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Alpha-Structures and Ladders in Logical Geometry

Abstract. Aristotelian diagrams, such as the square of opposition and other, more complex diagrams, have a long history in philosophical logic. Alpha-structures and ladders are two specific kinds of Aristotelian diagrams, which are often studied together because of their close interactions. The present paper builds upon this research line, by reformulating and investigating alpha-structures and ladders in the contemporary setting of logical geometry, a mathematically sophisticated framework for studying Aristotelian diagrams. In particular, this framework allows us to formulate well-defined functions that construct alpha-structures and ladders out of each other. In order to achieve this, we point out the crucial importance of imposing an ordering on the elements in the diagrams involved, and thus formulate all our results in terms of ordered versions of alpha-structures and ladders. These results shed interesting new light on the prospects of developing a systematic classification of Aristotelian diagrams, which is one of the main ongoing research efforts within logical geometry today.

Keywords: Alpha-structure, Ladder, Aristotelian diagram, Square of opposition, Logical geometry.

1. Introduction

Ever since the great mind of Aristotle walked the face of the Earth, people have been concerned with the logical relations holding among various sets of statements. Such constellations can be visualized using so-called *Aristotelian diagrams*, which have statements as vertices and the relations holding between them as edges.[1] By far the most well-known example is the so-called square of opposition for the categorical statements from syllogistics [44]. Not only do Aristotelian diagrams have a rich history in philosophy and logic, today they are also ubiquitous in a wide array of other application contexts, e.g., in disciplines such as linguistics, psychology and knowledge representation (see the introduction of [14] for bibliographic references to various historical and contemporary applications of Aristotelian diagrams).

[1] Note, however, that Aristotle himself never drew such a diagram [23, 39].

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A special class of Aristotelian diagrams that has received some interest of its own is the class of *α-structures*, which are also called 'logical bi-simplexes' or 'n-oppositions'. They were extensively studied by Moretti, who also coined the term 'α-structure' [42,43]. In other research, it has been shown that α-structures are closely related to Euler and partition diagrams [16,18,56], and they occur frequently in scholarship on the logic of Arthur Schopenhauer [12,35,36] and in ethical research on supererogation [25,27,28]. In Moretti's work, we find not only α-structures, but also so-called 'β- and γ-structures'. Moretti called the latter also 'modal $n(m)$-graphs', and used them to search for certain β-structures and to exhibit all α-structures [43]. Subsequently, Pellissier [45] has demonstrated that Moretti's framework can be improved in several ways. For example, he proved that there are two kinds of α-structures, namely *strong* and *weak* ones (while Moretti only focused on strong α-structures). Pellissier also showed that there is no need for general modal $n(m)$-graphs (i.e., γ-structures) to exhibit all α-structures, but that we can reach the same goal using only the modal $3(m)$-graphs. He called the latter 'simplicial ladder graphs', and we will simply call them *ladders*. Such ladders occur frequently in historical scholarship on William of Sherwood [31,34] and in research on the logic of singular statements [5,38,40] and proportional quantifiers [46–48]. Finally, using a set-theoretic approach, Pellissier [45] figured out a way to construct α-structures out of ladders (called 'decorating'), which also allowed him to find the β-structures Moretti was looking for.

Next to having various applications (e.g., Schopenhauer, supererogation, Sherwood and proportional quantifiers), α-structures and ladders are also important for more theoretical reasons: they are naturally associated to two distinct perspectives on the square of opposition. Historically speaking, the categorical square from syllogistics was viewed as exhibiting a theory of *negation* as well as a theory of *logical consequence*. These two perspectives are emphasized in distinct commentary traditions: the former is primarily found in commentaries on Aristotle's *De interpretatione*, while the latter mainly occurs in commentaries on his *Analytica Priora* [4,54]. If we focus on negation, the natural generalization of a square of opposition turns out to be the notion of α-structure; by contrast, if we focus on logical consequence, the natural generalization is the notion of ladder. Since α-structures and ladders constitute two distinct classes of Aristotelian diagrams, it is clear that the negation and consequence perspectives are conceptually quite distinct from each other. However, since the classical square of opposition is simultaneously an α-diagram and a ladder, it is equally clear that this square simultaneously exhibits the negation and consequence perspectives.

Over the past 15 years, Aristotelian diagrams in general have been studied in a thorough and systematic way under the heading of *logical geometry* [9, 14, 15, 53, 54]. One of the main aims of this research program is to develop a comprehensive theoretical framework for Aristotelian diagrams, which is capable of explaining their behaviour in a mathematically satisfying way. A major breakthrough of logical geometry in this respect is the insight that classical Aristotelian diagrams always reside within a certain Boolean algebra, and that we can therefore use bitstring semantics to investigate them in a simple yet powerful fashion [14, 53, 55]. In this paper, we build upon the work of Moretti and Pellissier, by revisiting the α-structures and ladders from the perspective of logical geometry. In particular, we define these two classes of Aristotelian diagrams in the general setting of Boolean algebra, and we show how the work of Pellissier can be formalized using the tools of logical geometry. Moreover, we describe the elegant interaction between these classes of diagrams, by defining several functions that allow us to easily construct weak α-structures, strong α-structures and ladders in terms of each other. Overall, the paper thus illustrates the fruitful interplay and continuity between Moretti and Pellissier's pioneering insights (for example, the latter's set-theoretical approach uses a prototypical example of a Boolean algebra [45]) and the systematicity and mathematical sophistication of logical geometry.

The paper is organized as follows. In Section 2, we provide the necessary background from logical geometry that is needed for the rest of the paper. In Sections 3 and 4, we define α-structures and ladders in the setting of logical geometry, and we explain how weak α-structures and strong α-structures can be constructed in terms of each other. Finally, in Section 5, we relate the α-structures to the ladders, by defining functions that construct ladders out of (weak/strong) α-structures, and vice versa.

2. Background from Logical Geometry

Logical geometry systematically studies Aristotelian diagrams, such as the square of opposition and many others. An informal example of a square of opposition, involving the categorical statements from syllogistics, is given in Figure 1.

However, recent developments in logical geometry suggest that the most natural (and general) setting in which to define Aristotelian diagrams is that of Boolean algebra [7, 11, 22]. Following this line of research, we formally define these diagrams as follows.

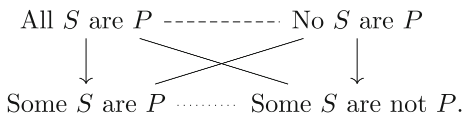

Figure 1. An informal square of opposition (the edges are drawn according to the legend given in Figure 2)

Definition 1. An Aristotelian diagram D is a pair $(\mathcal{F}, B)$, where B is a Boolean algebra $(B, \wedge_B, \vee_B, \neg_B, 1_B, 0_B)$ and $\mathcal{F}$ is a fragment of B, i.e., $\mathcal{F} \subseteq B$.[2] Furthermore, a σ-diagram is an Aristotelian diagram $(\mathcal{F}, B)$ such that $\mathcal{F}$ is closed under $\neg_B$, i.e., for all $x \in \mathcal{F}$, we have $\neg_B x \in \mathcal{F}$ as well. When the Boolean algebra B is clear from context, it is usually omitted as a subscript to $\wedge$, $\vee$, etc.

To interpret the arrows and edges in the square diagram above, we need the following four relations.

Definition 2. Given a Boolean algebra B, we say that $x, y \in B$ are:

- B-contradictory (CD_B) iff $x \wedge_B y = 0_B$ and $x \vee_B y = 1_B$,
- B-contrary (C_B) iff $x \wedge_B y = 0_B$ and $x \vee_B y \neq 1_B$,
- B-subcontrary (SC_B) iff $x \wedge_B y \neq 0_B$ and $x \vee_B y = 1_B$,
- in B-subalternation (SA_B) iff $\neg_B x \vee_B y = 1_B$ and $x \vee_B \neg_B y \neq 1_B$.

These four relations are called the Aristotelian relations for B; we also write $\mathcal{AR}_B := \{CD_B, C_B, SC_B, SA_B\}$. When no confusion is possible, B is usually omitted as a prefix and subscript.

Even though these relations are not explicitly mentioned in Definition 1, they will always appear in visualizations of Aristotelian diagrams. A common way of visualizing these four relations is indicated in Figure 2.

Example 1. Let us look again at the aforementioned informal example of an Aristotelian diagram (which is also a σ-diagram) from Figure 1. How is this informal example captured by Definition 1? First of all, we need a Boolean algebra to work in. For any logical system S that has Boolean connectives $\wedge$, $\vee$ and $\neg$, there exists a Boolean algebra $\mathbb{B}(\mathsf{S})$ whose underlying set is $\mathbb{B}(\mathsf{S}) := \{[\varphi] \mid \varphi \text{ is a well-formed formula in } \mathsf{S}\}$. Here, the notation $[\varphi]$ stands for the equivalence class of φ with respect to the relation $\equiv_{\mathsf{S}}$ of logical equivalence in S. The meet $\wedge_{\mathbb{B}(\mathsf{S})}$ in this algebra is given by $[\varphi] \wedge_{\mathbb{B}(\mathsf{S})} [\psi] :=$

[2] Note that we tacitly identify a Boolean algebra with its underlying set, which is common practice in the literature on Boolean algebra [22].

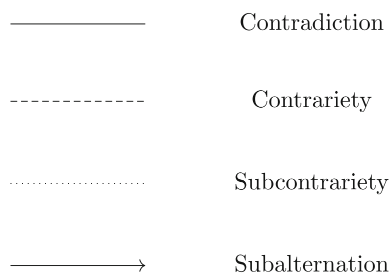

Figure 2. The code for visualizing the Aristotelian relations

$[\varphi \wedge \psi]$. The operations of $\vee_{\mathbb{B}}$ and $\neg_{\mathbb{B}}$ are defined in a similar way. Finally, we define $0_{\mathbb{B}(\mathsf{S})} := [\bot]$ and $1_{\mathbb{B}(\mathsf{S})} := [\top]$. It is not hard to check that all of this gives rise to a well-defined Boolean algebra, which is usually called the Lindenbaum-Tarski algebra of the logic S [19].

Now, consider the system of syllogistics, SYL, and its Lindenbaum-Tarski algebra $\mathbb{B}(\mathsf{SYL})$.[3] We define the fragment $\mathcal{F}^{cat} \subset \mathbb{B}(\mathsf{SYL})$ as

$$\mathcal{F}^{cat} := \{[\forall x(Sx \to Px)], [\exists x(Sx \wedge Px)], [\forall x(Sx \to \neg Px)], [\exists x(Sx \wedge \neg Px)]\}.$$

This fragment consists of (the SYL-equivalence classes of) the categorical statements from syllogistics, hence the name $\mathcal{F}^{cat}$. It is now clear from Definition 1 that we have an Aristotelian diagram, and even a σ-diagram, $(\mathcal{F}^{cat}, \mathbb{B}(\mathsf{SYL}))$, which is visualized in Figure 3 (we drop the equivalence class brackets for notational simplicity). Note that these formal definitions capture precisely the more well-known characterizations from the ancient history of logic [11]. For example, we say that $p := [\forall x(Sx \to Px)]$ and $q := [\forall x(Sx \to \neg Px)]$ are $\mathbb{B}(\mathsf{SYL})$-contrary to each other, because they 'cannot be true together' (i.e., $p \wedge_{\mathbb{B}(\mathsf{SYL})} q = 0_{\mathbb{B}(\mathsf{SYL})}$), but they 'can be false together' (i.e., $p \vee_{\mathbb{B}(\mathsf{SYL})} q \neq 1_{\mathbb{B}(\mathsf{SYL})}$).

