The Journal of Supercomputing (2019) 75:5583–5595
https://doi.org/10.1007/s11227-019-02817-6

One-to-one disjoint path covers in hypercubes with faulty edges

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Published online: 20 March 2019

Abstract
A one-to-one k-disjoint path cover $\{P_1, P_2, \dots, P_k\}$ of a graph G is a collection of k internally vertex disjoint paths joining source with sink that cover all vertices of G. In this paper, we investigate the problem of one-to-one disjoint path cover in hypercubes with faulty edges and obtain the following results: Let $u, v \in V(Q_n)$ be such that $p(u) \neq p(v)$ and $1 \leq k \leq n$. Then there exists a one-to-one k-disjoint path cover $\{P_1, P_2, \dots, P_k\}$ joining vertices u and v in Q_n. Moreover, when $1 \leq k \leq n-2$, the result still holds even if removing $n-2-k$ edges from Q_n.

Keywords Hypercubes · Vertex disjoint paths · Path covers · One-to-one · Fault edges

1 Introduction

A topological structure of an interconnection network can be modeled by a graph $G = (V(G), E(G))$, where the vertex set $V(G)$ represents the set of processors and the edge set $E(G)$ represents the set of links joining processors. One of the most central issues in various interconnection networks is to find vertex disjoint paths concerned with a routing among vertices [1–6, 33, 34].

A u, v-*path* is a path with endpoints u and v, denoted by $P_{u,v}$ when we specify a particular such path. We say that k paths are *vertex disjoint* in a graph G if any two

This work is supported by NSFC (Grant Nos. 11501282 and 11861032).

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of them have no common vertex. Given any two disjoint sets of k labeled vertices $S = \{s_1, s_2, \dots, s_k\}$ and $T = \{t_1, t_2, \dots, t_k\}$ in a graph G, are there k vertex disjoint paths $P_{s_1,t_1}, P_{s_2,t_2}, \dots, P_{s_k,t_k}$ which cover all vertices of G?

It has been investigated with respect to various special graphs such as hypercubes [8, 12, 17, 18, 21, 24, 25], k-ary n-cubes [30, 35, 36], and hypercube-like interconnection networks [13, 28, 29].

A path (respectively, cycle) in a graph G is a *hamiltonian path* (respectively, *hamiltonian cycle*) if every vertex in G appears exactly once in the path (respectively, cycle). One of the core subjects in hamiltonian graph theory is to develop sufficient conditions for a graph to have a hamiltonian path/cycle [14, 15, 19, 20, 26, 27, 31].

Fault tolerance is an important index of the stability of the network. It is useful to consider faulty networks because node faults or link faults may occur in networks. In this regard, the fault-tolerant capacity of a network is a critical issue in parallel computing. It motivated the study of various networks with faulty elements [10, 11, 16, 17, 21].

The n-dimensional hypercube, denoted by Q_n, is one of the most popular and efficient interconnection networks. Hypercubes play an important role in many areas of computer science. Motivated by the disjoint path cover problem and Hamilton problem, we consider the problem of one-to-one k-disjoint path cover in hypercubes with faulty edges.

A *one-to-one k-disjoint path cover* $\{P_1, P_2, \dots, P_k\}$ of a graph G is a collection of k internally vertex disjoint paths joining source with sink that cover all vertices of G. Denote the source and sink by u and v, respectively. Then the one-to-one k-disjoint path cover joining u and v sometimes is called by u-to-v k-disjoint path cover.

Studies about one-to-one disjoint path cover problem of some networks or graphs can be found in the literature [9, 30]. In this paper, we investigate the problem of one-to-one disjoint path cover in hypercubes and obtain the following results: Let $u, v \in V(Q_n)$ be such that $p(u) \neq p(v)$ and $1 \leq k \leq n$. Then there exists a u-to-v k-disjoint path cover $\{P_1, P_2, \dots, P_k\}$ in Q_n. Moreover, when $1 \leq k \leq n-2$, the result still holds even if removing $n-2-k$ edges from Q_n.

2 Definitions and preliminaries

Terminology and notation used in this paper but undefined below can be found in [7]. Let G be a graph. For a set $F \subseteq E(G)$, let $G - F$ denote the resulting graph after removing all edges in F from G. For a set $S \subseteq V(G)$, let $G - S$ denote the graph removing all vertices in S and all the edges incident with S from G.

Let $P_{x,y} = (x, \dots, v_i, \dots, v_j, \dots, y)$ be a path. We use $P_{x,y}[v_i, v_j]$ to denote the subpath $(v_i, \dots, v_j)$ of $P_{x,y}$ joining v_i and v_j. For two paths P_1 and P_2 with only one common endpoint, we use $P_1 + P_2$ to denote the path P such that $V(P) = V(P_1) \cup V(P_2)$ and $E(P) = E(P_1) \cup E(P_2)$.

Let $[n]$ denote the set $\{1, \dots, n\}$. The *n-dimensional hypercube* Q_n is a graph whose vertex set consists of all binary strings of length n, i.e., $V(Q_n) = \{u : u = \delta_1 \cdots \delta_n \text{ and } \delta_i \in \{0, 1\} \text{ for every } i \in [n]\}$, with two vertices being adjacent whenever the corresponding strings differ in just one position. The

Hamming distance between two vertices u and v in Q_n, denoted by $d(u, v)$, is the number of different bits of u and v.

