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Wavelength Assignment for Realizing Parallel FFT on Regular Optical Networks

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Abstract. Routing and wavelength assignment (RWA) is a central issue to increase efficiency and reduce cost in Wavelength Division Multiplexing (WDM) optical networks. In this paper, we address the problem of wavelength assignment for realizing parallel FFT on a class of regular optical WDM networks. We propose two methods for sequential mapping and shift-reversal mapping of FFT communication pattern to the optical WDM networks concerned. By sequential mapping, the numbers of wavelengths required to realize parallel FFT with 2^n nodes on WDM linear arrays, rings, 2-D meshes and 2-D tori are 2^{n-1}, 2^{n-1}, $2^{\max(k,n-k)-1}$ and $2^{\max(k,n-k)-1}$ respectively. By shift-reversal mapping, the numbers of wavelengths required are $\max(3 \times 2^{n-3}, 2)$, 2^{n-2}, $\max(3 \times 2^{\max(k,n-k)-3}, 2)$ and $2^{\max(k,n-k)-2}$. These results show that shift-reversal mapping outperforms sequential mapping. Our results have a clear significance for applications because FFT represents a common computation pattern shared by a large class of scientific and engineering problems and WDM optical networks as a promising technology in networking has an increasing popularity.

Keywords: parallel FFT, wavelength assignment, optical networks, Wavelength Division Multiplexing (WDM), network embedding

1. Introduction

Fast Fourier Transform (FFT) plays an important role in various scientific and technical applications including image processing, communications, cellular phones and digital control systems [1]. While the application fields of FFT are growing rapidly, the amount of data to be transformed is also increasing tremendously. Hence, there has been a great interest in implementing FFT on parallel computers and some parallel computers have been specially designed to perform FFT computations [2, 3, 6, 9, 13]. With the increasing computation power of parallel computers, interprocessor communication has become an important factor that limits the performance of supercomputing systems. Optical communication, in particular, Wavelength Division Multiplexing (WDM) technique, has become a promising technology for many emerging networking and parallel/distributed computing applications because of its huge bandwidth [15]. In [7], parallel computing using optical interconnection is introduced.

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Due to its topological properties, parallel FFT is often implemented on dense interconnection networks such as hypercube and shuffle-exchange networks [6], instead of simple connected networks such as linear arrays and rings. Since WDM divides the bandwidth of an optical fiber into multiple wavelength channels so that multiple devices can transmit on distinct wavelengths through the same fiber concurrently, these dense networks can be simplified to simple regular topologies by realizing connections of parallel FFT in optical lightpaths.

The problem of routing and wavelength assignment (RWA) is critical for increasing the efficiency of wavelength-routed all-optical networks [10]. Commercial wavelength-division multiplexers combining up to 16 wavelengths were introduced in 1996, and 40-chanel systems were made available in 1998. Recent laboratory experiments have achieved 1000 channels or more per fiber. How to make full use of these channels efficiently has attracted a lot of attentions. A review of routing and wavelength assignment approaches for wavelength-routed optical WDM networks was given in [16]. In [12], multicasting in multi-hop optical WDM networks with limited wavelength conversion was surveyed. In [17], some results in wide-sense nonblocking multicast in a class of regular optical WDM networks were given. In [14], optimal routing and channel assignments for hypercube communication on optical mesh–like processor arrays were studied. In [11], off-line permutation embedding and scheduling in multiplexed optical networks with regular topologies were discussed.

In this paper, we study the wavelength assignment problem for realizing parallel FFT on a class of regular optical WDM networks including linear arrays, rings, 2-D meshes and 2-D tori. We derive the numbers of wavelengths required to embed the parallel FFT communication pattern on the optical networks by sequential mapping and shift-reversal mapping. Our results show that shift-reversal mapping outperforms sequential mapping. A preliminary version of this paper appeared in [8].

The rest of this paper is organized as follows. In Section 2, we provide necessary background on wavelength assignment on optical WDM networks and parallel FFT. We then define the problem and provide some results of wavelength assignment for realizing parallel FFT on linear arrays, rings, meshes and tori by sequential mapping and shift-reversal mapping in Sections 3. In Sections 4, comparisons between the two embeddings are given. Finally, we conclude the paper in Section 5.

2. Preliminaries

In this section, we introduce some general concepts and definitions that are used in this paper.

