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Free Constructions in Hoops via ℓ-Groups

Abstract. Lattice-ordered abelian groups, or *abelian ℓ-groups* in what follows, are categorically equivalent to two classes of 0-bounded hoops that are relevant in the realm of the equivalent algebraic semantics of many-valued logics: liftings of cancellative hoops and perfect MV-algebras. The former generate the variety of product algebras, and the latter the subvariety of MV-algebras generated by perfect MV-algebras, that we shall call DLMV. In this work we focus on these two varieties and their relation to the structures obtained by forgetting the falsum constant 0, i.e., product hoops and DLW-hoops. As main results, we first show a characterization of the free algebras in these two varieties as particular weak Boolean products; then, we show a construction that freely generates a product algebra from a product hoop and a DLMV-algebra from a DLW-hoop. In other words, we exhibit the free functor from the two algebraic categories of hoops to the corresponding categories of 0-bounded algebras. Finally, we use the results obtained to study projective algebras and unification problems in the two varieties (and the corresponding logics); both varieties are shown to have (strong) unitary unification type, and as a consequence they are structurally and universally complete.

Keywords: Hoops, Free algebras, Product algebras, Perfect MV-algebras, Unification.

1. Introduction

This work aims at deepening our understanding of some relevant many-valued logics that are deeply connected to lattice-ordered abelian groups (abelian ℓ-groups in what follows), and the role that the falsum constant 0 plays in them. Beyond their purely algebraic interest, abelian ℓ-groups have surely played an important role in the understanding and study of some well-known varieties of algebras that constitute the equivalent algebraic semantics in the sense of Blok–Pigozzi [12] of many-valued logics. In particular MV-algebras, that are the equivalent algebraic semantics of infinite-valued Łukasiewicz logic [19], can be seen as *intervals* of lattice-ordered abelian groups; the latter connection extends to Mundici's categorical equivalence between abelian ℓ-groups with strong unit and MV-algebras [41]. In [32] the

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reader can find a general algebraic approach to Mundici's construction that delves into the connections between ℓ-groups and algebras of many-valued logics. Moreover, it is of particular interest for this work that the algebraic category of abelian ℓ-groups is equivalent to the algebraic categories associated with the following classes of algebras: cancellative hoops (since they are term-equivalent to negative cones of abelian ℓ-groups [9,22]), *liftings* of cancellative hoops [21], and perfect MV-algebras [27]; all these classes play an important role in the study of many-valued logics, and are examples of *hoops*, ordered algebras related to logic first defined in an unpublished manuscript by Büchi and Owens and later on investigated in particular in [2,11,28]. The lifting of a cancellative hoop $\mathbf{C}$, usually written as $\mathbf{2} \oplus \mathbf{C}$, is obtained by adding a new bottom element to $\mathbf{C}$; these algebras generate the variety of product algebras, which is the algebraic semantics of one of the main logics in Hájek's framework of many-valued logics arising from continuous t-norms [35], product logic. Perfect MV-algebras are instead an interesting class of MV-algebras with *infinitesimals* and *co-infinitesimals*, that can be described as the *disconnected rotations* [8,36] of negative cones of ℓ-groups. The first example of a perfect MV-algebra has been given by Chang in [17], which is essentially the disconnected rotation of the negative cone of the integers seen as an ℓ-group. Chang's algebra generates the whole variety generated by perfect MV-algebras, which we will call in what follows DLMV since it was first axiomatized by Di Nola and Lettieri in [27]. Both liftings and disconnected rotations can be seen as instances of the more general construction of *generalized rotation* as studied in the papers [8,16,23] with different levels of generality. In this setting, the two constructions can be thought of as ways to obtain 0-bounded structures starting from abelian ℓ-groups, and preserving all their categorical properties; moreover they constitute two opposed and extreme ways to define *negations*: in the lifting, all non-zero elements have as negation the new bound, 0, while disconnected rotations are involutive structures ($\neg\neg x = x$ for all elements x). These two choices both result in structures that have a *Boolean retraction term* [21]; this entails that they can be described by means of their Boolean skeleton (i.e. the Boolean subalgebra given by the complemented elements) and their radical (i.e. the intersection of their maximal filters), which is the negative cone of an ℓ-group.

The main aim of this work is to deepen the understanding of the variety of product algebras P and the variety DLMV generated by perfect MV-algebras, investigating in particular the role of the falsum constant 0. In order to do so, we study the relationship between product algebras and DLMV-algebras and the varieties of residuated structures that one obtains when forgetting the 0; that is, the varieties of their 0-free subreducts: product hoops and DLW-hoops. From the logical point of view, via algebraizability, we are studying the falsum-free (and therefore also negation-free) fragment of the corresponding logics. We observe that, while the two varieties of product and DLMV-algebras have been deeply studied, and their relationship with cancellative hoops has been quite well investigated (see for instance [21,40]), their 0-free subreducts have received less attention. While DLW-hoops seem absent from the literature, product hoops appear in the seminal work [2] and the two recent works [34,38], that inspired this investigation; in the former it is shown that product hoops coincide with the class of maximal filters of product algebras seen as residuated lattices, and the latter gives a functional representation of finitely generated free product hoops in terms of real-valued functions.

In this work, a first main result is a characterization of the free algebras over an arbitrary set of generators in the two varieties of product and DLW-hoops; the latter are obtained as particular subreducts of the corresponding free algebras in the 0-bounded varieties. More precisely, we obtain a representation in terms of *weak Boolean products* of which we characterize the factors; while this kind of description for 0-bounded residuated lattices is present in the literature, we are not aware of analogous results for varieties of residuated structures with just the constant 1. We observe that in a variety that is the equivalent algebraic semantics of a logic, free (finitely generated) algebras are isomorphic to the Lindenbaum–Tarski algebras of formulas of the logic; thus their study is important from both the perspective of algebra and logic.

As the next main outcome of this work, we go back from hoops to the corresponding 0-bounded varieties, and we exhibit the free functor from the varieties of hoops of interest to the corresponding 0-bounded varieties. In other words, we show a construction that *freely* adds the falsum constant 0: starting from a product hoop (or a DLW-hoop) we obtain the product algebra (DLMV-algebra) freely generated by it. The construction for DLW-hoops is shown to coincide with the *MV-closure* introduced in [1].

Finally, we use the results obtained on free product hoops and free DLW-hoops to characterize finitely generated projective algebras in the two varieties, which turn out to be exactly the finitely presented algebras. From the point of view of the associated logics, via Ghilardi algebraic approach to unification problems [33], this implies that their unification type is (strongly) unitary: there is always a *best solution* to a unification problem, and it is represented algebraically by the identity homomorphism; this is in parallel to the case of product algebras and DLMV-algebras studied in [5]. The study of unification problems is strongly connected to the study of admissible rules (or, in the algebraic setting, admissible quasiequations); a rule is said to be *admissible* in a logic if every substitution that makes the premises a theorem of the logic, also makes the conclusion a theorem of the logic. As a consequence of our results, we get that the logics associated with both product hoops and DLW-hoops are structurally complete, i.e. the admissibility of rules coincides with their derivability; using results in [6], we can actually conclude that the two logics are *universally* complete, that is, admissibility coincides with derivability also for multiple-conclusion rules.

We also anticipate that the results we obtained do not only hold for the varieties of product hoops and DLW-hoops; more precisely, we will carry out the investigation in the setting of the 0-free subreducts of varieties with a Boolean retraction term that are generated by the (previously mentioned) generalized rotation construction. From a methodological point of view, our novel approach shows how to transfer results and techniques from the 0-bounded varieties (that are usually better known) to their 0-free subreducts. It will become clear that, in order to be able to do so fruitfully, a key role is played by the radical of the algebras being a cancellative hoop, or, equivalently, the negative cone of an ℓ-group.

2. Preliminaries

2.1. Residuated Lattices and Hoops

Lattice-ordered abelian groups, or abelian ℓ-groups for short, are abelian groups with a lattice order, with the group operation distributing over the lattice operations. Here we will consider abelian ℓ-groups as their term-equivalent analogue within the theory of (commutative) residuated lattices. For all the unexplained notions of universal algebra we refer to [14], and for the theory of residuated lattices to [31].

A *commutative residuated lattice* is an algebra $\mathbf{A} = (A, \cdot, \to, \wedge, \vee, 1)$ of type $(2, 2, 2, 2, 0)$ such that:

(1) $(A, \cdot, 1)$ is a commutative monoid;

(2) $(A, \wedge, \vee)$ is a lattice;

(3) the residuation law holds: $x \cdot y \leq z$ iff $y \leq x \to z$.

We will often write xy for $x \cdot y$, and x^n for $x \cdot \ldots \cdot x$ (n times). A commutative residuated lattice is: *integral* if the monoidal unit 1 is also the top of the lattice structure; 0-*bounded* if there is an extra constant 0 that is the bottom element of the lattice. In 0-bounded structures we will consider a negation operator defined as $\neg x = x \to 0$, and with that a sum operation $x + y = \neg(\neg x \cdot \neg y)$. Then we also write nx for $x + \ldots + x$ (n times).

Commutative residuated lattices are a variety (see [13]), that we denote with CRL; we denote its subvariety given by integral algebras by CIRL, and sometimes we refer to the algebras therein as CIRLs. 0-bounded commutative integral residuated lattices will be abbreviated as BCIRLs, while their variety is usually denoted with FL_{ew}, since they are the equivalent algebraic semantics of the Full Lambek Calculus with the structural rules of exchange and weakening (see the textbook reference [31] for the connection between residuated structures and substructural logics).

In this framework, abelian ℓ-groups can be defined as commutative residuated lattices satisfying:

$$x(x \to 1) = 1. \tag{2.1}$$

Given an abelian ℓ-group $\mathbf{G}$, its *negative cone* $\mathbf{G}^-$ is the integral residuated lattice with domain $G^- = \{x \in G : x \leq 1\}$, and with operations defined starting from the ones of $\mathbf{G}$ as $x * y = x *_G y$ for $* \in \{\cdot, \wedge, \vee\}$ and $x \to y = (x \to_G y) \wedge 1$ for all $x, y \in G^-$. Negative cones of ℓ-groups constitute a variety LG^- that can be axiomatized relatively to CIRL by divisibility:

$$x \wedge y = x(x \to y) \tag{div}$$

and cancellativity, expressed by the following identity:

$$x \to xy = y. \tag{canc}$$

We remark that satisfying the divisibility equation is equivalent to saying that the order $\leq$ induced by the lattice operations is the inverse divisibility ordering, that is, $x \leq y$ if and only if there exists z such that $x = yz$.

Both LG and LG^- are *semilinear* varieties, i.e. they are generated by totally ordered algebras. In fact, they are generated respectively by $\mathbf{Z}$ and its negative cone $\mathbf{Z}^-$, where $\mathbf{Z}$ is the residuated lattice $\mathbf{Z} = (\mathbb{Z}, +, \ominus, \min, \max, 0)$ given by the integers where the product is the sum, the unit is 0, the implication is the difference ($x \ominus y = y - x$), and the order is the usual one.

Semilinearity in CIRLs is characterized by the prelinearity equation:

$$(x \to y) \vee (y \to x) = 1. \tag{prel}$$

Observe that in CIRLs that satisfy divisibility the meet operation can be defined by means of product and implication (by the very definition of divisibility), and it is well-known that if we add prelinearity as well, then also the join can be rewritten by means of $\{\cdot, \to\}$ as

$$x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x). \tag{2.2}$$

Thus such algebras can be written in the reduced language of *hoops*, $\{\cdot, \to, 1\}$, and are in fact usually called *basic hoops*. More precisely, a *hoop* is an algebra $\mathbf{A} = (A, \cdot, \to, 1)$ of type $(2, 2, 0)$ such that $(A, \cdot, 1)$ is a commutative monoid and for all $x, y, z \in A$,

$$x \to x = 1; \quad x(x \to y) = y(y \to x); \quad xy \to z = x \to (y \to z).$$

In a hoop the relation defined by

$$x \leq y \text{ if and only if } x \to y = 1$$

is a partial order that is not in general a lattice order but a $\wedge$-semilattice order; the meet operation can be defined as $x \wedge y = x(x \to y)$. In fact, hoops could be defined as divisible commutative integral residuated $\wedge$-semilattices. See [2, 11] for more details on the theory of hoops.

We call *cancellative hoops* basic hoops satisfying cancellativity, we denote their variety with CH. Cancellative hoops are term-equivalent to negative cones of ℓ-groups; for simplicity, in what follows we will be speaking of cancellative hoops instead of negative cones of ℓ-groups. Other relevant varieties of hoops are given by *Gödel hoops*, that are basic hoops satisfying idempotency:

$$x^2 = x, \tag{2.3}$$

and Wajsberg hoops. A *Wajsberg hoop* is a hoop which further satisfies

$$(x \to y) \to y = (y \to x) \to x. \tag{2.4}$$

Any cancellative hoop is also a Wajsberg hoop, and all Wajsberg hoops are basic hoops. Finally, a *generalized Boolean algebra* is an idempotent Wajsberg hoop (thus it is also a Gödel hoop). Notice once again that all these varieties are term-equivalent to varieties of CIRLs.

Let us now move to 0-bounded hoops, which are particularly relevant in the realm of many-valued logics in Hájek's framework [35]. Starting again from residuated lattices, a *BL-algebra* is a BCIRL that further satisfies divisibility and prelinearity; BL-algebras can once again be seen in the reduced signature, in this case of 0-bounded hoops, $\{\cdot, \to, 0, 1\}$. BL-algebras constitute a variety, BL, that is the equivalent algebraic semantics in the sense of Blok–Pigozzi of Hájek's basic logic, the logic of continuous t-norms [20]. The most relevant subvarieties of BL are Gödel, product, and MV-algebras.

MV-algebras are BL-algebras satisfying involutivity:

$$x = \neg\neg x; \tag{2.5}$$

Gödel algebras are BL-algebras satisfying idempotency; product algebras constitute the subvariety P of BL-algebras satisfying the following identity:

$$\neg x \vee ((x \to (x \cdot y)) \to y) = 1, \tag{2.6}$$

which in chains (i.e., totally ordered algebras) corresponds to cancellativity for non-zero elements. Adding the equation $x \vee \neg x = 1$ to any of these varieties defines the variety of Boolean algebras BA in this signature. The latter can equivalently be defined as BCIRLs where $x \cdot y = x \wedge y$ and $x \to y = \neg x \vee y$. Given a 0-bounded BCIRL $\mathbf{A}$, a 0-*free subreduct* of $\mathbf{A}$ is a subalgebra of the 0-free reduct of $\mathbf{A}$. Given a variety of BCIRLs, the class of its 0-free subreducts is a variety of CIRLs. We observe in passing that in general the class of subreducts of algebras in a (quasi)variety is always a quasivariety as shown by Mal'cev [37]; the fact that the class of 0-free subreducts V_0 of a variety of BCIRLs is in fact a variety can be easily shown using the fact that commutative residuated lattices have the congruence extension property (for a detailed proof see [44, Proposition 3.1]).

Proposition 2.1. *Basic hoops, Wajsberg hoops, Gödel hoops, and generalized Boolean algebras are the* 0*-free subreducts of, respectively, BL-algebras, MV-algebras, Gödel algebras, and Boolean algebras.*

Cancellative hoops in turn are not the subreducts of a variety of bounded hoops; indeed cancellativity implies that there are no nontrivial bounded (and in particular no finite) models. In the next subsection we introduce two constructions that allow to obtain bounded structures from unbounded ones, and that preserve all the categorical properties.

Before moving on, we stress that since all the structures we are interested in actually do have a term-definable lattice reduct, we will usually consider them in the signature of (B)CIRLs (instead of the one of hoops).

2.2. Generalized Rotations

Given a CIRL $\mathbf{A}$, its *lifting* is the algebra $\mathbf{2} \oplus \mathbf{A}$, having $A \cup \{0\}$ as domain, and the operations extending those of $\mathbf{A}$ in the obvious way: for $x \in A$, $x \cdot 0 = 0 \cdot x = x \to 0 = 0$, and $0 \to x = 1$. Let us mention for the interested reader that the notation makes it apparent that the lifting of $\mathbf{A}$ corresponds to the *ordinal sum* construction of the two-element Boolean algebra $\mathbf{2}$ and $\mathbf{A}$. For a pictorial intuition, see Figure 1.

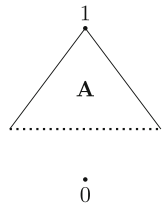


Figure 1. The algebra $\mathbf{2} \oplus \mathbf{A}$, given a CIRL $\mathbf{A}$

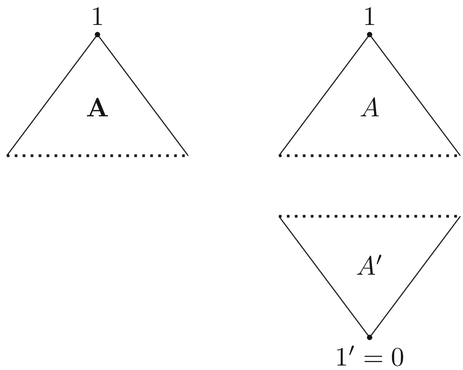


Figure 2. On the left we have $\mathbf{A} \in \mathsf{CIRL}$ and on the right its disconnected rotation

Consider again a CIRL $\mathbf{A}$; its *disconnected rotation* is a bounded involutive structure obtained by gluing below $\mathbf{A}$ its rotated copy (see Figure 2).

