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On Weak Lewis Distributive Lattices

Abstract. In this paper we study the variety WL of bounded distributive lattices endowed with an implication, called weak Lewis distributive lattices. This variety corresponds to the algebraic semantics of the $\{\vee, \wedge, \Rightarrow, \bot, \top\}$-fragment of the arithmetical base preservativity logic iP^-. The variety WL properly contains the variety of bounded distributive lattices with strict implication, also known as weak Heyting algebras. We introduce the notion of WL-frame and we prove a representation theorem for WL-lattices by means of WL-frames. We extended this representation to a topological duality by means of Priestley spaces endowed with a special neighbourhood relation between points and closed upsets of the space. These results are applied in order to give a representation and a topological duality for the variety of weak Heyting–Lewis algebras, i.e., for the algebraic semantics of the arithmetical base preservativity logic iP^-.

Keywords: Subintuitionistic logic, Preservativity logics, Weak Heyting algebras, Heyting algebras, Neighbourhood frames, Duality theory.

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1. Introduction

In this paper we study, from an algebraic and topological point of view, an implication, called weak Lewis implication or weak strict implication, that extends two known implications: the intuitionistic implication and the strict Lewis implication, which also appears in certain extensions of intuitionistic logic, called Preservativity logics.

Let us first analyze this weak strict implication in the framework of Preservativity logics. In [32] Visser defines and axiomatizes the preservativity logic iPH as an extension of the intuitionistic logic with a new connective of strict implication. This logic is studied in detail by Iemhoff in [17,18] (see also [19,23]). Iemhoff defines the basic logic iP^-, called the arithmetical base preservativity logic, and the extension of iP^- with the axiom Dp called

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Disjunctive Principle, which is denoted as iP. The logic iP is called the *semantic base preservativity logic* for reasons that will be explained below [18, p. 232]. Note that the logic iP is called Heyting–Lewis Logic in [23] and it is denoted by iA. The relational semantics for iP is given by the iP-frames or Heyting Lewis frames (see [18, Section 3.2] and [23, Section 3]). An iP-frame a is a triple $\langle X, \leq, R\rangle$ where $\leq$ is a partial order and R is a binary relation such that $\leq \circ R \subseteq R$. In [18, Proposition 7] or [17, Proposition 4.1.1] Iemhoff proves that iP is complete with respect to the class of all iP-frames. This completeness can be also deduced from the well known results on representation for Heyting algebras, and the results on representation for weak Heyting algebras given in [6], or applying the techniques developed in [5] for subintuitionistic logics. The algebraic semantics of iP is the class of algebras $\langle A, \vee, \wedge, \to, \Rightarrow, 0, 1\rangle$ where $\langle A, \vee, \wedge, \to, 0, 1\rangle$ is a Heyting algebra [2] and $\langle A, \vee, \wedge, \Rightarrow, 0, 1\rangle$ is a weak Heyting algebra [6]. These algebras are called *Heyting–Lewis algebras* in [23].

The algebraic semantics for IP^- is defined in [24, Section 3.1], although it was not studied in detail. As far as we know there is no relational semantics for IP^-. In this paper we propose an algebraic semantics and a relational semantics for the $\{\bot, \top, \vee, \wedge, \Rightarrow\}$-fragment of the logic iP^-. We introduce the class of bounded distributive lattices with a strict implication weaker than the strict implication considered for the weak Heyting algebras. These algebras will be called *weak Lewis distributive lattices*, or WL-lattices, and they correspond to weak subintuitionistic logics in the same sense that weak Heyting algebras correspond to subintuitionistic logics. Thus, if we take Heyting algebras with weak strict implication, then we obtain the variety of iA^--algebras defined in [24].

The relational semantics for iP^- is more general that the relational semantics for subintuitionistic logics. We introduce a class of intuitionistic Kripke frames with a neighbourhood relation, called *weak Lewis neighbourhood frames*. The neighbourhood relation is used to interpret the weak strict implication and the partial order is used to interpret the intuitionistic implication. Using the standard techniques of canonical models, or using the representation theorem for WL-lattices (Theorem 14), we can prove that a formula ϕ is a theorem of iP^- if and only if ϕ is valid in every weak Lewis neighbourhood frame. That is, the logic iP^- is complete with respect to the weak Lewis neighbourhood frames. These results along with a more detailed study of some extensions of iP^- will be presented in a future paper. Here we will only dedicate ourselves to the study of the $\{\bot, \top, \vee, \wedge, \Rightarrow\}$-fragment of the logic iP^- from an algebraic and topological point of view.

Subintuitionistic logics can be defined as those logics in the intuitionistic language that arise from weakening the frame conditions of the Kripke semantics for the intuitionistic logic. As far as we know, the first study on subintuitionist logics using Kripke models are the papers of Visser [31] and Corsi [7]. Later, subintuitionistic logics were also investigated by Restall [29], Došen [11], Celani and Jansana [5], and Maleki and De Jongh [25].

Modal logic was initially conceived as an extension of the classical propositional logic with an implication called Lewis strict implication [20–22]. In the classical setting, the Lewis strict implication is interdefinable with a normal modal box. From the algebraic point of view this fact is equivalent to say that in Boolean algebras the notion of a normal modal operator $\Box$ is interdefinable with the notion of the strict Lewis implication $\Rightarrow$ (see Lemma 4 and also [22]). However, this interdefinability is not valid in the intuitionistc case. Since the intuitionistic negation is not involutive, the operator $\Box$ (defined as $\Box p := \top \Rightarrow p$) is not interdefinable with the strict Lewis implication.

There exists a strong connection between subintuitionistic logics and normal modal logics. In Kripke frames, the semantic interpretation of the subintuitionistic implication coincides with the interpretation of Lewis strict implication. Recall that the Lewis strict implication $\Rightarrow$ is interpreted in the modal propositional language $\{\vee, \wedge, \supset, \Box\}$ as $\varphi \Rightarrow \alpha := \Box(\varphi \supset \alpha)$. Semantically we have the following interpretation. If $\langle X, R\rangle$ is a Kripke frame, i.e., X is a set and R a binary relation on X, then the strict implication $\Rightarrow$ is defined as

$$U \Rightarrow_R V = \Box_R(U^c \cup V) = \{x \in X \colon R(x) \cap U \subseteq V\},$$

for each $U, V \in \mathcal{P}(X)$, where $\mathcal{P}(X)$ denotes the powerset of X. This definition coincides with the definition of subintuitionistic implication given in [7,11,29].

Recall that a *modal operator* in a Boolean algebra A is a function $\Box \colon A \to A$. We say that $\Box$ is *monotone* if $\Box(a \wedge b) \leq \Box a \wedge \Box b$, for every $a, b \in A$, and we say that $\Box$ is *normal* if $\Box 1 = 1$ and $\Box(a \wedge b) = \Box a \wedge \Box b$, for every $a, b \in A$. A *monotone modal algebra* is a pair $\langle A, \Box\rangle$ where A is a Boolean algebra and $\Box$ is a monotone modal operator. Analogously, a *modal algebra* is a pair $\langle A, \Box\rangle$ where A is a Boolean algebra and $\Box$ is a normal modal operator. As we have mentioned before, the class of modal algebras is interdefinable with the the class of Boolean algebras with a strict implication. More precisely, a *Boolean weak Heyting algebra*, or *Boolean algebra with a strict implication*, is an algebra $\langle A, \vee, \wedge, \Rightarrow, \neg, 0, 1\rangle$ such that $\langle A, \vee, \wedge, \Rightarrow, 0, 1\rangle$ is a weak Heyting algebra [6] and $\langle A, \vee, \wedge, \neg, 0, 1\rangle$ is a Boolean algebra. It is not very difficult to prove that the class of Boolean weak Heyting algebras is term equivalent to the class of Boolean algebras with a normal modal operator $\Box$ (see Lemma 4 or [6]).

From an algebraic point of view we could interpret the connection between subintuitionistic logics and normal modal logics in the following way. By the representation theorems given in [6] we have that the variety of weak Heyting algebras is the variety generated by the $\{\vee, \wedge, \Rightarrow, 0, 1\}$-subreducts of the variety generated by the following class of Boolean algebras with strict implication:

$$\mathsf{KA} = \{\langle \mathcal{P}(X), \Rightarrow_R \rangle \colon \langle X, R \rangle \text{ is a Kripke frame}\}.$$

Since the semantic interpretation of the connective of subintuitionistic implication coincides with the semantic interpretation of Lewis strict implication, it is fair to also call the weak Heyting algebras as *Lewis distributive lattices*.

The interpretation of the subintuitionistic implication $\Rightarrow$ depends on the relational semantics chosen and also of the notion of valuation. In the case of neighbourhood semantics we have more options, as we will explain below.

Recall that a *neighbourhood frame* is a pair $\langle X, M \rangle$, where X is a non-empty set and M is a relation between X and $\mathcal{P}(X)$. A *monotone neighbourhood frame* is a neighbourhood frame $\langle X, M \rangle$ such that for every $x \in X$ and $U, V \in \mathcal{P}(X)$, if $U \subseteq V$ and $U \in M(x)$, then $V \in M(x)$ [27] , where $M(x) = \{Y \subseteq X : (x, Y) \in M\}$. In $\mathcal{P}(X)$ we can define a monotone modal operator m_M as

$$m_M(U) = \{x \in X \colon U \in M(x)\}.$$

It can be seen that $\langle \mathcal{P}(X), m_M \rangle$ is a monotone modal algebra [4,27]. The variety generated by the class of algebras of the form $\langle \mathcal{P}(X), m_M \rangle$ is the variety MonBA of monotone Boolean algebras. By means of the operator m_M we can define a binary operation $\mapsto_M$ on $\mathcal{P}(X)$ as

$$U \mapsto_M V = m_M(U^c \cup V),$$

for each $U, V \in \mathcal{P}(X)$. Since $m_M(U) = X \mapsto_M U$, for all $U \in \mathcal{P}(X)$, it is easy to see that the monotone modal operator m_M is interdefinable with $\mapsto_M$. The algebra $\langle \mathcal{P}(X), \mapsto_M \rangle$ is an example of *monotone Lewis algebra* [10, Definition 7.2]. By the representation results given in [10] we can ensure that the variety MLA of monotone Lewis algebras is the variety generated by the following class of algebras

$$\{\langle \mathcal{P}(X), \mapsto_M \rangle \colon \langle X, M \rangle \text{ is a monotone neighbourhood frame}\}.$$

Moreover, the variety MonBA is term equivalent to the variety MLA [10].

Now, we consider a neighbourhood frame $\langle X, M\rangle$, not necessarily monotone. In $\mathcal{P}(X)$ we define a modal operator $\Box_M$ as follows:

$$\Box_M(U) = \{x \in X : \forall Y \in M(x)(Y \subseteq U)\}. \tag{1.1}$$

It can be checked that $\langle \mathcal{P}(X), \Box_M\rangle$ is a normal modal algebra, and that the variety MA of normal modal algebras is also generated by the class

$$\{\langle \mathcal{P}(X), \Box_M\rangle : \langle X, M\rangle \text{ is a neighbourhood frame}\}.$$

The interpretation of a normal modal operator $\Box$ in neighbourhood frames by means of the clause (1.1) is not new. The same interpretation is used, for example, in [3,15]. With this interpretation we can define another type of implication $\Rightarrow_M$ as follows:

$$U \Rightarrow_M V = \{x \in X : \forall Y \in M(x)(Y \subseteq U \text{ implies } Y \subseteq V)\},$$

for all $U, V \in \mathcal{P}(X)$. Note that $\Box_M(U) = X \Rightarrow_M U$, for all $U \in \mathcal{P}(X)$. Also note that this definition is the same that the definition given in [26, Definition 2.4] to interpret intuitionistic logic by using a special kind of neighbourhood frames. Later, in Section 7, we will compare these neighbourhood frames with the results of this paper.

An important difference between the class KA and the class

$$\mathsf{NA} = \{\langle \mathcal{P}(X), \Rightarrow_M\rangle : \langle X, M\rangle \text{ is a neighbourhood frame}\}$$

is the following. Even though the modal operator $\Box_M$ is normal, the implication $\Rightarrow_M$ may not produce a subintuitionistic implication, i.e., $\langle \mathcal{P}(X), \Rightarrow_M\rangle$ is not necessarily a weak Heyting algebra, as we show in the following example.

Example 1. Take the set $X = \{x, y\}$ and the neighbourhood relation

$$M = \{(x, \emptyset), (y, \emptyset), (x, X), (y, X)\}.$$

Let $U = \{x\}$ and $V = \{y\}$. Then

$$(U \Rightarrow_M \emptyset) \cap (V \Rightarrow_M \emptyset) = X \nsubseteq (U \cup V) \Rightarrow_M \emptyset = X \Rightarrow_M \emptyset = \emptyset.$$

Thus, the algebra $\langle \mathcal{P}(X), \Rightarrow_M\rangle$ is not a weak Heyting algebra.

At this point we can formulate two problems:

- **Problem 1**: Study the $\{\vee, \wedge, \mapsto, \bot, \top\}$-subreducts of the variety MLA.
- **Problem 2**: Study the $\{\vee, \wedge, \Rightarrow, \bot, \top\}$-subreducts of the class NA.

Problem 1 is studied in [10]. The $\{\vee, \wedge, \mapsto, \bot, \top\}$-subreducts of monotone Lewis algebras are the distributive lattices with a monotone implication [10, Definition 3.6]. The variety of distributive lattices with a monotone implication is the algebraic semantics of the weak subintuitionistic logic $\mathcal{P}_{\mapsto}$ defined in [10]. The logic $\mathcal{P}_{\mapsto}$ can be seen as the positive fragment of monotone modal logic in the same way that the subintuitionistic logic **F** given by Corsi in [7] can be interpreted as the positive fragment of normal modal logic. By this fact the extensions of $\mathcal{P}_{\mapsto}$ are called *monotone subintuitionistics logics*.

Addressing **Problem 2** is the main goal of this paper. Although the variety of algebras generated by the algebras of the form $\langle \mathcal{P}(X), \Box_M \rangle$ is the variety MA of normal modal algebras, the variety generated by the $\{\vee, \wedge, \Rightarrow, \bot, \top\}$-reducts of the variety generated by the class NA is not the variety of weak Heyting algebras as we have shown above. In this paper we axiomatize and study this new variety, which coincides with the $\{\vee, \wedge, \Rightarrow, \bot, \top\}$-subreducts of the class NA. In a future work we are going to study the associated logics of this new variety following the research line developed in [5].