2.1. Optical WDM networks

Optical WDM networks are widely regarded as the best choice for providing the huge bandwidth required by future networks [15, 16]. Wavelength Division Multiplexing (WDM) divides the bandwidth of an optical fiber into multiple wavelength channels, so that multiple users can transmit at distinct wavelength channels through the same fiber concurrently. To efficiently utilize the bandwidth resources and eliminate the high cost and bottleneck caused by optoelectronic conversion and processing at intermediate nodes, end-to-end lightpaths are usually set up between each pair of source-destination nodes. A connection or a lightpath in a WDM network is an ordered pair of nodes (x,y) corresponding to that a packet is sent from source x to destination y. There are two approaches for establishing a connection in a network whose links are multiplexed with virtual channels. One is called *path multiplexing* (PM), in which the same wavelength has to be used on each link along a path, and the other is called *link multiplexing* (LM), in which different wavelengths may be used in a path [16]. In this paper, we assume that no wavelength converter facility is available in the network. Thus, a connection must use the same wavelength throughout its path by PM. In this case, the lightpath is said to satisfy the *wavelength-continuity constraint*.

2.2. Wavelength assignment

Given a physical network structure and the required connections, the problem of routing and wavelength assignment (RWA) is to select a suitable path and wavelength among many possible choices for each connection so that no two paths sharing a link are assigned the same wavelength. RWA tries to minimize the number of channels to realize a communication requirement by taking into consideration both routing options and channel assignment options which can be described as follows [14]. Given a set of all–optical connections, the problem is to (a) find routes from the source nodes to their respective destinations, and (b) assign channels to these routes so that the same channel is assigned to all the links of a particular route. (c) The goal of RWA is to minimize the number of assigned channels.

Numerous research studies have been conducted on the RWA problem [10, 11, 14, 16, 17]. A popular approach to tackle this problem is to apply integer programming technique [10], which, however, does not lead to efficient solution for special cases. In this paper, we discuss the RWA problem of realizing parallel FFT on a class of regular optical WDM networks.

2.3. Parallel FFT

The FFT developed by Cooley and Tukey [5] in the mid-60s is a method of computing the discrete Fourier transform which reduces the number of operations for an N-point complex vector from $O(N^2)$ to $O(N \log_2 N)$. The data-flow graph induced by an N-point FFT computation is usually described by means of the so-called *butterfly representation* [9]. The butterfly representation of FFT is a diagram made up of blocks representing identical computational units (*butterflies*) connected by arrows that show the flow of data between the blocks. Assuming that N is the length of the sequence to be transformed (N is an integer power of two), then the diagram with $N(\log_2 N+1)$ nodes arranged in N rows and $\log_2 N+1$ columns is made of $\log_2 N$stages of $N/2$ butterflies each. The nodes in column 0 are the problem inputs and those in column $\log_2 N$ are the outputs. Each non-input nodes represents an atomic computation [3]. Figure 1 shows the butterfly representation of an 8-point FFT. FFT can be easily implemented on a butterfly. In fact, the butterfly was first defined for the purpose of implementing FFT, and it is often referred to as the FFT network.

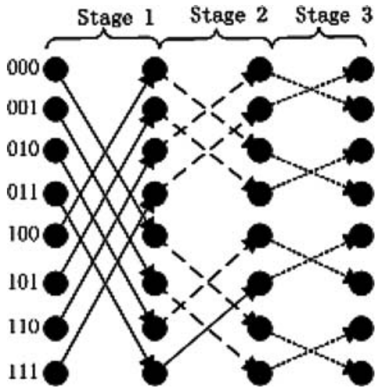


Figure 1. Butterfly computations of an 8-point FFT.

It is well known that FFT uses a very regular structure and most of the FFT algorithms are designed based on the butterfly computation. The butterfly representation of FFT computation pattern clearly shows the great potential of FFT for parallel processing. Generally, the FFT is implemented stage by stage, i.e. any stage of calculation cannot proceed until all the results of its previous stage have been completed. In this way, all operations within one stage can be performed in parallel and all the stages have to be handled sequentially. We call this form of implementation parallel FFT. In this paper, we consider one dimensional data sequence of size $N = 2^n$. If each data is assigned a binary representation, the communications during the ith $(1 \leq i \leq n)$ stage of the butterfly must take place between the nodes whose binary representations differ in the ith position [13]. As shown in Figure 1, processor 000 communicates with processor 100 in the first stage, with processor 010 in the next stage, and finally with processor 001 in the last stage. Obviously, if the butterfly representation is viewed as a process graph, i.e. each row of the butterfly is implemented by a process and each arrow by a communication channel, the butterfly communication pattern can map onto a WDM hypercube perfectly those links connecting the nodes having an address that differs by only one bit at each stage. However, if a WDM hypercube is used, only the ith dimensional links are used with one wavelength during the ith stage whereas other $(n-1) \times 2^{n-1}$ links are vacant during this stage, which may lead to wasting of wavelength channels.