More precisely, given a CIRL $\mathbf{A}$, its disconnected rotation $\mathbf{DR}(\mathbf{A})$ is a BCIRL whose lattice reduct is given by the union of A and its disjoint copy $A' = \{a' : a \in A\}$ with dualized order, placed below A: for all $a, b \in A$,

$$a' < b, \text{ and } a' \leq b' \text{ iff } b \leq a.$$

In particular, the top element of $\mathbf{DR}(\mathbf{A})$ is the top 1 of $\mathbf{A}$ and the bottom element of $\mathbf{DR}(\mathbf{A})$ is the copy $0 := 1'$ of the top 1. $\mathbf{A}$ is a subreduct, the products in A' are all defined to be the bottom element $0 = 1'$, and furthermore, for all $a, b \in A$,

$$a \cdot b' = (a \to b)'; \quad a \to b' = (b \cdot a)'; \quad a' \to b' = b \to a.$$

Liftings and disconnected rotations can be seen uniformly via the *generalized rotation construction* (see [8,43] for details, and [30] for the non-commutative version of this construction). The idea is to use *nuclei* to express both constructions uniformly. A nucleus on a residuated structure $\mathbf{A}$ is a closure operator δ on $\mathbf{A}$ such that $\delta(x)\delta(y) \leq \delta(xy)$, for all $x, y \in A$.

Definition 2.2. Let $\mathbf{A}$ be a CIRL; a *rotation operator* for $\mathbf{A}$ is a nucleus that preserves the lattice operations.

The *generalized rotation* $\mathrm{Rot}^{\delta}(\mathbf{A})$ of a CIRL $\mathbf{A}$ with respect to a rotation operator δ on $\mathbf{A}$ differs from the disconnected rotation above in that it replaces A' with $\delta[A]' = \{\delta(a)' : a \in A\}$, where $\delta(a)'$ is short for $(\delta(a))'$. It is easy to see that then with respect to the above order we have $\delta(a)' \wedge \delta(b)' = \delta(\delta(a) \vee \delta(b))'$. Moreover, for all $a \in A$, $b \in \delta[A]$,

$$a \to b' = (\delta(ba))'.$$

Then the reader can easily check that

(1) if $\delta(x) = \bar{1}(x) = 1$ we get the lifting construction;

(2) if $\delta(x) = x$, i.e. it is the identity map, we get the disconnected rotation.

The latter two operators are examples of *term-defined* rotation operators.

Definition 2.3. Let V be a variety of CIRLs; a *term-defined rotation operator* for V is a unary term $\delta(x)$ in the language of CIRLs that defines a rotation operator for each $\mathbf{A}$ in V.

Remark 2.4. In some relevant cases, the two introduced rotation operators (the identity and the map $\bar{1}$) are the only ones that matter. For instance, as observed in [3, §4], they are the only term-defined rotation operators for the variety of cancellative hoops. Note that this does not mean that they are the only rotation operators on such structures: in every distributive CIRL $\mathbf{A}$ (so in particular in every cancellative hoop), fix an element $a \in A$, then the map $\delta(x) = a \vee x$ is a rotation operator.

Following [5] we identify a class of particularly well behaved varieties generated by generalized rotations, which include all the relevant examples.

Definition 2.5. We say that a variety of BCIRLs V is a *term-defined variety of generalized rotations* if V is generated by generalized rotations such that the rotation operator is a fixed term-defined rotation $\delta(x)$.

If we start from a cancellative hoop $\mathbf{C}$, then its lifting is a product algebra, and its disconnected rotation is a *perfect MV-algebra*; in fact, perfect MV-algebras can be defined as disconnected rotations of cancellative hoops. Perfect MV-algebras do not form a variety, and the variety they generate is axiomatized relatively to MV-algebras by

$$2x^2 = (2x)^2 \tag{DL}$$

as shown by Di Nola and Lettieri in [27] (in the same paper they also show that perfect MV-algebras and abelian ℓ-groups are categorically equivalent). We call this variety DLMV. Similarly, liftings of cancellative hoops generate the variety of product algebras. Both product algebras and DLMV-algebras are term-defined varieties of generalized rotations. From all that we have seen above, the following is easy to see. Recall that given a class of algebras K, its associated algebraic category has the elements in K as objects, and the homomorphisms as morphisms.

THEOREM 2.6. *The following algebraic categories are equivalent:* $\mathsf{LG}, \mathsf{LG}^-$, *perfect MV-algebras, liftings of cancellative hoops.*

Thus perfect MV-algebras and liftings of cancellative hoops are 0-bounded structures with the same categorical properties of abelian ℓ-groups. We will now be mostly interested in the class of structures obtained by forgetting the 0 from the algebras in the variety of product algebras and DLMV-algebras, generated respectively by liftings and disconnected rotations of cancellative hoops. In particular, the 0-free subreducts of product algebras and DLMV-algebras constitute, respectively, the varieties of *product hoops* and of *DL-Wajsberg hoops* (DLW-hoops for short).

Product hoops are axiomatized in [2] as basic hoops satisfying:

$$(y \to z) \vee ((y \to xy) \to x) = 1; \tag{2.7}$$

DLW-hoops, since Łukasiewicz sum can be rewritten in a 0-free language as $x + y = (x \to (xy)) \to y$, can be axiomatized with respect to basic hoops by the translation of the equation characterizing DLMV-algebras with respect to MV, and thus by:

$$(x^2 \to x^4) \to x^2 = ((x \to x^2) \to x)^2. \tag{2.8}$$

2.3. Structure Theory and Boolean Retraction Terms

With respect to their structure theory, all the algebras we have introduced are quite well behaved. In particular, they are congruence permutable and 1-regular, that is, the congruences are totally determined by their 1-blocks (i.e., the equivalence class of 1), which we will call *congruence filters* (or *filters* for short). It can be shown that a filter of a hoop (or a CIRL) $\mathbf{A}$ is a nonempty subset of $\mathbf{A}$ closed under multiplication and upwards.

Filters form an algebraic lattice isomorphic with the congruence lattice of $\mathbf{A}$ and if $X \subseteq A$ then the filter generated by X in $\mathbf{A}$ is

$$\mathrm{Fil}_{\mathbf{A}}(X) = \{a \in A : x_1 \cdot \ldots \cdot x_n \leq a, \text{ for some } n \in \mathbb{N} \text{ and } x_1, \ldots, x_n \in X\}. \tag{2.9}$$

We will drop the subscript $\mathbf{A}$ whenever there is no danger of confusion. The isomorphism between the filter lattice and the congruence lattice is given by the maps:

$$\theta \longmapsto 1/\theta = \{a \in A : (a, 1) \in \theta\}$$
$$F \longmapsto \theta_F = \{(a, b) : a \to b,\ b \to a \in F\},$$

where θ is a congruence and F a filter. In what follows, if $F = 1/\theta$ is a filter of an algebra $\mathbf{A}$, we shall write the corresponding quotient with either $\mathbf{A}/\theta$ or $\mathbf{A}/F$. We call a filter F *maximal* if it is not properly contained in any proper filter. Moreover, we call *radical* of an algebra $\mathbf{A}$ the intersection of its maximal filters. The radical is a filter itself, and it is the domain of a CIRL with the operations inherited from $\mathbf{A}$, which we will denote by $\mathrm{Rad}(\mathbf{A})$.

Given a BCIRL $\mathbf{A}$, another subset of its domain that will be relevant in what follows is the *Boolean skeleton* of $\mathbf{A}$. We call an element $x \in A$ *complemented* if $x \vee \neg x = 1, x \wedge \neg x = 0$.

Definition 2.7. Let $\mathbf{A}$ be a BCIRL, its Boolean skeleton $\mathrm{Bool}(\mathbf{A})$ is the Boolean algebra of all the complemented elements of $\mathbf{A}$. We call an element x of $\mathbf{A}$ *Boolean* if $x \in \mathrm{Bool}(\mathbf{A})$.

Notice how the Boolean skeleton always contains 0 and 1; moreover, $\mathrm{Bool}(\mathbf{A})$ is the maximum Boolean algebra contained in $\mathbf{A}$. The following lemma is derived from [15, Lemma 2.5], and essentially follows from the well-known fact that in directly indecomposable BCIRLs (and therefore in subdirectly irreducible ones) the only Boolean elements are 0 and 1.

Lemma 2.8. *Let V be a variety of BCIRLs, and let $\boldsymbol{A} \in \mathsf{V}$. For any $b, b' \in \mathrm{Bool}(\boldsymbol{A})$ and $x, y \in A$ the following hold:*

(1) $b \cdot x = b \wedge x$;

(2) $b \vee x = \neg b \to x$;

(3) $x \vee (b \wedge b') = (x \vee b) \wedge (x \vee b')$ and $x \wedge (b \vee b') = (x \wedge b) \vee (x \wedge b')$;

(4) if $b \vee x = b \vee y$ and $b \to x = b \to y$ then $x = y$.

Definition 2.9. Given a variety V of BCIRLs, a *Boolean retraction term* for V is a unary term $t(x)$ in the language of BCIRLs such that for every $\mathbf{A} \in \mathsf{V}$ the interpretation of t in $\mathbf{A}$ defines a retraction from $\mathbf{A}$ onto $\mathrm{Bool}(\mathbf{A})$, i.e. an idempotent homomorphism from $\mathbf{A}$ onto $\mathrm{Bool}(\mathbf{A})$.

These last notions are quite relevant in varieties generated by generalized rotations; indeed, all such varieties have a Boolean retraction term (given by $2x^2$ or $(2x)^2$ [8,24]). This also implies that such algebras are characterized by their radical, their Boolean skeleton, and an operator that intuitively defines the join between elements of the Boolean skeleton and the radical, as we will discuss more in detail later on (but see [8]). For now, we observe that, in term-defined varieties of generalized rotations (such as product algebras and DLMV-algebras) the radicals of the algebras in V constitute a variety of CIRLs; we observe in passing that the latter property does not hold in general (see [5, Example 2.6]). Given a variety V, we call the *radical class of* V and denote it with R_{V} the class

$$\mathsf{R}_{\mathsf{V}} = \{\mathrm{Rad}(\mathbf{A}) : \mathbf{A} \in \mathsf{V}\}. \tag{2.10}$$

PROPOSITION 2.10. *[5, Proposition 2.10] Let* V *be a term-defined variety of generalized rotations. The radical class of V is a variety.*

We also recall that in a variety with a Boolean retraction term t the radical of an algebra $\mathbf{A} \in \mathsf{V}$ is defined by:

$$\mathrm{Rad}(\mathbf{A}) = \{x \in A : t(x) = 1\}, \tag{2.11}$$

while the Boolean skeleton is given by

$$\mathrm{Bool}(\mathbf{A}) = \{x \in A : t(x) = x\} = \{t(x) : x \in \mathbf{A}\}. \tag{2.12}$$

We observe that both product algebras and DLMV-algebras are term defined varieties and their radical class is the variety of cancellative hoops. Both product algebras and DLMV-algebras are then subvarieties of the variety of all generalized rotations of cancellative hoops. We will call the latter RCH. This variety is semilinear (since prelinearity is preserved by generalized rotations), and it can be axiomatized by prelinearity, the equation (DL) characterizing semilinear varieties with a Boolean retraction term, and the following identity

$$(t(x) \wedge t(y)) \to ((x \to xy) \to y) \wedge (y \to (x \to xy)) = 1, \tag{2.13}$$

where $t(x) = 2x^2$, which entails that the radical of the algebras is a cancellative hoop (see [3]).

REMARK 2.11. By Remark 2.4, the identity and $\bar{1}$ are the only term-defined rotation operators for cancellative hoops; moreover, the trivial variety is the only proper subvariety of cancellative hoops. Thus the only subvarieties of RCH that are term-defined varieties of generalized rotations are: product algebras (generated by rotations of cancellative hoops with the operator $\bar{1}$), DLMV-algebras (generated by rotations of cancellative hoops with the identity) and Boolean algebras (since the 2-element Boolean algebra is the rotation of the trivial algebra by any of those two operators).

2.4. Free Algebras and Weak Boolean Products

In the most relevant varieties of bounded residuated lattices with a Boolean retraction term, free algebras can be described by means of free Boolean algebras and free algebras in the radical class [5,24]. In particular, this representation is in terms of weak Boolean products indexed by the Stone space of the Boolean skeleton of the free algebra, and the factors are generalized rotations of free algebras in the radical class.

The underlying idea is that all varieties of BCIRLs are congruence distributive, and thus they have the *Boolean Factor Congruence property* or BFC: the set of factor congruences of any algebra is a distributive sublattice of its congruence lattice; this notion has been introduced by Chang, Jónsson and Tarski [25]. The BFC implies that one can, to some extent, use the Stone representation Theorem for Boolean algebras to characterize algebras in less manageable varieties. Indeed algebras with constants in varieties with the BFC are representable as *weak Boolean products* of directly indecomposable algebras [26]. Precisely, a weak Boolean product of a family $\{\mathbf{A}_i\}_{i\in I}$ of algebras is a subdirect product $\mathbf{A} \leq \prod_{i\in I} \mathbf{A}_i$, where I can be endowed with a Boolean space topology such that: the set $\{i \in I : a_i = b_i\}$ is open for all $a, b \in A$; if $a, b \in A$ and $N \subseteq I$ is clopen, then the element c, defined by $c_i = a_i$ for $i \in N$ and $c_i = b_i$ for $i \in I\backslash N$, belongs to A.

Thus, each algebra in a subvariety of BCIRLs can be represented as a weak Boolean product of directly indecomposable algebras over the Stone space of its Boolean skeleton. More precisely:

THEOREM 2.12. *[24] Let $\boldsymbol{A}$ be a BCIRL. Then $\boldsymbol{A}$ is representable as the weak Boolean product of the family*

$$\{\boldsymbol{A}/Fil_{\boldsymbol{A}}(U) : U \textit{ is an ultrafilter of } \mathrm{Bool}(\boldsymbol{A})\},$$

over the Boolean space given by the Stone topology on the ultrafilters of $\mathrm{Bool}(\boldsymbol{A})$.

In order to use this description more fruitfully, one wishes to characterize the factors of the decomposition; this can be done effectively in varieties with a Boolean retraction term that are term-defined, and that satisfy another extra condition; the following definitions are also used in [5].

DEFINITION 2.13. Let V be a term-defined variety of generalized rotations, with rotation operator $\delta(x)$. Let $\mathbf{A}, \mathbf{B} \in \mathsf{R}_{\mathsf{V}}$, and let X be a set of generators of $\mathbf{A}$; $f : X \to B$ *respects the rotation operator* if $x = \delta(x)$ implies $\delta(f(x)) = f(x)$. Then V is a *radical-determined variety of generalized rotations* if for all $\mathbf{A}, \mathbf{B} \in \mathsf{R}_{\mathsf{V}}$ and X generating set of $\mathbf{A}$, any map $f : X \to B$ respects the rotation operator.

It can be easily seen that product algebras and DLMV-algebras (and all term-defined varieties whose rotation operator is either the identity or the constant function $\bar{1}$) are radical-determined varieties of generalized rotations. For such varieties, the factors of the weak Boolean product representation are generalized rotations of free algebras in the radical class, as we will now discuss; the reader can check [5, Theorem 3.9] for a representation in the more general setting of term-defined varieties.

In what follows, given a variety V and a set $X = \{x_1, \ldots, x_n\}$, we write $\mathbf{F}_{\mathsf{V}}(X)$ for the free algebra over X in V. Now, notice that given a free Boolean algebra over some set X, $\mathbf{F}_{\mathsf{BA}}(X)$, its ultrafilters are in one-one correspondence with subsets of X, determining which generators belong to the ultrafilter. Let us fix some notation we will be using in the rest of the section. Let V be a variety of BCIRLs with a Boolean retraction term t, and consider a set X. For each $S \subseteq X$, let U_S be the filter generated in $\mathrm{Bool}(\mathbf{F}_{\mathsf{V}}(X))$ in the following way:

$$U_S = \mathrm{Fil}_{\mathrm{Bool}(\mathbf{F}_{\mathsf{V}}(X))}(\{t(x) : x \in S\} \cup \{\neg t(x) : x \in X \setminus S\}); \qquad (2.14)$$

U_S is an ultrafilter of $\mathrm{Bool}(\mathbf{F}_{\mathsf{V}}(X))$, and it generates a filter F_S of $\mathbf{F}_{\mathsf{V}}(X)$:

$$F_S = \mathrm{Fil}_{\mathbf{F}_{\mathsf{V}}(X)}(U_S). \qquad (2.15)$$

The following results about the Boolean skeleton and the radical of free algebras of varieties generated by generalized rotations are respectively from [24, Theorem 4.1] and [5, Corollary 3.12].

THEOREM 2.14. *Let V be a variety of BCIRLs with a Boolean retraction term $t(x)$. Given any set X, $\mathrm{Bool}(\boldsymbol{F}_{\mathsf{V}}(X))$ is the free Boolean algebra over the set of generators $\{t(x) : x \in X\}$.*

THEOREM 2.15. *Let V be a radical-determined variety of generalized rotations with Boolean retraction term $t(x)$ and rotation operator $\delta(x)$. Given any set X, $\boldsymbol{F}_{\mathsf{V}}(X)$ is representable as the weak Boolean product of the family $\{\boldsymbol{F}_{\mathsf{V}}(X)/F_S : S \subseteq X\}$. In particular, for each $S \subseteq X$:*

$$\boldsymbol{F}_{\mathsf{V}}(X)/F_S \cong \mathrm{Rot}^{\delta}(\boldsymbol{F}_{\mathsf{R}_{\mathsf{V}}}(X_S)),$$

where $X_S = \{x/F_S : x \in S\} \cup \{\neg x/F_S : x \in X \backslash S\}$.

We observe that each factor $\mathbf{F}_{\mathsf{V}}(X)/F_S$ in the above representation is a directly indecomposable algebra that is a generalized rotation of its radical with the rotation operator coinciding with its double negation $\neg\neg$ (see [8]); the radical in particular is the free algebra in R_{V} generated by X_S. For the varieties we are interested in, we get that:

(1) if the rotation operator is the identity, $\mathbf{F}_{\mathsf{V}}(X)/F_S$ is the disconnected rotation of the free algebra in R_{V} with $|X|$ many generators;

(2) if the rotation operator is the map $\bar{1}$ constantly equal to 1, $\mathbf{F}_{\mathsf{V}}(X)/F_S$ is the lifting of the free algebra in R_{V} with $|S|$ many generators.