In most Aristotelian diagrams $(\mathcal{F}, B)$ that are found in the extant literature, the fragment $\mathcal{F}$ is indeed closed under B-complementation, so these diagrams are σ-diagrams [7]. (The term 'σ-diagram' derives from the fact

[3]The system SYL has the same language as ordinary first-order logic (FOL), but is axiomatized by adding $\exists x Sx$, for all unary predicate symbols S, as additional axioms to FOL. This logical system is naturally interpreted on first-order models $\langle D, I\rangle$ (with domain D and interpretation function I) such that $I(S) \neq \emptyset$ [14]. It is not closed under uniform substitution (for example, $\exists x Sx$ is a tautology but $\exists x(Sx \wedge \neg Sx)$ is not), just like many of the recently developed systems of dynamic epistemic logic (for example, in public announcement logic, $[!p]p$ is a tautology, but $[!(p \wedge \neg Kp)](p \wedge \neg Kp)$ is not [24,57]). The system of SYL has also been called $\mathsf{FOL}_{\exists}$, and shown to be intertranslatable with Ben-Yami's Quantified Argument Calculus (QUARC) [1,49].

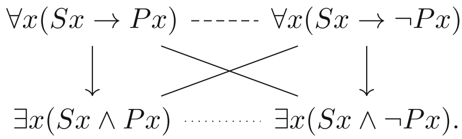

Figure 3. A classical square of opposition $(\mathcal{F}^{cat}, \mathbb{B}(\mathsf{SYL}))$

that $\neg_B$ is usually visualized by means of central symmetry.) For example, every classical square of opposition is a σ-diagram. Other interesting examples of σ-diagrams include the so-called Jacoby-Sesmat-Blanché (JSB) hexagons [2,3,26,51], unconnectedness-4 (U4) hexagons [13,21,33], Buridan octagons [10,32,50] and Keynes-Johnson octagons [17,29,30,41]; see [20] for precise definitions and further examples.[4]

The Aristotelian relations are naturally part of two other collections of logical relations [54], namely the opposition and implication relations:

Definition 3. Given a Boolean algebra B, we say that $x, y \in B$ are:

- B-contradictory (CD_B) iff $x \wedge_B y = 0_B$ and $x \vee_B y = 1_B$ (i.e., $x = \neg_B y$),
- B-contrary (C_B) iff $x \wedge_B y = 0_B$ and $x \vee_B y \neq 1_B$ (i.e., $x <_B \neg_B y$),
- B-subcontrary (SC_B) iff $x \wedge_B y \neq 0_B$ and $x \vee_B y = 1_B$ (i.e., $x >_B \neg_B y$),
- B-non-contradictory (NCD_B) iff $x \wedge_B y \neq 0_B$ and $x \vee_B y \neq 1_B$,
- in B-bi-implication (BI_B) iff $x \leq_B y$ and $x \geq_B y$ (i.e., $x = y$),
- in B-left-implication (LI_B) iff $x \leq_B y$ and $x \not\geq_B y$ (i.e., $x <_B y$),
- in B-right-implication (RI_B) iff $x \not\leq_B y$ and $x \geq_B y$ (i.e., $x >_B y$),
- in B-non-implication (NI_B) iff $x \not\leq_B y$ and $y \not\geq_B x$.

The first four relations are called the opposition relations for B and the last four are called the implication relations for B. We also write $\mathcal{OR}_B := \{CD_B, C_B, SC_B, NCD_B\}$ and $\mathcal{IR}_B := \{BI_B, LI_B, RI_B, NI_B\}$. We define $\mathcal{R}_B$ as the set of relations on B given by $\mathcal{R}_B := \{CD_B, C_B \cup SC_B, BI_B, LI_B \cup RI_B\}$.[5] When no confusion is possible, B is usually omitted as a prefix and subscript.

[4] Next to this long list of σ-diagrams, do note that we also find quite a few (naturally occurring) examples of Aristotelian diagrams that are *not* closed under complementation, i.e., that are *not* σ-diagrams; for example, see [52].

[5] Unlike $\mathcal{AR}_B$, $\mathcal{OR}_B$ and $\mathcal{IR}_B$, the set $\mathcal{R}_B$ has not been defined as such in previous work. Its relevance for our current purposes will become clear later in this paper.

Note that SA from Definition 2 coincides with LI from Definition 3 and therefore, the opposition and implication relations together properly extend the Aristotelian relations. More concretely, we have that $\mathcal{AR} \subset \mathcal{OR} \cup \mathcal{IR}$.

There is a way of saying that two Aristotelian diagrams have the same Aristotelian structure, which is made precise in Definition 4 below. This definition makes use of the relabel function $\imath_B^{B'}$, which simply identifies every relation in B with the same relation in B'. For example, $\imath_B^{B'}(CD_B) = CD_{B'}$ and $\imath_B^{B'}(SA_B) = SA_{B'}$ (see Definition 7 of [7] for a complete characterization of $\imath_B^{B'}$).

Definition 4. Let $D = (\mathcal{F}, B)$ and $D' = (\mathcal{F}', B')$ be Aristotelian diagrams. We say that $f : \mathcal{F} \to \mathcal{F}'$ is an Aristotelian isomorphism from D to D' iff f is bijective and for all Aristotelian relations $R_B \in \mathcal{AR}_B$ and $x, y \in \mathcal{F}$, we have

$$R_B(x, y) \iff \imath_B^{B'}(R_B)(f(x), f(y)).$$

If such an f exists, we also say that D and D' are Aristotelian isomorphic.

There is also a way of saying that two Aristotelian diagrams have the same Boolean structure. In order to make this formally precise, we first need to introduce the notion of a 'Boolean closure'.

Definition 5. Let B be a Boolean algebra and $\mathcal{F} \subseteq B$ be a fragment of B. The Boolean closure of $\mathcal{F}$ in B is the smallest subalgebra of B that contains $\mathcal{F}$. It is denoted by $Cl_B(\mathcal{F})$. Furthermore, if $\mathcal{F}$ is finite, then $Cl_B(\mathcal{F})$ is isomorphic to $\{0, 1\}^n$ for some natural number n [22][6]; this number n is called the Boolean complexity of $\mathcal{F}$ in B.

Definition 6. Let $D = (\mathcal{F}, B)$ and $D' = (\mathcal{F}', B')$ be Aristotelian diagrams. We say that $f : \mathcal{F} \to \mathcal{F}'$ is a Boolean isomorphism from D to D' iff f is bijective and f extends to a Boolean algebra isomorphism $Cl_B(\mathcal{F}) \to Cl_{B'}(\mathcal{F}')$ from the Boolean closure of $\mathcal{F}$ in B to the Boolean closure of $\mathcal{F}'$ in B'. If such an f exists, we also say that D and D' are Boolean isomorphic.

The notions of Aristotelian and Boolean isomorphism were first used in a less general setting in logical geometry [9,14], and have recently also been shown to arise naturally from a category-theoretical perspective [6,7]. Using more classification-oriented terminology, when two Aristotelian diagrams

[6]This observation lies at the foundation of bitstring semantics [9,14]. Furthermore, note that diagrams of the form $Cl_B(\mathcal{F}) - \{0_B, 1_B\}$ correspond to what Moretti and Pellissier call 'β-structures' [43,45]. More specifically, if $Cl_B(\mathcal{F})$ is isomorphic to $\{0, 1\}^{n+1}$, then $Cl_B(\mathcal{F}) - \{0_B, 1_B\}$ is called a 'β_n-structure' (note the offset by 1 in the notation).

are Aristotelian isomorphic, we say they belong to the same *Aristotelian family*; when they are Boolean isomorphic, we say they belong to the same *Boolean family*. In logical geometry, there has been a lot of interest in the subtle interplay between Aristotelian and Boolean structure. This interest is based on the observation that when two Aristotelian diagrams are Boolean isomorphic to each other, they are also Aristotelian isomorphic to each other, but the converse need not be true [14]. For example, the Aristotelian family of JSB hexagons contains precisely two Boolean subfamilies, namely the so-called strong and weak JSB hexagons [9,45]. A strong and a weak JSB hexagon are Aristotelian isomorphic to each other (i.e., they belong to the same Aristotelian family, viz., the family of JSB hexagons), but they are not Boolean isomorphic to each other (i.e., they belong to different Boolean subfamilies of the family of JSB hexagons). In recent years, several well-known Aristotelian families have received a detailed treatment in which they are dissected into their different Boolean subfamilies. For example, the JSB hexagons, the U4 hexagons, the Buridan octagons and the Keynes-Johnson octagons have been studied in this way [10,13,14,17].

Aristotelian diagrams can thus be classified according to their Aristotelian families (and Boolean subfamilies). However, there are also exist patterns *across* certain Aristotelian families. Informally, we can have a countably infinite series of Aristotelian families (that have increasingly large $|\mathcal{F}|$) which all satisfy the same properties. The α-structures and ladders investigated by Moretti and Pellissier [42,43,45] are two major examples of such series.

3. Alpha-Structures

We are now in a position to define α-structures and ladders in a fully general and mathematically sophisticated way. We first focus our attention on α-structures, and will turn to ladders in the next section.

Definition 7. Let $n \in \mathbb{N}_0$ be a natural number. An α_n-structure is a σ-diagram $(\mathcal{F}, B)$ such that

- $|\mathcal{F}| = 2n$,
- $0_B, 1_B \notin \mathcal{F}$ and
- $\exists X \subseteq \mathcal{F}$ such that $|X| = n$ and $C_B(a, b)$ for all distinct $a, b \in X$.

An ordered α_n-structure is a pair (D, x) such that D is an α_n-structure and $x = (x_1, \ldots, x_n)$ is an n-tuple consisting of pairwise contrary elements of the fragment of D. Dropping reference to the specific number of elements, any (ordered) α_n-structure is more generally called an (ordered) α-structure.

Given an α_n-structure, it can easily be turned into an ordered α_n-structure by fixing an order on its pairwise contrary elements, i.e., turning the unordered n-element set X into an ordered n-tuple x. It should be clear that such tuples x are never unique: there are $n!$ permutations on X, i.e., $n!$ different ways to order the set X into a tuple x. Now, suppose $D = (\mathcal{F}, B)$ is an α_n-structure and $X = \{x_1, \ldots, x_n\} \subseteq \mathcal{F}$ is its set of n pairwise contrary elements. Since (i) D is a σ-diagram, (ii) CD_B is a function[7] and (iii) CD_B and C_B are mutually exclusive, it now follows that $\mathcal{F} = \{x_1, \ldots, x_n, \neg x_1, \ldots, \neg x_n\}$.[8] From these considerations, together with Example 1, it can be seen that Definition 7 corresponds to the notions of logical bi-simplex [42] and n-opposition [45].

Proposition 1 below shows that the α-structures really constitute an infinite series of Aristotelian families, and determines their Boolean subfamilies. (This result first occurred as Theorem 2.6 in [12], where it was stated without proof.)