As we know, a connection in the hypercube is called a *dimensional i connection* [14] if it connects two nodes that differ in the ith bit position, where $1 \leq i \leq n$. In a network of size 2^n, the set DIM_i is defined as the set of all dimension i connections and H_n is defined as the hypercube communication pattern which contains all connections in the hypercube. That is, $H_n = \bigcup_{i=1}^{n} DIM_i$ and $DIM_i = \{(j, j + (-1)^{\lfloor j/2^{n-i} \rfloor} \times 2^{n-i}) | 0 \leq j \leq 2^n - 1\}$.

With 2^n input data distributed on 2^n processors, the set of all communications during n stages of parallel FFT is equivalent to H_n, and the set of communications during the ith stage is equivalent to DIM_i. Clearly, parallel FFT has a regular communication pattern which we denote by $\mathrm{FFT}_n (n \geq 2)$. In fact, even if the number of processors is less than the number of input data and each processor is allocated a group of input data, the inter-processor communication pattern has the same regularity as FFT_n.

3. Wavelength assignment for realizing parallel FFT on optical networks

3.1. Problem definition

We model a network as a directed graph $G(V, E)$. Nodes in V are switches and edges in E are links. In general, an optical WDM network consists of routing nodes interconnected by point-to-point fiber links, which can support a certain number of wavelengths. A lightpath is implemented by selecting a path of physical links between the source and destination nodes, and reserving a particular wavelength on each of these links for the lightpath. In this paper, we assume each link in the network is bidirectional and composed of a pair of unidirectional links with one link in each direction. For FFT_n, if $(x,y) \in \mathrm{FFT}_n$, then $(y, x) \in \mathrm{FFT}_n$. Assuming that these two communications can be realized by two lightpaths in the same path of opposite directions passing through different fiber links, the same wavelength can be assigned to these two lightpaths. In this case, we can ignore the problem of communication directions in FFT_n.

Since the n stages of parallel FFT communications should be implemented stage by stage, the number of wavelengths required to realize FFT_n on optical WDM networks is the maximum number among the wavelengths required by the n stages. Let $W_e(G', G)$ denote the number of wavelengths to realize communication pattern G' on network G by embedding scheme e. Thus, $W_e(\mathrm{FFT}_n, G) = \max_{1\le i\le n}(W_e(\mathrm{DIM}_i, G))$. In the following, W_s and W_r denote the numbers of wavelengths required by sequential mapping and shift-reversal mapping respectively.

3.2. Linear arrays

Assume that the nodes of WDM linear arrays are numbered from left to right in ascending order starting from 0, and that the links are numbered from left to right starting from 1. If the ith node of FFT_n is mapped onto the ith processor of the optical WDM networks, we call such an embedding *sequential mapping*.

Theorem 1 *By sequential mapping, the number of wavelengths required to realize* FFT_n *on a WDM linear array with* 2^n *nodes is* 2^{n-1}.

Proof: Sequential mapping of FFT_n to a linear array with 2^n nodes will result that the kth node on the linear array communicates with node $k+2^{n-1}$ for $0 \le k \le 2^{n-1}-1$ and with node $k-2^{n-1}$ for $2^{n-1} \le k \le 2^n - 1$. Therefore, there are $\min(i, 2^n - i)$ lightpaths passing through the ith link of the linear array, as illustrated in Figure 2(a). So, the maximum number of wavelengths required during the first stage is 2^{n-1} when $i = 2^{n-1}$. Similarly, it can be seen from Figure 2(a) that during the jth$(1 \le j \le n)$ stage the kth node on the linear array communicates with either node $k + 2^{n-j}$ or node $k - 2^{n-j}$. Thus, there are $\min(i - 2^{n+1-j}\lfloor \frac{i}{2^{n+1-j}} \rfloor, 2^{n+1-j}\lceil \frac{i}{2^{n+1-j}} \rceil - i)$ lightpaths passing through the ith link of the linear array. So, the maximum number of wavelengths required during the jth stage is 2^{n-j} when $i = m \times 2^{n+1-j} + 2^{n-j} (m = 0, 1, \ldots, 2^{j-1} - 1)$. Hence, the maximum number of wavelengths required for all the stages is 2^{n-1}, which happens on the 2^{n-1}th link during the first stage. That is, $W_s(\mathrm{FFT}_n, \mathit{line}) = 2^{n-1}$. □

