The Journal of Supercomputing (2019) 75:783–807
https://doi.org/10.1007/s11227-018-2612-2

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An efficient algorithm for embedding exchanged hypercubes into grids

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Published online: 20 September 2018
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Abstract
Graph embedding is an important technology in simulating parallel structures and designing VLSI layout. Hypercube is one of the most significant interconnection networks in parallel computing systems. The exchanged hypercube is an important variant of the hypercube, which is obtained by systematically deleting edges from a hypercube. It not only retains several valuable and desirable properties of the hypercube, but also has lower hardware cost. In this paper, we first give an exact formula of minimum wirelength of exchanged hypercube layout into a grid. Furthermore, we propose an $O(N)$ algorithm for embedding exchanged hypercube into a gird with load 1, expansion 1 and minimum wirelength and derive a linear layout of exchanged hypercube with minimum number of tracks and efficient layout areas. Finally, we present simulation experiments of our embedding algorithm on network overhead and total wirelength, which conduce to estimate the total wirelength and chip area.

Keywords Interconnection networks · $EH_{s,t}$ · Graph embedding · Grids · Wirelength

1 Introduction

With the rapid development in deep submicron technology, layout problems become more crucial in physical design of VLSI chips. Effective VLSI layout of interconnected networks can increase the cost-effectiveness of parallel architectures. By reducing its cost (fewer chips, wirelength and components) and reducing various performance barriers such as signal propagation delay, drive power and fraction of data transmission to off-chip destinations [1,2].

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It becomes possible to realize high-complexity and large-scale interconnection networks due to the rapid development of VLSI technology [3–5]. Interconnection network is an important component in parallel computing systems [6,7]. One of the constraints in VLSI routing problems is minimizing wirelength, and efficient layouts for several interconnection networks can be found in [8,9]. Researchers focus on layout interconnection networks into linear arrays and grids, which are called linear layout problem. A linear layout f of a graph $G = (V, E)$ with n vertices is a bijective mapping $f : V \rightarrow \{1, 2, \ldots, n\}$. The set of all linear layouts of the graph G can be denoted by $f(G)$. A linear layout is also called a labeling, arrangement, layout or numbering of the vertices of a graph. A lot of relevant issues in different domains molded by graph layout problems include very large-scale integration (VLSI) circuit design, optimization of networks for parallel computer architecture, graph theory, information retrieval, etc. [10].

The minimum linear layout problem is first stated by Harper in 1964 and is proved to be NP-complete [11]. Nakano [12] proposed a linear layout of generalized hypercube network. Recently, Arockiaraj et al. [13] proved that the minimum linear layout of locally twisted cubes is equal to the minimum linear layout of hypercubes. Interconnection networks can also lay out into optical linear arrays. Liu [14] studied the embedding of exchanged hypercube network into optical linear array with optimal congestion. An embedding of 3-ary n-cube into an optical linear array with minimum congestion is given in [15].

Grid embeddings are used not only to study the simulation capabilities of a parallel architecture but also to design its VLSI layout. In [16], Bezrukov et al. obtained the approximate results and the estimation of lower bounds of wirelength on embedding hypercube network into a grid, and Bezrukov et al. also studied the exact congestion of embedding hypercube network into a rectangular grid in [17]. Manuel et al. [8] proposed an embedding of hypercube network into a grid with minimum wirelength. Recently, Abraham et al. [18] investigated the optimal embedding of locally twisted cubes into grids.

The problem of efficiently laying out VLSI can be formulated as the graph embedding problem. Embeddability is a critical metric to evaluate the performance of an interconnection network. Many applications, such as architecture simulation and processor allocation, can be modeled as a graph embedding problem [19,20]. Graph embedding is an important issue that maps a guest graph into a host graph. Given a guest graph G and a host graph H, an embedding f from G to H can be defined as an injective mapping from $V(G)$ to $V(H)$.

Most researches on graph embedding consider paths, cycles and meshes as guest graphs because they are the architectures widely used in parallel computing systems [21–26]. In [27], Fan et al. proved that the cycles of all possible lengths can be embedded into twisted cube, and Fan et al. [28] also studied the embedding of paths with all possible lengths between any two vertices into crossed cube. Wang et al. [22] studied the embedding of three different types of special meshes into twisted cubes.

The hypercube network is one of the most popular interconnection network structures in parallel computing and communication systems. This is partly because of many attractive properties of the hypercube network such as regularity, recursive structure, vertex and edge symmetry, and maximum connectivity, as well as the effective routing and broadcasting.

As a variant of the n-dimensional hypercube network, the exchanged hypercube network $EH_{s,t}$ was proposed by Loh et al. [29]. An exchanged hypercube network is formed by removing edges from an n-dimensional hypercube network Q_n where $n = s+t+1$. This is evident in the fact that even though the number of edges of an exchanged hypercube network is nearly half of that of a hypercube network, their diameters are similar. Therefore, $EH_{s,t}$ retains several desirable properties of the hypercube network such as a small diameter [29], bipancyclicity [30] and super connectivity [31] and has lower link costs than hypercubes. Zhang et al. [32] proposed a new type of data center network $ExCCC\text{-}DCN$, which combines exchanged hypercube and cube-connected cycles, and proved that it is a highly scalable, cost-effective and energy-efficient data center network structure. Furthermore, the lower link complexity of $EH_{s,t}$ can also directly reduce the costs of hardware and the implementation of VLSI.

The VLSI layout model assumed in this paper is the Thompson's model [33]. In this model, a network is viewed as a graph whose nodes correspond to processing elements and edges correspond to wires. In this study, the following rules for a graph layout on the grid are used:

- It is a one-to-one mapping for assigning the vertices of the graph to the grid;
- The wires are routed by two layers of interconnect. Horizontal wires are routed in one layer, while vertical wires are routed in the other;
- The wires could run either vertically or horizontally along grid lines, but could not cross or overlap with one another.

In this paper, we study the embedding of $EH_{s,t}$ into a grid and obtain the exact wirelength of $EH_{s,t}$ into a grid. Next, we derive a result from which the minimum area for a common VLSI layout of $EH_{s,t}$ in two-dimensional technologies can be determined. The major contributions of the paper are as follows:

1. We proved the minimum wirelength of $EH_{s,t}$ into a linear array with minimum wirelength;
2. We proposed a decomposition embedding of $EH_{s,t}$ into grid and proved that $EH_{s,t}$ can be embedded into the grid $M(2^{\lfloor \frac{s+t+1}{2} \rfloor}, 2^{\lceil \frac{s+t+1}{2} \rceil})$ with minimum wirelength.
3. It is proved that $EH_{s,t}$ can be implemented in a linear layout with minimum tracks, and it also showed that $EH_{s,t}$ can be laid out into an efficient grid.

The rest of this paper is organized as follows: Sect. 2 gives some definitions and notations. Section 3 derives a maximum induced subgraph of $EH_{s,t}$. Section 4 gives an embedding of $EH_{s,t}$ into a linear array and a grid with minimum wirelength. A collinear layout area of $EH_{s,t}$ into a linear array and a grid is proposed in Sect. 5. Section 6 reports simulation and experimental results. The final section concludes this paper.

2 Preliminaries

2.1 Definitions and notations

In this section, we will give some definitions and notations used in this paper. All graphs in this paper are simple undirected graphs, which can be generally denoted by $G = (V(G), E(G))$, where $V(G)$ is the vertex set and $E(G)$ is the edge set. For two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, G_2 is said to be a subgraph of G_1 if $V_2 \subseteq V_1$ and $E_2 \subseteq E_1$. If $V' \subseteq V(G)$, the subgraph of G induced by the vertex subset V' is denoted by $G[V']$. The subgraph induced by the vertex subset $V(G_1) \cup V(G_2)$ is denoted by $G_1 \cup G_2$. Let $\tau(V')$ denote the number of edges of $G[V']$. If G_1 is a subgraph of G_2 and $G_1 \neq G_2$, G_1 is said to be the proper graph of G_2 and denoted by $G_1 \subset G_2$. For a pair of disjoint vertex subset S_1 and S_2 of graph G, let $\tau(S_1, S_2)$ denote the number of edges with one vertex in S_1 and the other vertex in S_2. For any integer $n \geq 1$, a binary string x of length n will be written as $x_{n-1}x_{n-2} \cdots x_1x_0$, where $x_i \in \{0, 1\}$ for any integer $i \in \{0, 1, \ldots, n-1\}$. Given any $x = x_{n-1}x_{n-2} \cdots x_1x_0$, x_i is said to be the ith bit of x and $x_{n-1}x_{n-2} \cdots x_k$ $(0 \leq k \leq n-1)$ is called a prefix of x. Besides, x_0 is called the first bit of x and x_{n-1} is called the last bit of x. For a graph $G = (V, E)$, an (u, v)-path of length l from vertex u to vertex v is denoted by $P = (u_0, u_1, \ldots, u_l)$, where $u_0 = u$ and $u_l = v$ are called the two end vertices of path P, and all the vertices $u_0, u_1, \ldots, u_l$ are distinct.

If u and v are two adjacent nodes in graph G when $(u, v) \in E(G)$. The neighborhoods of a vertex v are denoted by $N_G(v)$ in graph G such that $\{x|(v, x) \in E(G)\}$. The degree $\delta_G(v)$ of a node v is the number of edges incident with v. Let u and v be the source node and the destination node, respectively. The length of a shortest (u, v)-path is denoted by $d(u, v)$, which is called the distance between u and v in G. A Hamiltonian path is defined as a path which traverses each vertex of graph G exactly once. If there exists a Hamiltonian path between any two distinct vertices of graph G, we say that graph G is a Hamiltonian-connected graph.

A graph G_1 is isomorphic to another graph G_2 (represented by $G_1 \cong G_2$) if and only if there exists a bijection $f : V(G_1) \to V(G_2)$, such that if $(u, v) \in E(G_1)$ then $(f(u), f(v)) \in E(G_2)$. For two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, and a subset $S \subseteq V_1$, let f be a mapping from V_1 to V_2. Let $T = \{x \in V(G_2)|\text{there is } y \in S, \text{ such that } y = f(x)\}$. Then, we write $T = f(S)$ and $S = f^{-1}(T)$.

Graph embedding can be formally defined as: Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, an embedding from G_1 to G_2 is an injective mapping $\psi : V_1 \to V_2$. We call G_1 the guest graph and G_2 the host graph. There are four common metrics used to measure the quality of an embedding, namely *congestion*, *dilation*, *expansion* and *load*. The *congestion* of an embedding ψ is defined as $cong(G_1, G_2, \psi) = \max\{cong(e)|e \in E_2\}$, which measures queuing delay of messages, where $cong(e)$ denotes the number of edges of G_1 whose image paths in G_2 include the edge e. The *dilation* of embedding ψ is defined as: $dil(G_1, G_2, \psi) = \max\{\text{dist}(G_2, \psi(u), \psi(v))\,|(u, v) \in E_1\}$, which measures the communication delay, where $\text{dist}(G_2, \psi(u), \psi(v))$ denotes the distance between the two vertices $\psi(u)$ and $\psi(v)$ in G_2. The *expansion* of an embedding ψ of G_1 into G_2 is defined as

$exp(G_1, G_2, \psi) = |V_1|/|V_2|$, which measures processors utilization. Obviously, the expansion of the embedding is at least one. To measure the processing time of tasks is referred to as the load in an embedding. The $load$ of an embedding ψ is denoted by $\text{load}(G_1, G_2, \psi) = \max\{\text{load}(v)|v = \psi(u), u \in V_1\}$, where $\text{load}(v)$ denotes the number of vertices of G_1 mapped on v. In addition to these parameters, $wirelength$ is another criterion in embedding and widely used in VLSI design [16]. The wirelength is the total wire length required to complete the entire VLSI layout. The wirelength problem is to find an embedding of G into H that induces the minimum wirelength and thought to be cost-effective.