Thus, for the varieties we are mostly interested in, the previous theorem reads as follows.

COROLLARY 2.16. *Let* V *be the variety of product algebras or of DLMV-algebras. Then given any set* X*,* $\mathrm{Bool}(\boldsymbol{F}_{\mathsf{V}}(X))$ *is the free Boolean algebra over the set* $\{2x^2 : x \in X\}$*. Moreover* $\boldsymbol{F}_{\mathsf{V}}(X)$ *is representable as a weak Boolean product indexed by the ultrafilters of* $\mathrm{Bool}(\boldsymbol{F}_{\mathsf{V}}(X))$*, and the factors are isomorphic to:*

(1) $\boldsymbol{2} \oplus \boldsymbol{F}_{\mathsf{CH}}(S)$*, for each* $S \subseteq X$*, if* $\mathsf{V} = \mathsf{P}$*;*

(2) $\mathbf{DR}(\boldsymbol{F}_{\mathsf{CH}}(X))$ *if* $\mathsf{V} = \mathsf{DLMV}$*.*

Finally, we observe that if X is a finite set, the weak Boolean product representation actually yields an isomorphism onto the direct product; therefore, if V is a variety generated by liftings of algebras in a variety R_{V}, if X is a finite set

$$\mathbf{F}_{\mathsf{V}}(X) \cong \prod_{S \subseteq X} \mathbf{2} \oplus \mathbf{F}_{\mathsf{R}_{\mathsf{V}}}(S) \tag{2.16}$$

and if W is a variety generated by disconnected rotations of algebras in a variety R_{W}, with X a finite set

$$\mathbf{F}_{\mathsf{W}}(X) \cong \prod_{S \subseteq X} \mathbf{DR}(\mathbf{F}_{\mathsf{R}_{\mathsf{W}}}(X)). \tag{2.17}$$

Thus in particular $\mathbf{F}_{\mathsf{P}}(X) \cong \prod_{S \subseteq X} \mathbf{2} \oplus \mathbf{F}_{\mathsf{CH}}(S)$ and $\mathbf{F}_{\mathsf{DLMV}}(X) \cong \prod_{S \subseteq X} \mathbf{DR}(\mathbf{F}_{\mathsf{CH}}(X))$.

3. Free Algebras and Weak Boolean Products

We have seen that free algebras in some varieties of BCIRLs with a Boolean retraction term can be nicely described in terms of weak Boolean products of directly indecomposable algebras characterizable as some particular generalized rotations.

In varieties of non 0-bounded residuated lattices one does not have Boolean subalgebras anymore, and while a weak Boolean product representation exists by general results (CIRLs are still congruence distributive), we are not aware of any existing effective description in varieties with just the constant 1 in the signature. In this section we will show how one can obtain a weak Boolean product representation of free algebras in the varieties of product hoops and DLW-hoops.

Let us start by observing that what makes these varieties special in this scenario is the fact that the Boolean retraction term can be equivalently written in the 0-free language. Recall that RCH is the semilinear variety generated by generalized rotations of cancellative hoops.

PROPOSITION 3.1. *Let* V *be a subvariety of* RCH*. Then* $t(x) = (x^2 \to x^4) \to x^2$ *is a Boolean retraction term for* V*.*

PROOF. The term $2x^2$ is a Boolean retraction term for semilinear varieties with a Boolean retraction term [24, Corollary 5.9]. We show that in RCH it holds that $2x^2 = (x^2 \to x^4) \to x^2$.

Let us observe that in varieties generated by generalized rotations all directly indecomposable algebras (and thus in particular all subdirectly irreducible algebras) are generalized rotations of algebras in the radical class ([24, Lemma 3.5], [8, Theorem 2.8]). Thus it suffices to check that $2x^2 = (x^2 \to x^4) \to x^2$ in all generalized rotations of cancellative hoops. Let $\mathbf{A}$ be such an algebra. Then given $a \in A$, either a is in the radical of $\mathbf{A}$ or it is in its rotated copy. In the first case, $2a^2 = 1$, and since the radical is a cancellative hoop we get

$$(a^2 \to a^4) \to a^2 = a^2 \to a^2 = 1.$$

In the second case, $2a^2 = 0$, and

$$(a^2 \to a^4) \to a^2 = (0 \to 0) \to 0 = 0.$$

Thus the two terms $2x^2$ and $t(x) = (x^2 \to x^4) \to x^2$ are equivalent in RCH, and t is therefore a Boolean retraction term for V. ∎

We will now see how this fact is essential to show a characterization of free algebras in the varieties of product hoops and DLW-hoops as particular weak Boolean products. The starting idea comes from the following fact; the proof is standard but for details see [44, Theorem 3.2] where the same is used to show a representation of free Wajsberg hoops as subreducts of free MV-algebras.

THEOREM 3.2. *Let* V *be a variety of BCIRLs, and* V_0 *the variety of CIRLs given by its* 0*-free subreducts. Then, for any set* X*,* $\boldsymbol{F}_{\mathsf{V}_0}(X)$ *is isomorphic to the subalgebra of the* 0*-free reduct of the free algebra* $\boldsymbol{F}_{\mathsf{V}}(X)$ *generated by* X*.*

The previous fact allows one to give a representation for $\mathbf{F}_{\mathsf{V}_0}(X)$ starting from a representation of $\mathbf{F}_{\mathsf{V}}(X)$, by properly characterizing the 0-free subreduct of $\mathbf{F}_{\mathsf{V}}(X)$ generated by X. Recall that we can see the elements of a free algebra $\mathbf{F}_{\mathsf{V}}(X)$ as equivalence classes of terms in the language of V written over variables in X; thus characterizing $\mathbf{F}_{\mathsf{V}_0}(X)$ corresponds to characterizing the equivalence classes of the 0-free terms.

Definition 3.3. Let V be a variety of bounded hoops, or BCIRLs. We will call *positive* an element of $\mathbf{F}_{\mathsf{V}}(X)$ that is the equivalence class of a 0-free term written with variables in X.

We now proceed to show how to "rewrite" elements of free algebras in varieties with a Boolean retraction term, in order to be able to describe the positive ones. Let X be a set of variables, and let $p \in \mathbf{F}_{\mathsf{V}}(X)$; notice that there are $x_1, \ldots, x_n \in X$ such that $p = u(x_1, \ldots, x_n)$ for some term u in the language of V. Let $X_n = \{x_1, \ldots, x_n\}$, then p actually belongs to the subalgebra $\mathbf{F}_{\mathsf{V}}(X_n)$ of $\mathbf{F}_{\mathsf{V}}(X)$. Let us now consider the atoms of the Boolean skeleton of $\mathbf{F}_{\mathsf{V}}(X_n)$, $\mathbf{F}_{\mathsf{BA}}(t(X_n))$ (where by $t(X_n)$ we mean the set $\{t(x_1), \ldots, t(x_n)\}$). It is well known that the atoms of a finitely generated free Boolean algebra can be written as all the possible meets of the generators and their negations; that is, given any subset S of a finite set X_n, we can associate an atom

$$a_S = \bigwedge_{x \in S} t(x) \wedge \bigwedge_{y \in X_n \setminus S} \neg t(y), \tag{3.1}$$

and all atoms can be written in this way.

Notation 1. *We shall write a_+ for a_{X_n} to simplify the notation, since this particular atom will play an important role in what follows.*

One can use the atoms of finitely generated free algebras to rewrite the elements of possibly infinitely generated algebras; let X be any set, and let $p \in \mathbf{F}_{\mathsf{V}}(X)$ as above, $p = u(x_1, \ldots, x_n)$. Then since $\bigvee_{S \subseteq X_n} a_S = 1$ and $\bigwedge_{S \subseteq X_n} \neg a_S = 0$,

$$p = p \wedge \bigvee_{S \subseteq X_n} a_S = \bigvee_{S \subseteq X_n} p \wedge a_S \tag{3.2}$$

and

$$p = p \vee \bigwedge_{S \subseteq X_n} \neg a_S = \bigwedge_{S \subseteq X_n} p \vee \neg a_S = \bigwedge_{S \subseteq X_n} a_S \to p; \tag{3.3}$$

note that we have used Lemma 2.8 for the last equality and for the distributivity over a finitary join and meet of Boolean elements. We also observe that if we consider for each $S \subseteq X_n$ an element $p_S \in \mathbf{F}_{\mathsf{V}}(X_n)$ such that $a_S \to p = a_S \to p_S$ then

$$p = \bigwedge_{S \subseteq X_n} a_S \to p_S. \tag{3.4}$$

REMARK 3.4. With respect to the weak Boolean product representation

$$\mathbf{F}_{\mathsf{V}}(X) \leq \prod_{S \subseteq X} \mathbf{F}_{\mathsf{V}}(X)/F_S,$$

we observe that an element $p \in \mathbf{F}_{\mathsf{V}}(X)$ embeds into the tuple

$$(p/F_S)_{S \subseteq X} = ((a_S \to p)/F_S)_{S \subseteq X},$$

since $a_S \in U_S \subseteq F_S$ and then $a_S/F_S = 1/F_S$. We will often use this representation in proofs.

One can then reduce the analysis of an element p in some free algebra to its "components" $a_S \to p = a_S \to p_S$. While the previous observations seem promising, we are not aware of general methods to rewrite the elements of the kind $a_S \to p$ in order to check whether they are equivalent to positive terms; however, we will see that a uniform approach is possible whenever the Boolean retraction term t is positive, which is the case for product algebras and DLMV-algebras (and actually for algebras in RCH in general). The following is easy to prove.

LEMMA 3.5. *If b is an element of a free Boolean algebra, b is either positive or $b = \neg b'$ and b' is positive.*

We can then prove the next key lemma.

LEMMA 3.6. *Let V be a variety of BCIRLs with a positive Boolean retraction term t. Then $b \in \mathrm{Bool}(\boldsymbol{F}_{\mathsf{V}}(X))$ is positive if and only if $b \in U_X$.*

PROOF. We start by showing the left-to-right direction by contrapositive; assume that $b \notin U_X$. Then, since U_X is an ultrafilter, $\neg b \in U_X$, i.e., there are $x_1, \ldots, x_n \in X$ such that $t(x_1) \wedge \ldots \wedge t(x_n) \leq \neg b$. Then we get that $b \wedge t(x_1) \wedge \ldots \wedge t(x_n) \leq b \wedge \neg b = 0$, and thus $b \wedge t(x_1) \wedge \ldots \wedge t(x_n) = 0$. Since we are assuming that t is positive, if b were to be positive then 0 would be positive as well; the latter yields a contradiction, since 0 is not positive (to see it, observe that $0 = \neg 1$ and so it is not positive by Lemma 3.5). Thus if $b \notin U_X$, b is not positive.

Assume now that b is not positive, we show that $b \notin U_X$. Notice that since $b \in \mathrm{Bool}(\mathbf{F}_{\mathsf{V}}(X))$, b is in the equivalence class of some $p(t(x_1), \ldots, t(x_n))$, p being some term. By Lemma 3.5, either p is positive or it is equivalent to the negation of a 0-free term; but since t is positive and b is not, we get that necessarily $p = \neg p'$ for some p' written in the 0-free language. Now, each $t(x_i) \in U_X$ by definition, and (ultra)filters are closed under the operations $\{\cdot, \to, \wedge, \vee\}$, thus $p'(t(x_1), \ldots, t(x_n)) \in U_X$, which implies that $b = \neg p'(t(x_1), \ldots, t(x_n)) \notin U_X$. This completes the proof. ∎

It is now immediate to see that given a finite set $X_n = \{x_1, \ldots, x_n\}$, the only positive atom of $\mathbf{F}_{\mathsf{BA}}(t(X_n))$ is a_+ (since it is the only one in U_{X_n}), while all other atoms a_S for $S \subsetneq X_n$ are not positive.

Lemma 3.7. *Let* V *be a variety of BCIRLs with a positive Boolean retraction term* t*. Let* X *be any set of variables, and consider* $p(x_1, \ldots, x_n) \in \boldsymbol{F}_{\mathsf{V}}(X)$*; then given* $X_n = \{x_1, \ldots, x_n\}$*, for every* $S \subsetneq X_n$ *the Boolean element*

$$a_S = \bigwedge_{x \in S \subseteq X_n} t(x) \wedge \bigwedge_{y \in X_n \setminus S} \neg t(y)$$

is such that $a_S \to p$ *is positive.*

Proof. As observed right above, $a_S \notin U_{X_n}$ and then $\neg a_S \in U_{X_n}$, thus the latter is positive by Lemma 3.6. Consider the term p' obtained starting from p and substituting each occurrence of 0 with $\neg a_S$ (if p has no occurrences of 0, it means that it is positive, and we get $p' = p$); p' is 0-free and then positive by construction.

We show that $p \vee \neg a_S = p' \vee \neg a_S$. Let us consider their representative in the weak Boolean product representation $\mathbf{F}_{\mathsf{V}}(X) \leq \prod_{S' \subseteq X} \mathbf{F}_{\mathsf{V}}(X)/F_{S'}$, i.e. respectively the tuples

$$(p \vee \neg a_S/F_{S'})_{S' \subseteq X} \text{ and } (p' \vee \neg a_S/F_{S'})_{S' \subseteq X}.$$

We distinguish two cases: $\neg a_S \in U_{S'}$ and $\neg a_S \notin U_{S'}$. In the former case, we get that $p \vee \neg a_S, p' \vee \neg a_S \in F_{S'}$ and then $(p \vee \neg a_S)/F_{S'} = (p' \vee \neg a_S)/F_{S'} = 1/F_{S'}$. In the latter case, $\neg a_S \notin U_{S'}$ implies that $\neg a_S/F_{S'} = 0/F_{S'}$ and then by the definition of p' we get that $(p \vee \neg a_S)/F_{S'} = (p' \vee \neg a_S)/F_{S'}$. Thus $p \vee \neg a_S$, which coincides with $a_S \to p$ by Lemma 2.8, is positive. ∎

The previous result is extremely useful for our purpose, since it yields that in the case in which the Boolean retraction term is positive, the positivity of a term p is completely determined by its "component" $a_+ \to p$. We will now show that such an element is positive if and only if its component $(a_+ \to p)/F_X$ in the weak Boolean product representation belongs to the radical of the directly indecomposable factor.

Theorem 3.8. *Let* V *be a radical-determined variety of generalized rotations with a positive Boolean retraction term* t*. Let* X *be any set of variables,* *and consider* $p(x_1,\dots,x_n) \in \boldsymbol{F}_{\mathsf{V}}(X)$*; let* $a_+ = t(x_1) \wedge \dots \wedge t(x_n)$*. Then* p *is positive if and only if* $a_+ \to p$ *is positive if and only if* $t(a_+ \to p) \in U_X$*.*

Proof. The first equivalence follows directly from Lemma 3.7. We now show that if $a_+ \to p$ is positive then $t(a_+ \to p) \in U_X$. Let us consider the representation of the element $a_+ \to p$ in the weak Boolean product, i.e., the tuple $((a_+ \to p)/F_S)_{S \subseteq X}$. Since we assume that $a_+ \to p$ is positive, there is a 0-free term over variables $x_1,\dots,x_n$ in X such that $a_+ \to p = q(x_1,\dots,x_n)$. Since t is a Boolean retraction term, by definition it induces a homomorphism, thus $t(q(x_1,\dots,x_n)) = q(t(x_1),\dots,t(x_n))$; since each $t(x_i) \in U_X$, and q is written with operations in $\{\cdot,\to,\wedge,\vee\}$ under which U_X is closed, $t(a_+ \to p) = q(t(x_1),\dots,t(x_n)) \in U_X$.

Conversely, suppose that $t(a_+ \to p) \in U_X$, we will show that $a_+ \to p$ is positive by finding a 0-free term q such that $a_+ \to p = a_+ \to q$; first we assume without loss of generality that $a_+ \to p = p'(x_1,\dots,x_n)$ for some term p' in the language of V. Notice that $t(a_+ \to p)$ is an element of the free Boolean algebra $\mathbf{F}_{\mathsf{BA}}(t(X)) = \mathrm{Bool}(\mathbf{F}_{\mathsf{V}}(X))$ and therefore it can be written in canonical normal form as a disjunction of conjunctions of literals over variables $t(x_1),\dots,t(x_n)$; it follows that if we consider the subalgebra of $\mathbf{F}_{\mathsf{V}}(X)$ given by the finitely generated free algebra $\mathbf{F}_{\mathsf{V}}(X_n)$ (where we mean $X_n = \{x_1,\dots,x_n\}$), then

$$t(a_+ \to p) \in U_X \subseteq \mathbf{F}_{\mathsf{V}}(X) \text{ if and only if } t(a_+ \to p) \in U_{X_n} \subseteq \mathbf{F}_{\mathsf{V}}(X_n),$$

since both statements are fully determined by the fact that $t(x_1),\dots,t(x_n) \in U_{X_n} \subseteq U_X$.