Proposition 1. *Let $n \in \mathbb{N}_0$ be a natural number. The family of all α_n-structures is an Aristotelian family with*

- *a single Boolean subfamily (with Boolean complexity $n+1$) if $n \in \{1, 2\}$,*
- *two Boolean subfamilies (with Boolean complexities n and $n+1$) if $n \geq 3$.*

Proof. Let us first show that the α_n-structures constitute an Aristotelian family. We do this by proving that all Aristotelian relations in a given α_n-structure $(\mathcal{F}, B)$ with a set of pairwise contrary elements $X = \{x_1, \ldots, x_n\}$ are fully determined. It is a well-known fact in logical geometry that when 0_B and 1_B are not in play, two elements in a Boolean algebra B can be in at most one Aristotelian relation [8]. It therefore suffices to consider the following five cases:

- For two distinct elements $x_i, x_j \in X$, we have by definition that $C(x_i, x_j)$.

[7] A function $f : X \to Y$ can be viewed as a special kind of relation R_f on $X \times Y$ for which the following holds: for every $x \in X$ there exists exactly one $y \in Y$ such that $R_f(x, y)$. In this case, $X = Y = B$, $R_f = CD_B$ and $y = \neg_B x$.

[8] Using terminology from [7], $\mathcal{F}$ is thus the *negation closure* of X in B. Note that it is easy to show that $x_1, \ldots, x_n$ are pairwise contrary to each other iff $\neg x_1, \ldots, \neg x_n$ are pairwise subcontrary to each other [54]. Consequently, we can equivalently characterize α_n-structures by replacing C_B with SC_B in the third bulletpoint of Definition 7.

- For two distinct elements $\neg x_i$ and $\neg x_j$ (where $x_i, x_j \in X$), we have that $\neg x_i \wedge \neg x_j = \neg(x_i \vee x_j) \neq \neg 1 = 0$ and $\neg x_i \vee \neg x_j = \neg(x_i \wedge x_j) = \neg 0 = 1$, which means that $SC(\neg x_i, \neg x_j)$. (Also cf. Footnote 8).
- For two elements x_i and $\neg x_j$ with $i \neq j$ (where $x_i, x_j \in X$), we have that $\neg x_i \vee \neg x_j = 1$ and $x_i \vee \neg\neg x_j = x_i \vee x_j \neq 1$, which means that $LI(x_i, \neg x_j)$.
- For all $x_i \in X$, we have by definition that $CD(x_i, \neg x_i)$.
- An element that is not 0 or 1 does not stand in any Aristotelian relation to itself.

These cases determine all the Aristotelian relations in all α_n-structures. Now, note that Definition 4 essentially says that two Aristotelian diagrams are Aristotelian isomorphic to each other whenever they have the same configuration of Aristotelian relations. Therefore, we have now proven that, for any fixed n, all α_n-structures are Aristotelian isomorphic to each other, and thus constitute a single Aristotelian family.

Next, we analyze the Boolean subfamilies of the Aristotelian family of α_n-structures. Firstly, if $n = 1$, we can write $\mathcal{F} = \{x_1, \neg x_1\}$. It is then not hard to check that $Cl_B(\mathcal{F}) = \{0, x_1, \neg x_1, 1\} \cong \{0, 1\}^2$. Secondly, if $n = 2$, we can write $\mathcal{F} = \{x_1, x_2, \neg x_1, \neg x_2\}$ (with $C(x_1, x_2)$). Then it is again not hard to check that $Cl_B(\mathcal{F}) = \{0, x_1, x_2, \neg x_1, \neg x_2, x_1 \vee x_2, \neg(x_1 \vee x_2), 1\} \cong \{0, 1\}^3$. These observations, together with Definition 6, show that for $n \in \{1, 2\}$, the α_n-structures constitute a single Boolean family, with Boolean complexity $n + 1$.

Finally, suppose that $n \geq 3$. We first need to make an observation. Given $I, J \subset \{1, \dots, n\}$ such that $I \neq J$, it follows that $\bigvee_{i \in I} x_i \neq \bigvee_{j \in J} x_j$. Suppose, toward a contradiction, that $\bigvee_{i \in I} x_i = \bigvee_{j \in J} x_j$ after all. Since $I \neq J$, we assume without loss of generality that $I \backslash J \neq \emptyset$, and consider some $k \in I \backslash J$. Since the elements of $\{x_1, \dots, x_n\}$ are pairwise contrary, we have

$$x_k = \bigvee_{i \in I} (x_k \wedge x_i) = x_k \wedge \bigvee_{i \in I} x_i = x_k \wedge \bigvee_{j \in J} x_j = \bigvee_{j \in J} (x_k \wedge x_j) = 0,$$

which is the contradiction we were looking for. (The first equality in the chain above holds because $k \in I$, while the last one holds because $k \notin J$.) With this observation under our belt, we now make the following case distinction:

- Case 1: $\bigvee_{i=1}^{n} x_i = 1$.[9] Note that $Cl_B(\mathcal{F}) = Cl_B(X)$, since $\mathcal{F}$ is itself just the negation closure of X in B (recall Footnote 8). We now show that $Cl_B(X) = \left\{ \bigvee_{i\in I} x_i \mid I \subseteq \{1,\ldots,n\} \right\}$. Since $Cl_B(X)$ is a Boolean algebra, it must contain the latter set. To prove the other inclusion, we show that this set is closed under the Boolean operations. Consider $J, J' \subseteq \{1,\ldots,n\}$. Since the elements of $\{x_1,\ldots,x_n\}$ are pairwise contrary, it is clear that

$$\begin{aligned}\neg \bigvee_{j\in J} x_j &= \bigwedge_{j\in J} \neg x_j = \bigwedge_{j\in J} (\neg x_j \wedge 1) = \bigwedge_{j\in J} (\neg x_j \wedge \bigvee_{i=1}^{n} x_i) \\ &= \bigwedge_{j\in J} \bigvee_{i=1}^{n} (\neg x_j \wedge x_i) = \bigwedge_{j\in J} \bigvee_{i\neq j} x_i = \bigvee_{j\in J^C} x_j,\end{aligned}$$

where $J^C = \{1,\ldots,n\} \backslash J \subseteq \{1,\ldots,n\}$. It is also clear that

$$\left(\bigvee_{j\in J} x_j\right) \vee \left(\bigvee_{j'\in J'} x_{j'}\right) = \bigvee_{j\in J\cup J'} x_j,$$

with $J \cup J' \subseteq \{1,\ldots,n\}$ and that[10]

$$\left(\bigvee_{j\in J} x_j\right) \wedge \left(\bigvee_{j'\in J'} x_{j'}\right) = \bigvee_{(j,j')\in J\times J'} x_j \wedge x_{j'} = \bigvee_{j\in J\cap J'} x_j,$$

again with $J \cap J' \subseteq \{1,\ldots,n\}$. Also recall that $1 = \bigvee_{i\in I} x_i$ for $I = \{1,\ldots,n\}$, and that $0 = \neg 1 = \bigvee_{i\in I} x_i$ for $I = \emptyset$, which proves that $0, 1 \in \left\{ \bigvee_{i\in I} x_i \mid I \subseteq \{1,\ldots,n\} \right\}$. This set is therefore closed under all Boolean operations, and thus coincides with $Cl_B(X)$. Because of the observation we made above (viz., that $\bigvee_{i\in I} x_i \neq \bigvee_{j\in J} x_j$ for distinct $I, J \subseteq \{1,\ldots,n\}$), we thus have $Cl_B(\mathcal{F}) = Cl_B(X) = \left\{ \bigvee_{i\in I} x_i \mid I \subseteq \{1,\ldots,n\} \right\} \cong \wp(\{1,\ldots,n\}) \cong \{0,1\}^n$. Combining this with Definition 6, we find that for $n \geq 3$, the α_n-structures such that $\bigvee_{i=1}^{n} x_i = 1$ constitute one Boolean family, which has Boolean complexity n.

[9] Note that for $n < 3$, this case simply cannot obtain. In particular, for $n = 1$, the condition that $\bigvee_{i=1}^{n} x_i = 1$ would mean that $x_1 = 1$, which violates the second bulletpoint of Definition 7; for $n = 2$, it would mean that $x_1 \vee x_2 = 1$, which violates the third bulletpoint of that same definition.

[10] Given the definability of $\wedge$ in terms of $\neg$ and $\vee$, this last chain of equalities is actually redundant. However, we have decided to include it for clarity of exposition.

- Case 2: $\bigvee_{i=1}^{n} x_i \neq 1$. Then we can define a new element $x_{n+1} := \neg \bigvee_{i=1}^{n} x_i \notin \{0,1\}$. It clearly holds that $\bigvee_{i=1}^{n+1} x_i = 1$, so an analogous argument as in the previous case now shows that $Cl_B(\mathcal{F}) = \left\{ \bigvee_{i \in I} x_i \mid I \subseteq \{1,\dots,n{+}1\} \right\} \cong \{0,1\}^{n+1}$, which again proves that we have one Boolean subfamily of the α_n-structures for which $\bigvee_{i=1}^{n} x_i \neq 1$. In this case, the Boolean closure of $\mathcal{F}$ in B has exactly 2^{n+1} elements. Combining this with Definition 6, we find that for $n \geq 3$, the α_n-structures such that $\bigvee_{i=1}^{n} x_i \neq 1$ constitute one Boolean family, which has Boolean complexity $n+1$. ■

From the proof above it is clear that the two Boolean subfamilies of α_n-structures can be distinguished by whether or not $\bigvee_{i=1}^{n} x_i = 1$. If this equality holds, we say that the diagram is a *strong* α_n-structure (which has Boolean complexity n), otherwise we say that it is a *weak* α_n-structure (which has Boolean complexity $n{+}1$). We introduce the following notational conventions.

Notation 1. *We denote by S and W the classes of all strong α-structures and of all weak α-structures, respectively. A subscript n means we restrict ourselves to α_n-structures and a superscript o means we restrict ourselves to ordered α-structures. For example, we denote by W_n^o the class of all ordered weak α_n-structures (D, x).*

It is easy to see that for all $n \in \mathbb{N}_0$, there exist concrete examples of α_n-structures. For $n \geq 3$, both weak and strong α_n-structures exist, while for $n \in \{1,2\}$, we only have weak α_n-structures (recall Footnote 9).[11] For $n = 1,2,3$, the α_n-structures are the pairs of contradictories (PCDs), the classical squares of opposition and the JSB hexagons, respectively. The α_4-structures are the Moretti octagons (which Moretti himself drew as cubes [43]). In Figure 4, we show an example of a strong Moretti octagon in the Boolean algebra $\{0,1\}^4$. In order not to overcomplicate the diagram, we leave out all the subalternations, but it is not hard to see where they should go. Adding a 0 after each of the lower four elements and a 1 after each of

[11]This observation about α-structures is a special case of a more general point: an Aristotelian family can have distinct Boolean subfamilies only if its diagrams contain *at least 3* distinct elements (and their negations, in case of σ-diagrams). Only from that cutoff point onwards, a diagram's Boolean properties are no longer fully captured by its Aristotelian relations (which are all *binary* in nature) [9]. Similarly, in classical propositional logic we have that $\{p \vee q, \neg p, \neg q\}$ is inconsistent, even though all of its 2-element subsets are consistent.

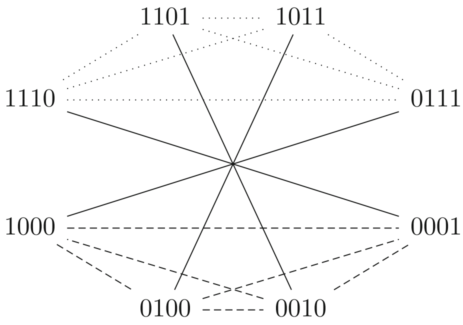

Figure 4. A strong α_4-structure in $\{0,1\}^4$

the upper four elements would yield an example of a weak Moretti octagon in the Boolean algebra $\{0,1\}^5$.

