad = (n+1-2)m_0 - ex_{m_0}(FQ_n) \\
&\quad = (n-1)m_0 - ex_{m_0}(FQ_n) \\
&\quad = (n-1)m_0 - ex_{m_0}(Q_{n-1}) \\
&\quad = \xi_{m_0}(Q_{n-1}) \\
&\quad \geq 0.
\end{aligned}
$$

The above results hold because $m_0 \leq 2^{n-1}$, $ex_{m_0}(FQ_n) = ex_{m_0}(Q_{n-1})$ by Lemmas 1 and 2. The last inequality holds because of the connectedness of Q_{n-1}. The proof is done. $\square$

Lemma 5 [5] *For $A \subseteq V(Q_n)$, let $Q_n[A]$ be the induced subgraph of A with average d, then $|A| \geq 2^d$.*

Lemma 6 *For any nonnegative integer $0 \leq k \leq n-1$ and $1 \leq i \leq 3$, $n \geq 1$, $\lambda(\mathcal{P}_i^k, Q_n) \geq \xi_{2^k}(Q_n) = (n-k)2^k$; $\lambda(\mathcal{P}_i^k, FQ_n) \geq \xi_{2^k}(FQ_n) = (n-k+1)2^k$.*

Proof For any integer $0 \leq k \leq n-1$, $1 \leq i \leq 3$, let F be any $\mathcal{P}_i^k$-conditional edge-cut of Q_n. Removal this edge-cut F from Q_n results in p components $C_1, C_2, \dots, C_p$, $p \geq 2$. By the definition of $\lambda(\mathcal{P}_i^k, Q_n)$, each component should satisfy the property of $\mathcal{P}_i^k$.

Let C^* be the minimum component among them. If the minimum degree of C^* is at least k, then so is the average degree of C^*. By Lemma 5, C^* contains at least 2^k vertices; then, one can have $|F| \geq |[V(C^*), \overline{V(C^*)}]| \geq \xi_{|V(C^*)|}(Q_n) \geq \xi_{2^k}(Q_n) = (n-k)2^k$. So the result $\lambda(\mathcal{P}_i^k, Q_n) \geq (n-k)2^k$ holds.

As for any $m \leq 2^{n-1}$, $ex_m(Q_n) = 2|E(Q_n[L_m^n])| = 2|E(FQ_n[L_m^n])| = ex_m(FQ_n)$. Similarly, let C_0^* be the minimum component of FQ_n after deleting any minimum $\mathcal{P}_i^k$-conditional edge-cut F, then $|F| \geq |[V(C_0^*), \overline{V(C_0^*)}]| \geq \xi_{2^k}(FQ_n) = (n-k+1)2^k$. So for any $1 \leq i \leq 3$, $0 \leq k \leq n-1$, the result $\lambda(\mathcal{P}_i^k, FQ_n) \geq \xi_{2^k}(FQ_n) = (n-k+1)2^k$ also holds. The proof is finished. $\square$

4 The proof of main theorem for $\mathcal{P}_i^k$-conditional edge-connectivity of FQ_n and Q_n

Theorem 1 *For any nonnegative integer $0 \leq k \leq n-1$ and $1 \leq i \leq 3$, $n \geq 1$, $\lambda(\mathcal{P}_i^k, Q_n) = \xi_{2^k}(Q_n) = (n-k)2^k$; $\lambda(\mathcal{P}_i^k, FQ_n) = \xi_{2^k}(FQ_n) = (n-k+1)2^k$.*

Proof On the one hand, by the definition of $\lambda(\mathcal{P}_i^k, Q_n)$ and $\lambda(\mathcal{P}_i^k, FQ_n)$, for $0 \leq k \leq n-1$, $1 \leq i \leq 3$ and $n \geq 1$, and their minimality, Lemma 3 offers a $\mathcal{P}_i^k$

-conditional edge-cut of Q_n and FQ_n, $|[L^n_{2^k}, \overline{L^n_{2^k}}]|$, with $\xi_{2^k}(Q_n) = |[L^n_{2^k}, \overline{L^n_{2^k}}]| = (n-k)2^k$ and $\xi_{2^k}(FQ_n) = |[L^n_{2^k}, \overline{L^n_{2^k}}]| = (n-k+1)2^k$, respectively. So $\lambda(\mathcal{P}^k_i, Q_n) \le \xi_{2^k}(Q_n) = (n-k)2^k$; $\lambda(\mathcal{P}^k_i, FQ_n) \le \xi_{2^k}(FQ_n) = (n-k+1)2^k$.

On the other hand, by Lemma 6, for any $\mathcal{P}^k_i$-conditional edge-cut of Q_n and FQ_n, $|F| \ge \xi_{2^k}(Q_n) = (n-k)2^k$ in Q_n and $|F| \ge \xi_{2^k}(FQ_n) = (n-k+1)2^k$ in FQ_n, respectively.