Let us now consider the representation of $\mathbf{F}_{\mathsf{V}}(X_n)$ as a weak Boolean product, i.e.

$$\mathbf{F}_{\mathsf{V}}(X_n) \leq \prod_{S \subseteq X_n} \mathbf{F}_{\mathsf{V}}(X_n)/F_S.$$

Let us have a look at the factor $\mathbf{F}_{\mathsf{V}}(X_n)/F_{X_n}$; this is a directly indecomposable algebra that is a generalized rotation of its radical $\mathbf{F}_{\mathsf{R}_{\mathsf{V}}}(X_n)$. Since we are assuming that $t(a_+ \to p) \in U_{X_n} \subseteq F_{X_n}$, it follows that $t(a_+ \to p)/F_{X_n} = 1/F_{X_n}$, which implies that $a_+ \to p/F_{X_n}$ belongs to the radical $\mathbf{F}_{\mathsf{R}_{\mathsf{V}}}(X_n)$ (by definition of radical, see identity (2.11)). Therefore, there is a 0-free term q such that $(a_+ \to p)/F_{X_n} = q(x_1,\dots,x_n)/F_{X_n}$. We will finally show that $a_+ \to p = a_+ \to q$; since the latter is positive, this will conclude the proof. Consider $a_+ \to p$ and $a_+ \to q$ as their representatives given by the tuples $((a_+ \to p/F_S))_{S \subseteq X_n}$ and $((a_+ \to q)/F_S)_{S \subseteq X_n}$. Now if $S = X_n$, then

$$(a_+ \to p)/F_{X_n} = q(x_1, \dots, x_n)/F_{X_n} = (a_+ \to q(x_1, \dots, x_n))/F_{X_n}$$

since $a_+/F_{X_n} = 1/F_{X_n}$. If $S \subsetneq X_n$, then $a_+ \notin U_S$ and also $a_+ \notin F_S$, thus $a_+/F_S = 0/F_S$ and then

$$(a_+ \to p)/F_S = (a_+ \to q(x_1, \dots, x_n))/F_S = 1/F_S.$$

We have shown that $a_+ \to p = a_+ \to q$, and thus $a_+ \to p$ is positive, which concludes the proof. ■

Thus, we obtain the following.

THEOREM 3.9. *Let V be a radical-determined variety of generalized rotations with a positive Boolean retraction term t, and let X be any set of variables. The free algebra of V_0 over X is the 0-free subreduct of the weak Boolean product of $\boldsymbol{F}_{\mathsf{V}}(X)$ given by the elements $p \in \boldsymbol{F}_{\mathsf{V}}(X)$ such that $t(a_+ \to p) \in U_X$.*

REMARK 3.10. Notice that an element $p \in \mathbf{F}_{\mathsf{V}}(X)$ is such that $t(a_+ \to p) \in U_X$ if and only if $p/F_X = (a_+ \to p)/F_X \in \mathrm{Rad}(\mathbf{F}_{\mathsf{V}}(X)/F_X)$.

We can rephrase this result in terms of weak Boolean products in the case of radical-determined varieties of generalized rotations. In order to do so, we first restrict the filters F_S for each $S \subseteq X$ to the positive elements.

NOTATION 2. *From now on, given a BCIRL $\boldsymbol{A}$ we write $\boldsymbol{A}_0$ for its 0-free reduct.*

Let V be a radical-determined variety of generalized rotations with a positive Boolean retraction term t. Let X be any set, and for each $S \subseteq X$ set

$$G_S = \{t(x) : x \in S\} \cup \{t(x) \to (t(y) \wedge y) : x \in X \setminus S, y \in X\}; \quad (3.5)$$

and let F_S^+ be the filter generated by G_S in $\mathbf{F}_{\mathsf{V}_0}(X)$:

$$F_S^+ = \mathrm{Fil}_{\mathbf{F}_{\mathsf{V}_0}(X)}(G_S). \quad (3.6)$$

LEMMA 3.11. *Let V be a radical-determined variety of generalized rotations with a positive Boolean retraction term t. Let X be any set, then F_S^+ is the restriction to $\boldsymbol{F}_{\mathsf{V}_0}(X)$ of $F_S \subseteq \boldsymbol{F}_{\mathsf{V}}(X)$.*

PROOF. Recall that F_S is the filter of $\mathbf{F}_{\mathsf{V}}(X)$ generated by

$$\{t(x) : x \in S\} \cup \{\neg t(x) : x \in X \setminus S\}.$$

We will show that $F_S^+ = F_S \cap \mathbf{F}_{\mathsf{V}_0}(X)$, i.e. F_S^+ is the restriction of F_S to positive elements. First we prove that $x \in F_S^+$ implies $x \in F_S \cap \mathbf{F}_{\mathsf{V}_0}(X)$. Let then $x \in F_S^+$; by definition F_S^+ is a filter of $\mathbf{F}_{\mathsf{V}_0}(X)$ so we really only need to show that $x \in F_S$. By the definition of a filter generated by a set (see the identity (2.9)), $x \in F_S^+$ implies that there are two finite subsets I, J that select (possibly repeated) elements of G_S, say $P_I = \{t(x_i) : i \in I, x_i \in S\}$ and $Q_J = \{t(x_j) \to (t(y_j) \wedge y_j) : j \in J, x_j \in X \backslash S, y_j \in X\}$ such that

$$x \geq \bigodot_{i \in I} t(x_i) \cdot \bigodot_{j \in J} [t(x_j) \to (t(y_j) \wedge y_j)]$$

where we are using $\bigodot$ to denote the iteration of the usual monoidal operation. Now, each $t(x_i) \in P_I$ is also a generator of F_S. Moreover, for each $x_j \notin S$ we have that $\neg t(x_j) \in F_S$, and in $\mathbf{F}_{\mathsf{V}}(X)$

$$t(x_j) \to (t(y_j) \wedge y_j) \geq t(x_j) \to 0 = \neg t(x_j).$$

Therefore

$$x \geq \bigodot_{i \in I} t(x_i) \cdot \bigodot_{j \in J} t(x_j) \to (t(y_j) \wedge y_j) \geq \bigodot_{i \in I} t(x_i) \cdot \bigodot_{j \in J} \neg t(x_j) \in F_S,$$

and then $x \in F_S$.

For the other direction, let $x \in F_S \cap \mathbf{F}_{\mathsf{V}_0}(X)$, we show that then $x \in F_S^+$. We will consider x as its representative in the weak Boolean product $\mathbf{F}_{\mathsf{V}}(X) \leq \prod_{S' \subseteq X} \mathbf{F}_{\mathsf{V}}(X)/F_{S'}$, that is, as the string $(x/F_{S'})_{S' \subseteq X}$. Notice that since $x \in F_S$ there are finite subsets I', J' selecting (possibly repeated) elements among the generators of F_S, say $M_I = \{t(x_i) : i \in I, x_i \in S\}$ and $N_J = \{\neg t(y_j) : j \in J, y_j \in X \backslash S\}$ such that

$$x \geq \bigodot_{i \in I} t(x_i) \cdot \bigodot_{j \in J} \neg t(y_j).$$

Let us write $z = \bigodot_{i \in I} t(x_i) \cdot \bigodot_{j \in J} \neg t(y_j)$, then $x \geq z$. Now, $x = p(v_1, \ldots, v_n)$ for some positive term p and variables $\{v_1, \ldots, v_n\} \subseteq X$; we will show that there exists $m \in \mathbb{N}$ such that the following element $z' \in F_S^+$:

$$z' = \bigodot_{i \in I} t(x_i) \cdot \bigodot_{j \in J, k = 1 \ldots n} \big(t(y_j) \to (t(v_k) \wedge v_k)\big)^m$$

is such that $z' \leq x$. Note that this will conclude the proof. Precisely, let us consider the term $p(x_1, \ldots, x_n)$ as an element of $\mathbf{F}_{\mathsf{CH}}(\{x_1, \ldots, x_n\})$, and let m be such that

$$p(x_1, \ldots, x_n) \geq \bigodot_{k = 1 \ldots n} x_k^m;$$

such m exists since all elements of the free cancellative hoop can be written as a join of meets of powers of the generators ([7, Theorem 3.1.1]). In particular,

$$p(x_1,\dots,x_n) = \bigvee_{i\in I^*}\bigwedge_{j\in J^*} x_{ij}^{m_{ij}} \geq \bigwedge_{j\in J^*} x_{i'j}^{m_{i'j}} \geq \bigodot_{j\in J^*} x_{i'j}^{m_{i'j}}$$

for $i' \in I^*$, and then we take $m = \max\{m_{i'j} : j \in J^*\}$.

We proceed to prove that the desired inequality holds in each component, i.e. $z'/F_{S'} \leq x/F_{S'}$ for each $S' \subseteq X$. Notice first that $z'/F_S \leq x/F_S = 1/F_S$. We now consider the cases where $S' \subseteq X, S' \neq S$; for the next calculations we will see once again $\mathbf{F}_{\mathsf{V}_0}(X)$ as a subreduct of $\mathbf{F}_{\mathsf{V}}(X)$, in order to be able to use the elements of the kind $0/F_{S'}$.

(1) If for all $i \in I$ and $j \in J$, $x_i \in S'$ and $y_j \notin S'$, we have that $z/F_{S'} = 1/F_{S'}$. Thus since $z \leq x$ we get that also $x/F_{S'} = 1/F_{S'}$ and then clearly $z'/F_{S'} \leq x/F_{S'}$;

(2) if there exists $i \in I$ such that $x_i \notin S'$ we get that $z'/F_{S'} = 0/F_{S'} \leq x/F_{S'}$.

(3) If for all $i \in I$ $x_i \in S'$ and there exists $j \in J$ such that $y_j \in S'$ then $t(x_i)/F_{S'} = t(y_j)/F_{S'} = 1/F_{S'}$ and

$$z'/F_{S'} \leq \bigodot_{k=1\dots n} (v_k \wedge t(v_k))^m/F_{S'}.$$

If there is a $v_k \notin S'$ we get that $z'/F_{S'} = 0/F_{S'} \leq x/F_{S'}$. Otherwise, if all $v_1 \dots v_n$ are in S' then it holds that $v_1/F_{S'} \dots v_n/F_{S'} \in \mathrm{Rad}(\mathbf{F}_{\mathsf{V}}(X)/F_{S'})$ which is a free cancellative hoop. Thus by the choice of m we get that

$$x/F_{S'} = p(v_1,\dots,v_n)/F_{S'} \geq \bigodot_{k=1\dots n} v_k^m/F_{S'} = z'/F_{S'}.$$

Therefore, for each $S' \subseteq X$, $x/F_{S'} \geq z'/F_{S'}$, which shows that $x \in F_S^+$ and concludes the proof. ∎

Corollary 3.12. *Let* V *be a radical-determined variety of generalized rotations with a positive Boolean retraction term* t *and rotation operator* δ. $\boldsymbol{F}_{\mathsf{V}_0}(X)$ *is representable as the weak Boolean product over the Boolean space given by the Stone topology on the ultrafilters of* $\boldsymbol{F}_{\mathsf{BA}}(t(X))$, *given by the family* $\{\boldsymbol{F}_{\mathsf{V}_0}(X)/F_S^+ : S \subseteq X\}$. *Moreover the factors can be represented as follows:*

$$\boldsymbol{F}_{\mathsf{V}_0}(X)/F_S^+ \cong [\mathrm{Rot}^\delta(\boldsymbol{F}_{\mathsf{R}_\mathsf{V}}(X_S))]_0$$

for $S \subsetneq X$ *and where* $X_S = \{x/F_S^+ : x \in S\} \cup \{(x \to t(x))/F_S^+ : x \in X \backslash S\}$, *and*

$$\boldsymbol{F}_{\mathsf{V}_0}(X)/F_X^+ \cong \boldsymbol{F}_{\mathsf{R}_\mathsf{V}}(X).$$

PROOF. By Theorem 3.9, $\mathbf{F}_{\mathsf{V}_0}(X)$ embeds in the 0-free reduct of $\prod_{S \subseteq X} \mathbf{F}_\mathsf{V}(X)/F_S$ via the map $i : x \mapsto (x/F_S)_{S \subseteq X}$, which selects only the positive elements; in symbols

$$\mathbf{F}_{\mathsf{V}_0}(X) \hookrightarrow [\prod_{S \subseteq X} \mathbf{F}_\mathsf{V}(X)/F_S]_0 = \prod_{S \subseteq X} [\mathbf{F}_\mathsf{V}(X)/F_S]_0.$$

In particular, Theorem 3.9 (together with Remark 3.10) yields that the image of i is the product

$$\prod_{S \subsetneq X} [\mathbf{F}_\mathsf{V}(X)/F_S]_0 \quad \times \quad \mathrm{Rad}(\mathbf{F}_\mathsf{V}(X)/F_X).$$

Now, it follows from the fact that F_S^+ is the restriction of F_S to positive elements (Lemma 3.11) and the representation of positive elements in the weak Boolean product that for each $S \subsetneq X$

$$\mathbf{F}_{\mathsf{V}_0}(X)/F_S^+ \cong [\mathbf{F}_\mathsf{V}(X)/F_S]_0,$$

and

$$\mathbf{F}_{\mathsf{V}_0}(X)/F_X^+ \cong \mathrm{Rad}(\mathbf{F}_\mathsf{V}(X)/F_X).$$

In particular, the reader can check that the isomorphism is given by the maps $x/F_S^+ \mapsto x/F_S$. Therefore, $\mathbf{F}_{\mathsf{V}_0}(X)$ embeds into $\prod_{S \subseteq X} \mathbf{F}_{\mathsf{V}_0}(X)/F_S^+$.

This embedding actually gives a subdirect representation, since the composition of the embedding with the projection to the factor indexed by $S \subseteq X$, π_S, is onto. Moreover, it is easy to check that it is a weak Boolean product as a consequence of the fact that $\mathbf{F}_\mathsf{V}(X)$ had a weak Boolean product representation with respect to the same Boolean space. Indeed, the product is indexed over the Boolean space given by the ultrafilters of $\mathbf{F}_{\mathsf{BA}}(t(X))$ with the Stone topology, as in the 0-bounded case. Moreover, for all $a, b \in \mathbf{F}_{\mathsf{V}_0}(X)$,

$$\{S \subseteq X : a/F_S^+ = b/F_S^+\} = \{S \subseteq X : a/F_S = b/F_S\}$$

and then they are both open; finally, for all $a, b \in \mathbf{F}_{\mathsf{V}_0}(X)$ and $N \subseteq \mathcal{P}(X)$ clopen, we can see that the element c defined by $c/F_S^+ = a/F_S^+$ if $S \in N$ and $c/F_S^+ = b/F_S^+$ otherwise, belongs to $\mathbf{F}_{\mathsf{V}_0}(X)$. Indeed we know that c/F_X^+ belongs to $\mathbf{F}_\mathsf{V}(X)$; now, c/F_X^+ belongs to $\mathbf{F}_{\mathsf{V}_0}(X)$ if and only if $c/F_X^+ = p/F_X^+$

for some p positive; the latter follows from the fact that c/F_X^+ is either a/F_S^+ or b/F_S^+, and both a and b are positive elements.

Finally, it follows from Theorem 2.15, and the observation that $x \to t(x)/F_S = \neg x/F_S$ for all $x \in X\backslash S$, that for each $S \subsetneq X$:

$$\mathbf{F}_{\mathsf{V}_0}(X)/F_S^+ \cong [\mathbf{F}_{\mathsf{V}}(X)/F_S]_0 \cong [\mathrm{Rot}^\delta(\mathbf{F}_{\mathsf{R_V}}(X_S))]_0,$$

and

$$\mathbf{F}_{\mathsf{V}_0}(X)/F_X^+ \cong \mathrm{Rad}(\mathbf{F}_{\mathsf{V}}(X)/F_X) \cong \mathbf{F}_{\mathsf{R_V}}(X).$$

■

Applying this corollary to the two varieties we are interested in, PH and DLWH, we get the following.

Corollary 3.13. *The free algebra of* PH *over a set X is representable as the weak Boolean product over the Boolean space given by the Stone topology on the ultrafilters of $\boldsymbol{F}_{\mathsf{BA}}(t(X))$, where $t(x) = (x^2 \to x^4) \to x^2$, and the factors are given by the family*

$$\{(\boldsymbol{2} \oplus \boldsymbol{F}_{\mathsf{CH}}(S))_0 : S \subsetneq X\} \cup \{\boldsymbol{F}_{\mathsf{CH}}(X)\}.$$

Corollary 3.14. *The free algebra of* DLWH *over a set X is representable as the weak Boolean product over the Boolean space given by the Stone topology on the ultrafilters of $\boldsymbol{F}_{\mathsf{BA}}(t(X))$, where $t(x) = (x^2 \to x^4) \to x^2$, and the factors are given by the family*

$$\{\mathbf{DR}(\boldsymbol{F}_{\mathsf{CH}}(X))_0 : S \subsetneq X\} \cup \{\boldsymbol{F}_{\mathsf{CH}}(X)\}.$$

Once again, when considering a finite set of variables, the representation yields an isomorphism and we get the following representations.

Corollary 3.15. *Let* V *be a radical-determined variety of generalized rotations, with rotation operator δ, and positive Boolean retraction term t. For every finite set X it holds that*

$$\boldsymbol{F}_{\mathsf{V}_0}(X) \cong \Big(\prod_{S \subsetneq X} [\mathrm{Rot}^\delta(\boldsymbol{F}_{\mathsf{R_V}}(X_S))]_0\Big) \times \boldsymbol{F}_{\mathsf{R_V}}(X),$$

where $X_S = \{x/F_S^+ : x \in S\} \cup \{(x \to t(x))/F_S^+ : x \in X\backslash S\}$. In particular:

$$\boldsymbol{F}_{\mathsf{PH}}(X) \cong \Big(\prod_{S \subsetneq X} (\boldsymbol{2} \oplus \boldsymbol{F}_{\mathsf{CH}}(S))_0\Big) \times \boldsymbol{F}_{\mathsf{CH}}(X),$$

$$\boldsymbol{F}_{\mathsf{DLWH}}(X) \cong \Big(\prod_{S \subsetneq X} \mathbf{DR}(\boldsymbol{F}_{\mathsf{CH}}(X))_0\Big) \times \boldsymbol{F}_{\mathsf{CH}}(X).$$

We observe that the result for product hoops in the finitely generated case has also been obtained in [38] using different methods (more precisely, via a functional representation of finitely generated product hoops).