Let $j \in [n]$. An edge in Q_n is an *j-edge* if its endpoints differ in the jth position. The set of all j-edges in Q_n is denoted by E_j. Let $Q^0_{n-1,j}$ and $Q^1_{n-1,j}$, with the superscripts j being omitted when the context is clear, be the $(n-1)$-dimensional subcubes of Q_n induced by the vertex sets $\{u = \delta_1 \cdots \delta_n \in V(Q_n) : \delta_j = 0\}$ and $\{u = \delta_1 \cdots \delta_n \in V(Q_n) : \delta_j = 1\}$, respectively. Thus $Q_n - E_j = Q^0_{n-1,j} + Q^1_{n-1,j}$. We say that Q_n *splits* into two $(n-1)$-dimensional subcubes Q^0_{n-1} and Q^1_{n-1} at position j. See Fig. 1 for example.

Let $\alpha \in \{0, 1\}$. We use $\bar{\alpha}$ to denote $1 - \alpha$. Every vertex $x^\alpha \in V(Q^\alpha_{n-1})$ has in $Q^{\bar{\alpha}}_{n-1}$ a unique neighbor, denoted by $x^{\bar{\alpha}}$.

Moreover, we may split Q^α_{n-1} into two $(n-2)$-dimensional subcubes $Q^{\alpha 0}_{n-2}$ and $Q^{\alpha 1}_{n-2}$ at some position i. So $Q_n - E_j - E_i = Q^{00}_{n-2} + Q^{01}_{n-2} + Q^{10}_{n-2} + Q^{11}_{n-2}$. We say that Q_n *splits* into four $(n-2)$-dimensional subcubes $Q^{00}_{n-2}, Q^{01}_{n-2}, Q^{10}_{n-2}$ and Q^{11}_{n-2} at positions i and j.

Let $\beta \in \{0, 1\}$. For every vertex $x^{\alpha\beta}$ in $Q^{\alpha\beta}_{n-2}$, let $x^{\alpha\bar{\beta}}$ denote the unique neighbor of $x^{\alpha\beta}$ in $Q^{\alpha\bar{\beta}}_{n-2}$, and let $x^{\bar{\alpha}\beta}$ denote the unique neighbor of $x^{\alpha\beta}$ in $Q^{\bar{\alpha}\beta}_{n-2}$, and let $x^{\bar{\alpha}\bar{\beta}}$ denote the unique neighbor of $x^{\bar{\alpha}\beta}$ in $Q^{\bar{\alpha}\bar{\beta}}_{n-2}$. For a path $P^{\alpha\beta} = (x^{\alpha\beta}_1, \ldots, x^{\alpha\beta}_l)$ in $Q^{\alpha\beta}_{n-2}$, we say that $P^{\alpha'\beta'} = (x^{\alpha'\beta'}_1, \ldots, x^{\alpha'\beta'}_l)$ is the corresponding path of $P^{\alpha\beta}$ in $Q^{\alpha'\beta'}_{n-2}$.

For any $F \subseteq E(Q_n)$, let $F^\alpha = F \cap E(Q^\alpha_{n-1})$ and $F^{\alpha\beta} = F \cap E(Q^{\alpha\beta}_{n-2})$.

The *parity* $p(u)$ of a vertex $u = \delta_1 \cdots \delta_n$ in Q_n is defined by $p(u) = \sum_{i=1}^n \delta_i (\mathrm{mod}\ 2)$. Then there are 2^{n-1} vertices with parity 0 and 2^{n-1} vertices with parity 1 in Q_n. Vertices with parity 0 and 1 are called black vertices and white vertices, respectively. Observe that Q_n is bipartite and vertices of each parity form bipartite sets of Q_n.

We give some results which will be used in the proof of the main results.

Theorem 2.1 [19] *Q_n has a Hamiltonian cycle for every $n \geq 2$.*

Theorem 2.2 [20] *If $n \geq 1$ and $x, y \in V(Q_n)$ are such that $p(x) \neq p(y)$, then Q_n contains a hamiltonian path joining x and y.*

Lemma 2.3 [23] For $n \geq 2$, let x, y, u be pairwise distinct vertices in Q_n with $p(x) = p(y) \neq p(u)$. Then there exists a hamiltonian path joining x and y in $Q_n - u$.

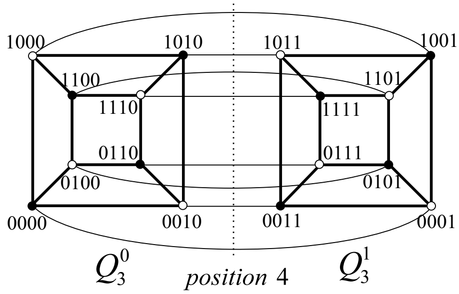


Fig. 1 Q_4 splits into two 3-dimensional subcubes Q^0_3 and Q^1_3 at position 4

Lemma 2.4 [22] Q_n has a hamiltonian cycle even if it has $(n-2)$ edge faults for every $n \geq 2$.

Lemma 2.5 [32] Assume that $n \geq 2$ and F is a subset of edges with $|F| \leq n-2$. Then there exists a hamiltonian path in $Q_n - F$ joining any two vertices of different colors.