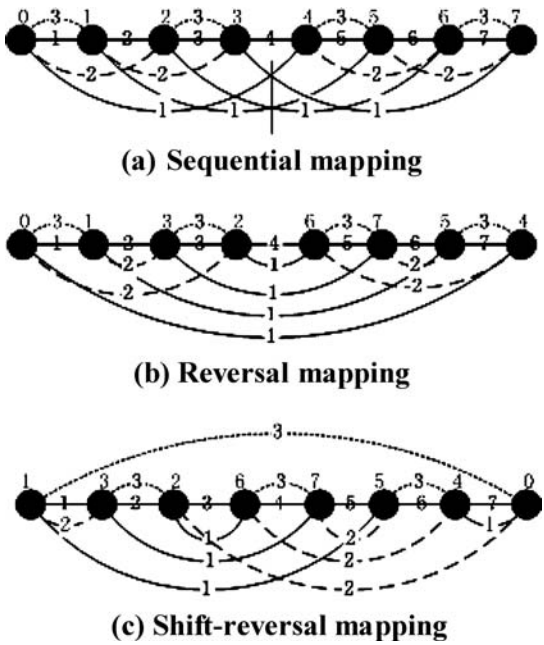


Figure 2. FFT_n embedded in 8-node linear array.

As shown in Figure 2(a), the maximum number of wavelengths required to realize FFT_3 on an 8-node linear array by sequential mapping is $2^2 = 4$.

Obviously, sequential mapping is not satisfactory because it requires a large number of wavelengths. In the following, we consider an alternative embedding, shift-reversal mapping, which is better than sequential mapping on the number of wavelengths.

At first, we introduce the definitions of reversal mapping and shift-reversal mapping.

Assume that X is an arrangement of binary representations, and X^{-1} is the reversal arrangement of these binary representations. For example, if $X = a, b, c, d$, then $X^{-1} = d, c, b, a$, and $(X_1X_2)^{-1} = X_2^{-1}X_1^{-1}$. The node arrangement of X_n is defined recursively as follows [4]:

$$
\begin{aligned}
X_1 &= 0, 1\\
X_2 &= 0X_1, (1X_1)^{-1} = 00, 01, (10, 11)^{-1} = 00, 01, 11, 10\\
&\ \ \vdots\\
X_n &= 0X_{n-1}, (1X_{n-1})^{-1}.
\end{aligned}
$$

We call X_n *reversal order* of 2^n nodes. For example, reversal order of 8 nodes $X_3 = 0X_2, (1X_2)^{-1} = 000, 001, 011, 010, 110, 111, 101, 100$.

If we move the node in position i of X_n towards its left (right) side to position $i-1$ $(i+1)$ of X_n, and node 0 (n) to position n (0), we define these movements left-shift (right-shift) operation on X_n. If we implement left-shift (right-shift) operations for $2^{n-3}(n \geq 3)$ times, we call the obtained arrangement *left shift-reversal order* (*right shift-reversal order*). Due to the symmetry between left shift-reversal order and right shift-reversal order, these two orders are equivalent for the results in this paper. In the following, we only use *shift-reversal order* to represent one of left shift-reversal order and right shift-reversal order. In addition, we assume that shift-reversal order is identical with its corresponding reversal order when $n = 2$.

If we map the ith node of reversal order or shift-reversal order of FFT_n onto the ith processor of the WDM network G, we establish the 1-1 mapping from the nodes of FFT_n to the nodes of G. We call these two embeddings *reversal mapping* and *shift-reversal mapping*.

Theorem 2 *By shift-reversal mapping, the number of wavelengths required to realize* FFT_n *on a WDM linear array with* 2^n *nodes is* $\max(3 \times 2^{n-3}, 2)$.

Proof: When $n = 2$, the nodes of 0, 1, 3 and 2 of FFT_2 are mapped onto the nodes of 0, 1, 2 and 3 on the linear array. Clearly, the number of wavelengths required is 2.

When $n \geq 3$, we consider reversal mapping firstly. By reversal mapping, during the first stage, the kth node on the linear array communicates with node $2^n - k - 1$. Therefore, there are $\min(i, 2^n - i)$ lightpaths passing through the ith link of the linear array. So, the maximum number of wavelengths required during the first stage is 2^{n-1} when $i = 2^{n-1}$. Similarly, during the jth $(1 \leq j \leq n)$ stage, there are $\min(i - 2^{n+1-j}\lfloor \frac{i}{2^{n+1-j}} \rfloor, 2^{n+1-j}\lceil \frac{i}{2^{n+1-j}} \rceil - i)$ lightpaths passing through the ith link of the linear array. So, the maximum number of wavelengths required during the jth stage is 2^{n-j} when $i = m \times 2^{n+1-j} + 2^{n-j} (m = 0, 1, \ldots, 2^{j-1})$. Therefore, the number of wavelengths required by reversal mapping is 2^{n-1}, which is the same as that by sequential mapping, as shown in Figure 2(b). □