The paper is organized as follows. In Section 2 we recall some results that will be useful for the rest of this paper. In particular, we recall Priestley duality and some properties about weak Heyting algebras [6]. In Section 3 we study the variety whose members are WL-lattices (see Definition 5), which properly contains the variety of weak Heyting algebras. We also give a representation theorem for the members of this variety; more precisely, we prove that every WL-lattice is isomorphic to a subalgebra of a WL-lattice whose underlying bounded distributive lattice is the bounded distributive lattice of upsets of a poset. We also show that the variety of WL-lattices coincides with the $\{\vee, \wedge, \Rightarrow, \bot, \top\}$-subreducts of NA. In Section 4 we give the main result of this paper, which is a Priestley style duality for the algebraic category of WL-lattices. In Section 5 we compare the dual categorical equivalence above mentioned with that developed in [6] for the algebraic category of weak Heyting algebras. More precisely, we study the link between WL-spaces and WH-spaces, which are the dual objects of WH-algebras [6]. In Section 6 some subvarieties of the variety of WL-lattices are introduced and studied. In Sectionrefsec:Connectionspswithspsintuitionistic we study the relation that there exists between certain class of WL-frames, introduced in Definition 6, in order to prove a representation theorem for WL-lattices, and the class of intuitionistic neighbourhood frames defined in [26, Definition 2.1]. In Section 8 we study the lattice of congruences of RWL-lattices, where a RWL-lattice is defined as a WL-lattice which satisfies the additional inequality $a \wedge (a \Rightarrow b) \leq b$, or equivalently, the following condition: for every a, b, c, if $a \Rightarrow b \leq c$ then $a \wedge b \leq c$. We finish this section by giving a description of the simple and subdirectly irreducible algebras of RWL-lattices, and also a characterization of the principal congruences of any RWL-lattice. In

Section 9 we study how the topological duality developed for WL-lattices in Section 4 can be applied to the case of Heyting algebras with a weak strict implication.

2. Preliminaries

In this section we recall some results which will be usefull in this paper. First we recall Priestley duality [28], which describes a dual categorical equivalence for the algebraic category of bounded distributive lattices by means of ordered topological spaces known as Priestley spaces, thus establishing that the mentioned algebraic category is dually equivalent to the category of Priestley spaces and continuous order preserving maps. Then we also recall some elemental properties of weak Heyting algebras [6].

2.1. Bounded Distributive Lattices and Priestley Duality

Let $\langle X, \leq \rangle$ be a poset. For each $Y \subseteq X$, let $[Y) = \{x \in X \colon \exists y \in Y (y \leq x)\}$ and $(Y] = \{x \in X \colon \exists y \in Y (x \leq y)\}$. We will say that Y is an *upset* of X (a *downset* of X) if $Y = [Y)$ ($Y = (Y]$). If $Y = \{y\}$, then we will write $[y)$ and $(y]$ instead of $[\{y\})$ and $(\{y\}]$, respectively. Note that $[y) = \{x \in X \colon x \geq y\}$ and $(y] = \{x \in X \colon x \leq y\}$. We also write $\mathcal{P}(X)$ and $\mathrm{Up}(X)$ for the set of all subsets and upsets of X, respectively. The complement of $Y \subseteq X$ will be denoted by Y^c.

Let $\langle A, \vee, \wedge, 0, 1 \rangle$ be a bounded distributive lattice. If there is no ambiguity, this algebra will be identified with its carrier set A. A subset $F \subseteq A$ is a *filter* of A if it is an upset, $1 \in F$ and $a \wedge b \in F$ whenever $a, b \in F$. The filter generated by a non-empty subset $X \subseteq A$, denoted by $\mathrm{Fig}(X)$, is the least filter (with respect to the inclusion) that contains X. It can be shown that

$$\mathrm{Fig}(X) = \{a \in A \colon \exists a_1, \ldots, a_n \in X (a_1 \wedge \cdots \wedge a_n \leq a)\}.$$

In particular, $\mathrm{Fig}(\{a\}) = [a)$. We denote by $\mathrm{Fi}(A)$ the set of filters of A. A filter F of A is said to be *proper* if $F \neq A$. A proper filter P is called *prime* if for every $a, b \in A$, $a \vee b \in P$ implies $a \in P$ or $b \in P$. We write $X(A)$ for the set of prime filters of A. If there is no ambiguity, this set will be identified with the poset $\langle X(A), \subseteq \rangle$. A set $I \subseteq A$ is an *ideal* of A if it is a downset, $0 \in I$ and $a \vee b \in I$ whenever $a, b \in I$. The ideal generated by a non-empty subset $X \subseteq A$, denoted by $\mathrm{Idg}(X)$, is the least ideal (with respect to the inclusion) that contains X. We have that

$$\mathrm{Idg}(X) = \{a \in A \colon \exists a_1, \ldots, a_n \in X (a \leq a_1 \vee \cdots \vee a_n)\}.$$

In particular, $\mathrm{Idg}(\{a\}) = (a]$. We write $\mathrm{Id}(A)$ to indicate the set of ideals of A. Let $\varphi\colon A \to \mathrm{Up}(X(A))$ be the map defined by

$$\varphi(a) = \{P \in X(A) \colon a \in P\},$$

which is an embedding of bounded distributive lattices.

In what follows we recall the definition of Priestley space.

Definition 2. A Priestley space is a triple $\langle X, \leq, \tau\rangle$ such that $\langle X, \leq\rangle$ is a poset, $\langle X, \tau\rangle$ is a compact topological space and the Priestley separation axiom is satisfied, which means that for every $x, y \in X$ such that $x \nleq y$ there exists a clopen upset U such that $x \in U$ and $y \notin U$.

If there is no ambiguity, a Priestley space $\langle X, \leq, \tau\rangle$ will be identified with its carrier set X. If X is a Priestley space, then the family of clopen upsets of X will be denoted by $D(X)$. It is known that $D(X)$ is a bounded distributive lattice. Let A be a bounded distributive lattice. Consider the map $\varphi : A \to \mathcal{P}(X(A))$ given by $\varphi(a) = \{P \in X(A) : a \in P\}$, for each $a \in A$. The triple $\langle X(A), \subseteq, \tau_A\rangle$ is a Priestley space, where τ_A is the topology generated by the subbase $\{\varphi(a)\colon a \in A\} \cup \{\varphi(a)^c\colon a \in A\}$. Moreover, since $D(X(A)) = \{\varphi(a)\colon a \in A\}$ we have that A and $D(X(A))$ are isomorphic. If X is a Priestley space, then the map $\epsilon\colon X \to X(D(X))$ defined by $\epsilon(x) = \{U \in D(X)\colon x \in U\}$ is a homeomorphism and an order-isomorphism. Let A and B be two bounded distributive lattices. If $h\colon A \to B$ is a homomorphism of bounded distributive lattices, then the map $h^*\colon X(B) \to X(A)$ defined by $h^*(P) = h^{-1}(P)$ is a continuous and monotone map. Conversely, if X and Y are Priestley spaces and $f\colon X \to Y$ is a continuous and monotone map, then the map $f^*\colon D(Y) \to D(X)$ defined by $f^*(U) = f^{-1}(U)$ is a homomorphism of bounded distributive lattices. Moreover, there exists a dual equivalence between the algebraic category of bounded distributive lattices and the category whose objects are Priestley spaces and whose morphisms are continuous and monotone maps.

Let A be a bounded distributive lattice. We denote by $\mathrm{CUp}(X(A))$ to the family whose members are the closed upsets of $X(A)$. It is part of the folklore of Priestley spaces that the lattices $\mathrm{Fi}(A)$ and $\mathrm{CUp}(X(A))$ are dually isomorphic [8]. More precisely, if $F \in \mathrm{Fi}(A)$, then $\widehat{F} = \{P \in X(A)\colon F \subseteq P\} \in \mathrm{CUp}(X(A))$. Conversely, if $Y \in \mathrm{CUp}(X(A))$, then the set $F_Y = \{a \in A\colon Y \subseteq \varphi(a)\} \in \mathrm{Fi}(A)$. Moreover, for every $F \in \mathrm{Fi}(A)$ and $Y \in \mathrm{CUp}(X(A))$ we get $F = F_{\widehat{F}}$ and $Y = \widehat{F_Y}$. Furthermore, for every $F, G \in \mathrm{Fi}(A)$, $F \subseteq G$ if and only if $\widehat{G} \subseteq \widehat{F}$. Note that for every $P \in X(A)$, $\widehat{P} = [P)$.

2.2. Heyting Algebras

We recall that a Heyting algebra (also known as pseudo-Boolean algebra) is a bounded lattice A equipped with a binary operation $\Rightarrow$ such that for every $a, b, c \in A$, $a \wedge b \leq c$ if and only if $a \leq b \Rightarrow c$ [2,13]. Every Boolean algebra is a Heyting algebra where $a \Rightarrow b$ coincides with $\neg a \vee b$. The class of Heyting algebras is a variety [2].

A very important example of Heyting algebra is the following. Let $\langle X, \leq \rangle$ be a poset. Then the algebra $\langle \mathrm{Up}(X), \cup, \cap, \Rightarrow_{\leq}, \emptyset, X \rangle$ is a Heyting algebra where the implication $\Rightarrow_{\leq}$ is defined as $U \Rightarrow_{\leq} V = \{x \in X \colon [x) \cap U \subseteq V\}$, for each $U, V \in \mathrm{Up}(X)$.

2.3. Weak Heyting Algebras

The variety of weak Heyting algebras, or WH-algebras for short, was introduced in [6] as the algebraic counterpart of the least subintuitionistic logic wK consider in [5]. A WH-algebra is a bounded distributive lattice with a binary operation which satisfies the properties of the strict implication in the modal logic K. Each one of the varieties of weak Heyting algebras studied in [6] corresponds to two propositional logics wK_σ and sK_σ defined in [5]. The logics wK_σ and sK_σ are the strict implication fragments of the local and global consequence relations defined by means of Kripke models, respectively.

Definition 3. An algebra $\langle A, \vee, \wedge, \Rightarrow, 0, 1 \rangle$ of type $(2, 2, 2, 0, 0)$ is a weak Heyting algebra, or WH-algebra for short, if $\langle A, \vee, \wedge, 0, 1 \rangle$ is a bounded distributive lattice and the following conditions are satisfied for every $a, b, c \in A$:

(1) $a \Rightarrow (b \wedge c) = (a \Rightarrow b) \wedge (a \Rightarrow c)$,

(2) $(a \vee b) \Rightarrow c = (a \Rightarrow c) \wedge (b \Rightarrow c)$,

(3) $(a \Rightarrow b) \wedge (b \Rightarrow c) \leq a \Rightarrow c$,

(4) $a \Rightarrow a = 1$.

Examples of WH-algebras that appear in the literature are Heyting algebras, Basic algebras introduced by Ardeshir and Ruitenburg in [1] and subresiduated lattices of Epstein and Horn in [12].

Let $\langle A, \vee, \wedge, \neg, 0, 1 \rangle$ be a Boolean algebra. A binary operation $\Rightarrow$ on A is called a *weak Heyting implication* if $\langle A, \vee, \wedge, \Rightarrow, 0, 1 \rangle$ is a weak Heyting algebra.

The following result is known (see [6, p. 224]). For details see also Lemma [10, Lemma 7.5].

Lemma 4. *Given a Boolean algebra A, there is a bijective correspondence between normal modal operators on A and weak Heyting implications on A, which is defined as follows. If $\Box$ is a normal modal operator, then the binary operation $\Rightarrow$ given by $a \Rightarrow b := \Box(\neg a \vee b)$ is a weak Heyting implication, and if $\Rightarrow$ is a weak Heyting implication, then the unary map $\Box$ given by $\Box a := 1 \Rightarrow a$ is a normal modal operator.*

3. Weak Lewis Distributive Lattices

In this section we introduce and study a variety, whose members will be called weak Lewis distributive lattices (WL-lattices for short), that properly contains the variety of WH-algebras. We give a representation theorem for WL-lattices. More precisely, we prove that every WL-lattice is isomorphic to a subalgebra of a WL-lattice whose underlying bounded distributive lattice is the bounded distributive lattice of upsets of a poset.

Definition 5. An algebra $\langle A, \vee, \wedge, \Rightarrow, 0, 1\rangle$ of type $(2, 2, 2, 0, 0)$ is a weak Lewis distributive lattice, or WL-lattice for short, if $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice and the following conditions are satisfied for every $a, b, c \in A$:

(1) $a \Rightarrow (b \wedge c) = (a \Rightarrow b) \wedge (a \Rightarrow c)$,

(2) $(a \vee b) \Rightarrow c \leq (a \Rightarrow c) \wedge (b \Rightarrow c)$,

(3) $(a \Rightarrow b) \wedge (b \Rightarrow c) \leq a \Rightarrow c$,

(4) $a \Rightarrow a = 1$.

The variety of WL-lattices will be denoted by WL. The difference between the definition of a WL-lattice and a WH-algebra is that in WL-lattices the inequality $(a \Rightarrow c) \wedge (b \Rightarrow c) \leq (a \vee b) \Rightarrow c$ is not necessarily satisfied (see Example 9).

We recall that a neighbourhood frame is a pair $\langle X, M\rangle$ where X is a set and M is a neighbourhood relation, i.e., $M \subseteq X \times \mathcal{P}(X)$. Neighbourhood semantics is used for the classical modal logics that are strictly weaker than the normal modal logic **K** [27]. Given a neighbourhood frame $\langle X, M\rangle$ we can define in the Boolean algebra $\mathcal{P}(X)$ an implication $\Rightarrow_M$ as $U \Rightarrow_M V = \{x \in X : \forall Y \in M(x)(Y \subseteq U \text{ implies } Y \subseteq V)\}$. In Corollary 15 we will prove that the variety WL coincides with the class of $\{\vee, \wedge, \Rightarrow, \bot, \top\}$-subreducts of the class $\mathsf{NA} = \{\langle \mathcal{P}(X), \Rightarrow_M\rangle : \langle X, M\rangle \text{ is a neighbourhood frame}\}$.

Below we define a generalization of this notion that will allow us to interpret the weak strict implication $\Rightarrow$.