The proof of Proposition 1 shows us that for $n \geq 2$, weak α_n-structures and strong α_{n+1}-structures have isomorphic Boolean closures: if $(\mathcal{F}, B) \in W_n$ and $(\mathcal{F}', B') \in S_{n+1}$, then $Cl_B(\mathcal{F}) \cong \{0,1\}^{n+1} \cong Cl_{B'}(\mathcal{F}')$. This suggests that we can create strong α_{n+1}-structures out of weak α_n-structures, and vice versa. In the following theorems, we define functions that formalize this insight. Before we turn to these results, we introduce some further handy notational conventions.

Notation 2. *When we need to denote an arbitrary n-tuple, we will use x or y, which are implicitly given by $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. If x is an (ordered) n-tuple, then we denote by $|x|$ the (unordered) set $\{x_1, \ldots, x_n\}$.*

Notation 3. *Let $\mathcal{F} \subseteq B$ be a fragment of a Boolean algebra. Then we denote by $\neg_B \mathcal{F}$ the fragment $\{\neg_B b \in B \mid b \in \mathcal{F}\}$ that contains all negations of elements in $\mathcal{F}$. When no confusion is possible, we omit B as a subscript.*

Theorem 1. *Let $n \geq 2$, then we have a well-defined function $Add_n : W_n^o \to S_{n+1}^o$, which for any $D = (\mathcal{F}, B)$ is given by:*

$$Add_n(D, x) := (Add_n(D), Add_n(x)),$$

with $(n+1)$-tuple $Add_n(x) := (x_1, \ldots, x_n, \neg \bigvee_{i=1}^{n} x_i)$ and strong α_{n+1}-structure $Add_n(D) := (|Add_n(x)| \cup \neg |Add_n(x)|, B)$.[12]

[12]Note that this involves a slight abuse of notation, as we write Add_n for three distinct functions, each with their own domain (viz., ordered weak α_n-structures, unordered weak α_n-structures and n-tuples) and their own codomain. We trust that it is clear from the context which function is being used where. Completely analogous remarks apply to many of the theorems that follow.

Proof. It is trivial that $\neg\bigvee_{i=1}^{n} x_i$ and $\neg\neg\bigvee_{i=1}^{n} x_i = \bigvee_{i=1}^{n} x_i$ are distinct from each other and from all elements of the fragment of D, so that the fragment of $Add_n(D)$ has $2n+2 = 2(n+1)$ elements. Since D is a weak α-structure, these two new elements are not equal to 0 or 1. We now prove that $C(x_k, \neg\bigvee_{i=1}^{n} x_i)$ for any $1 \leq k \leq n$:

$$\begin{aligned}
x_k \wedge \neg\bigvee_{i=1}^{n} x_i &= x_k \wedge \bigwedge_{i=1}^{n} \neg x_i \\
&= x_k \wedge \neg x_1 \wedge \cdots \wedge \neg x_k \wedge \cdots \wedge \neg x_n \\
&= 0, \\
x_k \vee \neg\bigvee_{i=1}^{n} x_i &= x_k \vee \bigwedge_{i=1}^{n} \neg x_i \\
&= \bigwedge_{i=1}^{n} (x_k \vee \neg x_i) \\
&\leq x_k \vee \neg x_\ell \\
&= \neg x_\ell \\
&\neq 1,
\end{aligned}$$

where ℓ is any number in $\{1,\ldots,n\}\backslash\{k\}$—note that since $n \geq 2$, the set $\{1,\ldots,n\}\backslash\{k\}$ is non-empty, so such an ℓ certainly exists. (The penultimate identity in the chain above holds because $C(x_k, x_\ell)$.) This already shows that $Add_n(D, x)$ is an α_{n+1}-structure. Moreover, we trivially have that

$$x_1 \vee \cdots \vee x_n \vee \neg\bigvee_{i=1}^{n} x_i = 1,$$

so $Add_n(D, x)$ is a *strong* α_{n+1}-structure, i.e., $Add_n(D, x) \in S_{n+1}^{o}$. ■

Theorem 2. *Let $n \geq 2$. Then we have a well-defined function $Drop_n : S_{n+1}^{o} \to W_n^{o}$, which for any $D = (\mathcal{F}, B)$ is given by:*

$$Drop_n(D, x) := (Drop_n(D), Drop_n(x)),$$

with n-tuple $Drop_n(x) := (x_1, \ldots, x_n)$ and weak α_n-structure $Drop_n(D) := (|Drop_n(x)| \cup \neg|Drop_n(x)|, B)$.[13]

[13]For the sake of clarity, we emphasize that since $(D, x) \in S_{n+1}^{o}$, the tuple x is an $(n+1)$-tuple, so that $Drop_n(x) = Drop_n(x_1, \ldots, x_n, x_{n+1}) = (x_1, \ldots, x_n)$ is itself an n-tuple.

PROOF. Note that since $n \geq 2$, there exist strong α_{n+1}-structures, so the domain of $Drop_n$, i.e., S^o_{n+1}, is non-empty. It is immediate by construction that $Drop_n(D)$ is an α_n-structure and that $Drop_n(x)$ is an appropriate n-tuple of pairwise contrary elements. Now suppose toward a contradiction that $\bigvee_{i=1}^n x_i = 1$. Since D is an α_{n+1}-structure, we have $C(x_{n+1}, x_i)$ for all $1 \leq i \leq n$, and hence

$$\begin{aligned} x_{n+1} = x_{n+1} \wedge 1 = x_{n+1} \wedge \bigvee_{i=1}^n x_i \\ = \bigvee_{i=1}^n (x_{n+1} \wedge x_i) \\ = \bigvee_{i=1}^n 0 = 0, \end{aligned}$$

which is the contradiction we were looking for. This shows that $\bigvee_{i=1}^n x_i \neq 1$, so $Drop_n(D, x)$ is a *weak* α_n-structure, i.e., $Drop_n(D, x) \in W^o_n$. ■

We now have well-defined functions Add_n and $Drop_n$ at our disposal, which allow us to construct strong α_{n+1}-structures out of weak α_n-structures, and vice versa. Furthermore, it is easy to show that each of these functions undoes the effect of the other one, i.e., they are each other's inverses.

THEOREM 3. *Let $n \geq 2$. Then $Drop_n \circ Add_n$ is the identity function on W^o_n and $Add_n \circ Drop_n$ is the identity function on S^o_{n+1}.*

PROOF. The first statement follows immediately because for any n-tuple x, we have $(Drop_n \circ Add_n)(x) = Drop_n(Add_n(x)) = Drop_n((x_1, \ldots, x_n, \neg \bigvee_{i=1}^n x_i)) = x$. For the second statement, note that for any $(n+1)$-tuple x, we have that

$$Add_n(Drop_n(x)) = Add_n((x_1, \ldots, x_n)) = (x_1, \ldots, x_n, \neg \bigvee_{i=1}^n x_i).$$

To prove that $(Add_n \circ Drop_n)(x) = Add_n(Drop_n(x)) = x = (x_1, \ldots, x_n, x_{n+1})$, it thus suffices to prove that $\neg \bigvee_{i=1}^n x_i = x_{n+1}$. Since the tuple x comes from a strong α_{n+1}-structure, we have that $x_{n+1} \vee \bigvee_{i=1}^n x_i = \bigvee_{i=1}^{n+1} x_i = 1$. Also, following the same reasoning as in the proof of Theorem 2, we have that $x_{n+1} \wedge \bigvee_{i=1}^n x_i = 0$. This proves the desired statement. ■

It might seem somewhat unsatisfactory that the functions Add_n and $Drop_n$ only concern *ordered* α-structures, since in practice, we primarily want to construct weak/strong α-structures out of each other, without having to take into consideration the specific ordering on their pairwise contrary elements. However, this is simply not possible to do in a canonical way, by the following argument.

As Proposition 1 shows, weak α_n-structures and strong α_{n+1}-structures both have Boolean complexity $n+1$. Thus, their Boolean closures are isomorphic to the Boolean algebra $\{0,1\}^{n+1}$. It is not hard to show that within this Boolean algebra, there is only one strong α_{n+1}-structure, whose set of pairwise contrary elements we will call X. However, there are exactly $n+1$ weak α_n-structures within $\{0,1\}^{n+1}$, whose sets of pairwise contrary elements we will call $X_1, \ldots, X_{n+1}$, respectively. Since $X = X_i \cup \{\neg \bigvee X_i\}$ for any $1 \leq i \leq n+1$, we can go canonically from W_n to S_{n+1}. On the other hand, to go from X to any X_i, we need to remove one element of X. Because there is no canonical way of choosing which element to remove, we cannot go canonically from S_{n+1} to W_n. The most reasonable solution to this problem is to order X.

The best we can hope for is thus canonicity on the level of ordered diagrams, which is provided by Add_n and $Drop_n$. To be able to construct (unordered) weak/strong α_n-structures out of each other, we compose these canonical functions with the functions $Choose$ (which 'essentially captures all non-canonicity') and $Forget$ (which is, again, canonical). The situation is summarized in Figure 5. Note that for any possible function $Choose : W_n \to W_n^o$, the composition $Forget \circ Add_n \circ Choose : W_n \to S_{n+1}$ is one and the same function, as expected, which we can denote by Add_n^*. On the other hand, if we start from any possible function $Choose : S_{n+1} \to S_{n+1}^o$, given any $D \in S_{n+1}$, there are $n+1$ possible outcomes $(Forget \circ Drop_n \circ Choose)(D)$, namely one for each element of the set of pairwise contrary elements of D that gets put in the last place of its corresponding tuple by $Choose$ (and thus subsequently gets deleted by $Drop_n$). We refer to these $n+1$ options as $Drop_n^{*1}(D), Drop_n^{*2}(D), \ldots, Drop_n^{*n}(D), Drop_n^{*n+1}(D)$.[14] We then have that $Add_n^*(Drop_n^{*i}(D)) = D$ for all i, but it is not true that $Drop_n^{*i}(Add_n^*(D)) = D$ for all i, as is shown by the following examples.

Example 2. Consider the (unordered) strong Moretti octagon $D_s := (\mathcal{F}_s, \{0,1\}^4) \in S_4$, with $\mathcal{F}_s := \{1000, 0100, 0010, 0001, 0111, 1011, 1101, 1110\}$. Let us first choose as an ordered 4-tuple of pairwise contrary elements the

[14] This notation suggests that we have $n+1$ different functions of the form $Drop_n^{*i} : S_{n+1} \to W_n$. However, the expression '$Drop_n^{*i}(D)$' denotes a specific weak α_n-structure related to D, but cannot be decomposed into a function $Drop_n^{*i}$ that gets applied to an argument D. More specifically, it is not possible to define such functions $Drop_n^{*i}$ on all of S_{n+1} at once. The only thing that comes close is to combine them into a single function from S_{n+1} to $\wp(W_n)$, but this is not the approach we take here.