The isoperimetric problem is to find a subset of vertices of a given graph, such that the edge cut separating this subset from its complement has minimum size among all subsets of the same cardinality. Mathematically, for a given positive integer m, if $\delta_G(m) = \min_{X \subseteq V, |X|=m} |[X, V - X]_G|$, where $[X, V - X]_G = \{(u, v) \in E | u \in X, v \in (V - X)\}$, then the problem is to find $X \subseteq V$ such that $|X| = m$ and $|[X, V - X]_G| = \delta_G(m)$, which is called an optimal set.

The maximum induced subgraph problem [34] is to find a subset of vertices of a given graph, such that the number of edges in the subgraph induced by this subset is maximum among all induced subgraphs with the same number of vertices. Mathematically, for a given positive integer m, if $I_G(m) = \max_{X \subseteq V, |X|=m} |T_G(X)|$, where $T_G(X) = \{(u, v) \in E | u, v \in X\}$, then the problem is to find $X \subseteq V$ such that $|X| = m$ and $|T_G(X)| = I_G(m)$. For regular graphs, the optimal set problem and maximum induced subgraph problem are equivalent.

The wirelength problem is solved by edge isoperimetric problem. The following two versions of the edge isoperimetric problem of a graph $G = (V, E)$ have been considered in the literature [34] and are NP-complete [11].

Definition 1 [8] Let f be an embedding from G to H. Let $EC_f(e)$ denote the number of edges (u, v) of G such that e is in the path $P_f(u, v)$ between the vertices $f(u)$ and $f(v)$ in H. Considering there possibly exist multiple paths between $(f(u), f(v))$ in H, we choose the shortest path as $P_f(f(u), f(v))$. The edge congestion f is given by

$$EC_f(G, H) = \max\left\{EC_f(e)|e \in E(H)\right\}.$$

Then, the minimum edge congestion of G into H is defined as

$$EC(G, H) = \min_f\{EC(G, H)|f \text{ is an embedding from } G \text{ to } H\}.$$

Definition 2 [8]. The wirelength of an embedding f of G into H is given by

$$WL_f(G, H) = \sum_{(u,v)\in G} d_H(f(u), f(v)),$$

where $d_H(f(u), f(v))$ denotes the length of the path $P_f(u, v)$ in H and $P_f(u, v)$ is the shortest path between $(f(u), f(v))$ in H.

Then, the minimum wirelength of G into H is defined as

$$WL(G, H) = \min_f WL(G, H),$$

where the minimum is taken over all embeddings f of G into H.

Lemma 1 [8] *Let G be an arbitrary graph and f be an embedding of G into H. Let S be an edge cut of H such that the removal of edges of S leaves H into two components H_1 and H_2. Let $G_1 = f^{-1}(H_1)$ and $G_2 = f^{-1}(H_2)$. Also S satisfies the following conditions:*

(i) *For every edge $(a, b) \in (G_i)$, $i = 1, 2$, $P_f(a, b)$ has no edges in S.*
(ii) *For every edge $(a, b) \in E(G)$ with $a \in V(G_1)$ and $b \in V(G_2)$, $P_f(a, b)$ has exactly one edge in S.*
(iii) *G_1 and G_2 are optimal sets.*
Then, $EC_f(S)$ is minimum and $EC_f(S) = \sum_{v \in V(G_1)} deg(v) - 2|E(G_1)| = \sum_{v \in V(G_2)} deg(v) - 2|E(G_2)|$.

Lemma 2 [8] Let $f : G \to H$ be an embedding. Let $S_1, S_2, \ldots, S_p$ be p edge cuts of H such that $S_i \cap S_j = \varnothing$, $i \neq j$, $1 \leq i, j \leq p$. Then,

$$WL_f(G, H) = \sum_{i=1}^{p} EC_f(S_i).$$

2.2 The exchanged hypercube

The definition of exchanged hypercubes $EH_{s,t}$ is presented as follows. The Hamming distance between two vertices u and v, denoted by $H(u, v)$, is the number of bits that are different in the corresponding strings for both vertices. Let $l \geq 1$ and $u = u_{l-1} \cdots u_0 \in \{0, 1\}^l$ be a binary string. Let $u_{j:i}$ be the substring $u_j u_{j-1} \cdots u_i$ of u for $0 \leq i \leq j < l$.

Definition 3 [29] The vertex set V of exchanged hypercube $EH_{s,t}$ ($s \geq 1$ and $t \geq 1$) is the set

$$\{u_{s+t} \cdots u_{t+1} u_t \cdots u_1 u_0 | u_i \in \{0, 1\} \text{ for } 0 \leq i \leq s + t\}.$$

Let $u_{s+t} u_{s+t-1} \cdots u_0$ and $v_{s+t} v_{s+t-1} \cdots v_0$ be two vertices in $EH_{s,t}$. E is the set of edges composed of three disjoint types E_1, E_2 and E_3:

$E_1 = \{(u, v) | u_0 \neq v_0$ and $u_i = v_i$ for $1 \leq i \leq s + t\}$,

$E_2 = \{(u, v) | u_0 = v_0 = 0, H(u, v) = 1$ with $u_i \neq v_i$ for some $t + 1 \leq i \leq s + t\}$,
and

$E_3 = \{(u, v) | u_0 = v_0 = 1, H(u, v) = 1$ with $u_i \neq v_i$ for some $1 \leq i \leq t\}$,
where $H(u, v)$ denotes the Hamming distance between two vertices u and v. The first set links node pairs that exhibit unity Hamming distance in the first t bits of their addresses, while the second set links node pairs that exhibit unity Hamming distance in the last s bits of their addresses.

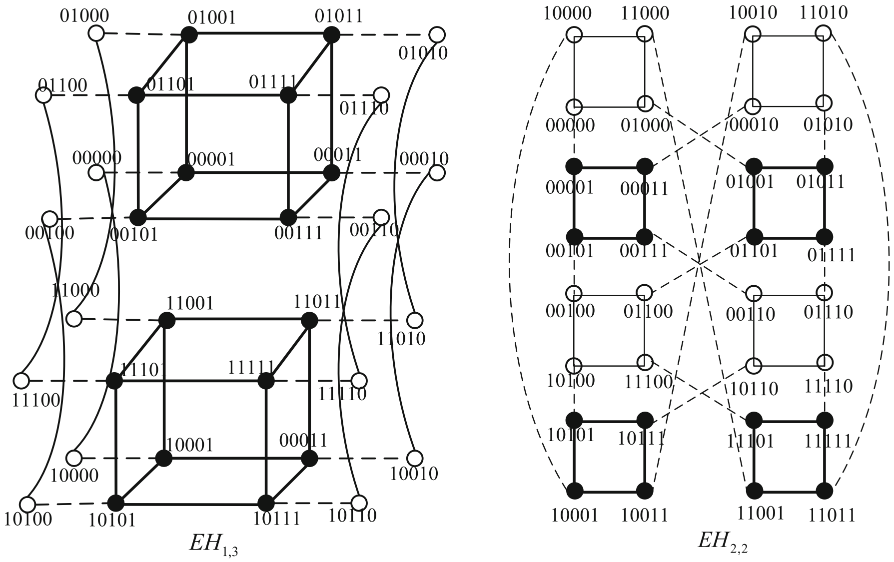


Fig. 1 Two exchanged hypercubes $EH_{1,3}$ and $EH_{2,2}$, where dashed links correspond to the edge set E_1, solid links correspond to the edge set E_2 and bold links correspond to the edge set E_3

From the definition of $EH_{s,t}$, the number of vertices is 2^{s+t+1} and the number of edges is $(s+t+2)2^{s+t-1}$ where $|E_1| = 2^{s+t}$, $|E_2| = s \cdot 2^{s+t-1}$ and $|E_3| = t \cdot 2^{s+t-1}$. For a vertex x with $x_0 = 0$, the vertex degree is $s + 1$, whereas the vertex degree with $x_0 = 1$ is $t + 1$. $EH_{s,t}$ is a subgraph of the $(s + t + 1)$-dimensional hypercube Q_{s+t+1}, and as a result, it is also a bipartite graph. Figure 1 illustrates the exchanged hypercubes $EH_{1,3}$ and $EH_{2,2}$.

In addition, the number of vertices in $EH_{s,t}$ is the same as the number of vertices in Q_{s+t+1}. The number of edges in $EH_{s,t}$ is only slightly over half of the number of edges in Q_{s+t+1}. A d-dimensional edge, or simply $(s + t + 1)$-edge, of $EH_{s,t}$ is an edge (u, v) such that the labels of x and y are contradictory at bit d but are identical at all previous bits. In this case, y is called the d-neighbor of x, denoted $v = N_d(x)$. Let DIM_d denote the set of all d-edges of $EH_{s,t}$. Then, $E(EH_{s,t}) = \bigcup_{d=0}^{s+t} DIM_d$. To be more precise, $|E(EH_{s,t})| = (s+t+2)2^{s+t-1} = (\frac{1}{2} + \frac{1}{2(s+t+1)})|E(Q_{s+t+1})|$ [35].

Lemma 3 [29] $EH_{s,t}$ and $EH_{t,s}$ are isomorphic.

Lemma 4 [29] $EH_{s,t}$ can be divided into 2^t copies as Q_s and 2^s copies as Q_t.

Lemma 5 [29] $EH_{s,t}$ can be partitioned into two copies of $EH_{s-1,t}$ or $EH_{s,t-1}$.

After deleting the edge set E_1 from $EH_{s,t}$, the vertex set of $EH_{s,t}$ can separated into two parts T and S, where T is the set of all vertices with rightmost bit being 1 and S is the set of all vertices with rightmost bit being 0. In other words,

$T = \{v_{s+t}v_{s+t-1}\cdots v_1 1 | v_i \in \{0, 1\}$ for $1 \leq i \leq s + t\}$ and
$S = \{u_{s+t}u_{s+t-1}\cdots u_1 0 | u_i \in \{0, 1\}$ for $1 \leq i \leq s + t\}$.

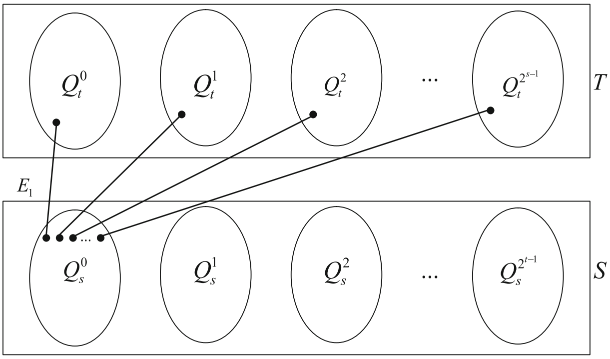


Fig. 2 Induced subgraphs Q_t and Q_s of $EH_{s,t}$

Each edge $e \in E_1$ has one endpoint in T and the other in S, which is illustrated in Fig. 2.