Definition 6. A weak Lewis neighbourhood frame, or WL-frame, is a structure $\langle X, \leq, M\rangle$ such that $\langle X, \leq\rangle$ is a poset, $M \subseteq X \times \mathcal{P}(X)$ and the following condition is satisfied

$$\forall x, y \in X, \forall Y \in \mathcal{P}(X)\, (x \leq y \text{ and } (y, Y) \in M \text{ imply } (x, Y) \in M)\,. \qquad (*)$$

Let $\langle X, \leq\rangle$ be a poset and $M \subseteq X \times \mathcal{P}(X)$. Note that with the notation $M(x)$, condition $(*)$ can be formulated as

$$\forall x, y \in X, \forall Y \in \mathcal{P}(X)\, (x \leq y \text{ implies } M(y) \subseteq M(x))\,. \qquad (*)$$

The following lemma will be used throughout this paper.

Lemma 7. *Let $\mathcal{F} = \langle X, \leq, M\rangle$ be a WL-frame. For every $U, V \in \mathrm{Up}(X)$, the set defined by*

$$U \Rightarrow_M V = \{x \in X \colon \forall Y \in M(x)\, (Y \subseteq U \textit{ implies } Y \subseteq V)\}$$

in an upset. Moreover, $A(\mathcal{F}) = \langle \mathrm{Up}(X), \cup, \cap, \Rightarrow_M, \emptyset, X\rangle$ is a WL-lattice.

Proof. It is immediate that $\langle \mathrm{Up}(X), \cup, \cap, \emptyset, X\rangle$ is a bounded distributive lattice. We will show that $\Rightarrow_M$ defines a binary operation on $\mathrm{Up}(X)$. Indeed, let $U, V \in \mathrm{Up}(X)$. In order to show that $U \Rightarrow_M V \in \mathrm{Up}(X)$, let $x, y \in X$ such that $x \leq y$ and $x \in U \Rightarrow_M V$. Let $Y \in M(y)$ such that $Y \subseteq U$. Then $(y, Y) \in M$, so by $(*)$ we get $(x, Y) \in M$. Since $x \in U \Rightarrow_M V$, we have $Y \subseteq V$. Thus, $y \in U \Rightarrow_M V$. Hence, $U \Rightarrow_M V \in \mathrm{Up}(X)$. We have proved that $\Rightarrow_M$ defines a binary operation on $\mathrm{Up}(X)$. Furthermore, a straightforward computation shows that $\langle \mathrm{Up}(X), \cup, \cap, \Rightarrow_M, \emptyset, X\rangle$ is a WL-lattice. ∎

Remark 8. Let A be a lattice with a binary operation $\Rightarrow$. A direct computation shows that the following quasi-identies are equivalent:

(1) If $a \leq b$ then $b \Rightarrow c \leq a \Rightarrow c$.

(2) $b \Rightarrow c \leq (a \wedge b) \Rightarrow c$.

(3) $(a \vee b) \Rightarrow c \leq (a \Rightarrow c) \wedge (b \Rightarrow c)$.

A direct computation also shows that the following quasi-identities are equivalent:

(1) If $a \leq b$ then $c \Rightarrow a \leq c \Rightarrow b$.

(2) $c \Rightarrow a \leq c \Rightarrow (a \vee b)$.

(3) $c \Rightarrow (a \wedge b) \leq (c \Rightarrow a) \wedge (c \Rightarrow b)$.

The class of WH-algebras forms a proper subvariety of WL, as is shown by the following example.

EXAMPLE 9. Let A be the Boolean algebra of four elements, where a and b are the atoms. Define the following binary operation $\Rightarrow$ on A:

$\Rightarrow$	0	a	b	1
0	1	1	1	1
a	1	1	1	1
b	1	1	1	1
1	0	0	0	1

A straightforward computation shows that A endowed with the binary operation $\Rightarrow$ is a WL-lattice which is not a WH-algebra because

$$(a \vee b) \Rightarrow 0 \neq (a \Rightarrow 0) \wedge (b \Rightarrow 0).$$

We know show how to construct a WL-frame from a WL-lattice.

Let $A \in \mathsf{WL}$. Recall that

$$\mathrm{CUp}(X(A)) = \{Y \subseteq X(A) \colon Y = \widehat{G} \text{ for some } G \in \mathrm{Fi}(A)\}.$$

We define the relation

$$M_A \subseteq X(A) \times \mathrm{CUp}(X(A))$$

by

$$(P, \widehat{G}) \in M_A \text{ iff } \forall a, b \in A \text{ (if } P \in \varphi(a \Rightarrow b) \text{ and } \widehat{G} \subseteq \varphi(a) \text{ then } \widehat{G} \subseteq \varphi(b)).$$

In other words,

$$(P, \widehat{G}) \in M_A \text{ iff } \forall a, b \in A \text{ (if } a \Rightarrow b \in P \text{ and } a \in G \text{ then } b \in G).$$

The proof of the following lemma follows from a direct computation.

LEMMA 10. *Let $A \in \mathsf{WL}$. Then $\langle X(A), \subseteq, M_A \rangle$ is a WL-frame.*

LEMMA 11. *Let $A \in \mathsf{WL}$. Then $\langle \mathrm{Up}(X(A)), \cup, \cap, \Rightarrow_{M_A}, \emptyset, X(A) \rangle$ is a WL-lattice.*

PROOF. It follows from Lemmas 7 and 10. ■

Let $A \in \mathsf{WL}$. We also define the relation

$$\overline{M}_A \subseteq \mathrm{Fi}(A) \times \mathrm{CUp}(X(A))$$

as the extension of M_A on $\mathrm{Fi}(A)$, i.e., for every $F, G \in \mathrm{Fi}(A)$,

$$(F, \widehat{G}) \in \overline{M}_A \text{ iff } \forall a, b \in A \text{ (if } a \Rightarrow b \in F \text{ and } a \in G \text{ then } b \in G).$$

The definition of $\overline{M}_A$ will be used in the next proposition and lemma respectively.

Let $A \in \mathsf{WL}$ and $F \in \mathrm{Fi}(A)$. Consider the operation $D_F \colon \mathcal{P}(A) \to \mathcal{P}(A)$ given by

$$D_F(X) = \left\{a \in A \colon \exists Y \subseteq X \text{ finite such that } \bigwedge Y \Rightarrow a \in F\right\},$$

where $\bigwedge Y$ is the infimum of Y.

The proof of the following result is similar to the proof of [6, Proposition 3.4] and [6, Proposition 3.5].

Proposition 12. *Let $A \in \mathsf{WL}$. Let $F \in \mathrm{Fi}(A)$ and $X \subseteq A$. The following conditions hold:*

(1) *D_F is a finitary closure operator.*

(2) *$D_F(X) \in \mathrm{Fi}(A)$.*

(3) *$(F, \widehat{D_F(X)}) \in \overline{M}_A$.*

(4) *$D_F(X)$ is the least filter G of A containing X such that $(F, \widehat{G}) \in \overline{M}_A$.*

(5) *$(F, \widehat{G}) \in \overline{M}_A$ if and only if $D_F(G) = G$, for all $F, G \in \mathrm{Fi}(A)$.*

The following lemma will play a fundamental role in this paper in order to give a representation theorem for WL-lattices.

Lemma 13. *Let $A \in \mathsf{WL}$. Let $a, b \in A$ and $F \in \mathrm{Fi}(A)$. Then $a \Rightarrow b \notin F$ if and only if there exists $G \in \mathrm{Fi}(A)$ such that $(F, \widehat{G}) \in \overline{M}_A$, $a \in G$ and $b \notin G$.*

Proof. Let $a, b \in A$. Suppose that $a \Rightarrow b \notin F$. Let $G = D_F([a))$, so $a \in G$. We will see that $(F, \widehat{G}) \in \overline{M}_A$. Let $c, d \in A$ such that $c \Rightarrow d \in F$ and $c \in G$. Then there exists $f \in [a)$ such that $f \Rightarrow c \in F$. Since $(f \Rightarrow c) \wedge (c \Rightarrow d) \leq f \Rightarrow d$ and F is a filter, $f \Rightarrow d \in F$. Hence, $d \in G$. Thus, $(F, \widehat{G}) \in \overline{M}_A$. Moreover, $b \notin G$. Indeed, suppose that $b \in G$. Then there exists $f \in [a)$ such that $f \Rightarrow b \in F$. Besides, $f \Rightarrow b \leq a \Rightarrow b$. Taking into account that F is an upset, we get $a \Rightarrow b \in F$, which is a contradiction. Then $b \notin G$. The converse is immediate. ■

Let $A \in \mathsf{WL}$. It follows from Lemma 11 that $\langle \mathrm{Up}(X(A)), \cup, \cap, \Rightarrow_{M_A}, \emptyset, X(A)\rangle$ is a WL-lattice. In the following theorem we will prove that every WL-lattice is isomorphic to a WL-lattice which is a subalgebra of the WL-lattice of upsets of some WL-frame.

Theorem 14. *Let $A \in \mathsf{WL}$. Then the map $\varphi \colon A \to \mathrm{Up}(X(A))$ defined by $\varphi(a) = \{P \in X(A) : a \in P\}$ is an embedding of WL-lattices.*

Proof. Let $A \in \mathsf{WL}$. Recall that φ is an embedding of bounded distributive lattices, so we only need to show that $\varphi(a \Rightarrow b) = \varphi(a) \Rightarrow_{M_A} \varphi(b)$, for every

$a, b \in A$. In order to prove it, let $a, b \in A$. Let $P \in X(A)$ and suppose that $a \Rightarrow b \notin P$. By Lemma 13 there exists $G \in \mathrm{Fi}(A)$ such that $(P, \widehat{G}) \in M_A$, $a \in G$ and $b \notin G$, i.e., $(P, \widehat{G}) \in M_A$, $\widehat{G} \subseteq \varphi(a)$ and $\widehat{G} \nsubseteq \varphi(b)$. So, $P \notin \varphi(a) \Rightarrow_{M_A} \varphi(b)$. Hence, $\varphi(a) \Rightarrow_{M_A} \varphi(b) \subseteq \varphi(a \Rightarrow b)$. The converse inclusion is immediate. ■

It follows from Theorem 14 that every WL-lattice is an isomorphic image of a subalgebra of the $\{\vee, \wedge, \Rightarrow, \bot, \top\}$-reduct of an algebra of NA, i.e., the variety WL is contained in the class of $\{\vee, \wedge, \Rightarrow, \bot, \top\}$-subreducts of NA. The converse inclusion is immediate. Therefore, we obtain the following result.

Corollary 15. *The variety* WL *coincides with the* $\{\vee, \wedge, \Rightarrow, \bot, \top\}$*-subreducts of* NA.

4. Categorical Duality

In this section we introduce the dual objects to WL-lattices, called WL-spaces. We focus on the construction of a topological duality for the algebraic category of WL-lattices. Since WL-lattices are a generalization of WH-algebras, in Section 5 we study the link between WL-spaces and WH-spaces, which are the dual objects of WH-algebras [6].

4.1. Topological Representation

Let us consider a structure $\langle X, M \rangle$ such that X is a Priestley space and $M \subseteq X \times \mathrm{CUp}(X)$. For each $U \in D(X)$, we take the set

$$L_U = \{Y \in \mathrm{CUp}(X) \colon Y \subseteq U\}.$$

It is clear that $L_{U \cap V} = L_U \cap L_V$ and $L_U \cup L_V \subseteq L_{U \cup V}$, for all $U, V \in \mathrm{CUp}(X)$. Also, $L_\emptyset = \{\emptyset\}$ and $L_X = \mathrm{CUp}(X)$. For every $U, V \in D(X)$ the set $U \Rightarrow_M V$ defined in Lemma 7 can be written as

$$U \Rightarrow_M V = \{x \in X \colon \forall Y \in M(x)\, (Y \in L_U \text{ then } Y \in L_V)\}.$$

Also note that $x \in U \Rightarrow_M V$ if and only if $M(x) \subseteq L_U^c \cup L_V$.

Definition 16. A structure $\langle X, M \rangle$ is a WL-space if X is a Priestley space, $M \subseteq X \times \mathrm{CUp}(X)$ and the following conditions are satisfied:

(WLS1) $U \Rightarrow_M V \in D(X)$, for every $U, V \in D(X)$.

(WLS2) $M(x) = \bigcap\{L_U^c \cup L_V \colon U, V \in D(X) \text{ and } x \in U \Rightarrow_M V\}$, for all $x \in X$.

Let $\langle X, M\rangle$ be a WL-space. Note that it follows from (WLS1) that $\Rightarrow_M$ defines a binary operation on $D(X)$.

REMARK 17. Let $\langle X, M\rangle$ be a WL-space. Then $\langle X, M\rangle$ is a WL-frame. In order to show it, let $x, y \in X$ such that $x \leq y$. Let $Y \in M(y)$ and suppose that $Y \notin M(x)$. It follows from condition (WLS2) that there exist $U, V \in D(X)$ such that $x \in U \Rightarrow_M V$ and $Y \notin L_U^c \cup L_V$. Moreover, it follows from (WLS1) that $U \Rightarrow_M V \in D(X)$, so $y \in U \Rightarrow_M V$. Thus, $M(y) \subseteq L_U^c \cup L_V$. Hence, $Y \in M(y) \subseteq L_U^c \cup L_V$, which is a contradiction. Therefore, $M(y) \subseteq M(x)$.

REMARK 18. Let X be a Priestley space and $M \subseteq X \times \mathrm{CUp}(X)$. It follows from Remark 17 and Lemma 7 that $\langle X, M\rangle$ is a WL-space if and only if $\langle X, M\rangle$ is a WL-frame which satisfies (WLS2) and the following additional condition: for every $U, V \in D(X)$, $U \Rightarrow_M V$ is clopen.

PROPOSITION 19. *Let $\langle X, M\rangle$ be a WL-space. Then $\langle D(X), \cup, \cap, \Rightarrow_M, \emptyset, X\rangle$ is a WL-lattice.*

PROOF. It follows from Remark 17 and Lemma 7. ■

Let X be a Priestley space. Taking into account that $\epsilon\colon X \to X(D(X))$, which is defined by $\epsilon(x) = \{U \in D(X)\colon x \in U\}$, is a homeomorphism and an order-isomorphism, we have that for every closed upset Z the set $\epsilon[Z]$ is a closed upset. If $\langle X, M\rangle$ is a pair where $M \subseteq X \times \mathrm{CUp}(X)$ and condition (WLS1) is satisfied, then we define the relation $M_{D(X)} \subseteq X(D(X)) \times \mathrm{CUp}(X(D(X)))$ in terms of the binary operation $\Rightarrow_M$, as it was done for the case of WL-lattices.