Next, we consider shift-reversal mapping on a linear array with 2^n nodes. Based on reversal mapping, implement the shift operations for 2^{n-3} times so that $\mathrm{FFT}_n (n \geq 3)$ is embedded in a linear array by shift-reversal mapping. After the shift operations, the kth node on the linear array during the first stage communicates with node $3 \times 2^{n-2} - 1 - k$ when $0 \leq k \leq 3 \times 2^{n-3} - 1$ and with node $7 \times 2^{n-2} - 1 - k$ when $3 \times 2^{n-2} \leq k \leq 7 \times 2^{n-3} - 1$, as shown in Figure 2(c). Thus, there are $\min(i, 3 \times 2^{n-2} - i)$ lightpaths passing through the ith link of the linear array when $1 \leq i \leq 3 \times 2^{n-2} - 1$, and $\min(i - 3 \times 2^{n-2}, 2^n - i)$ lightpaths passing through the ith link when $3 \times 2^{n-2} \leq i \leq 2^n - 1$. So, the maximum number of wavelengths required during the first stage is $3 \times 2^{n-3}$ when $i = 3 \times 2^{n-3}$, i.e. the number of wavelengths decreases by 2^{n-3} after the shift operations. Similarly, during the second stage, the maximum number of wavelengths is $3 \times 2^{n-3}$ when $i = 5 \times 2^{n-3}$, i.e. the number of wavelengths required increases by 2^{n-3} after the shift operations. Since the kth$(0 \leq k \leq 2^{n-3} - 1)$ node during the third stage communicates with node $2^{n-2} - 1 - k$ by reversal mapping, it will communicate with node $2^n - 1 - k$ after the shift operations by the shift-reversal mapping. As the other communications during the third stage requires 2^{n-3} wavelengths, it is easy to know the maximum number of wavelengths required after the shift operations is $2^{n-3} + 2^{n-3} = 2^{n-2}$. For each of the other stages$(4 \leq i \leq n)$, the number of wavelengths required is 2^{n-i}, which is the same as that by reversal mapping because the 2^{n-3} times of shift operations can't change the relative positions between the communication pairs during these stages. So, the maximum number of wavelengths required during all stages by shift-reversal mapping is $3 \times 2^{n-3}$ for $n \geq 3$, as shown in Figure 2(c).

Therefore, $W_r(\mathrm{FFT}_n, \mathit{line}) = \max(3 \times 2^{n-3}, 2)$.

From the above discussion, it can be concluded that realizing FFT_n on a WDM linear array with 2^n nodes by shift-reversal mapping requires $\lfloor 2^{n-3} \rfloor$ fewer wavelengths than that by sequential mapping.

3.3. Rings

Assume that the nodes of WDM rings are numbered clockwise starting from 0, and the links starting from 1.

Theorem 3 *By sequential mapping, the number of wavelengths required to realize* FFT_n *on a WDM ring with* 2^n *nodes is* 2^{n-1}.

Proof: As we know, the number of wavelengths required for a collection of connections in a WDM network under the wavelength-continuity constraint is determined using a graph G_c, the conflict graph [17], in which each connection in the network is represented by a vertex in G_c. An undirected edge connecting two vertices appears in if and only if the corresponding connections share a physical fiber link. Color the vertices of G_c such that no two adjacent vertices have the same color (a proper coloring). Then the minimum number of colors in a proper coloring of G_c (i.e., the chromatic number of G_c) is the minimum number of wavelengths required for the corresponding connections in the original network. For the communications on a ring during the first stage, node i ($0 \leq i \leq 2^n - 1$) communicates with node $(i + 2^{n-1}) \bmod 2^n$, that is, the communication distance in this stage is equal to the diameter of the ring. Each connection for both clockwise routing and anti-clockwise routing shares links with all the other connections. In this case, we know that the conflict graph of these connections during the first stage is an $N/2$-node complete graph. Clearly, the chromatic number of the conflict graph is $N/2$. Thus, at least $N/2 = 2^{n-1}$ wavelengths are required for this stage. It is easy to see that the number of wavelengths required during the other ith$(2 \leq i \leq n)$ stage is 2^{n-i} with communications routed in the shortest path, as shown in Figure 3(a). Therefore, $W_s(\mathrm{FFT}_n, \mathit{ring}) = 2^{n-1}$. □

Theorem 4 *By shift-reversal mapping, the number of wavelengths required to realize* FFT_n *on a WDM ring with* 2^n *nodes is* 2^{n-2}.