3 Maximum induced subgraph for $EH_{s,t}$

In this section, we mainly focus on finding the maximum induced subgraph of $EH_{s,t}$. There is a significant relationship between the maximum induced subgraph problem and the wirelength problem [8].

For any integer $m \geq 1$ and $S \subseteq V(G)$ with $|S| = m$, if $G[S]$ is the subgraph with the maximum number of edges among all induced subgraphs with m vertices, then $G[S]$ is called the maximum induced graph with m vertices in G.

For $1 \leq s \leq t$, we group $V(EH_{s,t})$ into eight disjoint subsets [14] as follows, $V_1 = \{1\underbrace{*\cdots*}_{t-1}01\}$, $V_2 = \{1\underbrace{*\cdots*}_{t-1}11\}$, $V_3 = \{0\underbrace{*\cdots*}_{t-1}01\}$, $V_4 = \{0\underbrace{*\cdots*}_{t-1}11\}$, $V_5 = \{1\underbrace{*\cdots*}_{t-1}00\}$, $V_6 = \{1\underbrace{*\cdots*}_{t-1}10\}$, $V_7 = \{0\underbrace{*\cdots*}_{t-1}00\}$, $V_8 = \{0\underbrace{*\cdots*}_{t-1}10\}$.

If $u = u_{s+t+1}\cdots u_{t+1}u_t\cdots u_1$ is a node in $Q_t^i(0 \leq i \leq 2^{s-1})$ and the decimal value of $u_{t:1}$ is j, then the node u can be denoted by $q_t^{i,j}$. The subgraph induced by $V_i(1 \leq i \leq 4)$ contains 2^{s-1} disjoint $(t-1)$-cubes, and the subgraph induced by $V_i(1 \leq i \leq 4)$ contains 2^{t-1} disjoint $(s-1)$-cubes. If $s \geq 2$, for the subgraph induced by $V_i(1 \leq i \leq 4)$, we denote the $(t-1)$-cube by $Q_{t-1}^{i,j}$, where $j(j \in [0, 2^{s-1}-1])$ is the decimal number of $u_{s+t-1,t+1}$, and the vertex u in $Q_{t-1}^{i,j}$ is represented by $q_{t-1}^{i,j,k}$, where $k(k \in [0, 2^{t-1}-1])$ is the decimal number of $u_{t-1,1}$. Similarly, for $(5 \leq i \leq 8)$, we can define the $(s-1)$-cube $Q_{s-1}^{i,j}$ and the vertex $q_{s-1}^{i,j,k}$, where $j \in [0, 2^{t-1}-1]$ and $k \in [0, 2^{s-1}-1]$. This labeling is denoted by lex.

Definition 4 [36] An incomplete hypercube on i vertices of Q_n is the subcube induced by $\{0, 1, \ldots, i-1\}$ and is denoted by L_i.

Theorem 1 [37] For $1 \leq i \leq 2^n$, L_i is an optimal set in the hypercube Q_n.

Lemma 6 [38] For $1 \leq i, j \leq 2^n$ such that $i+j \leq 2^n$, $|E(Q_n[L_i])|+|E(Q_n[L_j])|+$ {i,j} $\leq |E(Q_n[L_{i+j}])|$.

Lemma 7 [39] Let V be a vertex subset of graph G and $\{V_0, V_1\}$ be a partition of V. Then, $\tau(V) = \tau(V_0) + \tau(V_1) + \tau(V_0, V_1)$.

Lemma 8 Let K be a subgraph of $EH_{s,t}$ isomorphic to L_k where $1 \leq s \leq t$ and $k \leq 2^{s+t} + 2^s$. Let K_1 and K_2 be disjoint segments induced by k_1 and k_2 consecutive vertices on $\bigcup_{i=1}^{2^s} Q_t \cup Q_s^1$, respectively, such that $k_1 + k_2 = k$. Then, $|E(EH_{s,t}[K_1 \cup K_2])| \leq |E(EH_{s,t}[K])|$.

Proof We can denote Q_t^1, Q_t^2,…, and $Q_t^{2^s}$ as 2^s copies of Q_t who are composed of the edges E_3, and Q_s^1, Q_s^2,…, and $Q_s^{2^t}$ as 2^t copies of Q_s who are composed of the edges E_2. For simplicity, we denote u_1^1, u_1^2,…and $u_1^{2^t}$ as 2^t vertices of Q_t^1, u_2^1, u_2^2,…, and $u_2^{2^t}$ as 2^t vertices of Q_t^2, …, and $u_{2^s}^1$, $u_{2^s}^2$,…, and $u_{2^s}^{2^t}$ as 2^t vertices of $Q_t^{2^s}$. And we denote v_1^1, v_1^2,…and $v_1^{2^s}$ as 2^s vertices of Q_s^1, v_2^1, v_2^2,…and $v_2^{2^s}$ as 2^s vertices of Q_s^2,…, and $v_{2^t}^1$, $v_{2^t}^2$,…and $v_{2^t}^{2^s}$ as 2^s vertices of $Q_s^{2^t}$. Let $E(EH_{s,t}[K_1 \wedge K_2])$ denote the set of edges in $EH_{s,t}$ with one end in K_1 and the other end in K_2, and we have the following cases:

Case 1. $k_1, k_2 \leq 2^t$. We consider the following cases.

Case 1.1 $K_1 \subset Q_t^1$. Since Q_t is isomorphic the t-dimensional cube, by the definition of $EH_{s,t}$ and Theorem 1, $|E(EH_{s,t}[K_1 \cup K_2])| = |E(Q_t[K_1 \cup K_2])| \leq |E(Q_t[K])| = |E(EH_{s,t}[K])|$.

Case 1.2 $K_1 \subset Q_s^1$. The proof is similar to Subcase 1.2.

Case 2. $2^t < k_1 \leq 2^t + 2^s$. $K_1 \subset Q_t^1 \cup Q_s^1$. Let $2^t = k_1 + k_2$, where k_1 vertices lie in Q_t^1 and k_2 vertices lie in Q_s^1, inducing subgraphs K_1 and K_2 in Q_t^1 and Q_s^1, respectively. Since there are one edge joining vertices in K_1 and vertices in K_2, $|E(EH_{s,t}[K_1 \wedge K_2])| \leq k_2$. This implies that $|E(EH_{s,t}[K_1 \cup K_2])| = |E(EH_{s,t}[K_1])| + |E(EH_{s,t}[K_2])| + |E(EH_{s,t}[K_1 \wedge K_2])| \leq |E(EH_{s,t}[L_{k_1}])| + |E(EH_{s,t}[L_{k_2}])| + k_2$. By Lemma 1, we get $|E(EH_{s,t}[K_1 \cup K_2])| \leq |E(EH_{s,t}[L_{k_1} + k_2])| = |E(EH_{s,t}[K])|$.

Case 3. $2^t + 2^s < k_1 \leq 2^{s+t} + 2^s$. Let k_1, k_2 be the number of consecutive vertices in K_1, K_2 that lie in $\bigcup_{i=1}^{2^s} Q_t^i \cup Q_s^1$. Then, $|E(EH_{s,t}[K_1])| \leq |E(EH_{s,t}[L_{k_1}])|$, $|E(EH_{s,t}[K_2])| \leq |E(EH_{s,t}[L_{k_2}])|$ and $|E(EH_{s,t}[K_1 \wedge K_2])| \leq k_2 + k_2$. Hence, $|E(EH_{s,t}[K_1 \cup K_2])| \leq |E(EH_{s,t}[L_{k_1}])| + |E(EH_{s,t}[L_{k_2}])| + 2k_2$. Let $H_1 = L_{k_1}$. Then, $|E(EH_{s,t}[H_1])| = |E(EH_{s,t}[L_{k_1}])|$. Let H_2 be the subgraph of $EH_{s,t}$ induced by the vertices in Q_s^1 labeled $2^{s+t} - 1, 2^{s+t} - 2, \ldots, 2^{s+t} - k_2$. This implies $|E(EH_{s,t}[H_2])| = |E(EH_{s,t}[L_{k_2}])|$ and $|E(EH_{s,t}[H_1 \wedge H_2])| \geq k_2 + k_2$. Therefore $|E(EH_{s,t}[H_1 \wedge H_2])| \geq |E(EH_{s,t}[L_{k_1}])| + |E(EH_{s,t}[L_{k_2}])| + 2k_2$ and hence $|E(EH_{s,t}[K_1 \cup K_2])| \leq |E(EH_{s,t}[H_1 \cup H_2])|$. □

Theorem 2 The number of edges in a maximum subgraph induced by $2^{s+t}+m$ vertices of $EH_{s,t}$, $1 \leq s \leq t$, $1 \leq m \leq 2^{s+t+1}$, is given by

$$|E(EH_{s,t}[S])| = t \cdot 2^{s+t-1} + I_{EH_{s,t}}(m) + m.$$

Proof Let $I_m^{k_i}$ denote the k-dimensional subgraph of $EH_{s,t}$ on m vertices, which contains subcubes Q_t^1, Q_t^2,..., and Q_t^i and E_1 for $1 \leq i \leq 2^t$. This means that there are $k \cdot 2^{k_i - 1}$ edges between $\bigcup_{i=1}^{2^s} Q_t^i$ and $\bigcup_{i=1}^{2^s} Q_t^{i+1}$. Also, $\bigcup_{i=1}^{2^s} Q_t^i$ has $t_i 2^{t_i - 1}$ edges within itself. The maximum subgraph induced by I_m^k of $EH_{s,t}$ contains two components Q_k^i and $I_{m-2^{s+t}}^t$, where the vertices in Q_t^i are numbered as $0, 1, \ldots, 2^{s+t} - 1$ and the vertices in $I_{m-2^{s+t}}^t$ are numbered as $2^{s+t}, 2^{s+t} + 1, \ldots, 2^{s+t+1}$, for $t = \lceil \log(m - 2^{s+t}) \rceil$. Thus, I_m^k contains a set of Q_t^i and Q_s^i, and no two constituent cubes are of the same size. The number of edges induced by I_m^k in $EH_{s,t}$, $1 \leq s \leq t$ is given by $|E[I_m^k])| = t \cdot 2^{s+t-1} + I_{EH_{s,t}}(m) + m$. The lemma holds. □

Lemma 9 For $1 \leq s \leq t$ and $1 \leq i \leq 2^{s+t} + 2^s$, L_i is an optimal set.