LEMMA 20. *Let $\langle X, M\rangle$ be a pair such that X is a Priestley space, $M \subseteq X \times \mathrm{CUp}(X)$ and condition (WLS1) is satisfied. Then the following conditions are equivalent:*

(1) $M(x) = \bigcap\{L_U^c \cup L_V : U, V \in D(X)$ *and* $x \in U \Rightarrow_M V\}$, *for all* $x \in X$.

(2) *For every $x \in X$ and for every $Z \in \mathrm{CUp}(X)$, if $(\epsilon(x), \epsilon[Z]) \in M_{D(X)}$, then $(x, Z) \in M$.*

PROOF. Suppose that condition (1) is satisfied. Let $x \in X$ and $Z \in \mathrm{CUp}(X)$ such that $(\epsilon(x), \epsilon[Z]) \in M_{D(X)}$. In particular, there exists $F \in \mathrm{Fi}(D(X))$ such that $\epsilon[Z] = \widehat{F}$. Suppose that $(x, Z) \notin M$. Thus, it follows from hypothesis that there exist $U, V \in D(X)$ such that $x \in U \Rightarrow_M V$ and $Z \notin L_U^c \cup L_V$. Since $U \Rightarrow_M V \in D(X)$ we get $U \Rightarrow_M V \in \epsilon(x)$. On the other hand, $Z \in L_U \cap L_V^c$, i.e., $Z \subseteq U$ and $Z \nsubseteq V$. We will see that $U \in F$. Suppose that $U \notin F$. Then it follows from the prime filter theorem that there exists

$P \in X(D(X))$ such that $F \subseteq P$ and $U \notin P$. Taking into account that ϵ is onto we have that there exists $y \in X$ such that $\epsilon(y) = P$. Thus, $F \subseteq \epsilon(y)$ and $U \notin \epsilon(y)$. Then $\epsilon(y) \in \widehat{F} = \epsilon[Z]$, i.e., $y \in Z$ and $y \notin U$, which is a contradiction. So $U \in F$. Now we will prove that $V \notin F$. In order to show it, note that since $Z \nsubseteq V$, there exists $z \in Z$ such that $z \notin V$. Hence, $\epsilon(z) \in \epsilon[Z] = \widehat{F}$. Thus, $F \subseteq \epsilon(z)$ and $V \notin \epsilon(z)$, so $V \notin F$. In consequence, $U \Rightarrow_M V \in \epsilon(x)$, $U \in F$ and $V \notin F$ which is a contradiction because $(\epsilon(x), F) \in M_{D(X)}$. Hence, $(x, Z) \in M$. Thus, condition (2) is satisfied.

Conversely, assume that condition (2) is satisfied and let $x \in X$. Suppose that there exists $Z \in \mathrm{CUp}(X)$ such that

$$Z \in \bigcap \{L_U^c \cup L_V : U, V \in D(X) \text{ and } x \in U \Rightarrow_M V\}$$

and $Z \notin M(x)$, i.e., $(x, Z) \notin M$. It follows from hypothesis that $(\epsilon(x), \epsilon[Z]) \notin M_{D(X)}$. In particular, there exists $F \in \mathrm{Fi}(D(X))$ such that $\epsilon[Z] = \widehat{F}$. Then there exist $U, V \in D(X)$ such that $U \Rightarrow_M V \in \epsilon(x)$, $U \in F$ and $V \notin F$. Thus, $Z \in L_U^c \cup L_V$. We have two cases:

- Suppose that $Z \in L_U^c$, so $Z \nsubseteq U$. Thus, there is $z \in Z$ such that $z \notin U$. So, $\epsilon(z) \in \epsilon[Z] = \widehat{F}$, i.e., $F \subseteq \epsilon(z)$ and $U \notin \epsilon(z)$. Then $U \notin F$, which is a contradiction.
- Suppose that $Z \in L_V$, so $Z \subseteq V$. Since $V \notin F$, it follows from the prime filter theorem that there exists $P \in X(D(X))$ such that $F \subseteq P$ and $V \notin P$. Thus, there is $y \in X$ such $\epsilon(y) = P$. Hence, $F \subseteq \epsilon(y)$. So, $\epsilon(y) \in \widehat{F} = \epsilon[Z]$. Then $y \in Z$ and $y \notin V$, which is again a contradiction.

We conclude that $(x, Z) \in M$. Thus,

$$\bigcap \{L_U^c \cup L_V : U, V \in D(X) \text{ and } x \in U \Rightarrow_M V\} \subseteq M(x).$$

The other inclusion is immediate. Therefore, condition (1) is satisfied. ■

PROPOSITION 21. *Let $\langle X, M \rangle$ be a WL-space. Then for every $x \in X$ and $Z \in \mathrm{CUp}(X)$, $(x, Z) \in M$ if and only if $(\epsilon(x), \epsilon[Z]) \in M_{D(X)}$.*

PROOF. Let $x \in X$ and $Z \in \mathrm{CUp}(X)$. Suppose that $(x, Z) \in M$. Let $U, V \in D(X)$ such that $U \Rightarrow_M V \in \epsilon(x)$ and $U \in F_{\epsilon[Z]}$. We will see that $V \in F_{\epsilon[Z]}$. We have that $\epsilon[Z] \subseteq \varphi(U)$ if and only if $Z \subseteq U$, so $F_{\epsilon[Z]} = \{U \in D(X) : Z \subseteq U\}$. Since $Z \in M(x)$, $Z \subseteq U$ and $x \in U \Rightarrow_M V$ we get $Z \subseteq V$, i.e., $V \in F_{\epsilon[Z]}$. Hence, $(\epsilon(x), \epsilon[Z]) \in M_{D(X)}$. The fact that $(\epsilon(x), \epsilon[Z]) \in M_{D(X)}$ implies that $(x, Z) \in M$ follows from Lemma 20. ■

PROPOSITION 22. *Let $A \in \mathsf{WL}$. Then $\langle X(A), M_A \rangle$ is a WL-space.*

Proof. It follows from Theorem 14 that condition (WLS1) is satisfied. In order to show (WLS2), let $P \in X(A)$. We need to prove that

$$M_A(P) = \bigcap \left\{ L^c_{\varphi(a)} \cup L_{\varphi(b)} \colon a, b \in A \text{ and } a \to b \in P \right\}.$$

Let $F \in \mathrm{Fi}(A)$ such that $\widehat{F} \notin M_A(P)$, so there exist $a, b \in A$ such that $a \to b \in P$, $a \in F$ and $b \notin F$. In particular, $\widehat{F} \in L_{\varphi(a)}$. Besides, $\widehat{F} \notin L_{\varphi(b)}$. Indeed, taking into account that $b \notin F$ we get that there exists $P \in X(A)$ such that $b \notin P$ and $F \subseteq P$, so $P \in \widehat{F}$ and $P \notin \varphi(b)$. Thus, $\widehat{F} \notin L_{\varphi(b)}$. Hence, $\widehat{F} \notin L^c_{\varphi(a)} \cup L_{\varphi(b)}$. We have proved that

$$\bigcap \left\{ L^c_{\varphi(a)} \cup L_{\varphi(b)} \colon a, b \in A \text{ and } a \to b \in P \right\} \subseteq M_A(P).$$

The other inclusion follows from a straightforward computation. Thus, condition (WLS2) is satisfied. ■

4.2. Topological Duality

In this subsection we present a dual equivalence for the algebraic category of WL-lattices.

Note that if X and Y are Priestley spaces and $f \colon X \to Y$ is a continuous and monotone map, then for every $Z \in \mathrm{CUp}(X)$, $[f[Z]) \in \mathrm{CUp}(Y)$. Indeed, let $f \colon X \to Y$ be a continuous and monotone map between Priestley spaces. Consider Z a closed subset of X. It is known that $f[Z]$ is a closed subset of Y. Moreover, since $f[Z]$ is a closed subset of Y and Y is a Priestley space, then $[f[Z])$, which is the upset generated by $f[Z]$, is a closed subset of Y. Besides, $[f[Z])$ is an upset. Therefore, $[f[Z])$ is a closed upset of Y.

It is interesting to note that the previous property is not necessarily satisfied if we change $[f[Z])$ by $f[Z]$, as we show in the following example provided by one of the referees. Let $X = \{0, 1\}$ with $0 \neq 1$ and let τ be the discrete topology on X. Define the following two partial orders on X: $\leq = \{(0, 0), (1, 1)\}$ and $\leq^{'} = \{(0, 0), (0, 1), (1, 1)\}$. It is immediate that $\langle X, \leq, \tau \rangle$ and $\langle X, \leq^{'}, \tau \rangle$ are Priestley spaces. Let $f \colon \langle X, \leq, \tau \rangle \to \langle X, \leq^{'}, \tau \rangle$ be the identity map on X, which is a continuous and monotone map. We have that $\{0\}$ is a closed upset of $\langle X, \leq, \tau \rangle$. However, $f[\{0\}] = \{0\}$ is not a closed upset of $\langle X, \leq^{'}, \tau \rangle$.

Definition 23. Let $\langle X_1, M_1 \rangle$ and $\langle X_2, M_2 \rangle$ be two WL-spaces. A continuous and monotone map $f \colon X_1 \to X_2$ is a WL-morphism if the following conditions are satisfied:

(MF1) For every $x \in X_1$ and $Z \in \mathrm{CUp}(X_1)$, if $(x, Z) \in M_1$, then $(f(x), [f[Z])) \in M_2$.

(MF2) For every $x \in X_1$ and $Z \in \mathrm{CUp}(X_2)$, if $(f(x), Z) \in M_2$, then there exists $W \in \mathrm{CUp}(X_1)$ such that $W \in M_1(x)$ and $Z = [f[W])$.

Recall that if $h\colon A \to B$ is a homomorphism of bounded distributive lattices, then the map $h^*\colon X(B) \to X(A)$ is defined by $h^*(P) = h^{-1}(P)$.

The following elementary lemma will be used throughout this subsection.

Lemma 24. *Let A and B bounded distributive lattices. Let $h : A \to B$ be a homomorphism of bounded lattices. Let $G \in \mathrm{Fi}(B)$. Then $\widehat{h^{-1}(G)} = [h^*[\widehat{G}])$. Moreover, $F_{\widehat{h^{-1}(G)}} = h^{-1}(G)$.*

Proof. Let $G \in \mathrm{Fi}(B)$. A direct computation shows that

$$h^*[\widehat{G}] = \left\{h^{-1}(P)\colon P \in X(B) \text{ and } G \subseteq P\right\} \subseteq \widehat{h^{-1}(G)}.$$

Since $\widehat{h^{-1}(G)}$ is an upset, then $[h^*[\widehat{G}]) \subseteq \widehat{h^{-1}(G)}$. In order to prove the other inclusion, let $P \in X(A)$ such that $h^{-1}(G) \subseteq P$. Consider the set $h(P^c) = \{h(a) : a \notin P\}$. It is clear that $(h(P^c)]$ is an ideal. We will prove that $G \cap (h(P^c)] = \emptyset$. Suppose that there exist $f \in G$ and $a \notin P$ such that $f \leq h(a)$. Then $h(a) \in G$, so $a \in h^{-1}(G) \subseteq P$, which is a contradiction. Thus, it follows from the prime filter theorem that there exists $Q \in X(B)$ such that $G \subseteq Q$ and $h^{-1}(Q) \subseteq P$, so $P \in [h^*[\widehat{G}])$. ■

Proposition 25. *Let $A, B \in \mathsf{WL}$ and $h\colon A \to B$ a homomorphism of bounded lattices. Then:*

(1) *h^* satisfies (MF1) if and only if $h(a \Rightarrow b) \leq h(a) \Rightarrow h(b)$ for every $a, b \in A$.*

(2) *h^* satisfies (MF2) if and only if $h(a) \Rightarrow h(b) \leq h(a \Rightarrow b)$ for every $a, b \in A$.*

Proof. In order to prove (1), first assume that h^* satisfies the condition (MF1). Suppose that there exist $a, b \in A$ such that $h(a \Rightarrow b) \nleq h(a) \Rightarrow h(b)$. Then there is $P \in X(B)$ such that $h(a \Rightarrow b) \in P$ and $h(a) \Rightarrow h(b) \notin P$. It follows from Lemma 13 that there exists $G \in \mathrm{Fi}(B)$ such that $(P, \widehat{G}) \in M_B$, $h(a) \in G$ and $h(b) \notin G$, so it follows from hypothesis that $(h^*(P), [h^*[\widehat{G}])) \in M_A$. Moreover, it follows from Lemma 24 that $a \in F_{[h^*[\widehat{G}])}$ and $b \notin F_{[h^*[\widehat{G}])}$. Since $(h^*(P), [h^*[\widehat{G}])) \in M_A$, $a \Rightarrow b \in h^*(P)$ and $a \in F_{[h^*[\widehat{G}])}$ we get $b \in F_{[h^*[\widehat{G}])}$, which is a contradiction.

Conversely, suppose that $h(a \Rightarrow b) \leq h(a) \Rightarrow h(b)$ for every $a, b \in A$. Let $P \in X(B)$ and $G \in \mathrm{Fi}(B)$ such that $(P, \widehat{G}) \in M_B$. We need to show that

$(h^*(P), [h^*[\widehat{G}])) \in M_A$. In order to see it, let $a, b \in A$ such that $a \Rightarrow b \in h^*(P)$ and $a \in F_{[h^*[\widehat{G}])}$. It follows from Lemma 24 that $h(a) \in G$. Besides, taking into account that $h(a \Rightarrow b) \leq h(a) \Rightarrow h(b)$ and $h(a \Rightarrow b) \in P$ we get $h(a) \Rightarrow h(b) \in P$. But $(P, \widehat{G}) \in M_B$ and $h(a) \in G$, so $h(b) \in G$. Thus, it follows from Lemma 24 that $b \in F_{[h^*[\widehat{G}])}$. Hence, $(h^*(P), [h^*[\widehat{G}])) \in M_A$.