Proof: We embed FFT_n into a WDM ring by reversal mapping with all the communications routed in the shortest path, as shown in Figure 3(b). Since the kth node on the ring communicates with node $2^n - k - 1$ during the first stage, the lightpaths passing through the ith link, denoted by w_{i1}, can be calculated by the following equation:

$$w_{i1} = \begin{cases} |2^{n-2} - i|, & 1 \leq i < 2^{n-1} \\ |3 \times 2^{n-2} - i|, & 2^{n-1} \leq i \leq 2^n \end{cases}.$$

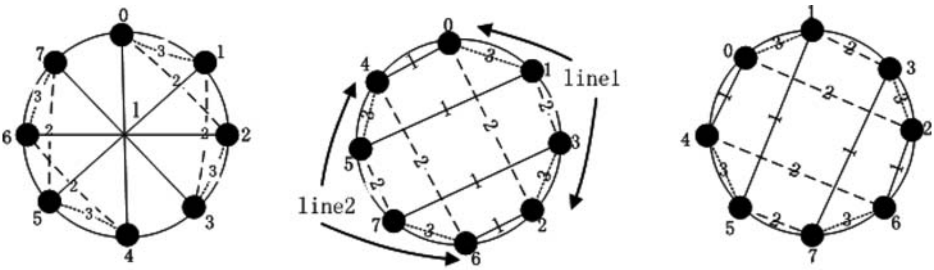


Figure 3. FFT_3 embedded in 8-node ring.

So, the maximum number of wavelengths is 2^{n-2} which can be achieved when $i = 2^{n-1}$ or 2^n. □

During the stages from 2 to n, there is no lightpath passing through the links of 2^{n-1} and 2^n. If we ignore these two links, the ring can be regarded as two 2^{n-1}-node linear arrays: *line*1(from node 0 to node $2^{n-1} - 1$ clockwise) and *line*2 (from node 2^{n-1} to node $2^n - 1$ clockwise), as shown in Figure 3(b). With all the communications routed in the shortest path, realizing the stages from 2 to n can be regarded as realizing FFT_{n-1} on each 2^{n-1}-node linear array. In the proving of Theorem 2, we know that the number of wavelengths required on an N-node linear array by reversal mapping is $N/2$. Therefore, the number of wavelengths required on *line*1 and *line*2 is 2^{n-2} during the stages from 2 to n.

Thus, the maximum number of wavelengths required to realize all the stages of parallel FFT by reversal mapping is 2^{n-2}. Since the relative positions of the nodes by reversal mapping and shift-reversal mapping are identical in the ring, the number of wavelengths required by shift-reversal mapping is the same as that by reversal mapping, as shown in Figure 3(b) and (c). That is, $W_r(\mathrm{FFT}_n, \mathit{ring}) = 2^{n-2}$.

From the above discussion, we know that realizing FFT_n in a WDM ring with 2^n nodes by shift-reversal mapping requires 2^{n-2} fewer wavelengths than that by sequential mapping.

3.4. Meshes and tori

In this section, we consider WDM meshes and tori of size $N = 2^k \times 2^{n-k}$, with nodes indexed sequentially in row major order: the node at position (x, y) has index $x + y \times 2^k$, for $0 \leq x \leq 2^{n-k} - 1$ and $0 \leq y \leq 2^k - 1$.

Theorem 5 *By sequential mapping, the number of wavelengths required to realize* FFT_n *on a* $2^k \times 2^{n-k}$ *mesh is* $2^{\max(k,n-k)-1}$.

Proof: With the ith node of FFT_n mapped onto the ith processor of a $2^k \times 2^{n-k}$ mesh, 1-1 mapping is established by sequential mapping. During the first k communications stages, communications takes place between processors in each column, and the communication steps are decreased from 2^{k-1} to 1. During the last $n - k$ stages, communications takes place between processors in each row, and the communication steps are decreased from 2^{n-k-1} to 1. In fact, communications on each column of a $2^k \times 2^{n-k}$ mesh can be regarded as those on a 2^k-node linear array and communications on each row as on an 2^{n-k}-node linear array. Since the maximum number of wavelengths on an N-node linear array by sequential mapping is $N/2$ by Theorem 1, the maximum numbers of wavelengths required in a $2^k \times 2^{n-k}$ mesh on each column and each row are 2^{k-1} and 2^{n-k-1} respectively. Thus, $W_s(\mathrm{FFT}_n, mesh(2^k \times 2^{n-k})) = \max(2^{k-1}, 2^{n-k-1}) = 2^{\max(k,n-k)-1}$.

In the following, we introduce the definition of shift-reversal mapping for 2-D meshes and 2-D tori with $2^k \times 2^{n-k}$ nodes.