Proof Let R be a subgraph of $EH_{s,t}$ isomorphic to L_k where $k \leq 2^{s+t} + 2^s$. Let N be a set of k non-consecutive vertices in $EH_{s,t}$. Then, $N = \bigcup_{i=1}^{p} X_i$ where $p \geq 2$, X_i's are mutually disjoint and each X_i is a set of consecutive vertices in $EH_{s,t}$ such that $\bigcup_{i=1}^{p} |X_i| = n$. If X_i contains vertices labeled $2^{s+t} + 2^s - 1$ and $2^{s+t} + 2^s$, then X_i is split into two sets such that one set ends with label $2^{s+t} + 2^s - 1$ and the other set begins with label $2^{s+t} + 2^s$. By Lemma 2, we get $|E(EH_{S,t}[N])| \leq |E(EH_{s,t}[R])|$. □

Theorem 3 For $1 \leq i \leq 2^{s+t+1}$, L_i is an optimal set in $EH_{s,t}$.

Proof By Lemma 4, after deleting the edge set E_1 from $EH_{s,t}$, $EH_{s,t}$ can be partitioned into $EH_{s-1,t}$ or $EH_{s,t-1}$. By Lemma 1, L_i is an optimal set for $1 \leq i \leq 2^{s+t} + 2^s$. Now let $i > 2^{s+t} + 2^s$. Then, we have $L_i' = EH_{s,t} - L_i \cong L_{2^{s+t-i}}$. Since $2^{s+t+1} - i < 2^{s+t+1} - 1$, by Lemma 1, L_i' is an optimal set in $EH_{s,t}$. Since $EH_{s-1,t} \cong EH_{s,t-1}$, L_i is an optimal set in $EH_{s,t}$. □

4 Embedding the exchanged hypercubes into grids

In this section, we consider the embeddings of exchanged hypercubes into linear arrays and grids, respectively.

4.1 Embedding exchanged hypercubes into linear arrays

In this section, we will give an embedding of $EH_{s,t}$ into a linear array with minimum wirelength. When H is a path, $WL(G, H)$ represents linear wirelength of G or minimum linear arrangement (MinLA) of G. The wirelength problem of a graph G into H is to find an embedding of G into H that induces the minimum wirelength $WL(G, H)$.

Linear arrangements are a particular case of embedding graphs in d-dimensional grids. The dilation of the embedding is most commonly called the bandwidth, which is NP-complete [11], and can be defined as follows:

Definition 5 [8] For any integer $n \geq 1$, the linear array of n vertices, denoted by P_n, is a graph such that $V(P_n) = \{1, 2, \ldots, n\}$ and where $E(P_n) = \{(i, i+1) | i \in [1, n-1]\}$.

Definition 6 Let $f : V(EH_{s,t}) \rightarrow V(P_{2^{s+t+1}})$ be an embedding, which is defined as follow: Label the vertices of $P_{2^{s+t+1}}$ as $0, 1, \ldots, 2^{s+t+1} - 1$. Then, for any $v \in V(EH_{s,t})$, let $f(v) = lex(v)$.

Let G be a graph and L_n be a linear array with n vertices. Let f be an embedding from G to L_n. The bandwidth of the embedding f of G into L_n is defined as

$$B_f(G) = \max\{|f(v) - f(u)|(u, v) \in E(G)\}.$$

Furthermore, the minimum bandwidth from all embeddings from G to L_n is defined as

$$B(G) = \min\{B_f(G)|f \text{ is an embedding from } G \text{ to } L_n\}.$$

The bandwidth problem is to find an embedding of G into L_n, such that it has the minimum bandwidth.

Theorem 4 $EH_{s,t}$ can be embedded into $L_{2^{s+t+1}}$ with dilation $2^{s+t} + 1$.

Proof Let $f = lex$. By Lemma 4, $EH_{s,t}$ can be divided into 2^s copies of Q_t and 2^t copies of Q_s. Hence, we can denote Q_t^1, Q_t^2,…, and $Q_t^{2^s}$ as 2^s copies of Q_t who are composed of the edges E_3, and Q_s^1, Q_s^2,…, and $Q_s^{2^t}$ as 2^t copies of Q_s who are composed of the edges E_2. For simplicity, we denote $u^1 1$, u_1^2,…and $u_1^{2^t}$ as 2^t vertices of Q_t^1, u_2^1, u_2^2,…, and $u_2^{2^t}$ as 2^t vertices of Q_t^2, …, and $u_{2^s}^1$, $u_{2^s}^2$,…, and $u_{2^s}^{2^t}$ as 2^t vertices of $Q_t^{2^s}$. And we denote v_1^1, v_1^2,…and $v_1^{2^s}$ as 2^s vertices of Q_s^1, v_2^1, v_2^2,…and $v_2^{2^s}$ as 2^s vertices of Q_s^2,…, and $v_{2^t}^1$, $v_{2^t}^2$,…and $v_{2^t}^{2^s}$ as 2^s vertices of $Q_s^{2^t}$. Then, we may verify the result by the following cases below.

Case 1. $(u, v) \in E(Q_t^i)(1 \leq i \leq 2^s)$. Without loss of generality, suppose that $(u_1^1, v_1^1) \in E(Q_t^1)$. Let $(f(u_1^1), f(v_1^1))$ be the image of (u_1^1, v_1^1) in the linear array. Clearly, $\max\{\text{dist}(L_{2^{s+t+1}}, f(u_1^1), f(v_1^1))|(u_1^1, v_1^1) \in E(Q_t^1) = \max\{|f(v_1^1) - f(u_1^1)|(u_1^1, v_1^1) \in E(Q_t^1)\} = 2^{t-1}$.

Case 2. $(u, v) \in E(Q_s^i)(1 \leq j \leq 2^t)$. The proof is similar to Case 1. Thus, results can be obtained directly as 2^{s-1}.

Case 3. $(u, v) \in E(Q_t^i \bigcup Q_t^j)(1 \leq i \leq 2^s, 1 \leq i \leq 2^t)$. Without loss of generality, suppose that $(u_1^1, v_1^1) \in E(Q_t^1 \bigcup Q_s^1)$. Let $(f(u_1^1), f(v_1^1))$ be the image of (u_1^1, v_1^1) in the linear array. It is easy to verify that $\max\{\text{dist}(L_{2^{s+t+1}}, f(u_1^1), f(v_1^1))|(u_1^1, v_1^1) \in E(Q_t^1 \bigcup Q_s^1) = \max\{|f(v_1^1) - f(u_1^1)|(u_1^1, v_1^1) \in E(Q_t^1 \bigcup Q_s^1)\} = 2^{s+t} + 1$.

In summary, the theorem is proved. □

Lemma 10 $R_i^{lex} = \{1, ..., i2^{\lceil \frac{s+t+1}{2} \rceil}\}$ is an optimal set in $EH_{s,t}$ for $i = 1, 2, \ldots, 2^{\lfloor \frac{s+t+1}{2} \rfloor}$ and $\lfloor \frac{s+t+1}{2} \rfloor + \lceil \frac{s+t+1}{2} \rceil = s + t + 1$.

Proof This proof can be obtained directly from Theorem 4. □

Lemma 11 For $j = 1, 2, \ldots, 2^{\lfloor \frac{s+t+1}{2} \rfloor}$,

$$C_j^{lex} = \begin{Bmatrix} 1, & 1 \times 2^{\lfloor \frac{s+t+1}{2} \rfloor}, & 2 \times 2^{\lfloor \frac{s+t+1}{2} \rfloor}, & \ldots (2^{\lceil \frac{s+t+1}{2} \rceil}) \times 2^{\lfloor \frac{s+t+1}{2} \rfloor}, \\ 2, & 1 \times 2^{\lfloor \frac{s+t+1}{2} \rfloor} + 1, & 2 \times 2^{\lfloor \frac{s+t+1}{2} \rfloor} + 1, & \ldots (2^{\lceil \frac{s+t+1}{2} \rceil}) \times 2^{\lfloor \frac{s+t+1}{2} \rfloor} + 1, \\ \ldots & & & \\ j, & 1 \times 2^{\lfloor \frac{s+t+1}{2} \rfloor} + j - 1, & 2 \times 2^{\lfloor \frac{s+t+1}{2} \rfloor} + j - 1, & \ldots (2^{\lceil \frac{s+t+1}{2} \rceil}) \times 2^{\lfloor \frac{s+t+1}{2} \rfloor} + j - 1 \end{Bmatrix}$$

is an optimal set in $EH_{s,t}$ where $2^{\lceil \frac{s+t+1}{2} \rceil} + 2^{\lfloor \frac{s+t+1}{2} \rfloor} = s + t + 1$.

Proof Let $f : C_j^{lex} \rightarrow L_{j \times 2^{\lfloor \frac{s+t+1}{2} \rfloor}}$ with $f(k \times 2^{\lceil \frac{s+t+1}{2} \rceil} + l) = l \times 2^{\lfloor \frac{s+t+1}{2} \rfloor} + k$. We use $u_1 u_2 \ldots u_{t+1}$ in C_j^{lex} to denote the decimal string of $l \times 2^{\lfloor \frac{s+t+1}{2} \rfloor} + k$. Since the decimal string representations of two numbers u and v differ in exactly one bit, the same holds for $f(u)$ and $f(v)$. Thus, (u, v) is an edge in R_i and $(f(u), f(v))$ is an edge in L_{2^i}. Therefore, R_i is isomorphic to L_i. By Theorem 1, C_j^{lex} is an optimal set of $EH_{s,t}$. □

Lemma 12 [37] $WL(Q_n, P_{2^n}) = 2^{2n-1} - 2^{n-1}$.

Lemma 13 The lex embedding of exchanged hypercube $EH_{s,t}$ into a linear array $P_{2^{s+t+1}}$ induces a minimum wirelength.

Proof Let $f = lex$ and $G = EH_{s,t}$. For $1 \leq i \leq 2^{s+t+1}$, let S_i be ith edge of $P_{2^{s+t+1}}$. Removal of S_i leaves $P_{2^{s+t+1}}$ into two components X_i and $X_i^{'}$ where $V(X_i) = \{0, 1, \ldots, i\}$ and $V(X_i^{'}) = \{j+1, j+2, \ldots, 2^{s+t+1}\}$. Let G_i and $G_i^{'}$ be the inverse images of X_i and $X_i^{'}$ under f, respectively. By Lemma 8, G_i is an optimal set in $EH_{s,t}$. Thus, the edge cut S_i satisfies Lemma 8. It can be further verified that $\{(i-1, i)\}$ satisfies Lemma 8, and the edge congestion $EC_f(S_i)$ is minimum under embedding lex for $i = 1, 2, \ldots, 2^{s+t+1}$. Thus, the wirelength $WL_f(EH_{s,t}, P_{2^{s+t+1}})$ of embedding $EH_{s,t}$ into $P_{2^{s+t+1}}$ is minimum. □

Theorem 5 For $1 \leq s \leq t$, the wirelength of embedding $EH_{s,t}$ into a linear array $P_{2^{s+t+1}}$ is given by

$$WL(EH_{s,t}, P_{2^{s+t+1}}) = 2^{s+2t-1} - 2^{s+t-1} + 2^{2t} + 2^{2t+2}.$$

Proof Let $f = lex$. We first derive the exact wirelength of embedding the induced subgraphs $EH_{s,t}[E_1]$, $EH_{s,t}[E_2]$ and $EH_{s,t}[E_3]$ into $L_{2^{s+t+1}}$. Let the edge set $E_1 = \{(u, v)|u_0 \neq v_0, u_i = v_i \text{ for } 1 \leq i \leq s+t\}$. After deleting E_1 from $EH_{s,t}$, the vertex set S is decomposed into 2^t connected components. Each component is an s-dimensional hypercube Q_s; moreover, these 2^t hypercubes Q_s are pairwise disjoint, and there are no edges joining any two Q_s. Since each edge $e \in E_1$ has one endpoint in Q_t and the other in Q_s, E_1 is a perfect matching of $EH_{s,t}$ between Q_s and Q_t.