Theorem 2.6 [11] *Let $n > 2k \geq 4$ and $F \subset E(Q_n)$ with $|F| \leq n-2k-1$. Assume that $S = \{s_1, \dots, s_k\}$ and $T = \{t_1, \dots, t_k\}$ are distinct vertices of Q_n such that $S \cup T$ contains k vertices from each class of bipartition of Q_n. Then $Q_n - F$ have k vertex disjoint paths $P_{s_1,t_1}, \dots, P_{s_k,t_k}$ which cover all vertices of Q_n.*

3 The main results

Theorem 3.1 *Let $u, v \in V(Q_n)$ be such that $p(u) \neq p(v)$ and let $1 \leq k \leq n$. Then there exists a u-to-v k-disjoint path cover $\{P_1, P_2, \dots, P_k\}$ in Q_n.*

Proof When $k = 1$, by Theorem 2.2, there exists a hamiltonian path joining u and v in Q_n. This is a u-to-v 1-disjoint path cover in Q_n.

When $k = 2$, now $n \geq 2$. By Theorem 2.1, there is a hamiltonian cycle in Q_n. Let P_1 and P_2 be the two paths on the cycle joining the vertices u and v. Then $\{P_1, P_2\}$ is a u-to-v 2-disjoint path cover in Q_n.

Next, we consider the case $k \geq 3$. Now $n \geq k \geq 3$. We prove the theorem by induction on n.

For $n = 3$, now $k = 3$. Since $p(u) \neq p(v)$, without loss of generality, there are two cases of $\{u, v\}$ to consider. See Fig. 2. For the two cases, we can verify that the conclusion holds.

Suppose that the theorem holds for $n - 1(\geq 3)$. We are to show that it holds for $n(\geq 4)$.

Case 1 $d(u, v) < n$.

Since $d(u, v) < n$, there exists a position $j \in [n]$ such that $u, v \in V(Q^0_{n-1})$. Denote u by u^0 and v by v^0. Since $3 \leq k \leq n$, we have $2 \leq k-1 \leq n-1$. By induction, there exists a u-to-v $(k-1)$-disjoint path cover $\{P^0_1, P^0_2, \dots, P^0_{k-1}\}$ in Q^0_{n-1}. Since $p(u) \neq p(v)$, we have $p(u^1) \neq p(v^1)$. By Theorem 2.2, there exists a hamiltonian path

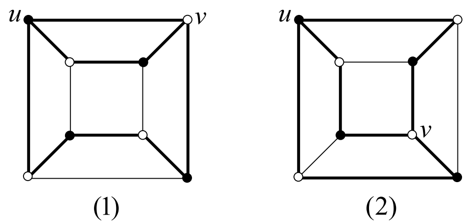
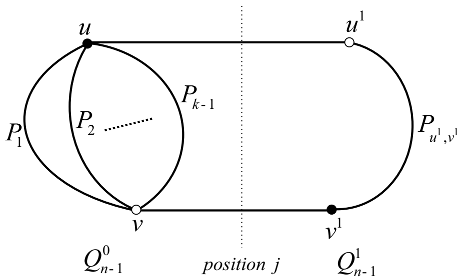


Fig. 2 The two cases of $\{u, v\}$ in Q_3

Fig. 3 The illustration for the contruction of Case 1



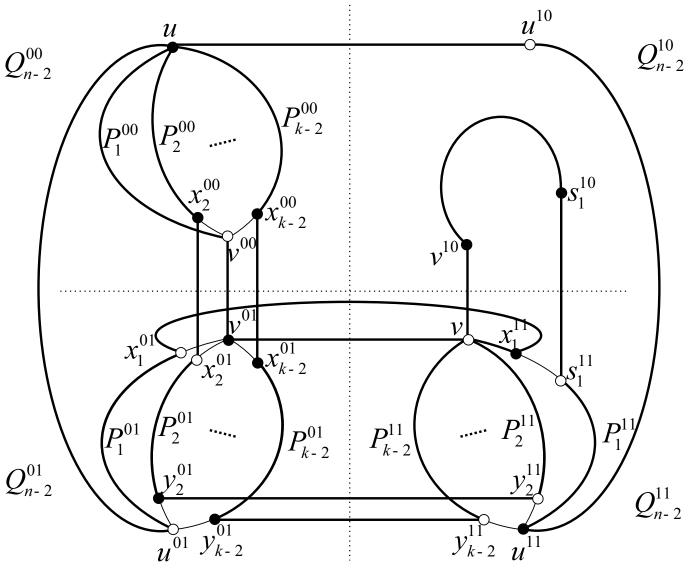


Fig. 4 The illustration for the contruction of Case 2

P_{u^1,v^1} in Q^1_{n-1}. Let $P_i = P^0_i$ for every $i \in [k-1]$, and let $P_k = uu^1 + P_{u^1,v^1} + v^1v$. Thus, $\{P_1, P_2, \dots, P_k\}$ is a u-to-v k-disjoint path cover in Q_n. See Fig. 3.

Case 2 $d(u, v) = n$. In this case, since $p(u) \neq p(v)$, we have n is odd and $n \geq 5$.

Arbitrarily split Q_n at some two positions to subcubes Q^{00}_{n-2}, Q^{01}_{n-2}, Q^{10}_{n-2} and Q^{11}_{n-2}. Without loss of generality, we may assume $u \in V(Q^{00}_{n-2})$. Then $v \in V(Q^{11}_{n-2})$. Denote u by u^{00} and v by v^{11}.