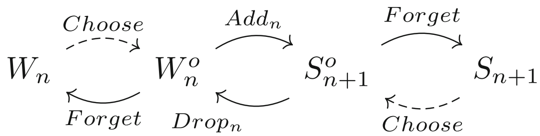

Figure 5. Going back and forth between weak and strong (ordered) α-structures. (Dashed lines indicate non-unique processes.)

tuple $x_s := (1000, 0100, 0010, 0001)$, thus obtaining $(D_s, x_s) \in S_4^o$. Both D_s and (D_s, x_s) are shown on the right-hand side of Figure 6 (the specific ordering on x_s is not visualized as such). Since $(D_s, x_s) \in S_4^o$, we can apply the function $Drop_3$ to it, which yields the ordered weak JSB hexagon $Drop_3(D_s, x_s) = (D_w, x_w) \in W_3^o$, with $x_w := (1000, 0100, 0010)$, $D_w := (\mathcal{F}_w, \{0,1\}^4)$ and $\mathcal{F}_w := \{1000, 0100, 0010, 0111, 1011, 1101\}$. By forgetting about x_w, we then arrive at the (unordered) weak JSB hexagon $D_w \in W_3$. Both D_w and (D_w, x_w) are shown on the left-hand side of Figure 6 (the specific ordering on x_w is not visualized as such). Likewise, we can go from D_w back to D_s: start from $D_w \in W_3$, choose the tuple x_w to obtain $(D_w, x_w) \in W_3^o$, apply Add_3 to obtain $Add_3(D_w, x_w) = (D_s, x_s) \in S_4^o$, and finally, forget about x_s to obtain $D_s \in S_4$. This example is an instantiation of the fact that $Add_3^*(Drop_3^{*i}(D_s)) = D_s$ for all $1 \leq i \leq 4$.

Example 3. Consider the classical square of opposition $D \in W_2$ with set of pairwise contraries $X := \{100, 010\}$. Then $Add_2^*(D) \in S_3$ is uniquely defined to be the strong JSB hexagon with set of pairwise contraries $\{100, 010, 001\}$. Next, we consider the three distinct classical squares of opposition $Drop_2^{*1}(Add_2^*(D))$, $Drop_2^{*2}(Add_2^*(D))$ and $Drop_2^{*3}(Add_2^*(D))$, with sets of pairwise contraries $X_1 := \{100, 010\}$, $X_2 := \{100, 001\}$ and $X_3 := \{010, 001\}$, respectively. Only the first one of these squares is identical to the original classical square D. This example shows that $Add_2^*(Drop_2^{*i}(D)) = D$ does not hold true for all $1 \leq i \leq 3$.

4. Ladders

Now, we turn our attention to ladders.

Definition 8. Let $n \in \mathbb{N}_0$ be a natural number. An n-ladder is a σ-diagram $(\mathcal{F}, B)$ such that

- $|\mathcal{F}| = 2n$,
- $0_B, 1_B \notin \mathcal{F}$ and

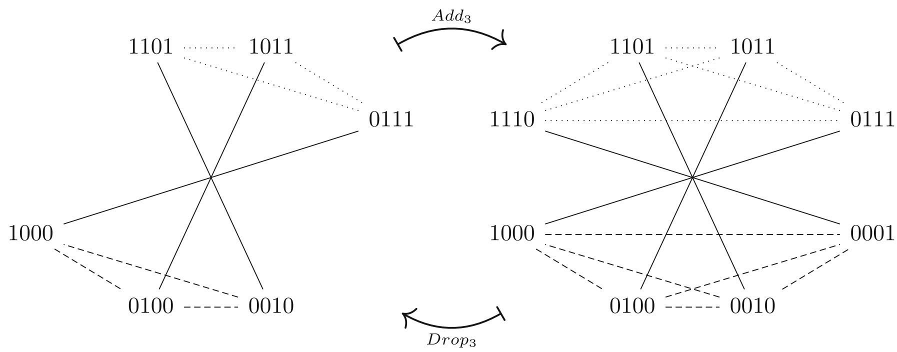

Figure 6. An instantiation of the fact that Add_n and $Drop_n$ are each other's inverses for all $n \geq 2$

- $\exists y = (y_1, \ldots, y_n) \in \mathcal{F}^n$ such that $LI_B(y_i, y_{i+1})$ for all $1 \leq i \leq n-1$.

An ordered n-ladder is a pair (D, y) such that D is an n-ladder and y is an n-tuple as described above. Dropping reference to the specific number of elements, any (ordered) n-ladder is more generally called an (ordered) ladder.

Given a ladder, it can easily be turned into an ordered ladder by fixing a specific tuple y. It should be clear that such tuples are never unique: if $(y_1, \ldots, y_n)$ meets the requirements stipulated in Definition 8, then $(\neg y_n, \ldots, \neg y_1)$ does so as well. Now, suppose $D = (\mathcal{F}, B)$ is an n-ladder and $(y_1, \ldots, y_n) \in \mathcal{F}^n$ is a tuple from Definition 8 with subalternations holding between its elements. Since (i) D is a σ-diagram, (ii) CD_B is a function and (iii) CD_B and LI_B are mutually exclusive when 0_B and 1_B are not in play, it now follows that $\mathcal{F} = \{y_1, \ldots, y_n, \neg y_1, \ldots, \neg y_n\}$, i.e., $\mathcal{F}$ is the negation closure of $|y| = \{y_1, \ldots, y_n\}$ in B. From these considerations, together with Example 1, it can be seen that Definition 8 corresponds to the notions of modal $3(m)$-graph [43] and simplicial ladder graph [45]. The following proposition shows that the ladders really constitute an infinite series of Aristotelian families.

Proposition 2. *Let $n \in \mathbb{N}_0$ be a natural number. The family of all n-ladders is an Aristotelian family with a single Boolean subfamily (with Boolean complexity $n+1$).*

Proof. Let us first show that the n-ladders constitute an Aristotelian family. Again, we do this by proving that all Aristotelian relations in a given n-ladder $(\mathcal{F}, B)$, with a tuple of elements $y = (y_1, \ldots, y_n)$ such that $LI(y_i, y_{i+1})$ for all $1 \leq i \leq n-1$, are fixed. It is well-known in logical geometry that when 0 and 1 are not in play, two elements in a Boolean algebra can be in at most one Aristotelian relation [8]. We thus distinguish six cases:

- For two elements $y_i, y_j \in |y|$ such that $i < j$, we have that $LI(y_i, y_j)$.
- For two elements $\neg y_i$ and $\neg y_j$ (where $y_i, y_j \in |y|$) such that $i < j$, we have that $LI(\neg y_j, \neg y_i)$.
- For two elements y_i and $\neg y_j$ (where $y_i, y_j \in |y|$) such that $i < j$, we have that $y_i \wedge \neg y_j = \neg(\neg y_i \vee y_j) = \neg 1 = 0$ and $y_i \vee \neg y_j \neq 1$, which means that $C(y_i, \neg y_j)$.
- For two elements y_i and $\neg y_j$ (where $y_i, y_j \in |y|$) such that $i > j$, we have that $y_i \wedge \neg y_j = \neg(\neg y_i \vee y_j) \neq \neg 1 = 0$ and $y_i \vee \neg y_j = 1$, which means that $SC(y_i, \neg y_j)$.
- For two elements y_i and $\neg y_j$ (where $y_i, y_j \in |y|$) such that $i = j$, we have that $CD(y_i, \neg y_j)$.
- An element that is not 0 or 1 does not stand in any Aristotelian relation to itself.

These cases determine all the Aristotelian relations in all n-ladders. It now follows from Definition 4 that all n-ladders are Aristotelian isomorphic to each other, and thus constitute a single Aristotelian family.

Next, we analyze the Boolean subfamilies of the Aristotelian family of n-ladders. First of all, note that $Cl_B(\mathcal{F}) = Cl_B(|y|)$, since $\mathcal{F}$ is itself just the negation closure of $|y|$ in B. Next, define the $(n+1)$-tuple $x := (y_1, \neg y_1 \wedge y_2, \ldots, \neg y_{n-1} \wedge y_n, \neg y_n)$[15]; we will show that $Cl_B(|y|) = Cl_B(|x|)$. The $\supseteq$-direction clearly holds by construction of x. For the $\subseteq$-direction, it suffices to see that $y_i = \bigvee_{j=1}^{i} x_j$ for all $1 \leq i \leq n$. Finally, we claim that $(|x| \cup \neg|x|, B)$ is a strong α_{n+1}-structure (this claim is proved below). It then follows from the proof of Proposition 1 that $Cl_B(\mathcal{F}) = Cl_B(|y|) = Cl_B(|x|) \cong \{0,1\}^{n+1}$. Combining this with Definition 6, we find that the n-ladders constitute one Boolean family, which has Boolean complexity $n + 1$.

We now prove our claim that $(|x| \cup \neg|x|, B)$ is a strong α_{n+1}-structure. We first check that any two elements $a, b \in |x|$ are contrary to each other. We distinguish between different cases depending on the form of a and b:

- $a = y_1$ **and** $b = \neg y_n$: In this case, $LI(y_1, y_n)$ implies that $C(a, b)$.
- $a = y_1$ **and** b **is of the form** $\neg y_i \wedge y_{i+1}$: We have that $a \wedge b = (y_1 \wedge \neg y_i) \wedge y_{i+1} = 0$, where the last equality follows because either $y_i = y_1$

[15] See Example 4 for a concrete illustration of this construction.

or $LI(y_1, y_i)$. We also have that $a \vee b = (y_1 \vee \neg y_i) \wedge (y_1 \vee y_{i+1}) \leq (y_1 \vee \neg y_1) \wedge (y_1 \vee y_{i+1}) = y_1 \vee y_{i+1} = y_{i+1} \neq 1$. Here, the second step holds because either $y_i = y_1$ or $LI(\neg y_i, \neg y_1)$, while the final equality holds since $LI(y_1, y_{i+1})$. We have now proven that $C(a, b)$.

- $a = \neg y_n$ **and** b **is of the form** $\neg y_i \wedge y_{i+1}$: Similar to the previous case.
- a **is of the form** $\neg y_i \wedge y_{i+1}$ **and** b **is of the form** $\neg y_j \wedge y_{j+1}$: Without loss of generality, we assume that $i < j$. Now $a \wedge b = \neg y_i \wedge (y_{i+1} \wedge \neg y_j) \wedge y_{j+1} = 0$ since either $y_{i+1} = y_j$ or $LI(y_{i+1}, y_j)$. Also, $a \vee b = (\neg y_i \wedge y_{i+1}) \vee (\neg y_j \wedge y_{j+1}) \leq y_{i+1} \vee y_{j+1} = y_{j+1} \neq 1$. Here, the penultimate step holds because $i < j$ and thus $LI(y_{i+1}, y_{j+1})$. We have now proven that $C(a, b)$.

Since the $n+1$ elements of $|x|$ are clearly distinct from each other, from their negations, and from 0 and 1, we have shown that $(|x| \cup \neg|x|, B)$ is an α_{n+1}-structure. To see that it is a strong one, recall that $y_n = \bigvee_{j=1}^{n} x_j$ and note that

$$\bigvee_{j=1}^{n+1} x_j = \bigvee_{j=1}^{n} x_j \vee x_{n+1} = y_n \vee \neg y_n = 1.$$

■

Again, we introduce a notational convention.

Notation 4. *We denote by L the class of all ladders. A subscript n means we restrict ourselves to n-ladders and a superscript o means we restrict ourselves to ordered ladders. For example, we denote by L_n^o the class of all ordered n-ladders.*

Just like in the case of α-structures, it might seem artificial to distinguish between L_n and L_n^o, since we are primarily concerned with ladders as such, without taking into consideration any specific tuple of consecutively *LI* elements. However, in the next section, we will construct weak/strong α-structures and ladders out of each other, and this can only be done (in a functional way) at the level of *ordered* diagrams. We therefore take the same approach here as in the previous section, and go back and forth between ladders and ordered ladders by simply choosing or forgetting a tuple. Once again, *forgetting* a tuple is canonical (i.e., there is a unique function $(D, y) \mapsto D$), but *choosing* a tuple is not: there are two distinct ways to choose a tuple,[16] i.e., to define a function $D \mapsto (D, y)$.