For $1 \leq i \leq 2^{s+t}$, S_j is an edge cut of P_{2^t}, which disconnects P_{2^t} into two linear arrays P_j and $P_j^{'}$, where $2 \leq j \leq 2^{s+t-1}$, $V(P_j) = \{1, 2, \ldots, j\}$, and $V(P_j^{'}) =$

$\{j+1, j+2, \ldots, 2^{s+t-1}\}$. Let $G_{j1} = f^{-1}(P_{j1})$ and $G_{j2} = f^{-1}(P_{j2})$. By Lemma 8, G_{j1} is an optimal set and each S_j satisfies conditions (i) and (ii) of Lemma 8. Therefore, $EC_f(S_j)$ is minimum. Let A_i be an edge cut of $P_{2^{s+t}}$ such that S_i disconnects $P_{2^{s+t}}$ into two components P_{i1} and P_{i2}. Let G_{i1} and G_{i2} be the inverse images of P_{i1} and P_{i2} under f, respectively. By Theorem 1, G_{i1} is an optimal set and each S_i satisfies conditions (i) and (ii) of Lemma 8. Therefore, the sum congestion of $G[\bigcup_{i=1}^{2^s-1} Q_t^i]$ is

$$\begin{aligned} WL_f(A_i) &= WL\left(G\left[\bigcup_{i=1}^{2^s-1} Q_t^i\right], P_{2^{s+t}}\right) \\ &= \sum_{i=1}^{2^{s+t-1}} EC_f(S_i) \\ &= 2^{s+2t-1} - 2^{s+t-1}. \end{aligned}$$

For $2^{s+t}+1 \le i \le 2^{s+t+1}$, S_i is an edge cut of $P_{2^{s+t-1}}$, which disconnects $P_{2^{s+t-1}}$ into two linear arrays P_i and $P_i^{'}$, where $2^{s+t-1}+1 \le i \le 2^{s+t+1}$, $V(P_i) = \{1, 2, \ldots, i\}$, and $V(P_i^{'}) = \{i+1, i+2, \ldots, 2^{s+t+1}-2\}$. Let $G_{i1} = f^{-1}(P_{i1})$ and $G_{i2} = f^{-1}(P_{i2})$. G_{i1} is an optimal set and each S_i satisfies conditions (i) and (ii) of Lemma 8. Therefore, $EC_f(S_i)$ is minimum. Let B_j be an edge cut of $P_{2^{s+t}}$ such that S_j disconnects $P_{2^{s+t}}$ into two components P_{j1} and P_{j2}. Therefore, the sum congestion of $G[\bigcup_{i=1}^{2^t-1} Q_s^i]$ is

$$\begin{aligned} WL_f(B_j) &= WL\left(G\left[\bigcup_{i=1}^{2^t-1} Q_s^i\right], P_{2^{s+t}}\right) \\ &= \sum_{j=2^{s+t}+1}^{2^{s+t+1}} EC_f S_j \\ &= 2^t(2^{2s-1} - 2^{s-1}). \end{aligned}$$

For $1 \le k \le 2^{s+t+1}$, let C_k be an edge cut of $P_{2^{s+t}}$ such that C_k disconnects $P_{2^{s+t+1}}$ into two components P_{k1} and P_{k2}. It is apparent that P_{kl} is symmetric about $l = 2^{s+t}$. So we need only consider the case for $1 \le l \le 2^{s+t}$ in computing the wirelength. Therefore, the sum congestion of E_1 is

$$\begin{aligned} EC_f(C_k) &= 2\sum_{k=1}^{2^{s+t}} S_k \\ &= 2(1 + 2 + \cdots + 2^{s+t} - 1) \\ &= 2^{s+t} \cdot \left(2^{s+t} - 1\right). \end{aligned}$$

Thus,

$$\begin{aligned}WL(EH_{s,t}, P_{2^{s+t+1}}) &= WL_f(EH_{s,t}, P_{2^{s+t+1}})\\ &= WL_f(A_i) + WL_f(B_j) + WL_f(C_k)\\ &= \sum_{i=1}^{2^{s+t}} A_i + \sum_{j=2^{s+t}+1}^{2^{s+t+1}} B_j + 2\sum_{k=1}^{2^{s+t}} C_k\\ &= 2^s\left(2^{2t-1} - 2^{t-1}\right) + 2^t\left(2^{2s-1} - 2^{s-1}\right) + 2^{s+t}\cdot(2^{s+t} - 1)\\ &= 2^{s+2t-1} + 2^{t+2s-1} + 2^{2(s+t)} - 2^{s+t+1}.\end{aligned}$$

□

4.2 Embedding the exchanged hypercube into a grid

In this section, we embed $EH_{s,t}$ into a grid with minimum wirelength. The proposed embedding of $EH_{s,t}$ into $P_{2^{s+t+1}}$ in Sect. 4.1 is actually an embedding of $EH_{s,t}$ into the special grid, which is a $1\times(s+t+1)$ grid. In the following, we will give an embedding of $EH_{s,t}$ into grid $M(2^{\lfloor(s+t+1)/2\rfloor}, 2^{\lceil(s+t+1)/2\rceil})$ with minimum wirelength. Firstly, the definition of grid is given as follows:

Notation 1 An $m\times n$ grid $M(m,n)$ is denoted by an $m\times n$ matrix

$$\begin{pmatrix}\alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n}\\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n}\\ \cdots & \cdots & \cdots & \cdots\\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn}\end{pmatrix},$$

where $V(M) = \{\alpha_{ij}\,|1\le i\le m$, and $1\le j\le n\}$, $\left(\alpha_{i,j},\alpha_{i,j+1}\right)\in E(M)$ for $1\le i\le m$ and $1\le j\le n-1$, and $\left(\alpha_{kl},\alpha_{k+1,l}\right)\in E(M)$ for $1\le k\le m-1$ and $1\le l\le n$. $\langle\alpha_{11},\alpha_{12},\ldots,\alpha_{1n}\rangle$ and $\langle\alpha_{m1},\alpha_{m2},\ldots,\alpha_{mn}\rangle$ are called the row borders, while $\langle\alpha_{11},\alpha_{21},\ldots,\alpha_{m1}\rangle$ and $\langle\alpha_{1n},\alpha_{2n},\ldots,\alpha_{mn}\rangle$ are called the column borders.

Definition 7 Let $\pi : V(EH_{s,t}) \to V(M(2^{\lfloor\frac{s+t+1}{2}\rfloor}, 2^{\lceil\frac{s+t+1}{2}\rceil})\})$ be an embedding, which is defined as follows: The first row is labeled from 1 to $2^{\lceil n/2\rceil}$ from top to bottom. The ith row is labeled as $(i-1)2^{\lfloor\frac{s+t+1}{2}\rfloor}+1$, $(i-1)2^{\lfloor\frac{s+t+1}{2}\rfloor}+2,\ldots,i2^{\lfloor\frac{s+t+1}{2}\rfloor}$ from left to right where $i=1,2,\ldots,2^{\lceil\frac{s+t+1}{2}\rceil}$. Then, for any $v\in V(EH_{s,t})$, let $\pi(v)=lex(v)$.

Then, we first prove the edge congestion problem and the wirelength problem of $EH_{s,t}$ into a grid can be solved by using the embedding π. Next, we will give the embedding of $EH_{s,t}$ into the grid $M(2^{\lfloor\frac{s+t+1}{2}\rfloor}, 2^{\lceil\frac{s+t+1}{2}\rceil})$ with minimum wirelength, for $1\le s\le t$.

Theorem 6 Let $G = EH_{s,t}$ and $H = M(2^{\lfloor\frac{s+t+1}{2}\rfloor}, 2^{\lceil\frac{s+t+1}{2}\rceil})\}$, for $1\le s\le t$. Let $S_i = \{S_1, S_2,\ldots,S_p\}$ be p edge cuts of each column in $M(2^{\lfloor\frac{s+t+1}{2}\rfloor}, 2^{\lceil\frac{s+t+1}{2}\rceil})\}$, which consists of edges between the rows i and $i+1$ of $M(2^{\lfloor\frac{s+t+1}{2}\rfloor}, 2^{\lceil\frac{s+t+1}{2}\rceil})$, where $1\le$

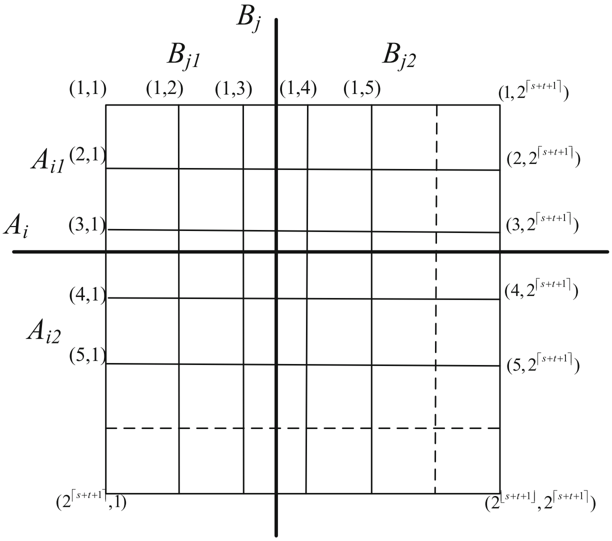


Fig. 3 Embedding $EH_{s,t}$ into $M(2^{\lfloor \frac{s+t+1}{2} \rfloor}, 2^{\lceil \frac{s+t+1}{2} \rceil})$

$p \leq 2^{\lceil \frac{s+t+1}{2} \rceil} - 1$, $1 \leq i \leq 2^{\lceil \frac{s+t+1}{2} \rceil} - 1$. Furthermore, let $f = \pi$. Then,

$$\sum_{i=1}^{2^{\lceil \frac{s+t+1}{2} \rceil}-1} EC_f(S_i) = 2^{\lfloor \frac{s+t+1}{2} \rfloor - 1} \left(2^{2t} - 2^t\right).$$

Proof Let H_{i1} and H_{i2} denote two connected components of $M(2^{\lfloor \frac{s+t+1}{2} \rfloor}, 2^{\lceil \frac{s+t+1}{2} \rceil}\} - S_i$, where $f(G_{i1}) = H_{i1}$ and $f(G_{i2}) = H_{i2}$, as depicted in Fig. 3. According to Theorem 2, the subgraph induced by $V(G_{i_1})$ is maximum. By Lemma 1, $EC_f(S_i)$ is minimum, $1 \leq j \leq 2^{\lceil \frac{s+t+1}{2} \rceil} - 1$. Thus, we have:

$$\begin{aligned}\sum_{i=1}^{2^{\lceil \frac{s+t+1}{2} \rceil}-1} EC_f(S_i) &= \sum_{i=1}^{2^{\lceil \frac{s+t+1}{2} \rceil}-1} EC_f(S_i)\\ &= \sum_{i=1}^{2^{\lceil \frac{s+t+1}{2} \rceil}-1} \lambda_G\left(i \cdot 2^{\lfloor \frac{s+t+1}{2} \rfloor}\right)\\ &= 2^{\lfloor \frac{s+t+1}{2} \rfloor - 1}\left(2^{2t} - 2^t\right).\end{aligned}$$

□

Theorem 7 For $1 \leq s \leq t$, $EH_{s,t}$ can be embedded into the grid $M(2^{\lfloor \frac{s+t+1}{2} \rfloor}, 2^{\lceil \frac{s+t+1}{2} \rceil})$ with minimum wirelength.