Since $3 \leq k \leq n$, we have $1 \leq k-2 \leq n-2$. In Q^{00}_{n-2}, since $p(u) \neq p(v^{00})$, by induction, there exists a u-to-v^{00} $(k-2)$-disjoint path cover $\{P^{00}_1, \dots, P^{00}_{k-2}\}$ of Q^{00}_{n-2}. For every $i \in [k-2]$, let P^{01}_i be the corresponding path of P^{00}_i in Q^{01}_{n-2}, and let P^{11}_i be the corresponding path of P^{00}_i in Q^{11}_{n-2}. Then $\{P^{01}_1, \dots, P^{01}_{k-2}\}$ is a u^{01}-to-v^{01} $(k-2)$-disjoint path cover of Q^{01}_{n-2}, and $\{P^{11}_1, \dots, P^{11}_{k-2}\}$ is a u^{11}-to-v $(k-2)$-disjoint path cover of Q^{11}_{n-2}. See Fig. 4.

For every $i \in [k-2]$, let x_i^{01} and y_i^{01} be the neighbors of v^{01} and u^{01} on the path P_i^{01}, respectively. Since $d(u, v) = n$, we have $d(v^{01}, u^{01}) = n - 2 \geq 3$. Thus x_i^{01} and y_i^{01} are distinct vertices and they are different from the vertices u^{01}, v^{01}. Next, we construct a u-to-v k-disjoint path cover $\{P_1, P_2, \dots, P_k\}$ of Q_n.

Step 1 Let $P_1 = P_1^{00} + v^{00}v^{01} + v^{01}v$.

Step 2 If $k-2=1$, then skip this step. If $k-2 \geq 2$, then for every $i \in \{2, \dots, k-2\}$, let $P_i = P_i^{00}[u, x_i^{00}] + x_i^{00}x_i^{01} + P_i^{01}[x_i^{01}, y_i^{01}] + y_i^{01}y_i^{11} + P_i^{11}[y_i^{11}, v]$.

Step 3 Let $P_{k-1} = uu^{01} + P_1^{01}[u^{01}, x_1^{01}] + x_1^{01}x_1^{11} + x_1^{11}v$. Let s_1^{11} be the neighbor of x_1^{11} on the path $P_1^{11}[x_1^{11}, u^{11}]$. Then $p(s_1^{11}) = p(v) \neq p(u)$. So $p(s_1^{10}) = p(v^{10}) \neq p(u^{10})$. By Lemma 2.3, there is a hamiltonian path $P_{s_1^{10}, v^{10}}$ in $Q_{n-2}^{10} - u^{10}$. Let $P_k = uu^{10} + u^{10}u^{11} + P_1^{11}[u^{11}, s_1^{11}] + s_1^{11}s_1^{10} + P_{s_1^{10}, v^{10}} + v^{10}v$.

Then $\{P_1, P_2, \dots, P_k\}$ is a u-to-v k-disjoint path cover in Q_n. □

Theorem 3.2 *Let $u, v \in V(Q_n)$ be such that $p(u) \neq p(v)$. Let $F \subseteq E(Q_n)$ and $1 \leq k \leq n-2$. If $|F| \leq n-2-k$, then there exists a u-to-v k -disjoint path cover $\{P_1, P_2, \dots, P_k\}$ in $Q_n - F$.*

Proof If $F = \emptyset$, then by Theorem 3.1, the theorem holds. So in the following, we only need to consider the case $|F| \geq 1$. Now $k \leq n-3$.

When $k=1$, now $|F| \leq n-3$. By Lemma 2.5, there exists a hamiltonian path joining vertices u and v in $Q_n - F$. This is a u-to-v 1-disjoint path cover in $Q_n - F$. In this case, the theorem holds.

When $k=2$, now $|F| \leq n-4$. By Lemma 2.4, there is a hamiltonian cycle in $Q_n - F$. Let P_1 and P_2 be the two paths on the cycle joining the vertices u and v. Then $\{P_1, P_2\}$ is a u-to-v 2-disjoint path cover in $Q_n - F$. In this case, the theorem holds.

So, we only need to consider the case $k \geq 3$. Now $n \geq k+3 \geq 6$. We prove the theorem by induction on n.

First we prove the basis of induction. When $n=6$, now $k=3$ and $|F|=1$. Let $F = \{f\}$. Since $p(u) \neq p(v)$, we have $d(u, v)$ is odd. So $d(u, v) < 6$. Hence there exists a position $j \in [6]$ such that $u, v \in V(Q_5^0)$. Denote u by u^0 and v by v^0.

If $f \in E(Q_5^0)$, then by Lemma 2.4 there is a hamiltonian cycle in $Q_5^0 - F$. Let P_1, P_2 be the two paths on the cycle joining vertices u and v. Since $p(u^1) \neq p(v^1)$, by Theorem 2.2, there is a hamiltonian path P_{u^1, v^1} in Q_5^1. Let $P_3 = uu^1 + P_{u^1, v^1} + v^1v$. Then $\{P_1, P_2, P_3\}$ is a u-to-v 3-disjoint path cover in $Q_6 - F$.