Now we will show condition (2). Suppose that h^* satisfies (MF2) and that there exist $a, b \in A$ such that $h(a) \Rightarrow h(b) \nleq h(a \Rightarrow b)$, so there exists $P \in X(B)$ such that $h(a) \Rightarrow h(b) \in P$ and $a \Rightarrow b \notin h^*(P)$. Then it follows from Lemma 13 that there exists $G \in \mathrm{Fi}(A)$ such that $(h^*(P), \widehat{G}) \in M_A$, $a \in G$ and $b \notin G$. In particular, it follows from hypothesis that there exists $H \in \mathrm{Fi}(B)$ such that $\widehat{H} \in M_B(P)$ and $\widehat{G} = [h^*(\widehat{H}))$. Since $F_{[h^*[\widehat{H}])} = G$ and $a \in G$, it follows from Lemma 24 that $h(a) \in H$. But $h(a) \Rightarrow h(b) \in P$ and $(P, \widehat{H}) \in M_B$, so $h(b) \in H$, i.e., $b \in G$, which is a contradiction. Hence, $h(a) \Rightarrow h(b) \leq h(a \Rightarrow b)$ for every $a, b \in A$.

Conversely, suppose that $h(a) \Rightarrow h(b) \leq h(a \Rightarrow b)$ for every $a, b \in A$. In order to prove that h^* satisfies (MF2), let $P \in X(B)$ and $G \in \mathrm{Fi}(A)$ such that $(h^*(P), \widehat{G}) \in M_A$. We define the set $H = D_P(h[G])$. It follows from Proposition 12 that $H \in \mathrm{Fi}(B)$ and $(P, \widehat{H}) \in M_B$. In what follows we will see that $h^{-1}(H) = G$. Let $c \in h^{-1}(H)$, so there exists $d \in G$ such that $h(d) \Rightarrow h(c) \in P$. But $h(d) \Rightarrow h(c) \leq h(d \Rightarrow c)$, so $d \Rightarrow c \in h^*(P)$. Taking into account that $d \in G$ and $(h^*(P), \widehat{G}) \in M_A$ we get $c \in G$, so $h^{-1}(H) \subseteq G$. The converse inclusion is immediate. Hence, $h^{-1}(H) = G$. Moreover, it follows from Lemma 24 that $F_{[h^*[\widehat{H}])} = h^{-1}(H) = G$, so $\widehat{G} = [h^*[\widehat{H}])$. Therefore, we have proved condition (MF2). ■

Corollary 26. *Let $A, B \in \mathsf{WL}$ and $h\colon A \to B$ a homomorphism of bounded lattices. Then h^* is a WL-morphism if and only if $h(a \Rightarrow b) = h(a) \Rightarrow h(b)$ for every $a, b \in A$.*

We abuse notation and also wite WL for the algebraic category whose members are WL-lattices. We write SWL for the category whose objects are WL-spaces and whose morphisms are WL-morphisms.

Theorem 27. *The assignment $A \mapsto \langle X(A), M_A\rangle$ and $h \mapsto h^*$ defines a functor $\mathrm{X}\colon \mathsf{WL} \to \mathsf{SWL}$, and the assignment $\langle X, M\rangle \mapsto \langle D(X), \cup, \cap, \Rightarrow_M, \emptyset, X\rangle$ and $f \mapsto f^*$ defines a functor $\mathrm{D}\colon \mathsf{SWL} \to \mathsf{WL}$. Moreover, the functors X and D establish a dual equivalence between the categories WL and SWL.*

Proof. It follows from Priestley duality, Theorem 14, Propositions 19, 21 and 22, and Corollary 26. ■

5. Connections Between WL-Spaces and WH-Spaces

The goal of this section is to compare the dual categorical equivalence given in Section 4 with that developed in [6] for the algebraic category of weak Heyting algebras. More precisely, we study the link between WL-spaces and WH-spaces, which are the dual objects of WH-algebras [6]. We start the section by giving necessary and sufficient conditions for a WL-lattice to be a WH-algebra. Then we recall the Priestley style duality for WH-algebras developed in [6] and we compare this with the dual categorical equivalence given in Theorem 27.

Proposition 28. *Let $A \in \mathsf{WL}$. Then the following conditions are equivalent:*

(1) $(a \Rightarrow c) \wedge (b \Rightarrow c) \leq (a \vee b) \Rightarrow c$ *for every* $a, b, c \in A$, *i.e.,* A *is a WH-algebra.*

(2) *For every* $a, b \in A$ *and* $P \in X(A)$, $a \Rightarrow b \notin P$ *if and only if there exists* $Q \in X(A)$ *such that* $(P, \widehat{Q}) \in M_A$, $a \in Q$ *and* $b \notin Q$.

(3) *For every* $P \in X(A)$, $F \in \mathrm{Fi}(A)$ *and* $c \in A$ *such that* $(P, \widehat{F}) \in M_A$ *and* $c \notin F$, *there exists* $Q \in X(A)$ *such that* $F \subseteq Q$, $c \notin Q$ *and* $(P, \widehat{Q}) \in M_A$.

Proof. First we will show the equivalence between (1) and (2). Suppose that (1) is satisfied. Since A is a WH-algebra, it follows from the proof of [6, Lemma 3.13] that condition (2) is satisfied. In order to prove that (2) implies (1), suppose that (2) is satisfied. Assume that (1) is not satisfied, so there exist $a, b, c \in A$ such that $(a \Rightarrow c) \wedge (b \Rightarrow c) \nleq (a \vee b) \Rightarrow c$. Thus, there is $P \in X(A)$ such that $a \Rightarrow c \in P$, $b \Rightarrow c \in P$ and $(a \vee b) \Rightarrow c \notin P$. So, it follows from the hypothesis that there exists $Q \in X(A)$ such that $(P, \widehat{Q}) \in M_A$, $a \vee b \in Q$ and $c \notin Q$. Since Q is prime, $a \in Q$ or $b \in Q$. If $a \in Q$, since $a \Rightarrow c \in P$ and $(P, \widehat{Q}) \in M_A$ we get $c \in Q$, which is a contradiction. If $b \in Q$ we obtain a contradiction too. Therefore, $(a \Rightarrow c) \wedge (b \Rightarrow c) \leq (a \vee b) \Rightarrow c$ for every $a, b, c \in A$.

Now we will prove that (1) implies (3). Assume that (1) is satisfied. Let $P \in X(A)$, $F \in \mathrm{Fi}(A)$ and $c \in A$ such that $(P, \widehat{F}) \in M_A$ and $c \notin F$. By Proposition 12, $F = D_P(F)$. Consider the family

$$\mathcal{F} = \{H \in \mathrm{Fi}(A) \colon F \subseteq H,\ (P, \widehat{H}) \in M_A \text{ and } c \notin H\}.$$

Since $F \in \mathcal{F}$, $\mathcal{F}$ is non-empty. It is immediate that the hypothesis of Zorn's Lemma is satisfied, so there exists a maximal element $Q \in \mathcal{F}$. In particular, Q is proper because $c \notin Q$. We will see that Q is prime. Let $a, b \in A$ such that

$a \vee b \in Q$. Suppose $a \notin Q$ and $b \notin Q$. Consider the filters $Q_a = \mathrm{Fig}(Q \cup \{a\})$ and $Q_b = \mathrm{Fig}(Q \cup \{b\})$. Since $F \subseteq Q \subseteq D_P(Q_a)$, $F \subseteq Q \subseteq D_P(Q_b)$ and $(P, \widehat{D_P(Q_a)}), (P, \widehat{D_P(Q_b)}) \in M_A$, the maximality of Q in $\mathcal{F}$ implies that $c \in D_P(Q_a)$ and $c \in D_P(Q_b)$. So, there exist $d_1 \in Q_a$ and $d_2 \in Q_b$ such that $d_1 \Rightarrow c, d_2 \Rightarrow c \in P$. Since $d_1 \in Q_a$ and $d_2 \in Q_b$ we get that there exist $d_3, d_4 \in Q$ such that $d_3 \wedge a \leq d_1$ and $d_4 \wedge b \leq d_2$. Hence, $d_1 \Rightarrow c \leq (d_3 \wedge a) \Rightarrow c$ and $d_2 \Rightarrow c \leq (d_4 \wedge b) \Rightarrow c$. Take $d = d_3 \wedge d_4 \in Q$. Then, $(d_3 \wedge a) \Rightarrow c \leq (d \wedge a) \Rightarrow c$ and $(d_4 \wedge a) \Rightarrow c \leq (d \wedge b) \Rightarrow c$. This implies that $(d \wedge a) \Rightarrow c, (d \wedge b) \Rightarrow c \in P$. Hence,

$$((d \wedge a) \Rightarrow c) \wedge ((d \wedge b) \Rightarrow c) \in P.$$

It follows from the hypothesis that

$$((d \wedge a) \Rightarrow c) \wedge ((d \wedge b) \Rightarrow c) = ((d \wedge a) \vee (d \wedge b)) \Rightarrow c = (d \wedge (a \vee b)) \Rightarrow c,$$

so $(d \wedge (a \vee b)) \Rightarrow c \in P$. Since $d \in Q$ and $a \vee b \in Q$ we get $d \wedge (a \vee b) \in Q$. Moreover, the fact that $(P, \widehat{Q}) \in M_A$ implies that $c \in Q$, which is a contradiction. Thus, Q is a prime filter such that $(P, \widehat{Q}) \in M_A$, $a \in Q$ and $b \notin Q$.

Finally we will see that (3) implies (1). Suppose that (3) is satisfied and that there exist $a, b, c \in A$ such that $(a \Rightarrow c) \wedge (b \Rightarrow c) \nleq (a \vee b) \Rightarrow c$. Then there is $P \in X(A)$ such that $a \Rightarrow c, b \Rightarrow c \in P$ and $(a \vee b) \Rightarrow c \notin P$. Thus, it follows from Lemma 13 that there exists $F \in \mathrm{Fi}(A)$ such that $(P, \widehat{F}) \in M_A$, $a \vee b \in F$ and $c \notin F$. By hypothesis, there is $Q \in X(A)$ such that $F \subseteq Q$, $c \notin Q$ and $(P, \widehat{Q}) \in M_A$. Then $a \vee b \in Q$ and since Q is prime, $a \in Q$ or $b \in Q$. Without loss of generality we can assume that $a \in Q$. Taking into account that $a \Rightarrow c \in P$ and $(P, \widehat{Q}) \in M_A$ we get $c \in Q$, which is a contradiction. Thus, (3) implies (1), which was our aim. ■

The following definition will be used in order to give the duality for WH-algebras developed in [6].

Definition 29. A structure $\langle X, R \rangle$ is a WH-space if X is a Priestley space and $R \subseteq X \times X$ satisfies the following conditions:

(WSH1) $U \Rightarrow_R V = \{x \in X \colon R(x) \cap U \subseteq V\} \in D(X)$, for every $U, V \in D(X)$.

(WHS2) $R(x)$ is a closed subset of X, for every $x \in X$.

Let $\langle X_1, R_1 \rangle$ and $\langle X_2, R_2 \rangle$ be two WH-spaces. A continuous and monotone map $f \colon X_1 \to X_2$ is said to be a WH-morphism if the following conditions are satisfied:

(1) For every $x, y \in X$, if $(x, y) \in R_1$, then $(f(x), f(y)) \in R_2$.

(2) For every $x \in X$ and $z \in Y$ such that $(f(x), z) \in R_2$ there exists $y \in X_1$ such that $(x, y) \in R_1$ and $f(y) = z$.

Note that a WH-morphism is a Priestley morphism that is also a p-morphism between the relational structures $\langle X_1, R_1 \rangle$ and $\langle X_2, R_2 \rangle$.

If $\langle X, R \rangle$ is a WH-space, then the structure $\langle D(X), \cup, \cap, \Rightarrow_R, \emptyset, X \rangle$ is a WH-algebra, where $\Rightarrow_R$ is the binary operation given in (WHS1). Conversely, if A is a WH-algebra, then $\langle X(A), R_A \rangle$ is a WH-space, where R_A is defined by

$$(P, Q) \in R_A \quad \text{iff} \quad \forall a, b \in A (a \Rightarrow b \in P \text{ and } a \in Q \text{ then } b \in Q).$$

Let A and B be two WH-algebras. A WH-homomorphism between A and B is a homomorphism $h \colon A \to B$ between bounded distributive lattices such that $h(a \Rightarrow b) = h(a) \Rightarrow h(b)$, for all $a, b \in A$. Let $h \colon A \to B$ be a WH-homomorphism. Then $h^* \colon X(B) \to X(A)$ is a WH-morphism. On the other hand, if $f \colon X_1 \to X_2$ is a WH-morphism, then the map $f^* \colon D(X_2) \to D(X_1)$ is a WH-homomorphism. Moreover, there exists a dual equivalence between the algebraic category whose objects are WH-algebras and the category whose objects are WH-spaces and whose morphisms are WH-morphisms (see [6] for more details).

In what follows we study the connection between WH-spaces and certain WL-spaces. In order to make it possible we give the following definition, which is inspired in Proposition 28.

Definition 30. A structure $\langle X, M \rangle$ is a MWL-space if $\langle X, M \rangle$ is a WL-space such that satisfies the following condition: for every $x \in X, Y \in M(x)$ and $U \in D(X)$, if $Y \nsubseteq U$ then there exists $y \in Y$ such that $y \notin U$ and $(x, [y)) \in M$.

Let $\langle X, M \rangle$ be a MWL-space. We define the binary relation $R_M \subseteq X \times X$ by

$$(x, y) \in R_M \quad \text{iff} \quad (x, [y)) \in M.$$

Lemma 31. *Let $\langle X, M \rangle$ be a MWL-space. Then $\langle X, R_M \rangle$ is a WH-space.*

Proof. Let $\langle X, M \rangle$ be a MWL-space. Without loss of generality we can assume that $\langle X, M \rangle$ is of the form $\langle X(A), M_A \rangle$ for some WL-lattice A. In order to show that $\langle X, R_M \rangle$ is a WH-space it is enough to see that $\langle X(A), R_{M_A} \rangle$ is a WH-space. First note that it follows from Proposition 28 that A is a WH-algebra. Besides, it is immediate that $R_A = R_{M_A}$. Therefore, $\langle X(A), R_{M_A} \rangle$ is a WH-space. ∎

Let $\langle X, R\rangle$ be a WH-space. We define $M_R \subseteq X \times \mathrm{CUp}(X)$ by

$$(x, Y) \in M_R \quad \text{iff} \quad \forall U \in D(X)(Y \notin L_U \text{ then } Y \cap U^c \cap R(x) \neq \emptyset).$$

Note that if $(x, y) \in R$ then $(x, [y)) \in M_R$, because if $[y) \notin L_U$, i.e., $y \notin U$, then $y \in [y) \cap U^c \cap R(x)$.