Remark 3.16. Note that given any set X, $\mathbf{F}_{\mathsf{V}_0}(X)$ is a maximal filter of $\mathbf{F}_{\mathsf{V}}(X)$. This is a direct consequence of Theorem 3.9 and Remark 3.10; indeed, in the representation, $\mathbf{F}_{\mathsf{V}_0}(X)$ is the algebra with domain:

$$\{p \in \mathbf{F}_{\mathsf{V}}(X) : t(a_+ \to p) \in U_X\} = \{p \in \mathbf{F}_{\mathsf{V}}(X) : p/F_X \in \mathrm{Rad}(\mathbf{F}_{\mathsf{V}}(X)/F_X)\}.$$

This set is a filter, since if $p/F_X, q/F_X \in \mathrm{Rad}(\mathbf{F}_{\mathsf{V}}(X)/F_X)$ then also $p \cdot q/F_X = p/F_X \cdot q/F_X \in \mathrm{Rad}(\mathbf{F}_{\mathsf{V}}(X)/F_X)$, and if $p/F_X \in \mathrm{Rad}(\mathbf{F}_{\mathsf{V}}(X)/F_X)$ and $p \leq q$ then $p/F_X \leq q/F_X \in \mathrm{Rad}(\mathbf{F}_{\mathsf{V}}(X)/F_X)$. Moreover it is maximal; indeed, let F be a proper filter of $\mathbf{F}_{\mathsf{V}}(X)$ and $\mathbf{F}_{\mathsf{V}_0}(X) \subsetneq F$, then there is $z \in F \backslash \mathbf{F}_{\mathsf{V}_0}(X)$; since $a_+ \in \mathbf{F}_{\mathsf{V}_0}(X) \subsetneq F$ also $t(a_+ \to z) \in F$ given that t is positive. But from the fact that $z \notin \mathbf{F}_{\mathsf{V}_0}(X)$ it follows that $t(a_+ \to z) \notin U_X$, hence $\neg t(a_+ \to z) \in U_X \subseteq \mathbf{F}_{\mathsf{V}_0}(X) \subsetneq F$, a contradiction.

To conclude this section let us finally comment that, intuitively, one can see that the whole free product algebra (or DLMV-algebra) can be recovered using the information encoded in its maximal filter given by the corresponding free product hoop (DLW-hoop). In the next section we will expand on this intuition, and show in generality how given any hoop that is in the variety RCH_0 of 0-free subreducts of RCH one can construct the RCH-algebra freely generated by it.

4. Free Functors

As anticipated, in this section we will give a description of the free functor from the relevant subvarieties of RCH_0 to their 0-bounded correspondent. In other words, we present a way to freely construct a product algebra starting from a product hoop and a DLMV-algebra starting from a DLW-hoop; we will call the first construction *product closure* and show that the latter corresponds to the MV-closure construction in [1] (introduced to freely construct an MV-algebra from a Wajsberg hoop). The construction we introduce can be seen as a way to introduce *freely* and *uniformly* the falsum constant 0 to both product hoops and DLW-hoops.

4.1. Representation in Triples and Quadruples

As briefly mentioned in the preliminaries, both product algebras and DLMV-algebras can be uniquely determined by a triple consisting of a Boolean algebra, a cancellative hoop, and a binary operator representing the join operation between their elements. In order to give more context to the reader, we mention that in lattice theory triples constructions date back to Chen and Grätzer's 1969 decomposition theorem for Stone algebras: each Stone algebra is characterized by the triple consisting of its lattice of complemented elements, its lattice of dense elements, and a map associating these structures [18]. Their approach has been generalized for instance in [42] to semigroups, and more recently it has been extended to account for residuated structures in [15,16]. Here we follow the slightly different (but equivalent in this context, see [43, §2.6]) approach independently initiated in [40] to describe product algebras, and later followed in [8] to represent the algebras in varieties generated by generalized rotations.

We now recall the representation in triples in more details, since we will use it to define the free constructions we are interested in. The starting idea is that the elements can be rewritten in terms of a *Boolean component* and a *radical component* (see [16, Corollary 4.11], [8, Proposition 2.10]).

Proposition 4.1. *Let* V *be a variety of BCIRLs generated by generalized rotations, and let* $\boldsymbol{A} \in \mathsf{V}$*. Then for every* $x \in A$*:*

$$x = (b(x) \vee \neg c(x)) \wedge (\neg b(x) \vee c(x))$$

where $b(x) = 2x^2 \in \mathrm{Bool}(\boldsymbol{A})$ *and* $c(x) = x \vee \neg x \in \mathrm{Rad}(\boldsymbol{A})$*.*

However, Boolean skeleton and radical are not sufficient to characterize an algebra in a variety with a Boolean retraction term, since one can find non-isomorphic algebras with the same Boolean skeleton and the same radical; see [40, Theorem 3.3] for a proof in the case of product algebras. The idea in [8,40] is to add a binary operator intuitively representing the join between a Boolean and a radical element. Let us be more precise.

Definition 4.2. Let $\mathbf{B}$ be a Boolean algebra and $\mathbf{C}$ be a CIRL, and let, for $b \in B$ and $c \in C$,

$$h_b(x) = b \vee_e x, \qquad k_c(x) = x \vee_e c.$$

A map $\vee_e : B \times C \to C$ is an *external join between* $\mathbf{B}$ *and* $\mathbf{C}$ if it satisfies the following:

- (V1) For fixed $b \in B$ and $c \in C$, h_b is an endomorphism of $\mathbf{C}$ and k_c is a lattice homomorphism from (the lattice reduct of) $\mathbf{B}$ into (the lattice reduct of) $\mathbf{C}$.
- (V2) h_0 is the identity on $\mathbf{C}$, and h_1 is constantly equal to 1.

(V3) For all $b, b' \in B$ and for all $c, c' \in C$, $h_b(c) \vee h_{b'}(c') = h_{b\vee b'}(c \vee c') = h_b(h_{b'}(c \vee c'))$.

(V4) For all $b \in B$ and for all $c, c' \in C$, $(b \vee_e c) \cdot c' = (\neg b \vee_e c') \wedge (b \vee_e (c \cdot c'))$.

Both product algebras and DLMV-algebras are equivalent to a category whose objects are triples of the form $(\mathbf{B}, \mathbf{C}, \vee_e)$, where $\mathbf{B}$ is a Boolean algebra, $\mathbf{C}$ is a cancellative hoop and $\vee_e$ is an external join between $\mathbf{B}$ and $\mathbf{C}$ (and therefore, they are also equivalent to each other [8]). In order to deal with different rotations operators uniformly, one has to also take into account the rotation operator as well.

Definition 4.3. A *Boolean quadruple* is a quadruple of the form $(\mathbf{B}, \mathbf{C}, \vee_e, \delta)$, where $\mathbf{B}$ is a Boolean algebra, $\mathbf{C}$ is a CIRL, $\vee_e$ is an external join between $\mathbf{B}$ and $\mathbf{C}$, and δ is a rotation operator on $\mathbf{C}$.

Boolean quadruples are the objects of a category BQ whose morphisms are called *good morphism pairs* and are defined as follows.

Definition 4.4. A good morphism pair from a quadruple $(\mathbf{B}, \mathbf{C}, \vee_e, \delta)$ to another quadruple $(\mathbf{B}', \mathbf{C}', \vee_e', \delta')$ is a pair (h, k) where:

(1) h is a homomorphism from $\mathbf{B}$ to $\mathbf{B}'$;

(2) k is a homomorphism from $\mathbf{C}$ to $\mathbf{C}'$ such that $k \circ \delta = \delta' \circ k$;

(3) (h, k) respects the external join, i.e. $k(x \vee_e y) = h(x) \vee_e' k(y)$ for all $x \in B$ and $y \in C$,.

We also mention that in the category of Boolean quadruples, given two good morphism pairs (h, k) and (h', k'), the composition is defined as $(h, k) \circ (h', k') = (h \circ h', k \circ k')$.

Boolean quadruples are categorically equivalent to (the algebraic category of) the variety generated by all generalized rotations, called MVR_2 and axiomatized in [16, 43]. In particular, given any algebra $\mathbf{A}$ in MVR_2, $\mathbf{A}$ corresponds to the quadruple $(\mathrm{Bool}(\mathbf{A}), \mathrm{Rad}(\mathbf{A}), \vee, \neg\neg)$. Vice versa, starting from a quadruple $(\mathbf{B}, \mathbf{C}, \vee_e, \delta)$ one can define a BCIRL in the following way. First, one fixes an equivalence relation on the direct product $B \times C$ intuitively describing which pairs (b, c) represent the same element: for every $(b, c), (b', c') \in B \times C$, $(b, c) \sim (b', c')$ if and only if

$$b = b', \quad \neg b \vee_e c = \neg b \vee_e c' \quad \text{and} \quad b \vee_e \delta(c) = b \vee_e \delta(c'). \tag{4.1}$$

We will write $[b, c]$ for $(b, c)/\sim$. Then the BCIRL corresponding to the quadruple $(\mathbf{B}, \mathbf{C}, \vee_e, \delta)$ is the algebra

$$\mathbf{B} \otimes_{\vee_e}^{\delta} \mathbf{C} = (B \times C/\sim, \odot, \Rightarrow, \sqcap, \sqcup, [0, 1], [1, 1])$$

where the operations are defined as follows (and $h_b(x) = b \vee_e x$ as above):

$$[b, c] \odot [b', c'] = [b \wedge b', h_{b\vee\neg b'}(c' \to c) \wedge h_{\neg b\vee b'}(c \to c') \wedge h_{\neg b\vee\neg b'}(c \cdot c')]$$

$$[b, c] \Rightarrow [b', c'] = [b \to b', h_{b\vee b'}(\delta(c) \to \delta(c')) \wedge h_{\neg b\vee b'}(\delta(c \cdot c')) \wedge h_{\neg b\vee\neg b'} (c \to c')]$$

$$[b, c] \sqcap [b', c'] = [b \wedge b', h_{b\vee b'}(c \vee c') \wedge h_{b\vee\neg b'}(c) \wedge h_{\neg b\vee b'}(c') \wedge h_{\neg b\vee b'}(c \wedge c')]$$

$$[b, c] \sqcup [b', c'] = [b \vee b', h_{b\vee b'}(c \wedge c') \wedge h_{b\vee\neg b}(c') \wedge h_{\neg b\vee b'}(c) \wedge h_{\neg b\vee\neg b'}(c \vee c')].$$

Moreover, any algebra $\mathbf{A}$ in a variety of generalized rotations is isomorphic to $\mathrm{Bool}(\mathbf{A}) \otimes_{\vee}^{\neg\neg} \mathrm{Rad}(\mathbf{A})$.

Remark 4.5. Despite their cumbersome look, the previous operations can be easily parsed if the reader thinks of them as a "definition by cases". The idea is to think of the Boolean and radical components of an element as their representative in a subdirect representation, where the Boolean component is either 0 or 1. Then, given an element x with components (b, c) and an element y with components (b', c'), e.g. the radical component of the product xy reads: if $b = 0$ and $b' = 1$ then the product is $c' \to c$; if $b = 1$ and $b' = 0$ it is $c \to c'$; if $b = b' = 1$ it is cc'; if $b = b' = 0$ it is 1.

We described the relationship between the objects of MVR_2 and BQ; with respect to the morphisms, given $\mathbf{A}, \mathbf{B} \in \mathsf{MVR}_2$ and a homomorphism $h : \mathbf{A} \to \mathbf{B}$, this corresponds to the good morphism pair given by the restrictions: $(h_{\restriction \mathrm{Bool}(\mathbf{A})}, h_{\restriction \mathrm{Rad}(\mathbf{A})})$; vice versa, given a good morphism pair $(h, k) : (\mathbf{B}, \mathbf{C}, \vee_e, \delta) \to (\mathbf{B}', \mathbf{C}', \vee_e', \delta')$, the corresponding homomorphism is given by the map $f : \mathbf{B} \otimes_{\vee_e}^{\delta} \mathbf{C} \to \mathbf{B}' \otimes_{\vee_e'}^{\delta'} \mathbf{C}$ defined by

$$f[b, c] = [h(b), k(c)]. \tag{4.2}$$

As shown in [5], the category of DLMV-algebras is equivalent to the full subcategory of BQ where the CIRLs are cancellative hoops and the rotation operator is $\delta = id$, and the category of product algebras is equivalent to the full subcategory of BQ where the CIRLs are cancellative hoops and the rotation operator is $\delta(x) = 1$. Thus any DLMV-algebra $\mathbf{A}$ can be seen as $\mathbf{B}(\mathbf{A}) \otimes_{\vee}^{id} \mathbf{C}(\mathbf{A})$ and any product algebra $\mathbf{A}$ as $\mathbf{B}(\mathbf{A}) \otimes_{\vee}^{\bar{1}} \mathbf{C}(\mathbf{A})$. We will use this construction in the next subsection.

4.2. Closure Construction

In order to be able to use the previous representation to talk about the 0-free subreducts, one needs to rewrite the Boolean and radical components in the 0-free language. In the previous section we have considered varieties with a positive Boolean retraction term; here this request does not suffice, indeed we need to be able to express the radical component $x \vee \neg x$ of an element x without the use of 0 (or $\neg$) as well.

While the latter tasks seems rather hopeless to tackle in general, it can easily be achieved if the radical is a cancellative hoop. Let us consider again the variety RCH, generated by generalized rotations of cancellative hoops. Recall by Proposition 4.1 that $b(x) = 2x^2 \in \mathrm{Bool}(\mathbf{A})$ and $c(x) = x \vee \neg x \in \mathrm{Rad}(\mathbf{A})$.

Lemma 4.6. *Let* V *be a subvariety of* RCH*. Then for any* $\boldsymbol{A} \in \mathsf{V}, x \in A$,

$$b(x) = (x^2 \to x^4) \to x^2, \qquad c(x) = x \to x^2.$$

Proof. The fact that the Boolean component is $b(x) = (x^2 \to x^4) \to x^2$ is shown in Proposition 3.1. For the second part, observe that it suffices to show that

$$x \to x^2 = x \vee \neg x$$

in directly indecomposable algebras; recall that the latter are generalized rotations of cancellative hoops. Let then $\mathbf{A}$ be a generalized rotation of a cancellative hoop $\mathbf{C}$, and let $x \in A$. Either $a \in \mathrm{Rad}(\mathbf{A}) = C$, or it is in its rotated copy.

If $x \in C$, then $x \to x^2 = x$ by cancellativity and $x \vee \neg x = x$ by construction. If x belongs to the rotated copy of $\mathbf{C}$, then by construction $\neg x \in C$ and then

$$x \vee \neg x = \neg x = x \to 0 = x \to x^2.$$

The proof is complete. ∎

The two terms above will be used in the rest of the section; let us denote them by β and γ:

$$\beta(x) = (x^2 \to x^4) \to x^2, \quad \gamma(x) = x \to x^2. \tag{4.3}$$

Combining Lemma 4.6 and the representation of the elements in Proposition 4.1 we get the following.

Proposition 4.7. *Let* V *be a subvariety of* RCH*. Then for any* $\boldsymbol{A} \in \mathsf{V}$ *and* $x \in A$

$$x = (\beta(x) \vee (\gamma(x) \to \beta(x))) \wedge (\beta(x) \to \gamma(x)).$$

In particular, if $\boldsymbol{A}$ *is a product algebra then*

$$x = \beta(x) \wedge \gamma(x).$$

PROOF. The fact that $b(x) = \beta(x)$, $c(x) = \gamma(x)$ is shown in Lemma 4.6. The representation of the elements in Proposition 4.1 can be rewritten as above given the fact that $b(x) \vee (c(x) \to b(x)) = b(x) \vee \neg c(x)$, as it can be easily shown using the subdirect representation. The fact that $b(x) \to c(x) = \neg b(x) \vee c(x)$ follows from Lemma 2.8.

Finally, for the result about product algebras, it can be shown again using the subdirect representation that

$$(b(x) \vee (c(x) \to b(x))) \wedge (b(x) \to c(x)) = b(x) \wedge c(x),$$

which concludes the proof. ∎

Observe that if one considers any algebra $\mathbf{A}$ in RCH_0 (e.g. a product hoop or a DLW-hoop), $\mathbf{A}$ is a 0-free subreduct of some bounded algebra $\mathbf{B}$ in RCH. Therefore any $x \in A$ can be seen as an element of $\mathbf{B}$, and thus it can be represented by means of its Boolean and radical component by Proposition 4.7. Since $b(x), c(x)$, and the representation itself is written in the 0-free language, the following result holds:

PROPOSITION 4.8. *Let* $\boldsymbol{A} \in \mathsf{RCH}_0$*, then for any* $x \in A$*:*

$$x = (\beta(x) \vee (\gamma(x) \to \beta(x))) \wedge (\beta(x) \to \gamma(x)).$$

If $\boldsymbol{A}$ *is a product hoop then*

$$x = \beta(x) \wedge \gamma(x).$$

We now follow some ideas used in [34], where a similar strategy has been used to show that all product hoops are maximal filters of product algebras; let $\mathbf{S} \in \mathsf{RCH}_0$, we consider the following two sets:

$$\mathbf{G}(\mathbf{S}) = \{\beta(x) \ : \ x \in S\}, \qquad \mathbf{C}(\mathbf{S}) = \{\gamma(x) \ : \ x \in S\}. \tag{4.4}$$

The proof of the following Lemma is the same as in [34, Lemma 3.1].