Much more importantly, it can be shown that for all $n \in \mathbb{N}_0$, there exist concrete examples of n-ladders. For $n \in \{1, 2\}$, the n-ladders are simply

[16] As stated before, if one tuple is $(y_1, \ldots, y_n)$, the other one is $(\neg y_n, \ldots, \neg y_1)$.

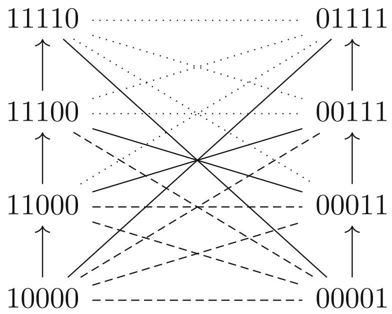

Figure 7. A 4-ladder in $\{0,1\}^5$

the PCDs and the classical squares of opposition, respectively, as is summarized in Theorem 4 below. It follows from this theorem, together with Propositions 1 and 2, that the classical square of opposition is the *largest* Aristotelian diagram that is simultaneously an α-structure and a ladder.[17]

THEOREM 4. *For $n \in \{1, 2\}$, the α_n-structures coincide with the n-ladders.*

PROOF. It follows trivially from the definitions that the Aristotelian families of α_1-structures and of 1-ladders both coincide exactly with the family of PCDs, and thus with each other. It is also not hard to check that the Aristotelian families of α_2-structures and 2-ladders both coincide exactly with the family of classical squares of opposition, and thus with each other. ■

Moving on, 3-ladders and 4-ladders are sometimes referred to in the literature as resp. 'Sherwood-Czeżowski hexagons' and 'Lenzen octagons' [5, 20, 31, 34, 37]. For example, Figure 7 shows a 4-ladder in the Boolean algebra $\{0,1\}^5$. In order not to overcomplicate the diagram, we leave out some of the subalternations on the far left and far right, but it is not hard to see where they should go.

Figure 7 clearly illustrates the intuition behind the terminology 'ladder', since we can use the subalternations to climb up in the diagram, rung by rung.[18] However, there is another way in which we could draw such a diagram, which will make it clearer why ladder diagrams are fruitfully studied

[17] In [54] it is suggested that considerations like these (albeit without using the specific terminology of 'α-structure' and 'ladder') might help to explain the widespread popularity of the classical square (especially in contrast with other, larger diagrams).

[18] If $n = 2k$ is even (like in Figure 7), then an n-ladder has k contrariety rungs and k subcontrariety rungs. If $n = 2k + 1$ is odd, then an n-ladder has k contrariety rungs, 1 contradiction rung and k subcontrariety rungs.

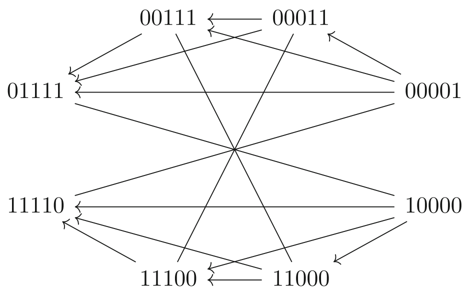

Figure 8. The same 4-ladder as in Figure 7, drawn in a different way

side by side with α-structures, like we do in this paper. Sticking with our example, the same ladder diagram could also be drawn as in Figure 8. This time, we leave out all of the (sub)contarieties in order not to overcomplicate the diagram. Comparing the 4-ladder from Figure 8 to the α_4-structure from Figure 4, we observe a lot of similarities. Informally, the roles of contrariety and subcontrariety in the α_4-structure *correspond in some way* to the role of subalternation in the 4-ladder. Put differently, the tuple $(x_1, \ldots x_n)$ from Definition 7 and the tuple $(y_1, \ldots, y_n)$ from Definition 8 *correspond in some way* to each other. This insight is made more precise in Propositions 3 and 4 below.

Proposition 3. *Let $n \in \mathbb{N}_0$ be a natural number, and consider a σ-diagram $(\mathcal{F}, B)$ such that $|\mathcal{F}| = 2n$ and $0_B, 1_B \notin \mathcal{F}$. Then the following are equivalent:*

1. *$\exists y = (y_1, \ldots, y_n) \in \mathcal{F}^n$ such that $LI_B(y_i, y_{i+1})$ for all $1 \leq i \leq n-1$,*
2. *$\exists Y \subseteq \mathcal{F}$ such that $|Y| = n$ and such that $LI_B(a, b)$ or $LI_B(b, a)$ for all distinct $a, b \in Y$,*
3. *$\exists Y \subseteq \mathcal{F}$ such that $|Y| = n$ and $(a, b) \in LI_B \cup RI_B$ for all distinct $a, b \in Y$.*

Proof. The third item is merely a slight reformulation of the second one, so we focus on proving that the first two items are equivalent. We first prove $1 \Rightarrow 2$. Suppose there exists an n-tuple $y = (y_1, \ldots, y_n)$ as described in item 1. Then it is easy to check that the set $Y := |y|$ satisfies the requirements of item 2. Finally, $2 \Rightarrow 1$ follows from the well-known result that every finite total order is well-ordered (in particular, it is easy to check that $(Y, \leq_B \cap Y^2)$ is a finite total order). ■

Proposition 4. *Let $(\mathcal{F}, B)$ be an α_n-structure with set X of pairwise B-contrary elements of $\mathcal{F}$. Let $(\mathcal{F}', B')$ be an n-ladder with a tuple y of elements of $\mathcal{F}'$ such that $LI_B(y_i, y_{i+1})$ for all $1 \leq i \leq n-1$. Let $f : \mathcal{F} \to \mathcal{F}'$ be* *any negation-preserving bijection such that* $f[X] = |y|$*. Then there exists a bijection* $F : \mathcal{R}_B \to \mathcal{R}_{B'}$ *(recall Definition 3) such that for all* $z, z' \in \mathcal{F}$ *and for all* $R_B \in \mathcal{R}_B$*, we have that* $R_B(z,z') \iff F(R_B)(f(z), f(z'))$*. Moreover, this bijection* F *does not depend on the concrete bijection* f*.*

PROOF. It is easy to check that F can be defined by $F(CD_B) := CD_{B'}$, $F(BI_B) := BI_{B'}$, $F(C_B \cup SC_B) := LI_{B'} \cup RI_{B'}$, and $F(LI_B \cup RI_B) := C_{B'} \cup SC_{B'}$. ■

Proposition 3 provides an alternative characterization of ladder diagrams, which coincides exactly with Definition 7 of α-structures, except for the fact that C_B is replaced with $LI_B \cup RI_B$.[19] Continuing along these lines, Proposition 4 formalizes the comparison between α_n-structures and n-ladders we sketched above.[20] The correspondence between the relations (given by F) and the correspondence between the sets $|x|$ and $|y|$ (given by f) together form the requirement that $R_B(z,z') \iff F(R_B)(f(z), f(z'))$ for all $z, z' \in \mathcal{F}$ and all $R_B \in \mathcal{R}_B$. Although these two propositions nicely highlight the similarities between both classes of Aristotelian families, it does not seem to have much practical use. It would be more interesting to have a way to construct ladders out of α-structures and vice versa. This is what we turn to in the next section.

5. Constructing α-Structures and Ladders

Propositions 1 and 2 show that α_n-structures and n-ladders diverge from each other as soon as $n \geq 3$. These propositions also tell us that n-ladders, weak α_n-structures and strong α_{n+1}-structures all have Boolean complexity $n+1$, i.e., they all have Boolean closures that are isomorphic to $\{0,1\}^{n+1}$, and thus to each other. Therefore, when given one of these three kinds of Aristotelian diagrams, it should be possible to construct both other kinds of diagrams out of it, using only the Boolean operators. In Theorems 1 and 2 we already showed that it is indeed possible to create strong α_{n+1}-structures

[19] Also recall Footnote 8 on the alternative characterization of α-structures that is obtained upon replacing C_B with SC_B in Definition 7.

[20] This comparison can also be expressed as an analogical proportion: *α-structures* are to the *opposition* relations (esp. $C, SC \in \mathcal{OR}$) like *ladders* are to the *implications* relations (esp. $LI, RI \in \mathcal{IR}$). (This is a slight oversimplification, as the opposition relations CD, NCD and the implication relations BI, NI occur across α-structures and ladders alike.) Finally, recall from Section 1 that α-structures and ladders are naturally associated with resp. the *negation* ($\sim$ opposition) and *consequence* ($\sim$ implication) perspectives on the categorical square of opposition.

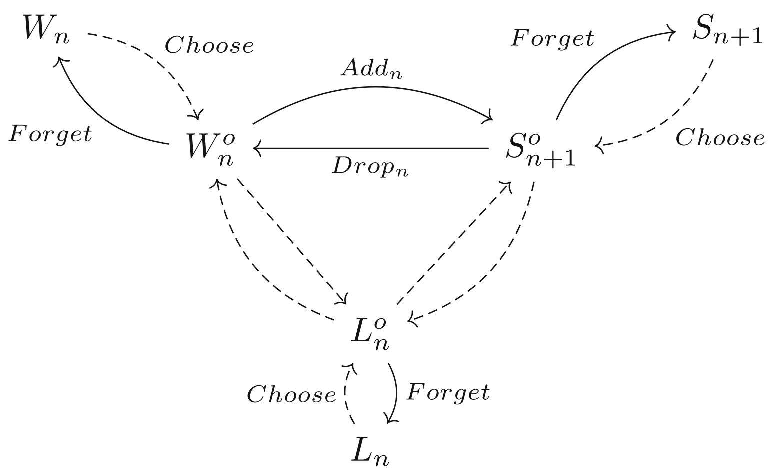

Figure 9. The blueprint for Boolean complexity $n+1$

out of weak α_n-structures, and vice versa. In the remainder of this section, we will extend this story by incorporating the n-ladders. In other words, we are going to define the dashed arrows in the central triangle of Figure 9, for every possible value of n.

5.1. From Ladders to α-Structures

In this subsection we investigate how we can create α-structures out of ladders, using nothing but the latter's elements and the Boolean operators. We first work out a simple example, and then move on to the general constructions. The proof of Proposition 2 already gives us some ideas.

EXAMPLE 4. Suppose we have a 2-ladder $(\{y_1, y_2, \neg y_1, \neg y_2\}, B)$ with tuple (y_1, y_2). Since $LI(y_1, y_2)$, we have that $C(y_1, \neg y_2)$. Now, the identities

$$y_1 \wedge (\neg y_1 \wedge y_2) = (y_1 \wedge \neg y_1) \wedge y_2 = 0$$

and

$$y_1 \vee (\neg y_1 \wedge y_2) = (y_1 \vee \neg y_1) \wedge (y_1 \vee y_2) = y_1 \vee y_2 = y_2 \neq 1$$

show that $C(y_1, \neg y_1 \wedge y_2)$. Similar identities show that also $C(\neg y_2, \neg y_1 \wedge y_2)$. Since we also have that $\neg y_1 \wedge y_2 = \neg(y_1 \vee \neg y_2) \neq \neg 1 = 0$ and $\neg y_1 \wedge y_2 \leq y_2 < 1$, we find that $(\{y_1, \neg y_1 \wedge y_2, \neg y_2, \neg y_1, y_1 \vee \neg y_2, y_2\}, B)$ is an α_3-structure, with set of pairwise contraries $\{y_1, \neg y_1 \wedge y_2, \neg y_2\}$. This α-structure is a strong one, since $y_1 \vee (\neg y_1 \wedge y_2) \vee \neg y_2 = 1$. The situation is summarized by Figure 10.