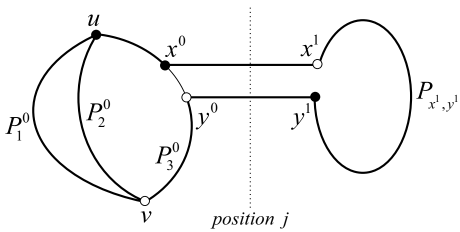


Fig. 5 The illustration for the contruction of the case $f \notin E(Q_5^0)$

If $f \notin E(Q_5^0)$, then by Theorem 3.1 there is a u-to-v 3-disjoint path cover $\{P_1^0, P_2^0, P_3^0\}$ in Q_5^0. See Fig. 5. Since $\sum_{i=1}^{3} |E(P_i^0)| = 2^5 - 2 + 3 > 3$, there exists an edge $x^0y^0 \in E(P_i^0)$ for some $i \in [3]$ such that $x^0x^1 \neq f$ and $y^0y^1 \neq f$. Without loss of generality, assume $i = 3$. Since $|F| \leq 1$, by Lemma 2.5, there is a hamiltonian path P_{x^1,y^1} in $Q_5^1 - F$. Let $P_1 = P_1^0$, $P_2 = P_2^0$, and $P_3 = P_3^0 - x^0y^0 + x^0x^1 + P_{x^1,y^1} + y^1y^0$. Then $\{P_1, P_2, P_3\}$ is a u-to-v 3-disjoint path cover in $Q_6 - F$.

By the above two cases, we know that the theorem holds for $n = 6$. Suppose that the theorem holds for $n - 1(\geq 6)$. We are to show that it holds for $n(\geq 7)$. We distinguish two cases (Cases 1 and 2) to consider.

Case 1 $d(u, v) < n$.

Since $d(u, v) < n$, there exists a position $j \in [n]$ such that $u, v \in V(Q_{n-1}^0)$. Denote u by u^0 and v by v^0. We distinguish two subcases (Subcases 1.1 and 1.2) to consider.

Subcase 1.1 $|F_0| \leq |F| - 1$.

Now $3 \leq k \leq n - 3 = (n - 1) - 2$, and $|F_0| \leq |F| - 1 \leq (n - 1) - 2 - k$. By induction, there exists a u-to-v k-disjoint path cover $\{P_1^0, P_2^0, \dots, P_k^0\}$ in $Q_{n-1}^0 - F^0$. Since $\sum_{i=1}^{k} |E(P_i^0)| = 2^{n-1} - 2 + k > 2k + 2(|F| - 2)$ for $n \geq 7$, there exists an edge $x^0y^0 \in E(P_i^0)$ for some $i \in [k]$ such that $x^0x^1 \notin F$ and $y^0y^1 \notin F$. Without loss of generality, we may assume $i = k$. Since $|F_1| \leq |F| \leq n - 5 < (n - 1) - 2$, by Lemma 2.5, there exists a hamiltonian path P_{x^1,y^1} in $Q_{n-1}^1 - F_1$. Let $P_i = P_i^0$ for every $i \in [k - 1]$, and let $P_k = P_k^0 - x^0y^0 + x^0x^1 + P_{x^1,y^1} + y^1y^0$. Thus, $\{P_1, P_2, \dots, P_k\}$ is a u-to-v k-disjoint path cover in $Q_n - F$.

Subcase 1.2 $F_0 = F$.

Since $3 \leq k \leq n - 3$ and $|F| \leq n - 2 - k$, we have $2 \leq k - 1 \leq n - 4 < (n - 1) - 2$ and $|F_0| = |F| \leq (n - 1) - 2 - (k - 1)$. By induction, there exists a u-to-v $(k - 1)$-disjoint path cover $\{P_1^0, P_2^0, \dots, P_{k-1}^0\}$ in $Q_{n-1}^0 - F^0$. Since $p(u^1) \neq p(v^1)$, by Theorem 2.2, there exists a hamiltonian path P_{u^1,v^1} in Q_{n-1}^1. Let $P_i = P_i^0$ for every $i \in [k - 1]$, and let $P_k = uu^1 + P_{u^1,v^1} + v^1v$. Thus, $\{P_1, P_2, \dots, P_k\}$ is a u-to-v k-disjoint path cover in $Q_n - F$.

By the above two subcases, we know that the theorem holds for Case 1.

Case 2 $d(u, v) = n$. In this case, since $p(u) \neq p(v)$, we have n is odd and $n \geq 7$.

Since $|F| \leq n - 2 - k \leq n - 5$, there exist two positions j_1 and j_2 such that $E_{j_1} \cap F = \emptyset$ and $E_{j_2} \cap F = \emptyset$. Split Q_n at positions j_1 and j_2 to subcubes Q_{n-2}^{00}, Q_{n-2}^{01}, Q_{n-2}^{10}, and Q_{n-2}^{11}. Without loss of generality, we may assume $u \in V(Q_{n-2}^{00})$. Then $v \in V(Q_{n-2}^{11})$. Denote u by u^{00} and v by v^{11}. Without loss of generality, we may assume $|F^{00}| \geq |F^{11}|$. Moreover, we may assume $|F^{01}| \geq |F^{10}|$. We distinguish two subcases (Subcases 2.1 and 2.2) to consider.

Subcase 2.1 $F^{00} = F$ or $F^{01} = F$.

Since $3 \leq k \leq n - 3$, we have $1 \leq k - 2 \leq n - 5 < (n - 2) - 2$.