Proof Let $f = \pi$. Let A_{ij} be an edge cut of each column in the grid $M(2^{\lfloor \frac{s+t+1}{2} \rfloor}, 2^{\lceil \frac{s+t+1}{2} \rceil})$ such that A_{ij} disconnects $M(2^{\lfloor \frac{s+t+1}{2} \rfloor}, 2^{\lceil \frac{s+t+1}{2} \rceil})$ into two components A_{i1} and A_{i2}, $i = 1, 2, \ldots, 2^{\lceil \frac{s+t+1}{2} \rceil} - 1$. Furthermore, let B_{ij} be an edge cut of each row in the grid $M(2^{\lfloor \frac{s+t+1}{2} \rfloor}, 2^{\lceil \frac{s+t+1}{2} \rceil})$ such that B_{ij} disconnects $M(2^{\lfloor \frac{s+t+1}{2} \rfloor}, 2^{\lceil \frac{s+t+1}{2} \rceil})$ into two components B_{j1} and B_{j2}, $j = 1, 2, \ldots, 2^{\lfloor \frac{s+t+1}{2} \rfloor} - 1$. Let G_{i1} and G_{i2} be the inverse images of A_{i1} and A_{i2} under f, respectively. Then, the edge cut A_{ij} satisfies conditions (i) and (ii) of Lemma 1. Further by Theorem 2, the subgraph G_i induced by the vertices of A_{ij} is maximum. Thus, by Lemma 1, $EC_f(A_{ij})$ is minimum for $i = 1, 2, \ldots, 2^{\lceil \frac{s+t+1}{2} \rceil} - 1$. Let G_{j1} and G_{j2} be the inverse images of B_{j1} and B_{j2} under f, respectively. The edge cut B_{ij} satisfies Lemma 1. Further by Theorem 2, the subgraph G_j induced by the vertices of B_j is maximum. Thus, by Lemma 1, $EC_f(B_{ij})$ is minimum for $j = 1, 2, \ldots, 2^{\lfloor \frac{s+t+1}{2} \rfloor} - 1$. The Lemma 2 implies that the wirelength of this embedding is minimum. □

Theorem 8 Let $G = EH_{s,t}$ and $H = M(2^{\lfloor \frac{s+t+1}{2} \rfloor}, 2^{\lceil \frac{s+t+1}{2} \rceil})$, $1 \leq s \leq t$. Let $f = \pi$, and let $S_1, S_2, \ldots, S_p$ be p edge cuts of $M(2^{\lfloor \frac{s+t+1}{2} \rfloor}, 2^{\lceil \frac{s+t+1}{2} \rceil})$. Furthermore, let H_{j1} and H_{j2} denote two connected components of $M(2^{\lfloor \frac{s+t+1}{2} \rfloor}, 2^{\lceil \frac{s+t+1}{2} \rceil}) - S_j$, where $f(G_{j1}) = H_{j1}$ and $f(G_{j2}) = H_{j2}$. For any $1 \leq j \leq p$, if $EC_f(H_{j1})$ is minimum, then $f^{-1}(H_{j1})$ is a maximum subgraph in G.

Proof Suppose $EC_f(\delta_H(j_1))$ is minimum with $V(H_{j1}) = m$. We will prove that the subgraph induced by $G_{j1} = f^{-1}(H_{j1})$ is maximum on m vertices of $EH_{s,t}$. Supposing not, there exists $V(G'_{j1}) \subseteq V(EH_{s,t})$ such that $E(G_{j1}) < |E(G'_{j1})|$. By Lemma 1, $EC_f(\delta_H(j_1)) = nm - 2|E(G_{j1})| > nm - 2|E(G'_{j1})| = EC_f(\delta_H(f(G'_{j1})))$, which is a contradiction to our assumption. Thus, $EC_f(\delta_H(j_1))$ is minimum. Therefore, $f^{-1}(H_{j1})$ is a maximum induced subgraph of $EH_{s,t}$. The theorem follows. □

Theorem 9 The minimum wirelength of embedding $EH_{s,t}$ into grid $M(2^{\lfloor \frac{s+t+1}{2} \rfloor}, 2^{\lceil \frac{s+t+1}{2} \rceil})$, where $1 \leq s \leq t$ is given by

$$
\begin{aligned}
WL(EH_{s,t}, M(2^{\lfloor \frac{s+t+1}{2} \rfloor}, 2^{\lceil \frac{s+t+1}{2} \rceil}) &= 2^{\lfloor \frac{s+t+1}{2} \rfloor - 1}\left(2^{2t} - 2^t\right) \\
&\quad + 2^{\lfloor \frac{s+t+1}{2} \rfloor}\left(2^{2s} - 2^s\right) + 2^{s+t}.
\end{aligned}
$$

Proof Let $f : V(EH_{s,t}) \to V(M(2^{\lfloor \frac{s+t+1}{2} \rfloor}, 2^{\lceil \frac{s+t+1}{2} \rceil}))$ be the embedding π. Let $C_i = \{(\alpha_{i,j}, \alpha_{i,j+1}) | 1 \leq j \leq 2^{\lceil \frac{s+t+1}{2} \rceil}\}$, $1 \leq i \leq 2^{\lfloor \frac{s+t+1}{2} \rfloor}$. Let H_{i1} and H_{i2} denote two connected components of $M(2^{\lfloor \frac{s+t+1}{2} \rfloor}, 2^{\lceil \frac{s+t+1}{2} \rceil}) - C_i$, where $f(G_{i1}) = H_{i1}$ and $f(G_{i2}) = H_{i2}$. Then, we have $EC_f(C_i) = \sum_{i=1}^{2^{s+t}} C_i = 2^{s+t}$.

Let $R_{ij} = \{(\alpha_{i,j}, \alpha_{i,j+}) | 1 \leq i \leq 2^{\lfloor \frac{s+t+1}{2} \rfloor}\}$, $1 \leq j \leq 2^{\lceil \frac{s+t+1}{2} \rceil}$. Furthermore, let H_{j1} and H_{j2} denote two connected components of $M(2^{\lfloor \frac{s+t+1}{2} \rfloor} \times 2^{\lceil \frac{s+t+1}{2} \rceil}) - R_j$, where $f(G_{j1}) = H_{j1}$ and $f(G_{j2}) = H_{j2}$. Obviously, each edge of R_i has the same edge congestion. Thus, the sum of edge congestion of each column is equal. By

Lemma 1, G_{j1} is the inverse images of R_{j1} under the embedding f. Clearly, G_{j1} is a subcube induced by the vertices of H_j. By Theorem 2, it is certain that G_{j1} is a maximum induced subgraph of $EH_{s,t}$. Thus, by Lemma 1, $EC_f(R_{j1})$ is minimum for $j = 1, 2, \ldots, 2^{\lceil \frac{s+t+1}{2} \rceil}$. By Lemma 2, the sum of edge congestion of each column of H_{i1} is $2^{\lceil \frac{s+t+1}{2} \rceil - 1}(2^{2t} - 2^t)$.

Let $R^{'}_{ij} = \{(\alpha_{i,j}, \alpha_{i,j+1}) | 1 \leq i \leq 2^{\lfloor \frac{s+t+1}{2} \rfloor}\}, 1 \leq j \leq 2^{\lceil \frac{s+t+1}{2} \rceil}$. It is easy to verify the sum of edge congestion of each column is $2^t(2^{2s-1} - 2^{s-1})$. By Theorem 2, $G^{'}_{j1}$ is a maximum subgraph induced by $R^{'}_{j1}$. Thus, by Lemma 2, $EC_f(R^{'}_{j1})$ is minimum, where $j = 1, 2, \ldots, 2^{\lceil \frac{s+t+1}{2} \rceil}$. By Lemma 2, the sum of edge congestion of each row of $R^{'}_{j1}$ is $2^{\lfloor \frac{s+t+1}{2} \rfloor}(2^{2s} - 2^s)$.

By Lemma 2, the wirelength of embedding $EH_{s,t}$ into $M(2^{\lfloor \frac{s+t+1}{2} \rfloor}, 2^{\lceil \frac{s+t+1}{2} \rceil})$ is:

$$\begin{aligned} WL\left(EH_{s,t}, M\left(2^{\lfloor \frac{s+t+1}{2} \rfloor}, 2^{\lceil \frac{s+t+1}{2} \rceil}\right)\right) &= WL_f\left(EH_{s,t}, M\left(2^{\lfloor \frac{s+t+1}{2} \rfloor}, 2^{\lceil \frac{s+t+1}{2} \rceil}\right)\right) \\ &= \sum_{i=1}^{2^{\lceil \frac{s+t+1}{2} \rceil}} C_i + \sum_{j=1}^{2^{\lfloor \frac{s+t+1}{2} \rfloor}} R_j + \sum_{j=1}^{2^{\lfloor \frac{s+t+1}{2} \rfloor}} R^{'}_j \\ &= 2^{\lceil \frac{s+t+1}{2} \rceil - 1}(2^{2t} - 2^t) \\ &\quad + 2^{\lfloor \frac{s+t+1}{2} \rfloor}(2^{2s} - 2^s) + 2^{s+t}. \end{aligned}$$

This completes the proof. □

Let $t(N)$ denote the running time of Algorithm 1, and $N = 2^{s+t+1}$ is the number of vertices of $EH_{s,t}$. By Theorem 9, the number of edge cuts of each column is $(2^{\lceil \frac{s+t+1}{2} \rceil} - 1)$, and deleting each edge cut needs one time unit and thus deleting all edge cuts takes $(2^{\lceil \frac{s+t+1}{2} \rceil} - 1)$ time units. Consequently, the total time for embedding $EH_{s,t}$ into $M(2^{\lceil \frac{s+t+1}{2} \rceil} \times 2^{\lceil \frac{s+t+1}{2} \rceil})$ with minimum wirelength is $t = O(N + (2^{\lceil \frac{s+t+1}{2} \rceil} - 1) - 1 + 1) \leq O(N)$, which is linear.

5 VLSI layout for $EH_{s,t}$

In this section, we propose a VLSI layout of $EH_{s,t}$ into a two-dimensional grid with minimum number of tracks. A track is a continuous horizontal or vertical line on which the wires are placed without overlapping any other wires. To build integrated $EH_{s,t}$, it is necessary to work within a few layers of two-dimensional integrated circuits.