LEMMA 4.9. *Let* $\boldsymbol{S}$ *be an algebra in* RCH_0*; then* $\boldsymbol{G}(\boldsymbol{S})$ *is a generalized Boolean algebra and* $\boldsymbol{C}(\boldsymbol{S})$ *is a cancellative hoop.*

Our aim is to use Boolean quadruples to define a product algebra starting from a product hoop and a DLMV-algebra starting from a DLW-hoop; the idea is to use the intuition stemming from the representation of free product hoops as subreducts of free product algebras (and free DLW-hoops as subreducts of free DLMV-algebras) shown in the previous section (Corollaries 3.13, 3.14 and 3.15). Let us focus for instance on the case of product hoops; the idea is to reconstruct a free product algebra $\mathbf{F}_{\mathsf{P}}(X)$ from its maximal filter given by the corresponding free product hoop $\mathbf{F}_{\mathsf{PH}}(X)$. With respect to the Boolean skeleton, a Boolean element is either in the maximal filter or its negation is; the radical of $\mathbf{F}_{\mathsf{P}}(X)$ is instead all included in $\mathbf{F}_{\mathsf{PH}}(X)$. The only piece of information left is the join between Boolean and cancellative elements; notice that if $b \in \mathbf{F}_{\mathsf{PH}}(X)$, then also $b \vee c \in \mathbf{F}_{\mathsf{PH}}(X)$ for all elements c in the radical; while if $b \notin \mathbf{F}_{\mathsf{PH}}(X)$, then $b \vee c = \neg b \to c$ (Lemma 2.8) and $\neg b \in \mathbf{F}_{\mathsf{PH}}(X)$. This circle of ideas leads to the following choices.

Let V be a term-defined subvariety of RCH, $\mathbf{S} \in \mathsf{V}_0$, and consider the generalized Boolean algebra $\mathbf{G}(\mathbf{S})$ defined above. We first obtain a Boolean algebra $\mathbf{B}(\mathbf{S})$ of which $\mathbf{G}(\mathbf{S})$ is isomorphic to a maximal filter by applying the MV-closure construction in [1]. Let $\mathbf{A}$ be a Wajsberg hoop, then its *MV-closure* is the MV-algebra $\mathbf{MV}(\mathbf{A})$ with domain $A \cup A^*$, with $A^* = \{a^* : a \in A\}$ and the operations extending those of $\mathbf{A}$ in the following way:

$$a \cdot b^* = (a \to b)^*; \quad a^* \cdot b^* = (a \oplus b)^*;$$
$$a \to b^* = (a \cdot b)^*; \quad a^* \to b = a \oplus b; \quad a^* \to b^* = b \to a$$

where $a \oplus b = (a \to ab) \to b$. Notice that the negation is then such that $\neg a = a^*$, $\neg(a^*) = a$, and then $\mathbf{MV}(\mathbf{A})$ is the disjoint union of A and $\neg A = \{\neg a : a \in A\} = A^*$. Since generalized Boolean algebras are idempotent Wajsberg hoops we can apply the MV-closure construction to them in particular, and we can observe the following.

Proposition 4.10. *[34] Let* $\boldsymbol{A}$ *be a generalized Boolean algebra, then its MV-closure* $\mathbf{MV}(\boldsymbol{A})$ *is a Boolean algebra of which* $\boldsymbol{A}$ *is a maximal filter.*

Thus, given $\mathbf{S}$ in RCH_0, let

$$\mathbf{B}(\mathbf{S}) = \mathbf{MV}(\mathbf{G}(\mathbf{S})). \tag{4.5}$$

Moreover, for any $b \in \mathbf{B}(\mathbf{S})$ and $c \in \mathbf{C}(\mathbf{S})$, we set

$$b \vee_f c = \begin{cases} b \vee c \text{ if } b \in \mathbf{G}(\mathbf{S}), \\ \neg b \to c \text{ otherwise} \end{cases} \tag{4.6}$$

We observe that $b \vee_f c : \mathbf{B}(\mathbf{S}) \times \mathbf{C}(\mathbf{S}) \longrightarrow \mathbf{C}(\mathbf{S})$, since the domain of $\mathbf{C}(\mathbf{S})$, that is the set $\{c(x) : x \in S\} = \{x \in S : b(x) = 1\}$, is closed upwards. Moreover, the following result holds:

Lemma 4.11. *Let* $\boldsymbol{S} \in \mathsf{RCH}_0$ *and let* δ *be a rotation operator for* $\boldsymbol{C}(\boldsymbol{S})$*, then* $(\boldsymbol{B}(\boldsymbol{S}), \boldsymbol{C}(\boldsymbol{S}), \vee_f, \delta)$ *is a Boolean quadruple.*

Proof. By definition $\mathbf{B}(\mathbf{S}) \cap \mathbf{C}(\mathbf{S}) = \{1\}$, and the facts that $\mathbf{B}(\mathbf{S})$ is a Boolean algebra and $\mathbf{C}(\mathbf{S})$ is a cancellative hoop follow from Lemma 4.9 and Proposition 4.10. It is left to show that $\vee_f$ is an external join, i.e. that properties $V1$-$V4$ in Definition 4.2 hold.

We will use once again the subdirect representation in order to show that some identities involving elements $b \in \mathbf{G}(\mathbf{S})$ and $c \in \mathbf{C}(\mathbf{S})$ hold. Notice that even though $\mathbf{S}$ is not a bounded structure, it is a 0-free subreduct of an algebra in RCH; moreover, since $\mathbf{G}(\mathbf{S})$ and $\mathbf{C}(\mathbf{S})$ are images of the domain of $\mathbf{S}$ via the terms β and γ, a 0-free identity involving elements $b \in \mathbf{G}(\mathbf{S})$ and $c \in \mathbf{C}(\mathbf{S})$ is satisfied in any algebra $\mathbf{S} \in \mathsf{V}_0 \subseteq \mathsf{RCH}_0$ if and only if the same equation is satisfied in (the 0-free reduct of) any totally ordered algebra $\mathbf{A}$ in V with $b \in \mathrm{Bool}(\mathbf{A}), c \in \mathrm{Rad}(\mathbf{A})$. Thus, in order to check some of the following identities we will restrict to considering the elements $b \in \mathbf{G}(\mathbf{S})$ to be either 1 or 0 (the latter seen as a bottom element rather than a constant operation).

Let us start with V1. It follows straightforwardly from the definition that for a fixed $b \in \mathbf{B}(\mathbf{S})$ the map h_b is an endomorphism of $\mathbf{C}(\mathbf{S})$. The fact that k_c is a lattice homomorphism from (the lattice reduct of) $\mathbf{B}(\mathbf{S})$ into (the lattice reduct of) $\mathbf{C}(\mathbf{S})$ can be shown by cases using the subdirect representation; we show that

$$k_c(b \wedge b') = k_c(b) \wedge k_c(b'), \quad k_c(b \vee b') = k_c(b) \vee k_c(b') \tag{4.7}$$

in the case where $b \in \mathbf{G}(\mathbf{S}), b' \notin \mathbf{G}(\mathbf{S})$ (thus $\neg b' \in \mathbf{G}(\mathbf{S})$), and we leave the other cases to the reader. Note that $b \wedge b' \notin \mathbf{G}(\mathbf{S})$ while $b \vee b' \in \mathbf{G}(\mathbf{S})$. Moreover, since $\mathbf{B}(\mathbf{S})$ is a Boolean algebra, $\neg(b \wedge b') = b \to \neg b' \in \mathbf{G}(\mathbf{S})$ and $b \vee b' = \neg b' \to b \in \mathbf{G}(\mathbf{S})$. Using this, the identities above in (4.7) become

$$(b \to \neg b') \to c = (b \vee c) \wedge (\neg b' \to c) \quad \text{and} \quad (\neg b' \to b) \vee c = b \vee c \vee (\neg b' \to c),$$

which can be easily shown using the subdirect representation of $\mathbf{S}$. The other cases are shown similarly.

V2 is easily seen to hold, i.e., h_0 is the identity on $\mathbf{C}(\mathbf{S})$, and h_1 is constantly equal to 1. Recall that V3 asks that for all $b, b' \in \mathbf{B}(\mathbf{S})$ and for all $c, c' \in \mathbf{C}(\mathbf{S})$,

$$h_b(c) \vee h_{b'}(c') = h_{b \vee b'}(c \vee c') = h_b(h_{b'}(c \vee c')).$$

If both $b, b' \in \mathbf{G}(\mathbf{S})$ then the identities hold since the external join is the join of the hoop, which satisfies the requested properties; if $b, b' \notin \mathbf{G}(\mathbf{S})$, then the external join behaves once again as a join, given the fact that $\mathbf{S}$ is a subreduct of some bounded algebra and by Lemma 2.8 $\neg b \to x = b \vee x$ in BCIRLs if b is Boolean. The other cases can be shown via the subdirect representation. For instance, if $b \in \mathbf{G}(\mathbf{S})$ and $b' \notin \mathbf{G}(\mathbf{S})$ (thus $\neg b' \in \mathbf{G}(\mathbf{S})$) one needs to show that

$$(b \vee c) \vee (\neg b' \to c') = (\neg b' \to b) \vee (c \vee c') = b \vee (\neg b' \to (c \vee c')),$$

which follows by trivial calculations by considering all the cases of $b, \neg b' \in \{0,1\}$ in the subdirect representation of $\mathbf{S}$. In particular if $b = \neg b'$ or $b = 1, \neg b' = 0$ then all members of the identities are 1, while if $b = 0, \neg b' = 1$ then they are all $c \vee c'$.

Finally, with the same technique one can show that V4 holds, that is, for all $b \in \mathbf{B}(\mathbf{S})$ and for all $c, c' \in \mathbf{C}(\mathbf{S})$, $(b \vee_f c) \cdot c' = (\neg b \vee_f c') \wedge (b \vee_f (c \cdot c'))$. If $b \in \mathbf{G}(\mathbf{S})$ the identity becomes

$$(b \vee c) \cdot c' = (b \to c') \wedge (b \vee (c \cdot c'))$$

which if $b = 0$ gives $cc' = cc'$ and if $b = 1$ it yields $c' = c'$; if $b \notin \mathbf{G}(\mathbf{S})$ (and then $\neg b \in \mathbf{G}(\mathbf{S})$) it is

$$(\neg b \to c) \cdot c' = (\neg b \vee c') \wedge (\neg b \to (c \cdot c'))$$

which holds since if $\neg b = 0$ it becomes $c' = c'$ and if $\neg b = 1$ it is $cc' = cc'$. This completes the proof. ∎

Recall that by Remark 2.11, the only subvarieties of RCH that are term-defined varieties of generalized rotations are product algebras, DLMV-algebras, and Boolean algebras (where the radical is always the trivial algebra). Thus in what follows we really only care about P and DLMV; we still write the results uniformly for simplicity. Let then V be either P or DLMV, and let δ be the corresponding rotation operator; given any $\mathbf{S} \in \mathsf{V}_0$, one can define the Boolean quadruple $(\mathbf{B}(\mathbf{S}), \mathbf{C}(\mathbf{S}), \vee_f, \delta)$. We now define the *closure of* $\mathbf{S}$ as the algebra

$$\mathbf{K}(\mathbf{S}) = \mathbf{B}(\mathbf{S}) \otimes^{\delta}_{\vee_f} \mathbf{C}(\mathbf{S}).$$

Let us specify the construction in the two separate cases of product hoops and DLW-hoops. In order to do so, first observe that since $\mathbf{G}(\mathbf{S})$ is a maximal filter of $\mathbf{B}(\mathbf{S})$, we can actually split the domain of $\mathbf{K}(\mathbf{S})$ in two parts:

$$\{[b, c] : b \in \mathbf{G}(\mathbf{S})\} \text{ and } \{[\neg b, c] : b \in \mathbf{G}(\mathbf{S})\}.$$

The first subset is the domain of the subreduct of $\mathbf{K}(\mathbf{S})$ that is isomorphic to $\mathbf{S}$. Let $[b, c]^\bullet = [\neg b, c]$. Thus, up to isomorphism, we can see the domain of $\mathbf{K}(\mathbf{S})$ as the disjoint union

$$\{[b, c] : b \in \mathbf{G}(\mathbf{S})\} \text{ and } \{[b, c]^\bullet : b \in \mathbf{G}(\mathbf{S})\}$$

With this in mind, let us rephrase the closure construction in the following way. Let $\mathbf{S}^\bullet = \{x^\bullet : x \in \mathbf{S}\}$, and consider the equivalence relation (induced by the one defined for the quadruple-construction in (4.1)) $s \equiv s'$ if and only if

$$\beta(s) = \beta(s'), \quad \neg\beta(s) \vee_f \gamma(s) = \neg\beta(s) \vee_f \gamma(s')$$
$$\text{and } \beta(s) \vee_f \delta(c(s)) = \beta(s) \vee_f \delta(\gamma(s')).$$

Let the domain of $\mathbf{K}(\mathbf{S})$ be

$$\mathbf{K}(\mathbf{S}) = \mathbf{S} \cup \mathbf{S}^{\bullet} / \equiv .$$

The operations of $\mathbf{K}(\mathbf{S})$ extend those of $\mathbf{S}$, and for the other elements the reader can adapt the operations from the definition of $\mathbf{B}(\mathbf{S}) \otimes^{\delta}_{\vee_f} \mathbf{C}(\mathbf{S})$ to this notation using the representation of the elements; however, given that they are quite complex, this does not add any further intuition. However, in the cases of both product hoops and DLW-hoops considered separately, the operations significantly simplify.

4.2.1. Product Closure Let us start from the case of $\mathbf{S}$ being a product hoop. Then

$$\mathbf{K}(\mathbf{S}) = \mathbf{S} \cup \mathbf{S}^{\bullet} / \equiv$$

where $s \equiv s'$ if and only if $\beta(s) = \beta(s')$ and $\neg\beta(s) \vee_f \gamma(s) = \neg\beta(s) \vee_f \gamma(s')$. The operations of $\mathbf{K}(\mathbf{S})$ extend those of $\mathbf{S}$ as follows; let $x, y \in \mathbf{S}$, and let us write for simplicity $\beta(x) = b, \gamma(x) = c, \beta(y) = b', \gamma(y) = c'$. Then:

$$\begin{aligned} xy^{\bullet} &= ((b \to b') \wedge cc')^{\bullet}; \\ x^{\bullet}y^{\bullet} &= ((b \vee b') \wedge cc')^{\bullet}; \\ x^{\bullet} \to y^{\bullet} &= (b' \to b) \wedge (b \vee (c \to c')); \\ x^{\bullet} \to y &= (b \vee b') \wedge (b \vee (c \to c')); \\ x \to y^{\bullet} &= (b \wedge b' \wedge (bc \to c'))^{\bullet}. \end{aligned}$$

The previous identities can be shown to hold using the usual trick of the subdirect representation of the elements. For instance, let us consider $x^{\bullet} \to y^{\bullet}$. Since the resulting structure is a product algebra,

$$x^{\bullet} = \beta(x^{\bullet}) \wedge \gamma(x^{\bullet}), \quad y^{\bullet} = \beta(y^{\bullet}) \wedge \gamma(y^{\bullet}).$$

By construction, $\beta(x^{\bullet}) = \neg b$, $\beta(y^{\bullet}) = \neg b'$, $\gamma(x^{\bullet}) = c$, and $\gamma(y^{\bullet}) = c'$. We then show that the identity $x^{\bullet} \to y^{\bullet} = (b' \to b) \wedge (b \vee (c \to c'))$ holds for all choices of $b, b' \in \{0, 1\}$. If $b = 1$, then

$$(\neg b \wedge c) \to (\neg b' \wedge c') = 0 \to (\neg b' \wedge c') = 1 = 1 \wedge 1 = (b' \to b) \wedge (b \vee (c \to c'));$$

If $b = 0, b' = 1$ then

$$(\neg b \wedge c) \to (\neg b' \wedge c') = c \to 0 = 0 = 0 \wedge (c \to c') = (b' \to b) \wedge (b \vee (c \to c'));$$

finally, if $b = b' = 0$ we get:

$$(\neg b \wedge c) \to (\neg b' \wedge c') = c \to c' = 1 \wedge (c \to c') = (b' \to b) \wedge (b \vee (c \to c')).$$

The proof for the other cases is left to the reader.

REMARK 4.12. In [34] it is presented a way to construct a product algebra from a product hoop that follows a similar line of thought, but with the aim of showing that product hoops coincide exactly with the maximal filters of product algebras. We remark that the product algebra we obtain here with the product closure is not isomorphic to the one constructed in [34], since the definition of external join is different.

4.2.2. (DL)MV-Closure We now show that if $\mathbf{S}$ is a DLW-hoop, then the closure $\mathbf{K}(\mathbf{S})$ is (isomorphic to) the MV-closure construction. The key observation is that $x^\bullet$ in this case is the negation of x, $\neg x$; indeed notice that in $\mathbf{B}(\mathbf{S}) \otimes^{id}_{\vee_f} \mathbf{C}(\mathbf{S})$:

$$\neg[b, c] = [\neg b, \delta(c)] = [\neg b, c] = [b, c]^\bullet.$$

The last observation does not hold in general; in product algebras, for instance, $\neg[b, c] = [\neg b, 1]$ which can be different from $[\neg b, c] = [b, c]^\bullet$.

We get $\mathbf{K}(\mathbf{S}) = \mathbf{S} \cup \mathbf{S}^\bullet / \equiv$ where $s \equiv s'$ if and only if

$$\beta(s) = \beta(s'), \quad \neg\beta(s) \vee_f \gamma(s) = \neg\beta(s) \vee_f \gamma(s') \quad \text{and} \quad \beta(s) \vee_f \gamma(s) = \beta(s) \vee_f \gamma(s').$$

Then by Lemma 2.8 we get that $s \equiv s'$ if and only if $\beta(s) = \beta(s')$ and $\gamma(s) = \gamma(s')$, i.e. $\equiv$ is the identity in this case. Let then $s, s' \in \mathbf{K}(\mathbf{S})$ such that $\beta(s) \in \mathbf{G}(\mathbf{S})$. The following can be shown via the subdirect representation, or using the fact that the resulting structure is involutive:

$$a \cdot b^\bullet = (a \to b)^\bullet; \quad a^\bullet \cdot b^\bullet = (a \oplus b)^\bullet;$$
$$a \to b^\bullet = (a \cdot b)^\bullet; \quad a^\bullet \to b = a \oplus b; \quad a^\bullet \to b^\bullet = b \to a$$

where $a \oplus b = (a \to ab) \to b$, that are exactly the operations of the MV-closure. This also shows that the MV-closure construction applied to a DLW-hoop yields a DLMV-algebra.