Lemma 32. *Let $\langle X, R\rangle$ be a WH-space. Then $\langle X, M_R\rangle$ is a MWL-space.*

Proof. Let $U, V \in D(X)$. First we will see that $U \Rightarrow_{M_R} V = U \Rightarrow_R V$. Let $x \in U \Rightarrow_{M_R} V$. In order to show that $R(x) \cap U \subseteq V$, let $y \in R(x) \cap U$. As $(x, [y)) \in M_R$, $[y) \subseteq U$ and $x \in U \Rightarrow_{M_R} V$, we have $[y) \subseteq V$ and $y \in V$. Thus, $x \in U \Rightarrow_R V$.

Conversely, let $x \in U \Rightarrow_R V$, i.e., $R(x) \cap U \subseteq V$. Let $Y \in M_R(x)$ such that $Y \subseteq U$. Suppose that $Y \nsubseteq V$, i.e., $Y \notin L_V$. Then there exists $y \in X$ such that $y \in Y \cap V^c \cap R(x)$. Since $Y \subseteq U$ we get $y \in R(x) \cap U \subseteq V$, so $y \in V$ which is a contradiction. Thus, $x \in U \Rightarrow_{M_R} V$. We have proved that $U \Rightarrow_{M_R} V = U \Rightarrow_R V$.

Now we will see that $M_R(x) = \bigcap\{L_U^c \cup L_V \colon x \in U \Rightarrow_{M_R} V \text{ and } U, V \in D(X)\}$, for every $x \in X$. Let $x \in X$. Let

$$Y \in \bigcap\{L_U^c \cup L_V \colon x \in U \Rightarrow_{M_R} V \text{ and } U, V \in D(X)\} \tag{5.1}$$

and suppose $Y \notin M_R(x)$. Then there exists $V \in D(X)$ such that

$$Y \notin L_V \tag{5.2}$$

and $Y \cap V^c \cap R(x) = \emptyset$, i.e., $R(x) \subseteq Y^c \cup V$. As $Y \in \mathrm{CUp}(X)$, $Y = \bigcap\{U \in D(X) \colon Y \in L_U\}$. Hence,

$$R(x) \subseteq \bigcup\{U^c \colon U \in D(X) \text{ and } Y \in L_U\} \cup V.$$

Since $R(x)$ is a closed subset of X, $R(x)$ is compact. Then there exist $U_1, \ldots, U_n \in D(X)$ such that $Y \in L_{U_1}, \ldots, Y \in L_{U_n}$ and

$$R(x) \subseteq U_1^c \cup \cdots \cup U_n^c \cup V = (U_1 \cap \cdots \cap U_n)^c \cup V = U^c \cup V,$$

where $U = U_1 \cap \cdots \cap U_n$. Then $x \in U \Rightarrow_R V = U \Rightarrow_{M_R} V$. As $Y \subseteq U$, we get $Y \notin L_U^c$. It follows from (5.1) that $Y \in L_V$, i.e., $Y \subseteq V$, which is a contradiction with (5.2). Thus, $\bigcap\{L_U^c \cup L_V \colon x \in U \Rightarrow_{M_R} V \text{ and } U, V \in D(X)\} \subseteq M_R(x)$.

In order to show the converse inclusion, let $Y \in M_R(x)$. Suppose that $Y \notin \bigcap\{L_U^c \cup L_V \colon x \in U \Rightarrow_{M_R} V \text{ and } U, V \in D(X)\}$. Then there exist $U, V \in D(X)$ such that $Y \notin L_U^c \cup L_V$ and $x \in U \Rightarrow_{M_R} V = U \Rightarrow_R V$. So, $Y \subseteq U$, $Y \nsubseteq V$ and $x \in U \Rightarrow_{M_R} V$. Since $Y \in M_R(x)$, $Y \subseteq U$ and

$x \in U \Rightarrow_{M_R} V$ we get $Y \subseteq V$, which is a contradiction. Thus,

$$M_R(x) = \bigcap \{L_U^c \cup L_V : x \in U \Rightarrow_{M_R} V \text{ and } U, V \in D(X)\}.$$

Hence, $\langle X, M_R \rangle$ is a WL-space. In order to show that it is a MWL-space, let $Y \in M_R(x)$ and $Y \nsubseteq U$. By definition of M_R there exists $y \in Y \cap U^c \cap R(x)$. As $(x, y) \in R$, we have $(x, [y)) \in M_R$. Therefore, $\langle X, M_R \rangle$ is a MWL-space. ■

Lemma 33. *Let $\langle X, R \rangle$ be a WH-space and $\langle X, M \rangle$ a MWL-space. Then $M_{R_M} = M$ and $R_{M_R} = R$.*

Proof. Let $\langle X, R \rangle$ be a WH-space. A direct computation shows that $R \subseteq R_{M_R}$. In order to see the converse inclusion, let $x, y \in X$ such that $(x, y) \in R_{M_R}$, i.e., $(x, [y)) \in M_R$. By definition of M_R, for each $W \in D(X)$ such that $y \notin W$ there exists $z \in [y) \cap W^c \cap R(x)$. Suppose that $y \notin R(x)$. As $R(x)$ is closed, there exists $U, V \in D(X)$ such that $x \in U \Rightarrow_R V$, $y \in U$ and $y \notin V$. By definition of M_R there exists $z \in V^c \cap [y) \cap R(x)$. Then $z \in U$, and since $z \in R(x)$ and $x \in U \Rightarrow_R V$ we get $z \in V$, which is a contradiction. Thus, $R_{M_R}(x) \subseteq R(x)$ for every $x \in X$. Hence, $R = R_{M_R}$.

Let $\langle X, M \rangle$ be a MWL-space. Let $Y \in \mathrm{CUp}(X)$ and $x \in X$ such that $Y \in M_{R_M}(x)$. Suppose that $Y \notin M(x)$. Then there exist $U, V \in D(X)$ such that $Y \notin L_U^c \cup L_V$ and $x \in U \Rightarrow_M V$. So, $Y \subseteq U$ and $Y \nsubseteq V$. Since $Y \in M_{R_M}(x)$, there exists $z \in Y \cap V^c \cap R_M(x)$. Thus, $z \in Y$, $z \notin V$ and $(x, [z)) \in M$. But as $x \in U \Rightarrow_M V$ and $[z) \subseteq U$, we have $[z) \subseteq V$, i.e., $z \in V$ which is a contradiction. Thus, $Y \in M(x)$ and $M_{R_M} \subseteq M$.

Finally, let $Y \in \mathrm{CUp}(X)$ and $x \in X$ such that $Y \notin M_{R_M}(x) = \bigcap \{L_U^c \cup L_V : x \in U \Rightarrow_{M_{R_M}} V \text{ and } U, V \in D(X)\}$. So, there exist $U, V \in D(X)$ such that $Y \subseteq U$, $Y \nsubseteq V$ and $x \in U \Rightarrow_{M_{R_M}} V = U \Rightarrow_{R_M} V = U \Rightarrow_M V$. This implies that $Y \notin M(x)$. Therefore, $M_{R_M} = M$. ■

The following result follows from Lemmas 31, 32 and 33.

Theorem 34. *There exists a categorical isomorphism between the full subcategory of* WL *whose objects are MWL-spaces and the category whose objects are WH-spaces and whose morphisms are WH-morphisms.*

6. Some Subvarieties of the Variety of WL-Lattices

Inspired by some of the varieties studied in [6], in this section we study some subvarieties of the variety of WL-lattices. First, we identify the classes of WL-frames that correspond to these subvarieties. So, we use Theorem 27 in order to give dually equivalences for the algebraic categories corresponding to these subvarieties. In the following section we study the connection between a particular class of WL-frames and the intuitionistic neighbourhood frames introduced in [26, Definition 2.1].

We introduce the following identities in the framework of WL-lattices:

(R) $a \wedge (a \Rightarrow b) \leq b$,

(T) $a \Rightarrow b \leq c \Rightarrow (a \Rightarrow b)$,

(B) $a \leq 1 \Rightarrow a$.

A direct computation shows that the inequality (T) is equivalent to the inequality $a \Rightarrow b \leq 1 \Rightarrow (a \Rightarrow b)$. We opt to introduce the inequality (T) in place of the equivalent inequality $a \Rightarrow b \leq 1 \Rightarrow (a \Rightarrow b)$ because our motivation cames from the subvarierty of WH-algebras whose members satisfy (T), which was introduced and studied in [6]. We note also that some of these axioms also appeared in [23] in the context of intuitionistic logic with a Lewis implication.

In what follows we introduce certain relational conditions defined in WL-frames that allow us to characterize some extensions of the variety of WL-lattices.

Let $\mathcal{F} = \langle X, \leq, M \rangle$ be a WL-frame. Let us consider the following relational conditions:

(WLR) For every $x \in X$, $(x, [x)) \in M$.

(WLT) For every $x, y \in X$ and $Y, Z \in \mathrm{Up}(X)$, if $(x, Y) \in M$, $y \in Y$ and $(y, Z) \in M$ then $(x, Z) \in M$.

(WLB) For every $x \in X$ and $Y \in \mathrm{Up}(X)$, if $(x, Y) \in M$ then $Y \subseteq [x)$.

Lemma 35. *Let $\mathcal{F} = \langle X, \leq, M \rangle$ be a WL-frame. Consider the WL-lattice $A(\mathcal{F}) = \langle \mathrm{Up}(X), \cup, \cap, \Rightarrow_M, \emptyset, X \rangle$. Then:*

(1) *If $\mathcal{F}$ satisfies* (WLR) *then $A(\mathcal{F})$ satisfies* (R).

(2) *If $\mathcal{F}$ satisfies* (WLT) *then $A(\mathcal{F})$ satisfies* (T).

(3) *If $\mathcal{F}$ satisfies* (WLB) *then $A(\mathcal{F})$ satisfies* (B).

(4) *If $\mathcal{F}$ satisfies* (WLB) *then $\mathcal{F}$ satisfies* (WLT).

Proof. We only prove (4). Let $x \in X$ and $Y, Z \in \mathrm{Up}(X)$ such that $(x, Y) \in M$, $y \in Y$ and $(y, Z) \in M$. Then $Y \subseteq [x)$ and $Z \subseteq [y)$. As $y \in Y \subseteq [x)$, we get $x \leq y$. So, by condition $(*)$ of Definition 6 we have $(x, Z) \in M$. ■

Proposition 36. *Let $A \in \mathsf{WL}$. Then the following conditions are equivalent:*

(1) *A satisfies* (R).

(2) *For every* $a, b, c \in A$*, if* $a \leq b \Rightarrow c$*, then* $a \wedge b \leq c$.

(3) *For every* $P \in X(A)$*,* $(P, [P)) \in M_A$.

(4) *For every* $P, Q \in X(A)$*, if* $P \subseteq Q$ *then* $(P, [Q)) \in M_A$.

Proof. The equivalence between (1) and (2) follows from a direct computation, and the fact that (2) implies (3) is immediate. In order to see that (3) implies (4), assume that (3) is satisfied and let $P, Q \in X(A)$ such that $P \subseteq Q$. Let $a, b \in A$ such that $a \Rightarrow b \in P$ and $a \in Q$. In particular, $a \Rightarrow b \in Q$. Since $(Q, [Q)) \in M_A$ we get $b \in Q$. Thus, $(P, [Q)) \in M_A$. Finally we will show that (4) implies (1). Suppose that (4) is satisfied and that (1) is not verified, so there exist $a, b \in A$ such that $a \wedge (a \Rightarrow b) \not\leq b$. Hence, there exists $P \in X(A)$ such that $a, a \Rightarrow b \in P$ and $b \notin P$. It follows from the hypothesis that $(P, [P)) \in M_A$, so $b \in P$, which is a contradiction. Therefore, condition (4) implies condition (1), which was our aim. ■

Proposition 37. *Let* $A \in \mathsf{WL}$*. Then the following conditions are equivalent:*

(1) *A satisfies* (T).

(2) *For every* $P, Q \in X(A)$ *and* $F, G \in \mathrm{Fi}(A)$*, if* $(P, \widehat{F}) \in M_A$*,* $Q \in \widehat{F}$ *and* $(Q, \widehat{G}) \in M_A$ *then* $(P, \widehat{G}) \in M_A$.

Proof. First we will see that (1) implies (2). Suppose that (1) is satisfied. Let $P, Q \in X(A)$ and $F, G \in \mathrm{Fi}(A)$ such that $(P, \widehat{F}) \in M_A$, $Q \in \widehat{F}$ and $(Q, \widehat{G}) \in M_A$. In particular, $F \subseteq Q$. We will see that $(P, \widehat{G}) \in M_A$. Let $a, b \in A$ such that $a \Rightarrow b \in P$ and $a \in G$. Since $a \Rightarrow b \leq 1 \Rightarrow (a \Rightarrow b)$ we get $1 \Rightarrow (a \Rightarrow b) \in P$. But $(P, \widehat{F}) \in M_A$, so $a \Rightarrow b \in F$. Since $a \Rightarrow b \in Q$, $a \in G$ and $(Q, \widehat{G}) \in M_A$ we get $b \in G$. Hence, $(P, \widehat{G}) \in M_A$.