Combining Lemmas 3 and 6, the value $\xi_{2^k}(Q_n) = (n-k)2^k$ offers both upper and lower bound for $\mathcal{P}^k_i$-conditional edge-connectivity of Q_n, $\lambda(\mathcal{P}^k_i, Q_n)$, and so does $\xi_{2^k}(FQ_n) = (n-k+1)2^k$ for the $\mathcal{P}^k_i$-conditional edge-connectivity of $\lambda(\mathcal{P}^k_i, FQ_n)$ of FQ_n. The proof of main theorem is completed. □

5 Application

The n-dimensional hypercube Q_n and folded hypercube FQ_n are applied to underlying topological structures of most potential interconnection networks of parallel and distributed system, such as ATM switches, 3D Fold-Noc networks and PM2I networks. Our unified method of the robustness of hypercube and folded hypercube is based on these systems in the presence of failing links. Based on our main result, we give some exact values for $\lambda(\mathcal{P}^k_i, Q_n)$ and $\lambda(\mathcal{P}^k_i, FQ_n)$ when $0 \le k \le 10$ in Table 2. Moreover, the visual images of both $\lambda(\mathcal{P}^k_i, Q_n)$ and $\lambda(\mathcal{P}^k_i, FQ_n)$ are shown in Fig. 3, for cases $n = 11$ and $n = 12$.

For $0 \le k \le n-1$, let $\mathcal{P}^k_1$, $\mathcal{P}^k_2$ and $\mathcal{P}^k_3$ be the property of having at least k neighbors for each processor, containing at least 2^k processors and admitting average neighbors at least k, respectively. For the n-dimensional hypercubes and folded hypercubes networks with $N = 2^n$ processors, we can find the minimum cardinality $(n-k)2^k$ and $(n-k+1)2^k$ of edge subsets of Q_n and FQ_n, respectively, whose removal disconnects the network with each component satisfying the property of $\mathcal{P}^k_i$, where $1 \le i \le 3$. In other words, similar to Menger's theorem, we want to find edge-disjoint paths to connect any two subnetworks satisfying the property $\mathcal{P}^k_i$ in Q_n,

Table 2 Some exact values of $\lambda(\mathcal{P}^k_i, Q_n)$ and $\lambda(\mathcal{P}^k_i, FQ_n)$

k	0	1	2	3	4	5	6
$\lambda(\mathcal{P}^k_i, Q_n)$	n	$2n-2$	$4n-8$	$8n-24$	$16n-64$	$32n-160$	$64n-384$
k	7	8	9	10	.	$n-2$	$n-1$
$\lambda(\mathcal{P}^k_i, Q_n)$	$128n-896$	$256n-2048$	$512n-4608$	$1024n-10240$	.	2^{n-1}	2^{n-1}
k	0	1	2	3	4	5	6
$\lambda(\mathcal{P}^k_i, FQ_n)$	$n+1$	$2n$	$4n-4$	$8n-16$	$16n-48$	$32n-128$	$64n-320$
k	7	8	9	10	.	$n-2$	$n-1$
$\lambda(\mathcal{P}^k_i, FQ_n)$	$128n-768$	$256n-1792$	$512n-4096$	$1024n-9216$	.	$3 \times 2^{n-2}$	2^n

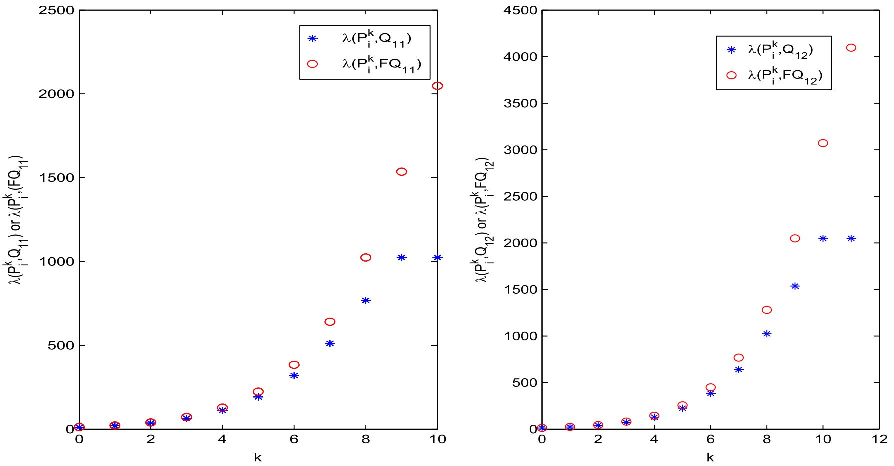

Fig. 3 The plots of $\lambda(\mathscr{P}_i^k, Q_n)$ and $\lambda(\mathscr{P}_i^k, FQ_n)$ for $0 \le k \le n-1$ and $11 \le n \le 12$