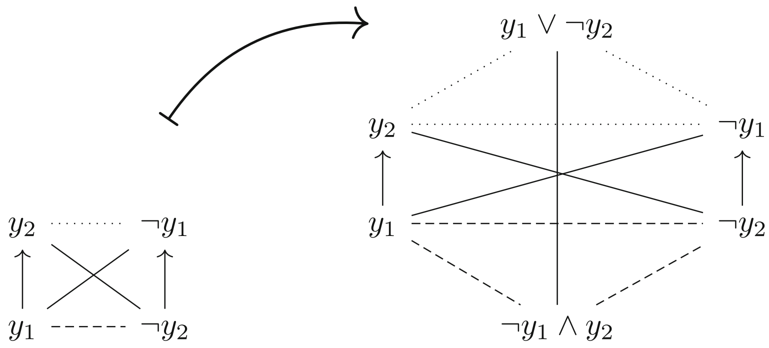

Figure 10. Going from a 2-ladder to a strong α_3-structure

It bears emphasizing that, modulo the extreme elements 0 and 1, this strong α_3-structure is closed under the Boolean operators, i.e., for all $z_1, z_2 \in H := \{y_1, \neg y_1 \wedge y_2, \neg y_2, \neg y_1, y_1 \vee \neg y_2, y_2\}$, we have that $z_1 \vee z_2, z_1 \wedge z_2, \neg z_1 \in H \cup \{0, 1\}$.[21] Furthermore, this α_3-structure is the *smallest* structure that has this property, while containing $\{y_1, y_2, \neg y_1, \neg y_2\}$. Using standard terminology from logical geometry, we say that the strong JSB hexagon on the right of Figure 10 is the *Boolean closure* of the classical square of opposition on its left [2, 14].

The previous example and the proof of Proposition 2 suggest that, given a ladder with tuple $(y_1, \ldots, y_n)$, we should consider elements of the form $\neg y_i \wedge y_{i+1}$ to create a set of pairwise contrary elements. In general, we have the following theorem.

THEOREM 5. *Let $n \geq 2$. Then we have a well-defined function $\alpha_n^s : L_n^o \to S_{n+1}^o$, which for any $D = (\mathcal{F}, B)$ is given by:*

$$\alpha_n^s(D, y) := (\alpha_n^s(D), \alpha_n^s(y)),$$

with $(n+1)$-tuple $\alpha_n^s(y) := (y_1, \neg y_1 \wedge y_2, \ldots, \neg y_{n-1} \wedge y_n, \neg y_n)$ and strong α_{n+1}-structure $\alpha_n^s(D) := (|\alpha_n^s(y)| \cup \neg|\alpha_n^s(y)|, B)$.

PROOF. Entirely analogous to part of the proof of Proposition 2. ■

---

[21] It is easy to check that (weak) α_1-structures, i.e., PCDs of the form $(\{x, \neg x\}, B)$, are also Boolean closed in this sense. Apart from (weak) α_1- and strong α_3-structures, no other α-structures are Boolean closed. In particular, for $n \geq 2$, a weak α_n-structure is not Boolean closed, since it omits, for example, the join of its n pairwise contrary elements ($0 \neq \bigvee_{i=1}^n x_i \neq 1$). For $n \geq 4$, a strong α_n-structure is not Boolean closed either, since it omits, for example, each binary join of its n pairwise contrary elements ($0 \neq x_i \vee x_j \neq 1$, for $1 \leq i \neq j \leq n$).

So far, we only have a way of constructing *strong* α-structures out of ladders. However, Theorem 2 gives us a way of going from strong to weak α-structures. This suggests the following theorem.

Theorem 6. *Let $n \geq 1$. Then we have a well-defined function $\alpha_n^w : L_n^o \to W_n^o$, which for any $D = (\mathcal{F}, B)$ is given by:*

$$\alpha_n^w(D, y) := (\alpha_n^w(D), \alpha_n^w(y)),$$

with n-tuple $\alpha_n^w(y) := (y_1, \neg y_1 \wedge y_2, \ldots, \neg y_{n-1} \wedge y_n)$ and weak α_n-structure $\alpha_n^w(D) := (|\alpha_n^w(y)| \cup \neg|\alpha_n^w(y)|, B)$.

Proof. The proof is analogous to part of the proof of Proposition 2, but now the final disjunct, $\neg y_n$, is missing in the chain of equalities at the very end of that proof. This shows that

$$y_1 \vee (\neg y_1 \wedge y_2) \vee (\neg y_2 \wedge y_3) \vee \cdots \vee (\neg y_{n-1} \wedge y_n) = y_n \neq 1,$$

which implies that $\alpha_n^w(D)$ is a weak α_n-structure.[22] ■

5.2. From α-Structures to Ladders

The final thing left to do is to find ways to create ladders out of α-structures. Of course, it would be nice if these arrows were in some sense the inverses of α_n^s and α_n^w, just like Add_n and $Drop_n$ are each other's inverses (recall Theorem 3). Let us first look at an example which, again, draws its inspiration from Proposition 2.

Example 5. Suppose we have a 3-ladder $D = (\{y_1, y_2, y_3, \neg y_1, \neg y_2, \neg y_3\}, B)$ with tuple $y = (y_1, y_2, y_3)$. Applying α_3^s to (D, y) yields an ordered strong α_4-structure that has $x := (y_1, \neg y_1 \wedge y_2, \neg y_2 \wedge y_3, \neg y_3)$ as its tuple of pairwise contrary elements. So how can we retrieve the original ladder from this constructed α-structure? We can retrieve y_1 by simply taking the first element from x, i.e., y_1. We can retrieve y_2 by taking the join of the first two elements from x, i.e., y_1 and $\neg y_1 \wedge y_2$. Finally, we can retrieve y_3 in two straightforward ways: one is by negating the final element from x, i.e., $\neg y_3$, and the other is by taking the join of the first three elements from x, i.e.,

[22] Note that in case $n = 1$, the function α_1^w maps a PCD (viewed as a 1-ladder) onto itself (but now viewed as a weak α_1-structure). In case $n = 2$, α_2^w maps a classical square of opposition (viewed as a 2-ladder) onto a different, but still Aristotelian isomorphic diagram, viz., onto a different classical square of opposition (but now viewed as a weak α_2-structure). From $n \geq 3$ onwards, α_n^w starts producing non-isomorphic diagrams; for example, α_3^w maps an SC hexagon onto a weak JSB hexagon, α_4^w maps a Lenzen octagon onto a weak Moretti octagon, etc.

y_1, $\neg y_1 \wedge y_2$ and $\neg y_2 \wedge y_3$. Even though the first option is simpler, the latter one is more natural in the sense that it extends the way in which we retrieve the other elements y_1 and y_2.

The previous example suggests the following way of constructing n-ladders out of strong α_{n+1}-structures.

THEOREM 7. *Let $n \geq 2$. Then we have a well-defined function $\lambda_n^s : S_{n+1}^o \to L_n^o$, which for any $D = (\mathcal{F}, B)$ is given by:*

$$\lambda_n^s(D, x) := (\lambda_n^s(D), \lambda_n^s(x)),$$

with n-tuple $\lambda_n^s(x) := (x_1, x_1 \vee x_2, \ldots, \bigvee_{i=1}^n x_i)$ and n-ladder $\lambda_n^s(D) := (|\lambda_n^s(x)| \cup \neg|\lambda_n^s(x)|, B)$.

PROOF. We have the following identities for all $1 \leq j \leq n-1$ that prove that there are subalternations between two consecutive elements in $\lambda_n^s(x)$:

$$\begin{aligned}\left(\bigvee_{i=1}^{j} x_i\right) \wedge \neg\left(\bigvee_{i=1}^{j+1} x_i\right) &= \left(\bigvee_{i=1}^{j} x_i\right) \wedge \left(\bigwedge_{i=1}^{j+1} \neg x_i\right)\\ &= \bigvee_{i=1}^{j}\left(x_i \wedge \bigwedge_{k=1}^{j+1} \neg x_k\right)\\ &= \bigvee_{i=1}^{j} 0 = 0,\end{aligned}$$

and

$$\begin{aligned}\neg\left(\bigvee_{i=1}^{j} x_i\right) \wedge \left(\bigvee_{i=1}^{j+1} x_i\right) &= \left(\bigwedge_{i=1}^{j} \neg x_i\right) \wedge \left(\bigvee_{i=1}^{j+1} x_i\right)\\ &= \bigvee_{i=1}^{j+1}\left(x_i \wedge \bigwedge_{k=1}^{j} \neg x_k\right)\\ &= \bigvee_{i=1}^{j+1} \bigwedge_{k=1}^{j} (x_i \wedge \neg x_k)\\ &= \bigwedge_{k=1}^{j} (x_{j+1} \wedge \neg x_k) = \bigwedge_{k=1}^{j} x_{j+1} = x_{j+1} \neq 0.\end{aligned}$$

In the above, the penultimate identity holds because for all $1 \leq k \leq j$, we have $C(x_k, x_{j+1})$, which implies $LI(x_{j+1}, \neg x_k)$. Since the elements from $\lambda_n^s(x)$ are clearly distinct from each other, from their negations, and from 0 and 1, we have also shown that $\lambda_n^s(D)$ is an n-ladder. ■

We now show that λ_n^s and α_n^s are each other's inverses, as desired.

THEOREM 8. *Let $n \geq 2$. Then $\lambda_n^s \circ \alpha_n^s$ is the identity function on L_n^o and $\alpha_n^s \circ \lambda_n^s$ is the identity function on S_{n+1}^o.*

PROOF. For any n-tuple y, we see that $(\lambda_n^s \circ \alpha_n^s)(y) = \lambda_n^s(\alpha_n^s(y)) = \lambda_n^s((y_1, \neg y_1 \wedge y_2, \ldots, \neg y_{n-1} \wedge y_n, \neg y_n)) = (y_1, y_1 \vee (\neg y_1 \wedge y_2), \ldots, y_1 \vee \bigvee_{i=1}^{n-1} (\neg y_i \wedge y_{i+1}))$. We need to show that this tuple is equal to y. Now, notice that we have the following identities for all $2 \leq j \leq n$:

$$\begin{aligned} y_1 \vee \bigvee_{i=1}^{j-1} (\neg y_i \wedge y_{i+1}) &= y_1 \vee (\neg y_1 \wedge y_2) \vee \bigvee_{i=2}^{j-1} (\neg y_i \wedge y_{i+1}) \\ &= (y_1 \vee \neg y_1) \wedge (y_1 \vee y_2) \vee \bigvee_{i=2}^{j-1} (\neg y_i \wedge y_{i+1}) \\ &= (y_1 \vee y_2) \vee \bigvee_{i=2}^{j-1} (\neg y_i \wedge y_{i+1}) \\ &\vdots \\ &= y_1 \vee y_2 \vee \cdots \vee y_j \\ &= y_j. \end{aligned}$$

The final equality holds since $LI(y_k, y_j)$ for all $1 \leq k \leq j-1$. This proves the first part of the theorem.