Finally we will prove that (2) implies (1). Assume that (2) is verified and suppose that there exist $a, b, c \in A$ such that $a \Rightarrow b \not\leq c \Rightarrow (a \Rightarrow b)$. Then there exists $P \in X(A)$ such that $a \Rightarrow b \in P$ and $c \Rightarrow (a \Rightarrow b) \notin P$. It follows from Lemma 13 that there exists $F \in \mathrm{Fi}(A)$ such that $(P, \widehat{F}) \in M_A$, $c \in F$ and $a \Rightarrow b \notin F$. Thus, there is $Q \in X(A)$ such that $a \Rightarrow b \notin Q$ and $F \subseteq Q$. It follows from Lemma 13 that there exists $G \in \mathrm{Fi}(A)$ such that $(Q, \widehat{G}) \in M_A$, $a \in G$ and $b \notin G$. We have proved that $(P, \widehat{F}) \in M_A$, $Q \in \widehat{F}$ and $(Q, \widehat{G}) \in M_A$. Then it follows from hypothesis that $(P, \widehat{G}) \in M_A$. Since $a \Rightarrow b \in P$ and $a \in G$ we get $b \in G$ which is a contradiction. ■

Proposition 38. *Let* $A \in \mathsf{WL}$*. Then the following conditions are equivalent:*

(1) *A satisfies* (B).

(2) *For every* $a, b, c \in A$*, if* $a \wedge b \leq c$*, then* $a \leq b \Rightarrow c$*.*

(3) *For every* $P \in X(A)$ *and* $F \in \mathrm{Fi}(A)$*, if* $(P, \widehat{F}) \in M_A$ *and* $Q \in \widehat{F}$ *then* $P \subseteq Q$*.*

Proof. The fact that (1) implies (2) can be proved as in [6, Proposition 4.20].

In order to show that (2) implies (3), suppose that (2) is satisfied. Let $P \in X(A)$ and $F \in \mathrm{Fi}(A)$ such that $(P, \widehat{F}) \in M_A$. Let $Q \in \widehat{F}$. We will prove that $P \subseteq Q$. Let $a \in P$. Then $a \wedge 1 \leq a$, so it follows from hypothesis that $a \leq 1 \Rightarrow a$. Thus, $1 \Rightarrow a \in P$. Since $(P, \widehat{F}) \in M_A$, $a \in F$. Taking into account that $F \subseteq Q$ we obtain that $a \in Q$. Hence, $P \subseteq Q$.

Finally we will see that (3) implies (1). Suppose that (1) is not satisfied. Assume that there is $a \in A$ such that $a \not\leq 1 \Rightarrow a$, so there exists $P \in X(A)$ such that $a \in P$ and $1 \Rightarrow a \notin P$. It follows from Lemma 13 that there exists $F \in \mathrm{Fi}(A)$ such that $(P, \widehat{F}) \in M_A$ and $a \notin F$. Since $a \notin F$, there is $Q \in X(A)$ such that $a \notin Q$ and $F \subseteq Q$. Hence, $Q \in \widehat{F}$, so it follows from hypothesis that $P \subseteq Q$. Thus, $a \in Q$, which is a contradiction. ■

By Propositions 36 and 38 we have that the variety of WL-lattices satisfying conditions (B) and (T) is the variety of Heyting algebras. For future reference we formulate this in a corollary.

Corollary 39. *Let* $A \in \mathsf{WL}$*. Then A is a Heyting algebra if and only if A satisfies* (B) *and* (T)*.*

Motivated by Propositions 36, 37 and 38 we consider the following conditions in the framework of a WL-space $\langle X, M\rangle$:

(SR) For every $x, y \in X$, if $x \leq y$ then $(x, [y)) \in M$.

(ST) For every $x, y \in X$ and $Z, W \in \mathrm{CUp}(X)$, if $(x, Z) \in M$, $y \in Z$ and $(y, W) \in M$ then $(x, W) \in M$.

(SB) For every $x, y \in X$ and $Z \in \mathrm{CUp}(X)$, if $(x, Z) \in M$ and $y \in Z$ then $x \leq y$.

The following result follows from Theorem 27 and Propositions 36, 37 and 38.

Theorem 40.

(1) *The full subcategory of* WL *whose objects satisfy the identity* (R) *and the full subcategory of* SWL *whose objects satisfy the condition* (SR) *are dually equivalent.*

(2) *The full subcategory of* WL *whose objects satisfy the identity* (T) *and the full subcategory of* SWL *whose objects satisfy the condition* (ST) *are dually equivalent.*

(3) *The full subcategory of* WL *whose objects satisfy the identity* (B) *and the full subcategory of* SWL *whose objects satisfy the condition* (SB) *are dually equivalent.*

7. Connection with Intuitionistic Neighbourhood Frames

In this section we study the relation that exists between a certain class of WL-frames and the class of intuitionistic neighbourhood frames defined in [26, Definition 2.1].

Let $\langle X, M\rangle$ be a neighbourhood frame [27]. Define a binary relation $\preceq$ on X by

$$x \preceq y \quad \text{iff} \quad M(y) \subseteq M(x).$$

It is clear that $\preceq$ is a reflexive and transitive binary relation. This relation is introduced in [26, Theorem 2.11] for intuitionistic neighbourhood frames. For each $x \in X$ let $[x)_{\preceq} = \{y \in X \colon x \preceq y\}$. We consider the family of subsets

$$\mathrm{Up}(X, \preceq) = \left\{Y \subseteq X \colon\ [x)_{\preceq} \subseteq Y,\ \text{for each } x \in Y\right\}.$$

Definition 41. An intuitionistic neighbourhood frame is a pair $\mathcal{I} = \langle X, M\rangle$ where X is a non-empty set and M is a (neighbourhood) relation between X and $\mathcal{P}(X)$, i.e., $M \subseteq X \times \mathcal{P}(X)$, such that for each $x \in X$ the following two conditions are satisfied:

(IN1) There exists $Y \in M(x)$ such that $x \in Y$.

(IN2) If $Y \in M(x)$ then $Y \subseteq [x)_{\preceq}$.

Note that this definition is the same that Definition 2.1 of [26] but written in terms of the relation $\preceq$.

Lemma 42. *Let $\mathcal{I} = \langle X, M\rangle$ be an intuitionistic neighbourhood frame. Then $U \in \mathrm{Up}(X, \preceq)$ if and only if for every $x \in U$ and $Z \in M(x)$ we have that $Z \subseteq U$.*

Proof. Let $U \in \mathrm{Up}(X, \preceq)$, $x \in U$ and $Z \in M(x)$. Then, $[x)_{\preceq} \subseteq U$. By (IN2) we have $Z \subseteq [x)_M$. Hence, $Z \subseteq U$.

Conversely, let $x, y \in X$ such that $x \preceq y$ and $x \in U$. So, $M(y) \subseteq M(x)$. By (IN1) there exists $Z \in M(y)$ such that $y \in Z$. Thus, $Z \subseteq U$. Therefore, $y \in U$. ∎

Lemma 43. *Let $\mathcal{I} = \langle X, M\rangle$ be an intuitionistic neighbourhood frame. Then $A(\mathcal{I}) = \langle \mathrm{Up}(X, \preceq), \cup, \cap, \Rightarrow_M, \emptyset, X\rangle$ is a Heyting algebra.*

Proof. Let $U, V, W \in \mathrm{Up}(X, \preceq)$. We only need to prove that $U \cap V \subseteq W$ if and only if $U \subseteq V \Rightarrow_M W$. In order to show it, suppose that $U \cap V \subseteq W$. Let $x \in U$ and $(x, Y) \in M$ such that $Y \subseteq V$. Then $Y \subseteq [x)_{\preceq}$. Let $y \in Y$, so $x \preceq y$. Taking into account that $U \in \mathrm{Up}(X, \preceq)$ we get $y \in U \cap V$, so $y \in W$. Thus, $Y \subseteq W$.

Conversely, assume that $U \subseteq V \Rightarrow_M W$. Let $x \in U \cap V$. It follows from (IN1) that there exists $Y \in M(x)$ such that $x \in Y$. As $x \in V$ and $V \in \mathrm{Up}(X, \preceq)$, it follows from Lemma 42 that $Y \subseteq V$. Since $x \in V \Rightarrow_M W$ we get $Y \subseteq W$. Therefore, $x \in W$. ■

The next two results study the connection between the class of WL-frames and a particular class of intuitionistic neighbourhood frames.

Lemma 44. *Let $\mathcal{F} = \langle X, \leq, M\rangle$ be a WL-frame satisfying conditions* (WLR) *and* (WLB). *Then $\mathcal{I}_{\mathcal{F}} = \langle X, M\rangle$ is an intuitionistic neighbourhood frame such that $\leq = \preceq$ and $U \Rightarrow_{\leq} V = U \Rightarrow_M V$, for all $U, V \in \mathrm{Up}(X)$, i.e., $A(\mathcal{F}) = A(\mathcal{I}_{\mathcal{F}})$. Thus, for every formula $\phi \in Form$, ϕ is valid in $\mathcal{F}$ if and only if ϕ is valid in $\mathcal{I}_{\mathcal{F}}$.*

Proof. Condition (IN1) follows from condition (WLR). In order to prove (IN2), let $x \in X$ and $Y \in \mathrm{Up}(X)$ such that $(x, Y) \in M$. Let $y \in Y$. We need to show that $x \preceq y$, i.e., $M(y) \subseteq M(x)$. In order to see it, let $Z \in M(y)$. By item (4) of Lemma 35 we have that $(x, Z) \in M$. Thus, $M(y) \subseteq M(x)$. Hence, (IN2) is satisfied.

Let $x, y \in X$. We need to prove that $x \leq y$ if and only if $M(y) \subseteq M(x)$. Suppose that $x \leq y$. By condition (*) of Definition 6 we have $M(y) \subseteq M(x)$. Conversely, suppose that $M(y) \subseteq M(x)$. It follows from condition (WLR) that $[y) \in M(y)$. Moreover, by condition (WLB) we get $[y) \subseteq [x)$, i.e., $x \leq y$.

Let $U, V \in \mathrm{Up}(X)$. Finally we will prove that

$$U \Rightarrow_{\leq} V = U \Rightarrow_M V.$$

Let $x \in U \Rightarrow_{\leq} V$, i.e., $[x) \cap U \subseteq V$. Let $(x, Y) \in M$ such that $Y \subseteq U$. Then $Y \subseteq [x)$ and $Y \subseteq U$ implies that $Y = Y \cap U \subseteq [x) \cap U \subseteq V$. Thus, $x \in U \Rightarrow_M V$. Conversely, let $x \in U \Rightarrow_M V$. Let $y \in X$ such that $x \leq y$ and $y \in U$. So, $M(y) \subseteq M(x)$. As $[y) \in M(y)$, $[y) \subseteq U$, and $x \in U \Rightarrow_M V$, we get $[y) \subseteq V$, i.e., $y \in V$. Thus, $x \in U \Rightarrow_{\leq} V$. ■

Lemma 45. *Let $\mathcal{I} = \langle X, M\rangle$ be a neighbourhood frame satisfying the following condition:*

$$\forall x \in X \forall Y \subseteq X \left(Y \in M(x) \text{ if and only if } Y \subseteq [x)_{\preceq}\right). \tag{7.1}$$

Then $\mathcal{I} = \langle X, M\rangle$ *is an intuitionistic neighbourhood frame and* $\mathcal{F}_{\mathcal{I}} = \langle X, \preceq, M\rangle$ *is a* WL*-frame satisfying conditions* (WLR) *and* (WLB)*. Moreover,* $U \Rightarrow_{\preceq} V = U \Rightarrow_M V$*, for all* $U, V \in \mathrm{Up}(X, \preceq)$*, i.e,* $A(\mathcal{I}) = A(\mathcal{F}_{\mathcal{I}})$*. Thus, for every formula* $\phi \in Form$*,* ϕ *is valid in* $\mathcal{I}$ *if and only if* ϕ *is valid in* $\mathcal{F}_I$*.*

Proof. It is immediate to see that $\mathcal{I} = \langle X, M\rangle$ is an intuitionistic neighbourhood frame and that $\mathcal{F}_{\mathcal{I}} = \langle X, \preceq, M\rangle$ is a WL-frame satisfying conditions (WLR) and (WLB). Let $U, V \in \mathrm{Up}(X, \preceq)$. We will prove that $U \Rightarrow_{\preceq} V = U \Rightarrow_M V$. Let $x \in U \Rightarrow_{\preceq} V$, i.e., $[x)_{\preceq} \cap U \subseteq V$. Let $(x, Y) \in M$ such that $Y \subseteq U$. Then $Y \subseteq [x)_{\preceq}$ and $Y \subseteq U$ implies that $Y = Y \cap U \subseteq [x)_{\preceq} \cap U \subseteq V$. Thus, $x \in U \Rightarrow_M V$. Conversely, let $x \in U \Rightarrow_M V$. Let $y \in [x)_{\preceq} \cap U$. Then $[y)_{\preceq} \subseteq [x)_{\preceq}$ and $[y)_{\preceq} \subseteq U$. Hence, it follows from (7.1) that $[y)_{\preceq} \in M(x)$. Taking into account that $x \in U \Rightarrow_M V$, we get $[y)_{\preceq} \subseteq V$, i.e., $y \in V$. Therefore, $x \in U \Rightarrow_{\preceq} V$. ■

Let **I** be a class of intuitionistic neighbourhood frames, and let **F** be a class of WL-frames. We shall say that **I** and **F** are *equivalent* if for each intuitionistic neighbourhood frame $\mathcal{I} \in \mathbf{I}$ there exists a WL-frame $\mathcal{F} \in \mathbf{F}$ such that $A(\mathcal{I})$ and $A(\mathcal{F})$ are isomorphic, and for each $\mathcal{F} \in \mathbf{F}$ there exists $\mathcal{I} \in \mathbf{I}$ such that $A(\mathcal{I})$ and $A(\mathcal{F})$ are isomorphic.

Corollary 46. *The class of neighbourhood frames satisfying condition* (7.1) *is equivalent to the class of WL-frames satisfying conditions* (WLR) *and* (WLB)*.*

Proof. It follows from Lemmas 44 and 45. ■

8. Congruences of WL-Lattices Which Satisfy (R)

A WL-lattice is called a RWL-lattice if the identity (R) is satisfied. We write RWL to indicate the subvariety of WL whose members are RWL-lattices. Our interest in RWL-lattices cames from the fact that these algebras are characterized as those WL-lattices such that for every a, b, c, if $a \leq b \to c$ then $a \wedge b \leq c$, which is a kind of modus pones rule. The aim of this section is to study the lattice of congruences of the members of the variety RWL. Since RWL is a variety, a good description of the congruences of RWL-lattices may be used as tool for the study of properties of these algebras, as for instance for the study of principal congruences, simple and subdirectly irreducible algebras respectively. First we use Proposition 36 in order to prove that for every $A \in \mathsf{RWL}$ there is a dual isomorphism between the congruences of A and a family of certain closed upsets of $X(A)$. Later we also prove that for every $A \in \mathsf{RWL}$ there is an order isomorphism between the lattice of congruences of A and the lattice open filters of A, where an open filter is defined as a filter F such that $1 \Rightarrow a \in F$ whenever $a \in F$. This last property is a generalization of [6, Theorem 6.12], where it was studied the case of RWH-algebras (i.e., RWL-lattices which are also WH-algebras). We finish this section by giving a description of the simple and subdirectly irreducible algebras of RWL respectively, and also a characterization of the principal congruences of RWL-lattices.