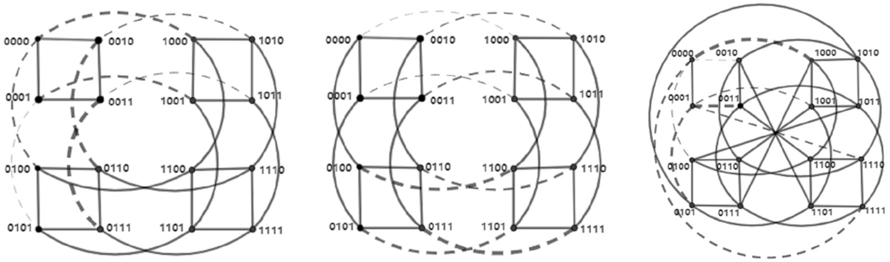

Fig. 4 Edge-disjoint paths in two subgraphs of Q_4 and FQ_4

the maximum number of such edge-disjoint paths in Q_n is $(n-k)2^k$. For FQ_n , such numerical value is $(n-k+1)2^k$.

In order to make this issue more intuitive, we take some examples. For $n = 4$, $k = 2$ and $k = 3$, under above three conditions, there are 8 edge-disjoint paths of Q_4 marked by dotted lines of different thicknesses in the first two picture in Fig. 4. For $n = 4$ and $k = 1$, under above three conditions, there are also 8 edge-disjoint paths of FQ_4 marked by dotted lines of different thicknesses in the last picture of Fig. 4. Our results enriched the theory of reliability and edge fault tolerance of cube-based interconnection networks.

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References

1. Archit JS (2002) Folded-crossed hypercube: a complete interconnection network. J Syst Archit 47:917–922
2. Cai XP, Vumar E (2019) The super connectivity of folded crossed cubes. Inf Process Lett 142:52–56
3. Chang NW, Tsai CY, Hsieh SY (2014) On 3-extra connectivity and 3-extra edge connectivity of folded hypercubes. IEEE Trans Comput 63:1594–1600
4. Cheng E, Qiu K, Shen ZZ (2017) On the restricted connectivity of the arrangement graph. J Supercomput 73(8):3669–3682
5. Chung FRK, Füredi Z, Graham RL, Seymour P (1998) On induced subgraphs of the cube. J Comb Theory 49(1):180–187
6. El-Amawy A, Latifi S (1991) Properties and performance of folded hypercubes. IEEE Trans Parallel Distrib Syst 2(1):31–42
7. Esfahanian AH (1989) Generalized measure of fault tolerance with application to n-cube networks. IEEE Trans Comput 38(11):1586–1591
8. Esmaeili T, Lak G, Rad AN (2012) 3D-FolH-NOC: a new structure for parallel processing and distributed systems. J Comput 4(6):163–168
9. Fàbrega J, Fiol MA (1996) On the extraconnectivity of graphs. Discret Math 155:49–57
10. Guo LT (2018) Reliability analysis of crossed cube networks on degree. J Comput Commun 6:129–134
11. Guo LT, Guo XF (2014) Fault tolerance of hypercubes and folded hypercubes. J Supercomput 68:1235–1240
12. Harary F (1983) Conditional connectivity. Networks 13(3):346–357
13. Harper LH (1964) Optimal assignments of numbers to vertices. J Soc Ind Appl Math 12(1):131–135
14. Hsieh SY, Huang HW, Lee CW (2016) $\{2, 3\}$-Restricted connectivity of locally twisted cubes. Theor Comput Sci 615:78–90
15. Latifi S (1991) Simulation of PM21 network by folded hypercube. IEEE Proc E Comput Digit Tech 138(6):397–400
16. Lee CW, Hsieh SY, Yang SS (2020) R_3-connectivity of folded hypercubes. Discret Appl Math 285:261–273
17. Li XJ, Xu JM (2013) Edge-fault tolerance of hypercube-like networks. Inf Process Lett 113(19–21):760–763
18. Li H, Yang WH (2013) Bounding the size of the subgraph induced by m vertices and extra edge-connectivity of hypercubes. Discret Appl Math 161(16–17):2753–2757
19. Park JS, Davis NJ (2001) Modeling the folded hypercube ATM Switches. In: The proceedings of OPNETWORK, Washington DC
20. Park PS, Davis NJ (2001) The folded hypercube ATM switches. In: International Conference on Networking 2094:370–379
21. Rajasingh I, Arockiaraj M (2020) Linear wirelength of folded hypercubes. Math Comput Sci 5:101–111
22. Wei CC, Hsieh SY (2017) H-restricted connectivity of locally twisted cubes. Discret Appl Math 217(2):330–339
23. Yang WH, Lin HQ (2014) Reliability evaluation of BC networks in terms of extra vertex- and edge-connectivity. IEEE Trans Comput 63(10):2540–2548
24. Zhang MZ, Zhang LZ, Feng X (2016) Reliability measures in relation to the h-extra edge-connectivity of folded hypercubes. Theor Comput Sci 615:71–77
25. Zhu Q, Xu JM (2006) On restricted edge connectivity and extra edge connectivity of hypercubes and folded hypercubes. J Univ Sci Technol China 36:246–253

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