For the second part, note that for any $(n+1)$-tuple x, we have that $(\alpha_n^s \circ \lambda_n^s)(x) = \alpha_n^s(\lambda_n^s(x)) = \alpha_n^s((x_1, x_1 \vee x_2, \ldots, \bigvee_{i=1}^n x_i)) = (x_1, \neg x_1 \wedge (x_1 \vee x_2), \ldots, \neg\left(\bigvee_{i=1}^{n-1} x_i\right) \wedge \left(\bigvee_{i=1}^n x_i\right), \neg \bigvee_{i=1}^n x_i)$. We need to show that this tuple is equal to x. Now notice that we have the following identities for all $1 \leq j \leq n-1$:

$$\begin{aligned} \neg\left(\bigvee_{i=1}^{j} x_i\right) \wedge \left(\bigvee_{i=1}^{j+1} x_i\right) &= \left(\bigwedge_{i=1}^{j} \neg x_i\right) \wedge \left(\bigvee_{i=1}^{j+1} x_i\right) \\ &= \bigvee_{i=1}^{j+1} \left(x_i \wedge \bigwedge_{k=1}^{j} \neg x_k\right) \\ &= x_{j+1} \wedge \bigwedge_{k=1}^{j} \neg x_k \\ &= x_{j+1}. \end{aligned}$$

The final equality holds since $C(x_k, x_{j+1})$ implies that $LI(x_{j+1}, \neg x_k)$, for all $1 \leq k \leq j$. The above chains of equalities already prove that $(\alpha_n^s \circ \lambda_n^s)(x) = (x_1, \ldots, x_n, \neg \bigvee_{i=1}^n x_i)$. Since the tuple x comes from a strong α_{n+1}-structure, the exact same line of reasoning that appears in the proof of Theorem 3 can be used here to prove that $x_{n+1} = \neg \bigvee_{i=1}^n x_i$, and hence, $(\alpha_n^s \circ \lambda_n^s)(x) = x$. ∎

So far, we only have a way of constructing ladders out of strong α-structures. However, Theorem 1 gives us a way of going from weak to strong α-structures. This suggests the following theorem.

Theorem 9. *Let $n \geq 1$. Then we have a well-defined function $\lambda_n^w : W_n^o \to L_n^o$, which for any $D = (\mathcal{F}, B)$ is given by:*

$$\lambda_n^w(D, x) := (\lambda_n^w(D), \lambda_n^w(x)),$$

with n-tuple $\lambda_n^w(x) := (x_1, x_1 \vee x_2, \ldots, \bigvee_{i=1}^n x_i)$ and n-ladder $\lambda_n^w(D) := (|\lambda_n^w(x)| \cup \neg|\lambda_n^w(x)|, B)$.

Proof. Entirely analogous to the proof of Theorem 7.[23] ∎

We now show that λ_n^w and α_n^w, too, are each other's inverses, as desired.

Theorem 10. *Let $n \geq 1$. Then $\lambda_n^w \circ \alpha_n^w$ is the identity function on L_n^o and $\alpha_n^w \circ \lambda_n^w$ is the identity function on W_n^o.*

Proof. Analogous to the proof of Theorem 8. ∎

5.3. Putting Everything Together

We now have all the necessary ingredients at our disposal to prove the following satisfying theorem.

Theorem 11. *Let $n \geq 2$. Then we have canonical ways to construct weak α_n-structures, strong α_{n+1}-structures and n-ladders out of each other, which are all compatible with each other. More formally, we have the commutative diagram in Figure 11.*

Proof. It is immediately clear by construction that $\alpha_n^w = Drop_n \circ \alpha_n^s$. Together with Theorems 3, 8 and 10, this is enough to ensure commutativity of the entire diagram. ∎

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[23] For low values of n, entirely analogous remarks apply to λ_n^w as to α_n^w; recall Footnote 22.

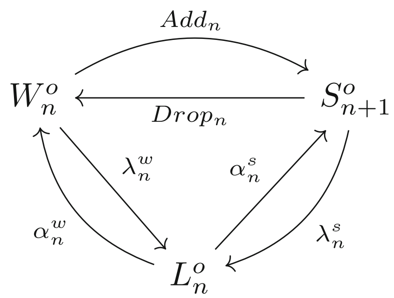

Figure 11. The commutative triangle for Boolean complexity $n+1 \geq 3$

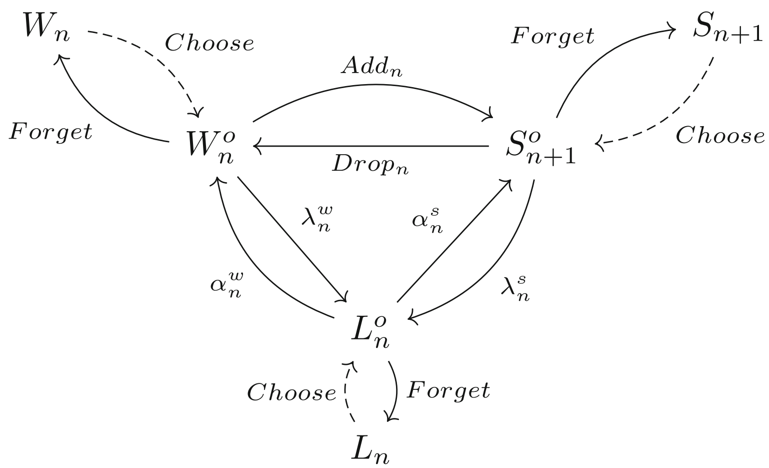

Figure 12. The complete situation for Boolean complexity $n+1 \geq 3$

It bears repeating that the weak α_n-structures, strong α_{n+1}-structures and n-ladders in this diagram all have the same Boolean complexity, $n+1$. For the sake of completeness, we draw the entire picture that covers both ordered and unordered diagrams in Figure 12. This figure shows canonical ways (up to choice of tuples) of constructing weak/strong α-structures and ladders out of each other, which was the main purpose of this paper.

Let us investigate the triangle in Figure 11 in some more detail, for different values of n. From $n \geq 3$ onwards, the classes L_n^o, W_n^o and S_{n+1}^o are all non-empty and pairwise distinct. For $n = 2$, Theorem 4 tells us that the classes W_2^o and L_2^o coincide with each other (both comprise the classical squares of opposition); however, it bears emphasizing that α_2^w and λ_2^w are not identity functions (cf. Footnotes 22 and 23). Finally, for $n = 1$, such a triangle technically does not exist since, by the proof of Proposition 1, S_2^o

is empty (cf. Footnote 9). Therefore, the functions Add_1, $Drop_1$, α_1^s and λ_1^s do not exist either. However, we still have W_1^o and L_1^o, and by Theorem 4, they even coincide (both comprise the PCDs). In this case, α_1^w and λ_1^w are simply identity functions (again cf. Footnotes 22 and 23). It is not hard to link all these triangles together, using the functions f_n from the following theorem.

THEOREM 12. *Let $n \geq 3$. Then we have a well-defined function $f_n : W_n^o \to S_n^o$, which for any $D = (\mathcal{F}, B)$ is given by:*

$$f_n(D, x) := (f_n(D), f_n(x)),$$

with n-tuple $f_n(x) := (x_1, x_2, \ldots, x_{n-1}, \neg \bigvee_{i=1}^{n-1} x_i)$ and α_n-structure $f_n(D) := (|f_n(x)| \cup \neg|f_n(x)|, B)$.

PROOF. Very similar to the proof of Theorem 1. ■

Note that in the spirit of this theorem, it would also be possible to treat the case $n = 2$ in exactly the same way, except for the fact that the range of f_2 is W_1^o instead of S_2^o. This leads to the chain of triangles in Figure 13. For the sake of completeness, we have also included the 'reduced' triangle, which includes W_1^o and L_1^o but lacks S_2^o, as the latter is empty. Above each triangle, we have also written the Boolean complexity (BC) shared by all diagrams mentioned in that triangle.[24] Finally, we could of course also have included all the classes of *unordered* diagrams, by adding *Choose* and *Forget* arrows everywhere (as we did in Figure 12). However, this would only have harmed the simplicity of the figure, so we leave these arrows out on purpose.

Our treatment of α-structures and ladders using the functions from Theorem 11 closely resembles the set-theoretic approach of Pellissier [45]. Given an n-ladder in some appropriate logic, Pellissier implicitly calculates a set-theoretic Boolean algebra that is isomorphic to the Boolean closure of the n-ladder inside the logic's Lindenbaum-Tarski algebra. He then proceeds to

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[24] Given the tight connection between a diagram's Boolean complexity and its Boolean closure (cf. Definition 5), the upper sequence of Boolean complexities in Figure 13 can equivalently be viewed as a sequence of Boolean closures (or yet equivalently, as a sequence of β-structures; cf. Footnote 6). In particular, for $n \geq 1$, every ordered diagram $((\mathcal{F}, B), x) \in L_n^o \cup W_n^o \cup S_{n+1}^o$ has Boolean complexity $n+1$, so its Boolean closure $Cl_B(\mathcal{F})$ is isomorphic to $\{0,1\}^{n+1}$ (or equivalently, $Cl_B(\mathcal{F}) - \{0_B, 1_B\}$ is a β_n-structure). It bears emphasizing, however, that the notion of Boolean closure is far more general than those of α-structure and ladder. Indeed, we can take the Boolean closure $Cl_B(\mathcal{F})$ of any Aristotelian diagram $(\mathcal{F}, B)$ whatsoever, and recent category-theoretical work on logical geometry has shown that this operation of taking the Boolean closure constitutes a (reflective) functor [6].

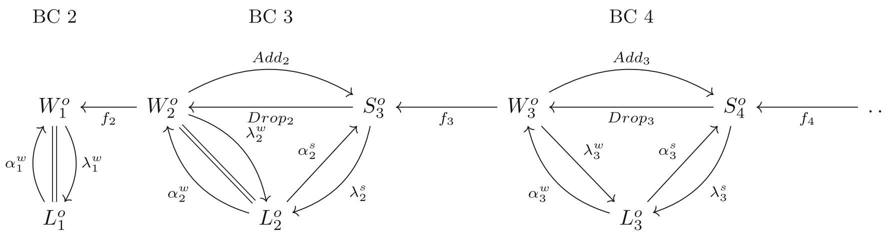

Figure 13. The chain of triangles

search for all possible α-structures inside this algebra. There is only a single strong α_{n+1}-structure, which can also be constructed using α_n^s. To find all n weak α_n-structures, it is not enough to just use α_n^w. Instead, we should first apply α_n^s, then allow for a change of the tuple x (i.e., *Forget* the old tuple and *Choose* a new one), and finally apply $Drop_n$. Anyway, it is clear that Pellissier's intuitions were enough to reach the appropriate conclusions, but that they can be formulated and proven more generally and systematically in the framework of logical geometry.

6. Conclusion

In this paper we combined previous work in n-opposition theory on α-structures and ladders with the contemporary research line of logical geometry, which provides a systematic mathematical framework for studying Aristotelian diagrams. Using this approach, we were able to define α-structures and ladders in a general way, using Boolean algebra. This setting then allowed us to classify these diagrams into infinite series of Aristotelian families and Boolean subfamilies, viz., L_n, W_n and S_n for $n \in \mathbb{N}_0$. Finally, we showed how to move back and forth between these various series, by defining mathematically elegant ways to construct weak/strong α-structures and ladders out of each other. Some of these constructions naturally generalize earlier results by Moretti and Pellissier [43, 45].

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