4.3. Free Functors

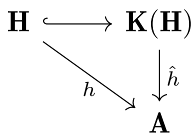
We will now show that the closure construction is free.

DEFINITION 4.13. Let V_0 be the variety of 0-free subreducts of a variety V of BCIRLs (or bounded hoops). Let $\mathbf{H} \in \mathsf{V}_0$. We say that an algebra $\mathbf{K} \in \mathsf{V}$ is *free over* $\mathbf{H}$ if $\mathbf{H}$ is (isomorphic to) a subreduct of $\mathbf{K}$ that generates $\mathbf{K}$, and given any CIRL (or hoop) homomorphism $h : \mathbf{H} \longrightarrow \mathbf{A} \in \mathsf{V}$, h extends uniquely to a bounded homomorphism $\hat{h} : \mathbf{K} \longrightarrow \mathbf{A}$.

Note that any CIRL homomorphism h commutes with the two rotation operators $\delta = id, \delta = \bar{1}$, i.e. $h\delta = \delta h$.

THEOREM 4.14. *Let* V *be either* P *or* DLMV. *Then given any* $\boldsymbol{H} \in \mathsf{V}_0$, $\boldsymbol{K}(\boldsymbol{H})$ *is free over* $\boldsymbol{H}$.

PROOF. We write the proof uniformly for both $\mathsf{V} = \mathsf{P}$ or $\mathsf{V} = \mathsf{DLMV}$. Let $\mathbf{H} \in \mathsf{V}_0$, $\mathbf{A} \in \mathsf{V}$, a hoop homomorphism $h : \mathbf{H} \longrightarrow \mathbf{A}$. We can identify $\mathbf{H}$ with the subalgebra of $\mathbf{K}(\mathbf{H})$ given by the elements $\{[b, c] : b \in \mathbf{G}(\mathbf{H})\}$. We will show that h extends uniquely to a bounded hoop-homomorphism $\hat{h} : \mathbf{K}(\mathbf{H}) \longrightarrow \mathbf{A}$.



By the categorical equivalence between algebras in RCH and the corresponding quadruples, any homomorphism between RCH-algebras is uniquely determined by its restrictions to the Boolean skeleton and the radical. More precisely, to define a homomorphism from $\mathbf{K}(\mathbf{H})$ to $\mathbf{A}$ it suffices to define a homomorphism $f : \mathrm{Bool}(\mathbf{K}(\mathbf{H})) \to \mathrm{Bool}(\mathbf{A})$, a homomorphism $g : \mathrm{Rad}(\mathbf{K}(\mathbf{H})) \to \mathrm{Rad}(\mathbf{A})$, such that

(1) $g\delta = \delta g$;

(2) $g(b \vee c) = f(b) \vee g(c)$, for $b \in \mathrm{Bool}(\mathbf{K}(\mathbf{H}))$, $c \in \mathrm{Rad}(\mathbf{K}(\mathbf{H}))$.

Now, given $h : \mathbf{H} \longrightarrow \mathbf{A}$, we consider the restrictions

$$h_{\restriction \mathbf{G}(\mathbf{H})} : \mathbf{G}(\mathbf{H}) \longrightarrow \mathbf{A} \qquad h_{\restriction \mathbf{C}(\mathbf{H})} : \mathbf{C}(\mathbf{H}) \longrightarrow \mathbf{A}.$$

Notice that $h_{\restriction \mathbf{G}(\mathbf{H})} : \mathbf{G}(\mathbf{H}) \longrightarrow \mathrm{Bool}(\mathbf{A})$, $h_{\restriction \mathbf{C}(\mathbf{H})} : \mathbf{C}(\mathbf{H}) \longrightarrow \mathrm{Rad}(\mathbf{A})$, since the two subsets of the domain are term-defined in the 0-free language (see identities (4.4)) and so preserved by homomorphisms.

We define $g = h_{\restriction \mathbf{C}(\mathbf{H})}$, while f is the extension of $h_{\restriction \mathbf{G}(\mathbf{H})}$ to $\mathbf{B}(\mathbf{H})$ defined as

$$f(b) = \begin{cases} h(b) \text{ if } b \in \mathbf{G}(\mathbf{H}), \\ \neg h(\neg b) \text{ otherwise.} \end{cases}$$

The map g is a homomorphism since it coincides with the restriction of h; it is easy to check that f is the unique extension of $h_{\restriction \mathbf{G}(\mathbf{H})}$ to a homomorphism from $\mathbf{B}(\mathbf{H})$ to $\mathrm{Bool}(\mathbf{A})$ (see [1, Theorem 3.6] for a proof).

We now show that $g(b \vee c) = f(b) \vee g(c)$ holds for any $b \in \mathbf{B}(\mathbf{H})$ and $c \in \mathbf{C}(\mathbf{H})$. If $b \in \mathbf{G}(\mathbf{H})$, then

$$g(b \vee c) = h(b \vee c) = h(b) \vee h(c) = f(b) \vee g(c).$$

If $b \notin \mathbf{G}(\mathbf{H})$, then $\neg b \in \mathbf{G}(\mathbf{H})$ and using Lemma 2.8

$$g(b \vee c) = h(\neg b \to c) = h(\neg b) \to h(c) = \neg h(\neg b) \vee h(c) = f(b) \vee g(c).$$

Since it is clear that in both cases ($\delta = id$ and $\delta = \bar{1}$) $h\delta = \delta h$, the pair of homomorphisms (f, g) define a homomorphism from $\mathbf{K}(\mathbf{H})$ to $\mathbf{A}$. Using the description of morphisms in (4.2) and the representation of the elements in Proposition 4.1, we can write

$$\hat{h}[b, c] = (f(b) \vee \neg g(c)) \wedge (\neg f(b) \vee g(c)).$$

It is clear that $\hat{h}$ extends h; the fact that it is the unique homomorphism that does so follows again from the fact that any homomorphism is fully determined by its restriction to the radical and Boolean skeleton, and the observation that a homomorphism on $\mathbf{B}(\mathbf{H})$ is fully determined by its restriction on $\mathbf{G}(\mathbf{H})$ (that is a maximal filter). Thus, the claim follows. ∎

Let us now show how our results translate to the categorical setting. That is to say, we exhibit the free functor from PH to PA and from DLWH to DLMV (seen as algebraic categories), i.e. the *left adjoint* to the *forgetful functor* $\Upsilon_{\mathsf{PA}} : \mathsf{PA} \to \mathsf{PH}$ and $\Upsilon_{\mathsf{DLMV}} : \mathsf{DLMV} \to \mathsf{DLWH}$ that forgets the falsum constant 0. We observe that such left adjoint always exists by categorical arguments, but there are no general techniques to actually exhibit the construction.

Let V be either P or DLMV, and correspondingly let V_0 be either PH or DLWH. With a slight abuse of notation, we denote in the same way the corresponding algebraic categories. We define the maps $\mathcal{K}_{\mathsf{V}_0} : \mathsf{V}_0 \longrightarrow \mathsf{V}$ in the following way; given any $\mathbf{A}, \mathbf{B} \in \mathsf{V}_0$, $h : \mathbf{A} \longrightarrow \mathbf{B}$,

$$\mathcal{K}_{\mathsf{V}_0}(\mathbf{A}) = \mathbf{K}(\mathbf{A});$$
$$\mathcal{K}_{\mathsf{V}_0}(h) = \hat{h},$$

where $\hat{h} : \mathcal{K}_{\mathsf{V}_0}(\mathbf{A}) \to \mathcal{K}_{\mathsf{V}_0}(\mathbf{B})$ is defined by $\hat{h}[b, c] = [f(b), h(c)]$ and

$$f(b) = \begin{cases} h(b) \text{ if } b \in \mathbf{G}(\mathbf{H}), \\ \neg h(\neg b) \text{ otherwise.} \end{cases}$$

Lemma 4.15. *Let* V *be either* P *or* DLMV. $\mathcal{K}_{\mathsf{V}_0}$ *is a functor from* V_0 *to* V.

Proof. Let $\mathbf{A}, \mathbf{B} \in \mathsf{V}_0$, $h : \mathbf{A} \longrightarrow \mathbf{B}$. Clearly $\mathbf{K}(\mathbf{A}) \in \mathsf{V}$. Now, seeing $\mathbf{A}$ and $\mathbf{B}$ as their copies in $\mathcal{K}_{\mathsf{V}_0}(\mathbf{A})$ and $\mathcal{K}_{\mathsf{V}_0}(\mathbf{B})$, we have that h is a hoop homomorphism from $\mathbf{A}$ to $\mathcal{K}_{\mathsf{V}_0}(\mathbf{B})$; thus within the proof of Theorem 4.14 we have shown that $\mathcal{K}_{\mathsf{V}_0}(h)$ is a homomorphism from $\mathcal{K}_{\mathsf{V}_0}(\mathbf{A})$ to $\mathcal{K}_{\mathsf{V}_0}(\mathbf{B})$. It is clear that $\mathcal{K}_{\mathsf{V}_0}$ preserves the identity; we need to show that it preserves compositions of morphisms. Consider $h : \mathbf{A} \longrightarrow \mathbf{B}$ and $k : \mathbf{B} \longrightarrow \mathbf{C}$ homomorphisms, then one needs to show that

$$\mathcal{K}_{\mathsf{V}_0}(k \circ h) = \mathcal{K}_{\mathsf{V}_0}(k) \circ \mathcal{K}_{\mathsf{V}_0}(h) = \hat{k} \circ \hat{h}.$$

The latter follows from the fact that $\mathcal{K}_{\mathsf{V}_0}(k \circ h)$, $\mathcal{K}_{\mathsf{V}_0}(k)$, and $\mathcal{K}_{\mathsf{V}_0}(h)$ are fully determined by their restrictions to $\mathbf{A}$ and $\mathbf{B}$, that is, by h and k. ■

We say that $\mathcal{K}_{\mathsf{V}_0}$ is the left adjoint to Υ_{V} if there is a natural isomorphism

$$\phi : \mathsf{hom}_{\mathsf{V}}(\mathcal{K}_{\mathsf{V}_0}-, -) \longrightarrow \mathsf{hom}_{\mathsf{V}_0}(-, \Upsilon-).$$

This means that for all objects $\mathbf{A} \in \mathsf{V}_0$ and $\mathbf{P} \in \mathsf{V}$, ϕ gives a *natural* bijection between the morphism sets $\mathsf{hom}_{\mathsf{V}}(\mathcal{K}_{\mathsf{V}_0}(\mathbf{A}), \mathbf{P}) \longrightarrow \mathsf{hom}_{\mathsf{V}_0}(\mathbf{A}, \Upsilon(\mathbf{P}))$, i.e. the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{hom}_{\mathsf{PA}}(\mathcal{K}_{\mathsf{V}_0}(\mathbf{A}_1), \mathbf{P}_1)) & \overset{\phi}{\longleftrightarrow} & \mathrm{hom}_{\mathsf{PH}}(\mathbf{A}_1, \Upsilon(\mathbf{P}_1)) \\
\downarrow & & \downarrow \\
\mathrm{hom}_{\mathsf{PA}}(\mathcal{K}_{\mathsf{V}_0}(\mathbf{A}_2), \mathbf{P}_2)) & \overset{\phi}{\longleftrightarrow} & \mathrm{hom}_{\mathsf{PH}}(\mathbf{A}_2, \Upsilon(\mathbf{P}_2))
\end{array}$$

for all homomorphisms $h : \mathbf{P}_1 \longrightarrow \mathbf{P}_2$ and $k : \mathbf{A}_2 \longrightarrow \mathbf{A}_1$, where the vertical arrows are induced by composition. In particular, given $\mathbf{A} \in \mathsf{V}_0, \mathbf{P} \in \mathsf{V}$, and $f : \mathcal{K}_{\mathsf{V}_0}(\mathbf{A}) \longrightarrow \mathbf{P}$, let

$$\phi(f) = f_{\restriction \mathbf{A}}.$$

The fact that ϕ is a bijection follows from Theorem 4.14, while the naturality condition can be checked by direct computation. The proof of the following theorem is then standard.

Theorem 4.16. *$\mathcal{K}_{\mathsf{PH}}$ is the free functor from PH to PA, and $\mathcal{K}_{\mathsf{DLWH}}$ is the free functor from DLWH to DLMV.*

5. Applications: Projectivity, Unification, and Structural Completeness

In this section we use our results to study projective algebras and unification problems; in particular, as a consequence of the results in this section, we will show that both the varieties of product and DLW-hoops have strong unitary unification type, and they are structurally (and actually universally) complete.

By *unification problem* for a logic one usually means the following: given two formulas (or terms) p, q, find a uniform replacement of the variables (i.e. a substitution) occurring in p and q, called a *unifier*, that makes p and q equal. If equality is given up to some equational theory, then one speaks about *equational unification*. The latter has been studied in the framework of algebraizable logics, and it is shown by Ghilardi to have a purely algebraic counterpart [33], which we are going to follow here; his approach is based on finitely presented and projective algebras in a variety.

We say that an algebra $\mathbf{A}$ in a variety V is *finitely presented* if, intuitively, it can be defined by a finite number of generators and finitely many identities; more precisely, we call $\mathbf{A} \in \mathsf{V}$ finitely presented if there exists a finite set X and a finitely generated congruence $\theta \in \mathrm{Con}(\mathbf{F}_{\mathsf{V}}(X))$ such that $\mathbf{F}_{\mathsf{V}}(X)/\theta \cong \mathbf{A}$. The other fundamental concept that we need is the definition of projective algebras: an algebra $\mathbf{P}$ is *projective* in a class of algebras K if for all $\mathbf{A}, \mathbf{B} \in \mathsf{K}$, homomorphism $f : \mathbf{P} \to \mathbf{A}$, surjective homomorphism $g : \mathbf{B} \to \mathbf{A}$, there is $k : \mathbf{P} \to \mathbf{B}$ such that $f = gk$. This notion is simplified if K is a (quasi)variety. Indeed, an algebra is projective in a (quasi)variety V if and only if it is a retract of a free algebra $\mathbf{F} \in \mathsf{V}$, that is, there exist homomorphisms $f : \mathbf{P} \to \mathbf{F}, g : \mathbf{F} \to \mathbf{P}$ such that $gf = id_{\mathbf{P}}$. Thus, in particular, all free algebras are projective.

Now, a unification problem for a variety V is a finitely presented algebra $\mathbf{A}$ in V and a solution for the said problem is a homomorphism $u : \mathbf{A} \to \mathbf{P}$ where $\mathbf{P}$ is a projective algebra in V; u is called a *unifier* for $\mathbf{A}$ and we say that $\mathbf{A}$ is *unifiable*. If u_1, u_2 are two different unifiers for $\mathbf{A}$ (with projective algebras $\mathbf{P}_1$ and $\mathbf{P}_2$) we say that u_1 is *more general* than u_2 if there exists a homomorphism $m : \mathbf{P}_1 \to \mathbf{P}_2$ such that $mu_1 = u_2$. The relation "being more general than" is a preordering on the set of all the unifiers of $\mathbf{A}$ thus, if we consider its associated equivalence relation, the equivalence classes form a partially ordered set $U_{\mathbf{A}}$ of *equally general* unifiers. We say that the unification type of $\mathbf{A}$ is: *unitary* if $U_{\mathbf{A}}$ has a maximum; *finitary* if it is not unitary and there are finitely many maximal elements; *infinitary* if it is not finitary and there are infinitely many maximal elements; finally, if $U_{\mathbf{A}}$ has no maximal elements we say that the type is *nullary*. The type of a variety is the worst type of its finitely presented algebras (considering the "best" case to be the unitary type, and the worst being the nullary one).

If $\mathbf{A}$ has unitary type then the maximum of $U_{\mathbf{A}}$ is called the *most general unifier* (*mgu*) of $\mathbf{A}$, which can be seen as the "best solution" for the unification problem. We say that $\mathbf{A}$ has *strong unitary type* if its *mgu* is the identity; a variety V has strong unitary type if every unifiable finitely presented algebra $\mathbf{A}$ in V has strong unitary type. We observe that the property of having strong unitary type in a variety can be reformulated in a simpler way.

PROPOSITION 5.1. *Let* V *be any variety, then the following are equivalent:*

(1) V *has strong unitary type;*

(2) for any finitely presented algebra $\boldsymbol{A} \in \mathsf{V}$, $\boldsymbol{A}$ *is unifiable if and only if it is projective.*

It is an easy observation that for an algebra $\mathbf{A}$ to be unifiable corresponds to having a homomorphism to the least free algebra $\mathbf{F}_{\mathsf{V}}$ (i.e. the 0-generated one if the variety has a constant, the 1-generated one otherwise). Indeed, such algebra is projective, and there is a homomorphism from every free (and therefore also from every projective) algebra to $\mathbf{F}_{\mathsf{V}}$ (see [6, Lemma 2.22]).