The wires can run either horizontally or vertically along grid lines. All vertices are placed on the same linear array in a collinear layout. We use bisection width to calculate the required number of tracks. Bisection width [40] is defined as the number of links interconnecting two subgraphs having the same number of vertices. The area of a layout is defined as the area of the smallest rectangle that contains all the nodes and wires. When there are two layers of wires, it is guaranteed that we can lay out the

Algorithm 1 Embedding $EH_{s,t}$ into $M(2^{\lfloor \frac{s+t+1}{2} \rfloor}, 2^{\lceil \frac{s+t+1}{2} \rceil})$, $1 \le s \le t$

Input: An exchanged hypercube $EH_{s,t}$ and a grid $M(2^{\lfloor \frac{s+t+1}{2} \rfloor}, 2^{\lceil \frac{s+t+1}{2} \rceil})$, $1 \le s \le t$.

Output: An embedding f of $EH_{s,t}$ into $M(2^{\lfloor \frac{s+t+1}{2} \rfloor}, 2^{\lceil \frac{s+t+1}{2} \rceil})$ with minimum wirelength.

Label the vertices of $EH_{s,t}$;
Set $count = 1$;
For each vertex in $u \in EH_{s,t}$, let the decimal value of $u = null$;
for $k = 1$ to 4 **do**
    for $i = 0$ to $2^{s-1} - 1$ **do**
        for $j = 1$ to $2^{t-1} - 1$ **do**
            $\text{num}(u_{t-1}^{m,i,j}) = count$;
            $count = count + 1$;
        end for
    end for
end for
for $k = 5$ to 8 **do**
    for $i = 0$ to $2^{t-1} - 1$ **do**
        for $j = 0$ to $2^{s-1} - 1$ **do**
            $\text{num}(u_{s-1}^{n,i,j}) = count$;
            $count = count + 1$;
        end for
    end for
end for
for j=0 to $2^{s+t+1} - 1$ **do**
    Label the ith row of $M(2^{\lfloor \frac{s+t+1}{2} \rfloor}, 2^{\lceil \frac{s+t+1}{2} \rceil})$ as $(i-1)2^{\lfloor \frac{s+t+1}{2} \rfloor} + 1, (i-1)2^{\lfloor \frac{s+t+1}{2} \rfloor} + 2, \ldots, i2^{\lfloor \frac{s+t+1}{2} \rfloor}$ from left to right where $i = 1, 2, \ldots, 2^{\lceil \frac{s+t+1}{2} \rceil}$.
end for
return f

network within the area. More precisely, it only uses one layer of wires to lay out all the horizontal segments of wires and the other layer to lay out all the vertical segments.

The first layout is to assign the vertices of Q^t with 2^s copies in the same line. By extending the horizontal space, we can assign the vertices of Q^s in the second layout which is adjacent to the vertices of Q^t.

Theorem 10 Let $EH_{s,t}$ be an exchanged hypercube for $1 \le s \le t$. The number of tracks required for the collinear layout of $EH_{s,t}$ is given by

$$t_{\text{num}}(EH_{s,t}) = \begin{cases} 2^t + \lfloor \frac{2^t}{3} \rfloor, & s < t, \\ 2^{s+t+1} + 2^{s+t} - t \cdot 2^s - s \cdot 2^t, & s = t. \end{cases}$$

Proof All vertices are placed on the same line in a collinear layout. To describe the layout of $EH_{s,t}$, we use a bottom-up approach, starting with $EH(1,1)$ and inductively moving to $EH_{s,t}$ of higher dimensions. See Fig. 4. A collinear layout of $EH(1,1)$ can be obtained by placing the eight nodes on a linear array, connecting vertex 0 with vertex 1, vertex 2 with vertex 3, vertex 4 with vertex 6, through wires in the first track, and then connecting vertex 0 with vertex 4, and vertex 5 with vertex 7, through wires in the second track. Then, connecting vertex 1 with vertex 5, vertex 2 with vertex 6,

Algorithm 2 Collinear layout of $EH_{s,t}$

Input: The exchanged hypercube $EH_{s,t}(1 \leq s \leq t)$ and grid $M(p,q)$, with $p = 2^{\lfloor \frac{s+t+1}{2} \rfloor}, q = 2^{\lceil \frac{s+t+1}{2} \rceil}$.
Output: A collinear layout h of $EH_{s,t}$ into $M(p,q)$ with minimum tracks.

For each vertex in $M(p,q)$;
for i=0 to 2^{s+t+1} **do**
  if $s < t$ **then**
    Assign the ith vertex of the second layout adjacent to the ith vertex of the first layout in horizonal direction. Connect the ith vertex of the second layout adjacent to the ith vertex of the first layout. The links in each Q_s could share tracks with Q_t.
  else
    Divide each Q_s into two equal sub-subcubes with 2^{t-1} vertices, and the obtain bisection width 2^{t-2} of this division. Double the number of tracks to connect the 2^t copies Q_s and 2^s copies Q_t into an $EH_{s,t}$.
  end if
end for
return h

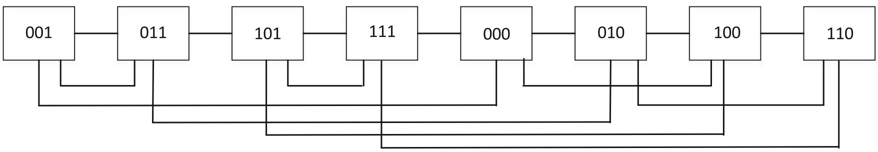


Fig. 4 Collinear layouts of $EH_{1,1}$

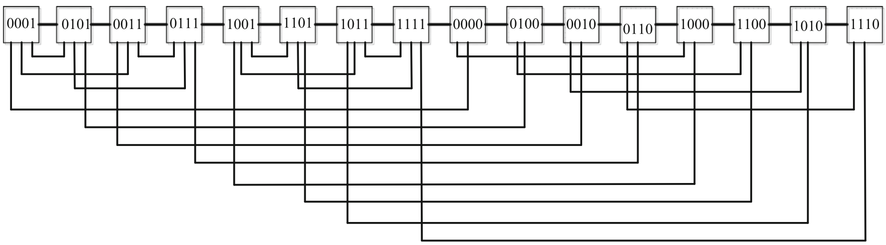


Fig. 5 VLSI physical design for the connection pattern of $EH_{1,2}$

and vertex 3 and vertex 7 in turn. Clear, this layout requires five tracks. For $1 \leq s \leq t$, we have the following cases:

Case 1. $s < t$. Let I_t^i denote the subgraph induced by the vertices in Q_t^i and all the vertices in E_1, $1 \leq i \leq 2^s$. To obtain the collinear layout of I_t^i, we start with the layouts of two subgraphs $\bigcup_{i=1}^{2^s} I_t^i$ and $\bigcup_{i=1}^{2^s} I_t^{i+1}$. By doubling the horizontal space, we can place the ith vertex of the second layout adjacent to the ith vertex of the first layout. For the collinear layout of Q_s, the links in it could share tracks with I_t^i, such that it will not need extra tracks. See Fig. 5. Since $EH_{s,t}$ and $EH_{t,s}$ are isomorphic, the same holds for subgraphs $\bigcup_{i=1}^{2^t} I_s^i$ and $\bigcup_{i=1}^{2^t} I_s^{i+1}$. According to the result in [17], the track number of Q_t into L_{2^t} is $\lfloor \frac{2^t}{3} \rfloor$ and the required track number of E_1 is 2^t.

Since $EH_{s,t}$ can be decomposed into 2^{s-1} copies Q_t^i, the total number of tracks for collinear layout of $EH_{s,t}$ is $t_{num}(EH_{s,t}) = 2^t + \lfloor \frac{2^t}{3} \rfloor$.

Case 2. $s = t$. Since $EH_{s,t}$ is Hamiltonian [29], we can construct a hamiltonian path in $EH_{s,t}$, and let this path be a base track. By Definition 3, from the center of the base track, $EH_{s,t}$ can be divided into two subcubes $EH_{s-1,t}$ with 2^{s+t} vertices. Then, the bisection width of the first partition is 2^{s+t}. After deleting the edge set E_1 from $EH_{s,t}$, the vertex set of $EH_{s,t}$ is separated into two parts T and S, where T is the set of all vertices with rightmost 0th bit being 1, and S is the set of all vertices with rightmost 0th bit being 0. Thus, the vertex set S is decomposed into 2^t connected components. Each component is an s-dimensional hypercube Q_s; moreover, these 2^t hypercubes Q^s are pairwise disjoint, and there are no edges joining any two Q_s. For 2^s copies Q_t, we continue to divide each subcube into two equal sub-subcubes with 2^{t-1} vertices, and the bisection width of this division is 2^{t-2}. Repeat this division t times. Then, the number of tracks is given by $t(Q_t) = \sum_{i=1}^{2^t} b_i - 1$. The same holds for 2^t copies Q_s, which denoted as $t(Q_s) = \sum_{j=1}^{2^s} b_j - 1$. Thus, the required number of tracks for $EH_{s,t}$ can be obtained by summing the bisection width in each procedure. Based on the above division, it can be obtained as follows: It needs one track for constructing the Hamiltonian path. Then, the first bisection needs 2^{s+t} tracks, the second bisection needs 2^{t-1}(resp. 2^{s-1}) tracks,..., the $(n-1)$th bisection needs $2 \cdot 2^1 - 1$ tracks, and the nth bisection needs $2^1 - 1$ tracks.

Thus, the required number of tracks is,

$$
\begin{aligned}
t_{\text{num}} &= 1 + 2^{s+t} - 1 + 2^s \left(\sum_{i=1}^{2^t} b_i - 1 \right) + 2^t \left(\sum_{j=1}^{2^s} b_j - 1 \right) \\
&= 2^{s+t} + 2^s \left(\sum_{i=1}^{2^t} 2^{i-1} - 1 \right) + 2^t \left(\sum_{j=1}^{2^s} 2^{j-1} - 1 \right) \\
&= 2^{s+t} + \left(2^{s+t}\right) - t \cdot 2^s + \left(2^{s+t} - s \cdot 2^t\right) \\
&= 2^{s+t+1} + 2^{s+t} - t \cdot 2^s - s \cdot 2^t.
\end{aligned}
$$

□

Theorem 11 Let $EH_{s,t}$ be an exchanged hypercube for $1 \leq s \leq t$ and A denote the VLSI layout area for $EH_{s,t}$. Then,

$$
A = \begin{cases} 2^{\lfloor \frac{s+t}{2} \rfloor + \lceil \frac{s+t+1}{2} \rceil} \left(\lfloor \frac{2^{t+1}}{3} \rfloor + \lfloor \frac{2^{s+1}}{3} \rfloor \right) \left(\lfloor \frac{2^{s+1}}{3} \rfloor + 1 \right), & s < t, \\ 2^{\lceil \frac{s+t+1}{2} \rceil + \lfloor \frac{s+t+1}{2} \rfloor} \left(\lfloor \frac{2^{t+1}}{3} \rfloor + \lfloor \frac{2^{s+1}}{3} \rfloor \right), & s = t. \end{cases}
$$

Proof Let $f = lex$. The layout $EH_{s,t}$ on a two-dimensional grid is performed by Algorithm 2. We use W and H to denote the numbers of vertical and horizontal