If $F^{00} = F$, then the other three subcubes have no faulty edges. Since $|F^{00}| = |F| \leq n - 2 - k = (n - 2) - 2 - (k - 2)$, by induction, there exists a u-to-v^{00} $(k - 2)$-disjoint path cover $\{P_1^{00}, \dots, P_{k-2}^{00}\}$ in $Q_{n-2}^{00} - F^{00}$. For every $i \in [k - 2]$, let

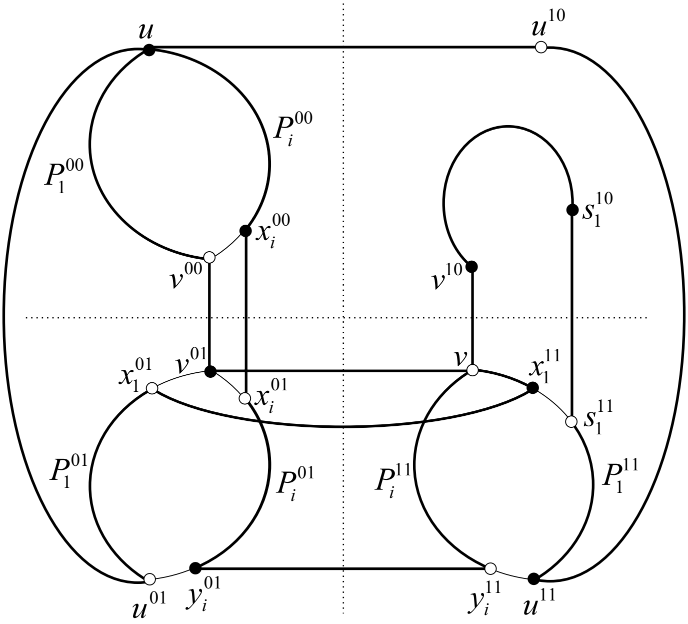


Fig. 6 The illustration for the contruction of Subcase 2.1

P_i^{01} and P_i^{11} be the corresponding paths of P_i^{00} in Q_{n-2}^{01} and Q_{n-2}^{11}, respectively. Then $\{P_1^{01}, \ldots, P_{k-2}^{01}\}$ is a u^{01}-to-v^{01} $(k-2)$-disjoint path cover of Q_{n-2}^{01}, and $\{P_1^{11}, \ldots, P_{k-2}^{11}\}$ is a u^{11}-to-v $(k-2)$-disjoint path cover of Q_{n-2}^{11}. See Fig. 6.

If $F^{01} = F$, then similarly, by induction there exists a u^{01}-to-v^{01} $(k-2)$-disjoint path cover $\{P_1^{01}, \ldots, P_{k-2}^{01}\}$ in $Q_{n-2}^{01} - F^{01}$. For every $i \in [k-2]$, let P_i^{00} and P_i^{11} be the corresponding paths of P_i^{01} in Q_{n-2}^{00} and Q_{n-2}^{11}, respectively. Then $\{P_1^{00}, \ldots, P_{k-2}^{00}\}$ is a u-to-v^{00} $(k-2)$-disjoint path cover of Q_{n-2}^{00}, and $\{P_1^{11}, \ldots, P_{k-2}^{11}\}$ is a u^{11}-to-v $(k-2)$ -disjoint path cover of Q_{n-2}^{11}.

In the above two cases, for every $i \in [k-2]$, let x_i^{01} and y_i^{01} be the neighbors of v^{01} and u^{01} on the path P_i^{01}, respectively. Since $d(u, v) = n$, we have $d(v^{01}, u^{01}) = n - 2 \geq 5$. Thus x_i^{01} and y_i^{01} are distinct vertices and they are different from the vertices u^{01}, v^{01}. Next, we construct a u-to-v k-disjoint path cover $\{P_1, P_2, \ldots, P_k\}$ in $Q_n - F$.

Step 1 Let $P_1 = P_1^{00} + v^{00}v^{01} + v^{01}v$.

Step 2 If $k-2=1$, then skip this step. If $k-2 \geq 2$, then for every $i \in \{2, \ldots, k-2\}$, let $P_i = P_i^{00}[u, x_i^{00}] + x_i^{00}x_i^{01} + P_i^{01}[x_i^{01}, y_i^{01}] + y_i^{01}y_i^{11} + P_i^{11}[y_i^{11}, v]$.

Step 3 Let $P_{k-1} = uu^{01} + P_1^{01}[u^{01}, x_1^{01}] + x_1^{01}x_1^{11} + x_1^{11}v$. Let s_1^{11} be the neighbor of x_1^{11} on the path $P_1^{11}[x_1^{11}, u^{11}]$. Then $p(s_1^{11}) = p(v) \neq p(u)$. So $p(s_1^{10}) = p(v^{10}) \neq p(u^{10})$. By Lemma 2.3, there is a hamiltonian path $P_{s_1^{10}, v^{10}}$ in $Q_{n-2}^{10} - u^{10}$. Let $P_k = uu^{10} + u^{10}u^{11} + P_1^{11}[u^{11}, s_1^{11}] + s_1^{11}s_1^{10} + P_{s_1^{10}, v^{10}} + v^{10}v$.

Then $\{P_1, P_2, \dots, P_k\}$ is a u-to-v k-disjoint path cover in $Q_n - F$. So in this case, the theorem holds.

Subcase 2.2 $|F^{00}| \le |F| - 1$ and $|F^{01}| \le |F| - 1$.

In this case, we may observe that $|F| \ge 2$. For the sake of discussion, we distinguish two subcases (Subcases 2.2.1 and 2.2.2) to consider.

Subcase 2.2.1 $|F^{01}| \le |F| - 2$.