Group the 2^n nodes of FFT_n into 2^k groups with each group consisting of 2^{n-k} nodes. Accordingly, the binary representation of each node, $r = (b_{n-1}, \ldots b_{n-k}, b_{n-k-1}, \ldots, b_0) = (B_1, B_2)$, can be decomposed into two parts: $B_1 = b_{n-1}, \ldots b_{n-k}$ and $B_2 = b_{n-k-1}, \ldots, b_0$. B_1 is the identification of each group and B_2 is the identification of each node within one group. Obviously, the nodes in the same group have the same value of B_1 and different values of B_2. The shift-reversal order for 2-D meshes and tori with $2^k \times 2^{n-k}$ nodes can be constructed by the following two steps:

1. According to B_2, sort the nodes within each group into shift-reversal order mentioned in Section 3.2;
2. According to B_1, sort the groups into shift-reversal order between groups.

We define the resulting arrangement of 2^n nodes shift-reversal order for 2-D meshes and tori with $2^k \times 2^{n-k}$ nodes.

Mapping the ith node of shift-reversal order defined above onto the ith processor of a $2^k \times 2^{n-k}$ mesh or torus, the 1-1 mapping is established from the nodes of FFT_n to the nodes of the 2-D mesh or the torus. Such embedding is defined as *shift-reversal mapping* for a 2-D mesh or a 2-D torus. □

Theorem 6 *By shift-reversal mapping, the number of wavelengths required to realize* FFT_n *on a* $2^k \times 2^{n-k}$ *mesh is* $\max(3 \times 2^{\max(k,n-k)-3}, 2)$.

Proof: By shift-reversal mapping on a $2^k \times 2^{n-k}$ mesh, we consider communications in each row and column as those on a linear array with 2^{n-k} and 2^k nodes by shift-reversal mapping respectively. According to Theorem 2, the numbers of wavelengths required on each row and column are $\max(3 \times 2^{n-k-3}, 2)$ and $\max(3 \times 2^{k-3}, 2)$ respectively. Therefore, $W_r(\mathrm{FFT}_n, mesh(2^k \times 2^{n-k})) = \max(3 \times 2^{\max(k,n-k)-3}, 2)$.

For a $2^k \times 2^{n-k}$ torus, communications on each row and column can be regarded as those on a WDM ring with 2^{n-k} and 2^k nodes respectively. The following results can be easily obtained according to Theorem 3 and Theorem 4. □

Theorem 7 *By sequential mapping, the number of wavelengths required to realize* FFT_n *on a* $2^k \times 2^{n-k}$ *torus is* $2^{\max(k,n-k)-1}$.

Theorem 8 *By shift-reversal mapping, the number of wavelengths required to realize* FFT_n *on a* $2^k \times 2^{n-k}$ *torus is* $2^{\max(k,n-k)-2}$.

Using Theorem 5, Theorem 6, Theorem 7 and Theorem 8, the following corollary is derived when the number of rows is equal to that of columns on the meshes and tori.

Corollary 1 *By sequential mapping and shift-reversal mapping, the numbers of wavelengths required to realize* FFT_n *on a* $2^{n/2} \times 2^{n/2}$ *mesh and torus satisfy:*

$$W_s(\mathrm{FFT}_n, mesh(2^{n/2} \times 2^{n/2})) = 2^{n/2-1} = \sqrt{N}/2,$$
$$W_r(\mathrm{FFT}_n, mesh(2^{n/2} \times 2^{n/2})) = \max(3 \times 2^{n/2-3}, 2) = \max(3\sqrt{N}/8, 2);$$
$$W_s(\mathrm{FFT}_n, torus(2^{n/2} \times 2^{n/2})) = 2^{n/2-1} = \sqrt{N}/2,$$
$$W_r(\mathrm{FFT}_n, torus(2^{n/2} \times 2^{n/2})) = 2^{n/2-2} = \sqrt{N}/4.$$

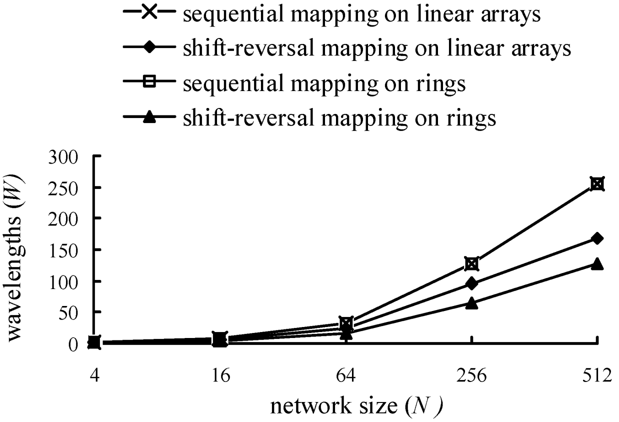
4. Comparisons between the two embeddings

It can be seen that shift-reversal mapping outperforms sequential mapping on the number of wavelengths, as shown in Figure 4(a). In the following, we give some comparisons between the two embeddings on the number of communication steps required during the n communication stages of parallel FFT.