Now, in a variety V of hoops or CIRLs (in which the only constant is 1), the smallest free algebra is the trivial one, i.e. $\mathbf{F}_{\mathsf{V}} = \mathbf{1}$. Clearly, every algebra has $\mathbf{1}$ as a homomorphic image, so, in a variety of CIRLs, every finitely presented algebra is unifiable, therefore Proposition 5.1 becomes:

PROPOSITION 5.2. *Let* V *be any variety of CIRLs (or hoops), then the following are equivalent:*

(1) V *has strong unitary type;*

(2) every finitely presented algebra $\boldsymbol{A} \in \mathsf{V}$ *is projective.*

Projectivity in varieties of hoops and (B)CIRLs has been studied in [4]; in the latter paper it is shown that all locally finite varieties of hoops and bounded hoops have strong unitary type. Moreover, local finiteness does not characterize this property, since the same holds for cancellative hoops. Indeed, projective and finitely generated ℓ-groups coincide with finitely presented ℓ-groups, as shown in [10]; since the properties of being projective, finitely presented, and finitely generated in every variety is categorical, i.e., it can be described in the abstract categorical setting by properties of morphisms, all these notions are preserved by categorical equivalences. Therefore, we have the following result.

PROPOSITION 5.3. *Finitely presented cancellative hoops are exactly the finitely generated and projective in their variety. Therefore, cancellative hoops have strong unitary unification type.*

Other non-locally finite varieties of (bounded) hoops however do not share this property; for instance, MV-algebras have nullary type [39], and the same holds for Wajsberg hoops [44]. All the last mentioned results have been obtained via geometrical methods using categorical dualities; we are not aware of general methods to study projectivity and the unification type in non-locally finite varieties of hoops or CIRLs.

In the rest of the section we will adapt the techniques introduced in [4] to study projectivity in varieties of BCIRLs generated by generalized rotations, in order to obtain analogous results for the corresponding varieties of 0-free subreducts; in particular, we will see that product hoops and DLW-hoops have strong unitary unification type, in analogy with product algebras and DLMV-algebras. In algebraic terms, we will see that in these classes of algebras finitely generated projective algebras coincide with finitely presented ones. In fact, this will be shown to hold in all varieties of basic hoops that are obtainable by means of generalized rotations from cancellative hoops. The following essentially adapts the proof of [5, Theorem 4.10]. Recall once again that the only radical-determined varieties of generalized rotations that are subvarieties of RCH are Boolean algebras, product algebras, and DLMV-algebras.

THEOREM 5.4. *Let* $\mathsf{V} \subseteq \mathsf{RCH}$ *be a radical-determined variety of generalized rotations. Every finitely presented algebra in* V_0 *is projective.*

PROOF. Let $\mathbf{A} \in \mathsf{V}_0$ be finitely presented, that is, there is a finite set X and a compact congruence $\theta \in \mathrm{Con}(\mathbf{F}_{\mathsf{V}_0}(X))$ such that $\mathbf{F}_{\mathsf{V}_0}(X)/\theta \cong \mathbf{A}$. We will prove that $\mathbf{A}$ is a retract of $\mathbf{F}_{\mathsf{V}_0}(X)$; by Corollary 3.15 we have that

$$\mathbf{F}_{\mathsf{V}_0}(X) \cong \big(\prod_{S \subsetneq X} [\mathrm{Rot}^{\delta}(\mathbf{F}_{\mathsf{CH}}(X_S))]_0 \big) \times \mathbf{F}_{\mathsf{CH}}(X).$$

Now, since CIRLs have distributive congruences, all congruences of a finite direct product are product congruences, thus we can see θ as $\prod_S \theta_S$, where each θ_S is exactly the restriction of the congruence θ on the component of the direct product indexed by $S \subseteq X$ (where θ_X is the restriction to the last component $\mathbf{F}_{\mathsf{CH}}(X)$). Then $\mathbf{A} \cong \mathbf{F}_{\mathsf{V}_0}(X)/\theta$ is isomorphic to

$$\Big(\big(\prod_{S \subsetneq X} [\mathrm{Rot}^{\delta}(\mathbf{F}_{\mathsf{CH}}(X_S))]_0 \big) \times \mathbf{F}_{\mathsf{CH}}(X) \Big)/\theta \cong \Big(\prod_{S \subsetneq X} [\mathrm{Rot}^{\delta}(\mathbf{F}_{\mathsf{CH}}(X_S))]_0/\theta_S \Big) \times \mathbf{F}_{\mathsf{CH}}(X)/\theta_X.$$

Since θ is finitely generated, each congruence θ_S is finitely generated as well. For the last coordinate, we get immediately that $\mathbf{F}_{\mathsf{CH}}(X)/\theta_X$ is a finitely presented cancellative hoop, hence, by Proposition 5.3, it is projective; this, by the definition of projectivity, implies that $\mathbf{F}_{\mathsf{CH}}(X)/\theta_X$ is a retract of the free algebra it is a quotient of, $\mathbf{F}_{\mathsf{CH}}(X)$. If $S \subsetneq X$, we distinguish two cases:

- if θ_S is the total congruence, then $[\mathrm{Rot}^{\delta}(\mathbf{F}_{\mathsf{CH}}(X_S))]_0/\theta_S$ is the trivial algebra, that is a retract of $[\mathrm{Rot}^{\delta}(\mathbf{F}_{\mathsf{CH}}(X_S))]_0$;
- if θ_S is not total, then it is completely determined by its restriction θ'_S to the radical $\mathbf{F}_{\mathsf{CH}}(X_S)$, since proper congruence filters of a generalized rotation are fully determined by their restriction to the radical; the latter can be easily observed by noticing that proper filters in a generalized rotation are filters of the radical. Now, $\mathbf{F}_{\mathsf{CH}}(X_S)/\theta'_S$ is again a finitely presented cancellative hoop, thus by Proposition 5.3 it is also projective, and then it follows again from the definition of projectivity that $\mathbf{F}_{\mathsf{CH}}(X_S)/\theta'_S$ is a retract of $\mathbf{F}_{\mathsf{CH}}(X_S)$; the previous retraction can be lifted to show that $[\mathrm{Rot}^{\delta}(\mathbf{F}_{\mathsf{CH}}(X_S)/\theta'_S)]_0 \cong [\mathrm{Rot}^{\delta}(\mathbf{F}_{\mathsf{CH}}(X_S))]_0/\theta_S$ (the latter isomorphism is easy to check but for details see [5, Lemma 2.3]) is a retract of $[\mathrm{Rot}^{\delta}(\mathbf{F}_{\mathsf{CH}}(X_S))]_0$ (for details, see the completely analogous proof of [5, Theorem 4.7]).

Thus we get that the quotient on the free algebra $\mathbf{F}_{\mathsf{V}_0}(X)$ induces a retraction on each component of the product; say that for each component the homomorphisms testifying the retraction are i_S, j_S, with $j_S \circ i_S = id$. Then, the maps

$$i : \Big(\prod_{S \subsetneq X} [\mathrm{Rot}^{\delta}(\mathbf{F}_{\mathsf{CH}}(X_S))]_0/\theta_S\Big) \times \mathbf{F}_{\mathsf{CH}}(X)/\theta_X \Big(\prod_{S \subsetneq X} [\mathrm{Rot}^{\delta}(\mathbf{F}_{\mathsf{CH}}(X_S))]_0\Big) \times \mathbf{F}_{\mathsf{CH}}(X)$$

$$j : \Big(\prod_{S \subsetneq X} [\mathrm{Rot}^{\delta}(\mathbf{F}_{\mathsf{CH}}(X_S))]_0\Big) \times \mathbf{F}_{\mathsf{CH}}(X) \to \Big(\prod_{S \subsetneq X} [\mathrm{Rot}^{\delta}(\mathbf{F}_{\mathsf{CH}}(X_S))]_0/\theta_S\Big) \times \mathbf{F}_{\mathsf{CH}}(X)/\theta_X$$

defined as

$$i((x/\theta_S)_{S \subseteq X}) = (i_S(x/\theta_S))_{S \subseteq X}, \quad j((y_S)_{S \subseteq X}) = (j_S(y_S))_{S \subseteq X},$$

testify that the quotient algebra $\big(\prod_{S \subsetneq X} [\mathrm{Rot}^{\delta}(\mathbf{F}_{\mathsf{CH}}(X_S))]_0/\theta_S\big) \times \mathbf{F}_{\mathsf{CH}}(X)/\theta_X$ is a retract of $\big(\prod_{S \subsetneq X} [\mathrm{Rot}^{\delta}(\mathbf{F}_{\mathsf{CH}}(X_S))]_0\big) \times \mathbf{F}_{\mathsf{CH}}(X)$, and thus $\mathbf{A}$ is a retract of the free algebra $\mathbf{F}_{\mathsf{V}_0}(X)$. Therefore in V_0 every finitely presented algebra is projective. ■

On the same lines, we adapt the proof of [5, Theorem 4.8] to show a converse to the previous theorem.

Theorem 5.5. *Let* $\mathsf{V} \subseteq \mathsf{RCH}$ *be a radical-determined variety of generalized rotations. Every finitely generated projective algebra in* V_0 *is finitely presented.*

Proof. Let $\mathbf{A}$ be a projective and finitely generated algebra in V_0, thus it is a retract of a finitely generated free algebra $\mathbf{F}_{\mathsf{V}_0}(X)$, with X finite. So $\mathbf{A} \cong \mathbf{F}_{\mathsf{V}_0}(X)/\theta$ for some congruence θ. We will show that θ is a compact (i.e. finitely generated) congruence. As in the proof of Theorem 5.4, we can write θ as $\prod_S \theta_S$, and then $\mathbf{A} \cong \mathbf{F}_{\mathsf{V}_0}(X)/\theta$ is isomorphic to

$$\Big((\prod_{S \subsetneq X} [\mathrm{Rot}^{\delta}(\mathbf{F}_{\mathsf{CH}}(X_S))]_0) \times \mathbf{F}_{\mathsf{CH}}(X) \Big)/\theta \cong \Big(\prod_{S \subsetneq X} [\mathrm{Rot}^{\delta}(\mathbf{F}_{\mathsf{CH}}(X_S))]_0/\theta_S \Big) \times \mathbf{F}_{\mathsf{CH}}(X)/\theta_X.$$

It suffices to show that each θ_S is compact, that is, θ_S is finitely generated for every $S \subseteq X$. Notice that if θ_S is the total congruence for $S \subsetneq X$, then it is finitely generated by the pair $(0,1)$ where 0 is the bottom of the rotation; while if θ_X is the total congruence it is generated by the finite set $\{(x,1) : x \in X\}$.

Suppose now that θ_S is not the total congruence, and consider $\mathbf{A}$ as its isomorphic copy $\Big(\prod_{S \subsetneq X} [\mathrm{Rot}^{\delta}(\mathbf{F}_{\mathsf{CH}}(X_S))]_0/\theta_S\Big) \times \mathbf{F}_{\mathsf{CH}}(X)/\theta_X$, and the free algebra $\mathbf{F}_{\mathsf{V}_0}(X)$ as its isomorphic copy $\prod_{S \subsetneq X} [\mathrm{Rot}^{\delta}(\mathbf{F}_{\mathsf{CH}}(X_S))]_0) \times \mathbf{F}_{\mathsf{CH}}(X)$; let us enumerate all subsets of X as $S_1, \dots, S_k$ (where $k = 2^{|X|}$), and let g be the homomorphism projecting from $\mathbf{F}_{\mathsf{V}_0}(X)$ onto every factor,

$$g(x_1, \dots, x_k) = (x_1/\theta_{S_1}, \dots, x_k/\theta_{S_k}).$$

Since $\mathbf{A}$ is projective, there is a (injective) homomorphism $f : \mathbf{A} \to \mathbf{F}_{\mathsf{V}_0}(X)$ such that $gf = id_{\mathbf{A}}$ i.e.

$$f(x_1/\theta_{S_1}, \dots, x_k/\theta_{S_k}) = (y_1, \dots, y_k) \text{ where } x_i \theta_{S_i} y_i \text{ for } i = 1 \dots k.$$

From f and g we can obtain maps that testify the retraction in each coordinate, i.e. maps f_S and g_S such that $g_S \circ f_S$ is the identity on the coordinate indexed by S. In particular, $g_{S_i}(x) = x/\theta_{S_i}$ and

$$f_{S_i}(x/\theta_{S_i}) = \pi_{S_i} \circ f(1/\theta_{S_1}, \dots, 1/\theta_{S_{i-1}}, x/\theta_{S_i}, 1/\theta_{S_{i+1}}, \dots, 1/\theta_{S_k})$$

where π_{S_i} is the natural projection onto the S_i-th component of the direct product. The map is well defined since f is, and direct computation shows it is a homomorphism and that $g_{S_i} \circ f_{S_i} = id$ on the S_i-th factor. It follows that each $[\mathrm{Rot}^{\delta}(\mathbf{F}_{\mathsf{CH}}(X_S))]_0/\theta_S$ and $\mathbf{F}_{\mathsf{CH}}(X)/\theta_X$ are also projective. In particular:

- $\mathbf{F}_{\mathsf{CH}}(X)/\theta_X$ is a projective cancellative hoop, hence, by Proposition 5.3, it is finitely presented and this means that θ_X is finitely generated;
- if $S \subsetneq X$, consider θ'_S to be the restriction of θ_S to the radical $\mathbf{F}_{\mathsf{CH}}(X_S)$, then given that $\mathrm{Rot}(\mathbf{F}_{\mathsf{CH}}(X_S)/\theta'_S) \cong [\mathrm{Rot}^{\delta}(\mathbf{F}_{\mathsf{CH}}(X_S))]_0/\theta_S$, we consider the restriction f'_S of f_S on $\mathbf{F}_{\mathsf{CH}}(X)/\theta'_S$ and the restriction g'_S of g_S on

$\mathbf{F}_{\mathsf{CH}}(X_S)$. Now $g'_S \circ f'_S$ is still the identity on $\mathbf{F}_{\mathsf{CH}}(X_S)/\theta'_S$, so $\mathbf{F}_{\mathsf{CH}}(X_S)/\theta'_S$ is projective and, by Proposition 5.3, this means that θ'_S is finitely generated. Finally, notice that, since θ_S is not total, it is determined by its restriction to the radical, and thus θ_S is generated by the same elements of θ'_S, hence θ_S is finitely generated too.

Therefore θ is a factor congruence determined by finitely many finitely generated congruences, hence it is finitely generated, which completes the proof. ∎

Summarizing, we get the following.

Corollary 5.6. *Let* V_0 *be one among the varieties of generalized Boolean algebras, product hoops, and DLW-hoops. Finitely presented algebras in* V_0 *are exactly the finitely generated and projective ones.*

5.1. Admissibility and Structural Completeness

The study of unification problems is strictly connected to the study of admissible rules (or, algebraically, admissible quasiequations). A rule $\Gamma \Rightarrow \varepsilon$ (where Γ is a finite set of formulas and ε is a single formula, all on the appropriate language) is said to be *admissible* in a logic if every substitution that makes the premises in Γ a theorem of the logic, also makes ε a theorem of the logic. On the algebraic side, a quasiequation $\Sigma \Rightarrow \delta$ is given by a finite set of equations Σ followed by a single equation δ. A quasiequation is said to be *admissible* in a (quasi)variety Q if every substitution that makes all the identities in the premises valid in Q also makes the conclusion valid in Q. A logic is said to be *structurally complete* if every admissible rule is derivable, and similarly a quasivariety is structurally complete if every admissible quasiequation is valid. A logic whose equivalent algebraic semantics is a quasivariety Q is structurally complete if and only if Q is. These notions can be extended to multiple-conclusion rules, and from the algebraic side to *clauses*. A clause $\Sigma \Rightarrow \Delta$ is given by two finite set of equations Σ, Δ; a clause $\Sigma \Rightarrow \Delta$ is valid in a quasivariety Q if the universal sentence $(\forall \bar{x})(\bigwedge \Sigma \Rightarrow \bigvee \Delta)$ is valid in Q, and it is admissible in Q if every substitution that makes all the identities in Σ valid in Q also makes at least one of the identities in Δ valid in Q. A quasivariety is *universally complete* if every admissible clause is valid (the reader can derive the corresponding notions for a logic). Of course universal completeness implies structural completeness, and both of them can be studied in the purely algebraic setting. In particular, [6, Corollary 3.6] shows that if all finitely presented algebras in a quasivariety are projective, the quasivariety is universally complete, and therefore also structurally complete. The following is then a consequence of Corollary 5.6.

Corollary 5.7. *Let* V_0 *be one among the varieties of generalized Boolean algebras, product hoops, and DLW-hoops.* V_0 *has strong unitary unification type and it is structurally and universally complete.*

6. Conclusions and Future Work

The results we have obtained in this work rely on the fact that the classes of hoops we considered are subreducts of varieties with a Boolean retraction term, whose radical class is the variety of cancellative hoops. In particular, we have seen that cancellativity allows one to *term-define* both Boolean elements and the elements of the radical within the 0-free language; this seems not to be the case in general. In future work, we plan to investigate whether one can obtain different term-defined rotations operators in different subvarieties of CIRLs, and we shall also explore weaker notions of *definability*, which might allow us to apply our methods to larger classes of structures. We also mention that our techniques will most likely apply to the more general case of algebras with an *MV-retraction term* studied in [16], again in the case of having a cancellative radical. A different and interesting line of work would be to investigate the dual representation of the triple constructions developed in [29], in order to apply it to the varieties of 0-free subreducts considered here.

Moreover, another research topic underlying this investigation involves the understanding of how to freely add the falsum constant in other interesting varieties of hoops; for instance, to the best of our knowledge, it is not known how to freely obtain a Gödel algebra from a Gödel hoop. Despite the fact that Gödel algebras have a Boolean retraction term, our methods cannot be directly applied there, since it seems not straightforward to write the elements of the Boolean skeleton and of the radical in the 0-free language (in fact, it might not be possible). In order to deepen the understanding of the falsum constant in Gödel logic, one might also wonder how to obtain a description of free Gödel hoops as subreducts of free Gödel algebras; the descriptions currently available in the literature are indeed not in these terms.

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