Since $3 \le k \le n-3$, we have $2 \le k-1 \le n-4 = (n-2)-2$. Since $|F| \le n-2-k$, we have $|F^{11}| \le |F^{00}| \le n-2-k-1 = (n-2)-2-(k-1)$. By induction, there exist a u-to-v^{00} $(k-1)$-disjoint path cover $\{P_1^{00}, P_2^{00}, \dots, P_{k-1}^{00}\}$ in $Q_{n-2}^{00} - F^{00}$ and a u^{11}-to-v $(k-1)$-disjoint path cover $\{P_1^{11}, P_2^{11}, \dots, P_{k-1}^{11}\}$ in $Q_{n-2}^{11} - F^{11}$. (Note that in this case, P_i^{11} are not necessarily the corresponding path of P_i^{00} in Q_{n-2}^{11}.) Let $k_1 = \lceil \frac{k-1}{2} \rceil$ and $k_2 = \lfloor \frac{k-1}{2} \rfloor$. Then $k_1 + k_2 = k - 1$.

For every $i \in \{2, \dots, k-1\}$, let x_i^{00} be the neighbor of v^{00} on the path P_i^{00}, and let y_i^{11} be the neighbor of u^{11} on the path P_i^{11}. Since $d(u, v) = n \ge 7$, we have $d(u, v^{00}) = d(u^{11}, v) = n-2 \ge 5$. So $x_i^{00} \ne u$ and $y_i^{11} \ne v$. Hence $u^{01}, v^{01}, x_i^{01}, y_i^{01}, i \in \{2, \dots, k_1\}$, are distinct vertices, and $p(x_i^{01}) = p(u^{01}) \ne p(v^{01}) = p(y_i^{01})$. Similarly, we have $u^{10}, v^{10}, x_t^{10}, y_t^{10}, t \in \{k_1+1, \dots, k-1\}$, are distinct vertices, and $p(x_t^{10}) = p(u^{10}) \ne p(v^{10}) = p(y_t^{10})$.

Note that $k_1 \ge 1$ and $k \ge 2k_1$. Then $|F^{01}| \le |F| - 2 \le n-2-k-2 \le n-2-2k_1 -2 < (n-2) - 2k_1 - 1$. If $k_1 = 1$, then by Lemma 2.5, there is a hamiltonian path $P_{u^{01},v^{01}}$ in $Q_{n-2}^{01} - F^{01}$. If $k_1 \ge 2$, then by Theorem 2.6, $Q_{n-2}^{01} - F^{01}$ have k_1 vertex disjoint paths $P_{u^{01},v^{01}}, P_{x_2^{01},y_2^{01}}, \dots, P_{x_{k_1}^{01},y_{k_1}^{01}}$ which cover all the vertices of Q_{n-2}^{01}.

Similarly, $k_2 \ge 1$ and $k \ge 2k_2 + 1$. Then $|F^{10}| \le |F| - 2 \le n-2-k-2 \le n-2-(2k_2+1)-2 = (n-2) - 2(k_2+1) - 1$. By Theorem 2.6, $Q_{n-2}^{10} - F^{10}$ have $k_2 + 1$ vertex disjoint paths $P_{u^{10},v^{10}}, P_{x_{k_1+1}^{10},y_{k_1+1}^{10}}, \dots, P_{x_{k-1}^{10},y_{k-1}^{10}}$ which cover all the vertices of Q_{n-2}^{10}.

Next, we construct a u-to-v k-disjoint path cover $\{P_1, P_2, \dots, P_k\}$ in $Q_n - F$.

Step 1 Let $P_1 = P_1^{00} + v^{00}v^{01} + P_{v^{01},u^{01}} + u^{01}u^{11} + P_1^{11}$.

Step 2 If $k_1 = 1$, then skip this step. If $k_1 \ge 2$, then for every $i \in \{2, \dots, k_1\}$, let $P_i = P_i^{00}[u, x_i^{00}] + x_i^{00}x_i^{01} + P_{x_i^{01},y_i^{01}} + y_i^{01}y_i^{11} + P_i^{11}[y_i^{11}, v]$.

Step 3 For every $t \in \{k_1+1, \dots, k-1\}$, let $P_t = P_t^{00}[u, x_t^{00}] + x_t^{00}x_t^{10} + P_{x_t^{10},y_t^{10}} + y_t^{10}y_t^{11} + P_t^{11}[y_t^{11}, v]$.

Step 4 Let $P_k = uu^{10} + P_{u^{10},v^{10}} + v^{10}v$.

Then $\{P_1, P_2, \dots, P_k\}$ is a u-to-v k-disjoint path cover in $Q_n - F$. See Fig. 7.

Subcase 2.2.2 $|F^{01}| = |F| - 1$.

Since $2 \le |F| \le n-2-k$, we have $3 \le k \le n-4$. So $2 \le k-1 \le n-5 < (n-2) -2$. Since $|F^{01}| = |F| - 1$, we have $|F^{00}| + |F^{10}| + |F^{11}| = 1$.