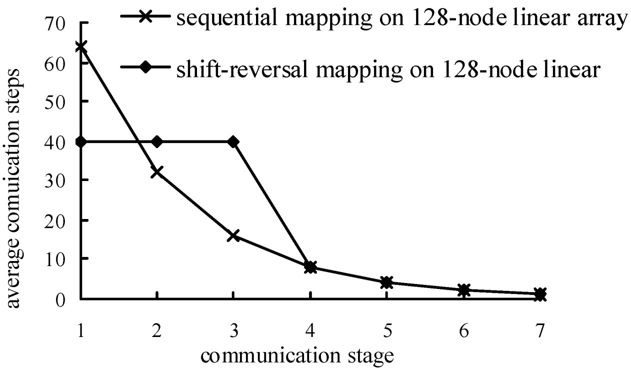
By sequential mapping on WDM linear array, the number of communication steps during the ith$(1 \le i \le n)$ stage is 2^{n-i}. By shift-reversal mapping on linear array, the average number of communication steps during stage 1, stage 2 and stage 3 is $5 \times 2^{n-4}$; For the other ith$(i \ge 3)$ stages, the average number of communication steps is 2^{n-i}. Figure 4(b) illustrates the average number of communication steps during the 7 stages on a linear array with 128 nodes by the sequential mapping and shift-reversal mapping. It can be seen that the average number of communication steps by the shift-reversal mapping is not more than the sequential mapping except during the stage 2 and stage 3 on the linear array.

By sequential mapping on WDM ring, the number of communication steps required during the ith stage is 2^{n-i}. By shift-reversal mapping, the average number of communication steps during the first stage is 2^{n-2}; During the other ith $(i \ge 2)$ stages, the average number of communication steps is 2^{n-i}. Figure 4(c) illustrates the average number of communication steps on a ring with 128 nodes, from which we can see that the average number of communication steps by shift-reversal mapping is not more than sequential mapping during all the stages. The analysis on 2-D meshes or 2-D tori is similar.

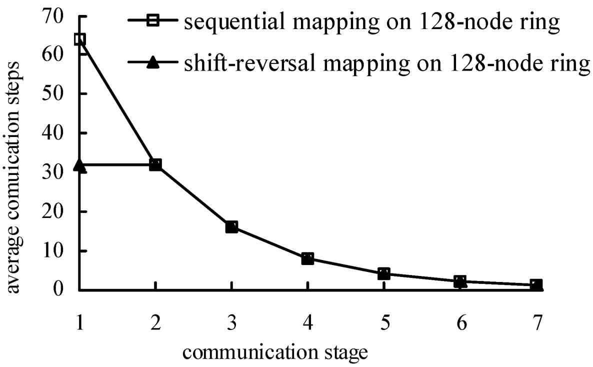
Because of the high transmission rate and low latency of optical communication, the influence of the number of communication steps is much smaller than the number of wavelengths required to realize the communication steps for the problems we are concerned.



(a) Comparisons on the number of wavelengths



(b) Average number of communication steps on 128-node linear array



(c) Average number of communication steps on 128-node ring

Figure 4. Comparisons between the two embeddings.

Table 1. Number of wavelengths required to realize parallel FFT on a class of regular optical networks by different embedding schemes.

	Embedding scheme	
Optical network	Sequential mapping	Shift-reversal mapping
Linear array	2^{n-1}	$\max(3 \times 2^{n-3}, 2)$
Ring	2^{n-1}	2^{n-2}
2-D mesh	$2^{\max(k,n-k)-1}$	$\max(3 \times 2^{\max(k,n-k)-3}, 2)$
2-D torus	$2^{\max(k,n-k)-1}$	$2^{\max(k,n-k)-2}$

5. Conclusions

In this paper, we discussed the wavelength assignment problem to embed the parallel FFT communication pattern on a class of regular optical WDM networks, including linear arrays, rings, 2-D meshes and 2-D tori. We derived the number of wavelengths required for each type of these networks to realize parallel FFT by sequential mapping and shift-reversal mapping. Our results show that shift-reversal mapping outperforms sequential mapping, as summarized in Table 1. Considering the number of processors and the capacity of the fiber links in practice, we can select the proper WDM networks to implement parallel FFT according to the results in this paper. Our results have a clear significance for applications for the widespread applications of FFT in scientific and engineering computations and the increasing popularity of the promising WDM technology in optical networks. Our proposed embedding method also provides a new approach to the hypercube layout problem considering connections dimension by dimension rather than all connections as in the traditional approach.

Since different parallel algorithms have different communication patterns, how to embed these communication patterns on optical networks is a key research problem. Our future work includes study on FFT realization in other types of optical networks and other RWA problems.

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