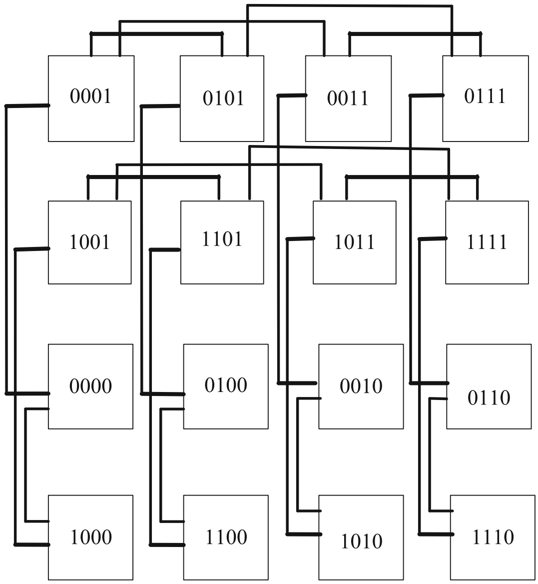


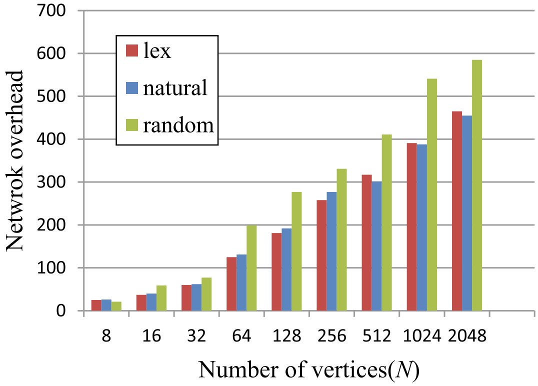
Fig. 6 A 2-D layout of $EH_{1,2}$

tracks, respectively, i.e., the width and the height of a layout. See Fig. 6. Let $C_{ij} = \{(\alpha_{i,j}, \alpha_{i,j+1}) | 1 \leq j \leq 2^{\lfloor \frac{s+t+1}{2} \rfloor}\}$ and $R_{ij} = \{(\alpha_{i,j}, \alpha_{i+1,j}) | 1 \leq i \leq 2^{\lceil \frac{s+t+1}{2} \rceil}\}$, where $1 \leq i \leq 2^{\lceil \frac{s+t+1}{2} \rceil}$ and $1 \leq j \leq 2^{\lfloor \frac{s+t+1}{2} \rfloor}$. Clearly, R_{ij} is an edge cut of each column in the grid $M(2^{\lfloor \frac{s+t+1}{2} \rfloor}, 2^{\lceil \frac{s+t+1}{2} \rceil})$ such that R_i disconnects $M(2^{\lfloor \frac{s+t+1}{2} \rfloor}, 2^{\lceil \frac{s+t+1}{2} \rceil})$ into two components R_1 and R_2 where $V(R_1) = \{0, 1, \ldots, 2^{s+t}\}$ and $V(R_2) = \{2^{s+t} + 1, 2^{s+t} + 2, \ldots, 2^{s+t+1}\}$. We have the following two cases to allocate the vertices on a grid.

Case 1. $s < t$. Let C_{ij} be an edge cut of each column in the grid $M(2^{\lfloor \frac{s+t+1}{2} \rfloor}, 2^{\lceil \frac{s+t+1}{2} \rceil})$ such that C_{ij} disconnects $M(2^{\lfloor \frac{s+t+1}{2} \rfloor}, 2^{\lceil \frac{s+t+1}{2} \rceil})$ into two components C_{i1} and C_{i2}, $i = 1, 2, \ldots, 2^{\lceil \frac{s+t+1}{2} \rceil} - 1$. Let each row in C_{i1} be the image of Q_t. By Lemma 1, $EC_f(R_1)$ is minimum. Thus, the required number of tracks for each row in C_{i1} is $\lfloor \frac{2^{t+1}}{3} \rfloor$. So the area for rows is $H = (\lfloor \frac{2^{t+1}}{3} \rfloor + \lfloor \frac{2^{s+1}}{3} \rfloor) 2^{\lfloor \frac{s+t}{2} \rfloor}$. The same holds for columns of $M(2^{\lfloor \frac{s+t+1}{2} \rfloor}, 2^{\lceil \frac{s+t+1}{2} \rceil})$. The required number of tracks for each column is $\lfloor \frac{2^{s+1}}{3} \rfloor + 1$. Thus, the area for columns is $W = 2^{\lceil \frac{s+t+1}{2} \rceil}(\lfloor \frac{2^{s+1}}{3} \rfloor + 1)$. Hence, the whole area of the grid is $W \times H = 2^{\lfloor \frac{s+t}{2} \rfloor + \lceil \frac{s+t+1}{2} \rceil}(\lfloor \frac{2^{t+1}}{3} \rfloor + \lfloor \frac{2^{s+1}}{3} \rfloor)(\lfloor \frac{2^{s+1}}{3} \rfloor + 1)$.

Case 2. $s = t$. Let R_{ij} be an edge cut of each row in the grid $M(2^{\lfloor \frac{s+t+1}{2} \rfloor}, 2^{\lceil \frac{s+t+1}{2} \rceil})$ such that R_{ij} disconnects $M(2^{\lfloor \frac{s+t+1}{2} \rfloor}, 2^{\lceil \frac{s+t+1}{2} \rceil})$ into two components R_{j1} and R_{j2}, $j = 1, 2, \ldots, 2^{\lfloor \frac{s+t+1}{2} \rfloor} - 1$. The sum of edge congestion of each column in R_{j1}(resp. R_{j2}) is equal. Let each row in R_{j1}(resp. R_{j2}) be the image of Q_t(resp. Q_s). By Lemma 1, $EC_f(R_{j1})$(resp. R_{j2}) is minimum. Thus, the required number of tracks for each row in R_{j1} is $\lfloor \frac{2^{t+1}}{3} \rfloor$ and for each row in R_{j2} is $\lfloor \frac{2^{s+1}}{3} \rfloor$. So the area for rows is

Fig. 7 Network overhead of three embedding schemes



$H = (\lfloor \frac{2^{t+1}}{3} \rfloor + \lfloor \frac{2^{s+1}}{3} \rfloor) 2^{\lfloor \frac{s+t+1}{2} \rfloor}$. Thus, the area for columns is $W = 2^{\lceil \frac{s+t+1}{2} \rceil}$. Hence, the required whole area for the grid is $W \times H = 2^{\lceil \frac{s+t+1}{2} \rceil + \lfloor \frac{s+t+1}{2} \rfloor} (\lfloor \frac{2^{t+1}}{3} \rfloor + \lfloor \frac{2^{s+1}}{3} \rfloor)$. □

6 Simulation and experiments

We have carried out experimental testing to verify that the proposed embedding performs better than another two embeddings. And we use a server with Nvidia GTX 1060 GPU, Intel Xeon E5-2670 as our experimental platform, the Intel processor with 16 processors running at 3.3 GHz. This server has 3 TB disk, 64GB physical memory and runs on Windows Server 2008 R2 Enterprise operating system.

Network overhead is the most crucial factor to measure an interconnection network. Network overhead is the usage rate parameter volume for different resources, and the definition is as follows:

$$\text{volume} = \frac{1}{1 - \text{mem}} * \frac{1}{1 - \text{cpu}} * \frac{1}{1 - \text{net}}.$$

In the formula, *mem* represents the memory utilization rate of the server, *cpu* indicates the utilization rate of CPU and *net* represents the bandwidth utilization ratio.

With the increment of the interconnection network scale, the delay of message passing seriously affects the communication efficiency between nodes. Particularly the redundant search messages will increase in an exponential way, which would seriously influence the efficiency of the interconnection network search schemes. Congestion and dilation directly affect the queuing delay of messages and communication delay in the embedding process. In the process of executing the algorithms, we monitor the status of server with Ganglia [41]. We analyze the algorithm's network overhead by monitoring the usage state of resources.

We compare our *lex* embedding algorithm with natural embedding [42] and random embedding [43], respectively. The natural embedding (short for natural)is a bijection $f : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$ such that the natural numberings of vertices increase one

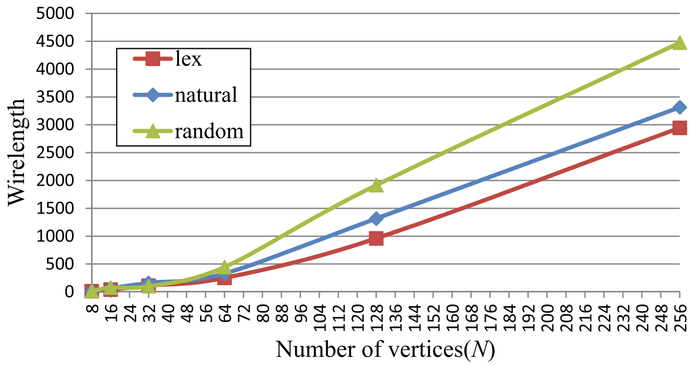


Fig. 8 Comparison of three embedding schemes

by one, and the random embedding (short for random) is the bijection $f : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$ is random.

Figure 7 illustrates the network overhead of three embedding schemes. When the number of nodes is less than 32, the overhead of the three algorithms is relatively close. As the number of nodes increases, the random overhead becomes larger than the other two algorithms. Due to the random mapping of nodes, the congestion and dilation of some links become quite large. This will increase the communication overhead.

As shown in Fig. 8, the lex embedding induces the lower wirelength compared with the other two embeddings. As the number of nodes increases, lex embedding has better performance than natural embedding.

Experimental results show that natural embedding is more suitable for regular networks, such as hypercube. With the increase in network size, the sum of the edge congestion would keep a uniform increase. However, natural embedding is not suitable for exchange hypercube. Due to its irregularity, it will cause local congestion and eventually lead to layout failure. Obviously, random embedding is the worst embedding because of the maximum wirelength required. That is mainly due to the randomness of the mapping. It not only requires a large wirelength, but also causes massive network overhead.

7 Conclusions

In this paper, we propose embeddings of exchanged hypercube into a grid. Firstly, we prove that exchanged hypercube can be embedded into a linear array with minimum wirelength and obtain the exact wirelength. Furthermore, we obtain the minimum wirelength of embedding exchanged hypercube into a grid and prove that the embedding algorithm is linear. Finally, we extend the embedding of exchanged hypercube into a grid with efficient VLSI layout area. To the best of our knowledge, this is the first result for layout exchanged hypercube into a grid.

Acknowledgements We would like to express our sincerest appreciation to Prof. Guoliang Chen for his constructive suggestions. This work is supported by National Key R&D Program of China (No.

2018YFB1003201), Natural Science Foundation of China under Grant (Nos. 61572337, 61602333, 61672296, 61702351 and 61872196), Scientific & Technological Support Project of Jiangsu Province (No. BE2016777, BE2016185), Natural Science Foundation of the Jiangsu Higher Education Institutions of China (Nos. 17KJB520036 and 18KJA520009) and Jiangsu High Technology Research Key Laboratory for Wireless Sensor Networks Foundation (No. WSNLBKF201701).

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