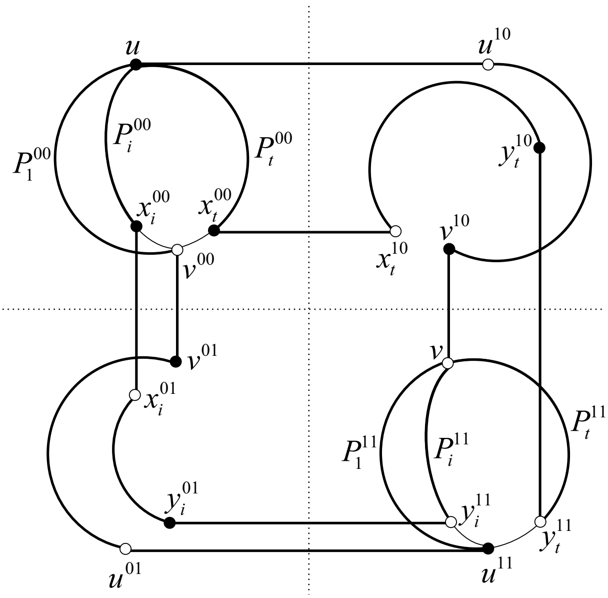


Fig. 7 The illustration for the contruction of Subcase 2.2.1

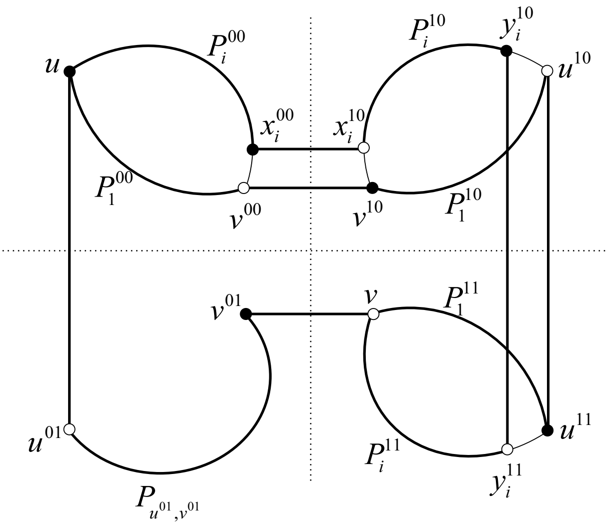


Fig. 8 The illustration for the contruction of Subcase 2.2.2

If $|F^{00}| = 1$, then $|F^{10}| = |F^{11}| = 0$. Since $|F^{00}| = 1 \leq (n-2) - 2 - (k-1)$, by induction, there exists a u-to-v^{00} $(k-1)$-disjoint path cover $\{P_1^{00}, P_2^{00}, \ldots, P_{k-1}^{00}\}$ in $Q_{n-2}^{00} - F^{00}$. For every $i \in [k-1]$, let P_i^{10} and P_i^{11} be the corresponding paths of P_i^{00}

in Q_{n-2}^{10} and Q_{n-2}^{11}, respectively. Then $\{P_1^{10},\dots,P_{k-2}^{10}\}$ is a u^{10}-to-v^{10} $(k-1)$-disjoint path cover of Q_{n-2}^{10}, and $\{P_1^{11},\dots,P_{k-2}^{11}\}$ is a u^{11}-to-v $(k-1)$-disjoint path cover of Q_{n-2}^{11}. Similarly, if $|F^{10}|=1$ or $|F^{11}|=1$, then the other two subcubes have no faulty edges. Hence we may also construct the above three path covers. See Fig. 8.

For every $i\in\{2,\dots,k-1\}$, let x_i^{10} and y_i^{10} be the neighbors of v^{10} and u^{10} on the path P_i^{10}, respectively. Since $d(u,v)=n$, we have $d(v^{10},u^{10})=n-2\geq 5$. Thus x_i^{10} and y_i^{10} are distinct vertices and they are different from the vertices u^{10},v^{10}. Next, we construct a u-to-v k-disjoint path cover $\{P_1,P_2,\dots,P_k\}$ in Q_n-F.

Step 1 Let $P_1=P_1^{00}+v^{00}v^{10}+P_1^{10}+u^{10}u^{11}+P_1^{11}$.

Step 2 For every $i\in\{2,\dots,k-1\}$, let $P_i=P_i^{00}[u,x_i^{00}]+x_i^{00}x_i^{10}+P_i^{10}[x_i^{10},y_i^{10}]+y_i^{10}y_i^{11}+P_i^{11}[y_i^{11},v]$.

Step 3 Since $|F^{01}|=|F|-1\leq n-2-k-1<(n-2)-2$, by Lemma 2.5, there is a hamiltonian path $P_{u^{01},v^{01}}$ in $Q_{n-2}^{01}-F^{01}$. Let $P_k=uu^{01}+P_{u^{01},v^{01}}+v^{01}v$.

Then $\{P_1,P_2,\dots,P_k\}$ is a u-to-v k-disjoint path cover in Q_n-F.

By the above two subcases (Subcases 2.2.1 and 2.2.2), we know that the theorem holds for Subcase 2.2.

To sum up, by the principle of the induction hypothesis, the theorem holds. □

4 Concluding remarks

Finding node-disjoint paths is one of the most important issues in various interconnection networks, which is concerned with routing among nodes and embedding of linear arrays.

In this paper, we investigate the problem of one-to-one disjoint path cover in hypercubes with faulty edges and obtain the following results: Let $u,v\in V(Q_n)$ be such that $p(u)\neq p(v)$ and $1\leq k\leq n$. Then there exists a one-to-one k-disjoint path cover $\{P_1,P_2,\dots,P_k\}$ joining vertices u and v in Q_n. Moreover, when $1\leq k\leq n-2$, the result still holds even if removing $n-2-k$ edges from Q_n.

Acknowledgements The authors would like to express their gratitude to the anonymous reviewers for their kind suggestions on the original manuscript.

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