Let us begin by giving an example of a RWL-lattice which is not a RWH-algebra.

Example 47. Consider the Boolean algebra of four elements A, where a and b are the atoms. Let $\Rightarrow$ be the binary operation on A defined by

$\Rightarrow$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	0	b	1

A direct computation shows that A is a RWL-lattice. However, this is not a RWH-algebra. Indeed,

$$(a \vee b) \Rightarrow a \neq (a \Rightarrow a) \wedge (b \Rightarrow a).$$

The previous example shows that the variety of RWH-algebras is a proper subvariety of RWL.

8.1. Congruences of RWL-Lattices Through the Topological Duality

Let A be an algebra. We write $\mathrm{Con}(A)$ in order to indicate the lattice of congruences of A. Let $\theta \in \mathrm{Con}(A)$. For every $a \in A$ we write a/θ for the equivalence class of a associated to θ. We also write A/θ in order to indicate the quotient algebra of A associated to θ.

It is part of the folklore of Priestley duality [14] that if A is a bounded distributive lattice A then for every $Y \in \mathrm{CUp}(X(A))$ the set

$$\theta(Y) = \{(a, b) \in A \times A \colon Y \cap \varphi(a) = Y \cap \varphi(b)\}$$

is a congruence of A. Moreover, the assignment $Y \mapsto \theta(Y)$ establishes a dual lattice isomorphism from the lattice of closed upsets of $X(A)$ to the lattice $\mathrm{Con}(A)$.

The following definition will be used later.

Definition 48. Let $\langle X, M\rangle$ be a WL-space and $Y \subseteq X$. We say that Y is M-closed if for every $x \in Y$ it holds that $M(x) \subseteq \{Z \in \mathrm{CUp}(X) \colon Z \subseteq Y\}$.

Lemma 49. *Let $A \in \mathsf{RWL}$. Let Y be a closed subset of $X(A)$. Then the following conditions are equivalent:*

(1) *$\theta(Y)$ is a congruence of A.*

(2) *Y is a M_A-closed subset of $X(A)$.*

Proof. Suppose that (1) is satisfied. First we will prove that $Y \in \mathrm{CUp}(X(A))$. Let $P, Q \in X(A)$ such that $P \subseteq Q$, $P \in Y$ and assume that $Q \notin Y$. As Y is a closed set we have that $Y^c = \bigcup_{i\in I} \varphi(a_i)^c \cap \varphi(b_i)$ for some $\{a_i\}_{i\in I} \subseteq A$ and $\{b_i\}_{i\in I} \subseteq A$. Then there exist $a \in \{a_i\}_{i\in I}$ and $b \in \{b_i\}_{i\in I}$ such that $a \notin Q$ and $b \in Q$. Moreover, a direct computation shows that $Y \cap \varphi(b) \subseteq Y \cap \varphi(a)$, i.e., $Y \cap \varphi(b) = Y \cap \varphi(a \wedge b)$. Since $(b, a \wedge b) \in \theta(Y)$, it follows from the hypothesis that $(b \Rightarrow a, (a \wedge b) \Rightarrow a) = (b \Rightarrow a, 1) \in \theta(Y)$, i.e, $Y \subseteq \varphi(b \Rightarrow a)$. But $P \in Y$, so $b \Rightarrow a \in P$. Since A is a RWL-lattice and $P \subseteq Q$, it follows from Proposition 36 that $(P, [Q)) \in M_A$. But $b \Rightarrow a \in P$ and $b \in Q$, so $a \in Q$, which is a contradiction. Hence, $Y \in \mathrm{CUp}(X(A))$.

Now we will show that Y is a M_A-closed subset of $X(A)$. Suppose that this condition is not satisfied, so there exists $P \in X(A)$ such that $P \in Y$ and $M_A(P) \nsubseteq \{Z \in \mathrm{CUp}(X(A)) \colon Z \subseteq Y\}$. Thus, there exists $F \in \mathrm{Fi}(A)$ such that $(P, \widehat{F}) \in M_A(P)$ and $\widehat{F} \nsubseteq Y$. Hence, there exists $Q \in X(A)$ such that $Q \in \widehat{F}$ and $Q \notin Y$. Since $Y \in \mathrm{CUp}(X(A))$, there exists $a \in A$ such that $Y \subseteq \varphi(a)$ and $Q \notin \varphi(a)$. But $P \in Y$, so $a \in P$. Besides, since $(a, 1) \in \theta(Y)$ it follows from the hypothesis that $(1 \Rightarrow a, 1) \in \theta(Y)$, i.e., $Y \subseteq \varphi(1 \Rightarrow a)$. Taking into account that $P \in Y$ we obtain that $1 \Rightarrow a \in P$. Moreover, the fact that $(P, \widehat{F}) \in M_A$, $1 \Rightarrow a \in P$ and $1 \in F$ implies that $a \in F$. However, $Q \in \widehat{F}$, i.e., $F \subseteq Q$, so $a \notin F$ because $a \notin Q$, which is a contradiction. Hence, Y is a M_A-closed subset of $X(A)$.

Conversely, suppose that Y is a M_A-closed subset of $X(A)$. In order to see that $\theta(Y)$ is a congruence, let $a, b, c \in A$ such that $(a, b) \in \theta(Y)$, i.e., $Y \cap \varphi(a) = Y \cap \varphi(b)$. We will show that $(a \Rightarrow c, b \Rightarrow c) \in \theta(Y)$. Let $P \in Y \cap \varphi(a \Rightarrow c)$ and suppose that $b \Rightarrow c \notin P$. Hence, it follows from Lemma 13 that there exists $G \in \mathrm{Fi}(A)$ such that $(P, \widehat{G}) \in M_A$, $b \in G$ and $c \notin G$. Since Y is M_A-closed, $P \in Y$ and $\widehat{G} \in M_A(P)$ we get $\widehat{G} \subseteq Y$. Besides, since $b \in G$ we get $\widehat{G} \subseteq \varphi(b)$, so $\widehat{G} \subseteq Y \cap \varphi(b) = Y \cap \varphi(a) \subseteq \varphi(a)$. Then $\widehat{G} \subseteq \varphi(a)$. A direct computation shows that $a \in G$. Then $a \Rightarrow c \in P$, $a \in G$ and $(P, \widehat{G}) \in M_A$ implies that $c \in G$, which is a contradiction. Then $Y \cap \varphi(a \Rightarrow c) \subseteq Y \cap \varphi(b \Rightarrow c)$. The other inclusion can be similarly showed, so $Y \cap \varphi(a \Rightarrow c) = Y \cap \varphi(b \Rightarrow c)$. An analogous argument proves the equality $Y \cap \varphi(c \Rightarrow a) = Y \cap \varphi(c \Rightarrow b)$. Therefore, $\theta(Y) \in \mathrm{Con}(A)$. ■

The following result follows from Lemma 49.

THEOREM 50. *Let $A \in \mathsf{RWL}$. The assignment $Y \mapsto \theta(Y)$ establishes a dual lattice isomorphism from the lattice of closed and M_A-closed subsets of $X(A)$ to the lattice* $\mathrm{Con}(A)$.

8.2. Congruences of RWL-Lattices in Terms of Open Filters and Some Applications

Let $A \in \mathsf{RWL}$. For every $a \in A$ we define $\Box a = 1 \Rightarrow a$. A filter F of A will be called open if $\Box a \in F$ whenever $a \in F$. The family of open filters of A will be denoted by $\mathrm{Fi}_{\Box}(A)$.

Let $A \in \mathsf{RWL}$. The first goal of this subsection is to show that there exists an order isomorphism between $\mathrm{Con}(A)$ and $\mathrm{Fi}_{\Box}(A)$.

Let $A \in \mathsf{RWL}$. For each $a, b \in A$, we define $a \Leftrightarrow b = (a \Rightarrow b) \wedge (b \Rightarrow a)$. We also define $\Box^0(a) = a$, $\Box^1(a) = \Box a$ and $\Box^{n+1}(a) = \Box(\Box^n(a))$ for every $n \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of natural numbers.

LEMMA 51. *Let $A \in \mathsf{RWL}$ and $a, b, c \in A$. Then*

$$\Box^2(c) \leq (a \Rightarrow b) \Leftrightarrow ((a \wedge c) \Rightarrow (b \wedge c)).$$

PROOF. Let $a, b, c \in A$. Since $a \Rightarrow b \leq (a \wedge c) \Rightarrow b$ and $(a \wedge c) \Rightarrow b = (a \wedge c) \Rightarrow (b \wedge c)$ we deduce that $a \Rightarrow b \leq (a \wedge c) \Rightarrow (b \wedge c)$, i.e.,

$$(a \Rightarrow b) \Rightarrow ((a \wedge c) \Rightarrow (b \wedge c)) = 1. \tag{8.1}$$

On the other hand, since

$$(a \Rightarrow (a \wedge c)) \wedge ((a \wedge c) \Rightarrow b) \leq a \Rightarrow b$$

we get

$$((a \wedge c) \Rightarrow b) \Rightarrow ((a \Rightarrow (a \wedge c)) \wedge ((a \wedge c) \Rightarrow b)) \leq ((a \wedge c) \Rightarrow b) \Rightarrow (a \Rightarrow b),$$

which is equivalent to

$$((a \wedge c) \Rightarrow b)) \Rightarrow (a \Rightarrow c) \leq ((a \wedge c) \Rightarrow b) \Rightarrow (a \Rightarrow b). \tag{8.2}$$

Since $(a \wedge c) \Rightarrow b \leq 1$ we get

$$\Box(a \Rightarrow c) \leq ((a \wedge c) \Rightarrow b) \Rightarrow (a \Rightarrow c). \tag{8.3}$$

Besides, $a \leq 1$ implies that $\Box c \leq a \Rightarrow c$. Then

$$\Box^2(c) \leq \Box(a \Rightarrow c). \tag{8.4}$$

Hence, it follows from (8.2), (8.3) and (8.4) that $\Box^2(c) \leq ((a \wedge c) \Rightarrow b) \Rightarrow (a \Rightarrow b)$. Moreover, since $(a \wedge c) \Rightarrow b = (a \wedge c) \Rightarrow (b \wedge c)$ we get

$$\Box^2(c) \leq ((a \wedge c) \Rightarrow (b \wedge c)) \Rightarrow (a \Rightarrow b). \tag{8.5}$$

Therefore, it follows from (8.1) and (8.5) that $\Box^2(c) \leq (a \Rightarrow b) \Leftrightarrow ((a \wedge c) \Rightarrow (b \wedge c))$, which was our aim. ■

Let $A \in \mathsf{RWL}$ and $F \in \mathrm{Fi}_{\Box}(A)$. We define the binary relation

$$\theta(F) = \{(a, b) \in A \times A \colon a \wedge f = b \wedge f \text{ for some } f \in F\}.$$

The following definition will be used later.

DEFINITION 52. [30, Definition 6] Let $A \in \mathsf{RWL}$ and $F \in \mathrm{Fi}(A)$. We say that F is congruent if $(a \Rightarrow b) \Leftrightarrow ((a \wedge f) \Rightarrow (b \wedge f)) \in F$ for every $a, b \in A$ and $f \in F$.

Let $A \in \mathsf{RWL}$. It follows from Lemma 51 and [30, Lemma 9] that $\mathrm{Fi}_{\Box}(A)$ coincides with the set of congruent filters of A. Thus, the following result follows from [30, Corollary 11].

THEOREM 53. *Let $A \in \mathsf{RWL}$. Then there exists an order isomorphism between* $\mathrm{Con}(A)$ *and* $\mathrm{Fi}_{\Box}(A)$*, which is established via the assignments $\theta \mapsto 1/\theta$ and $F \mapsto \theta(F)$.*

REMARK 54. Let $A \in \mathsf{RWL}$ and $F \in \mathrm{Fi}_{\Box}(A)$. Then

$$\theta(F) = \{(a, b) \in A \times A \colon a \Leftrightarrow b \in F\}.$$

Indeed, let $(a, b) \in \theta(F)$. Then there exists $f \in F$ such that $a \wedge f = b \wedge f$. Thus, $a \Rightarrow (a \wedge f) = a \Rightarrow (b \wedge f)$, i.e., $a \Rightarrow f = a \Rightarrow (b \wedge f)$. Since $\Box f \leq a \Rightarrow f$ we get $\Box f \leq a \Rightarrow (b \wedge f)$. But $(a \wedge f) \Rightarrow b = (b \wedge f) \Rightarrow b = 1$. So,

$$\Box f \leq (a \Rightarrow (b \wedge f)) \wedge ((b \wedge f) \Rightarrow b) \leq a \Rightarrow b.$$

Analogously, we get $\Box f \leq b \Rightarrow a$. Hence, $\Box f \leq a \Leftrightarrow b$. Since $\Box f \in F$, $a \Leftrightarrow b \in F$. Conversely, suppose that $a \Leftrightarrow b \in F$. It is immediate that $a \wedge (a \Leftrightarrow b) = b \wedge (a \Leftrightarrow b)$. Then $(a, b) \in \theta(F)$. Hence, $\theta(F) = \{(a, b) \in A \times A \colon a \Leftrightarrow b \in F\}$. Therefore, Theorem 53 is a generalization of [6, Theorem 6.12].

We finish this section by giving a characterization of simple algebras, subdirectly irreducible algebras and principal congruences in the framework of RWL-lattices. In order to make it possible, we need to give a description of the open filter generated by an arbitrary subset of a given RWL-lattice.

Let $A \in \mathsf{RWL}$. Given $X \subseteq A$ we write $\mathrm{F}^o(X)$ for the open filter generated by X. In particular, if $a \in A$ we write $\mathrm{F}^o(a)$ in place of $\mathrm{F}^o(\{a\})$.

Lemma 55. *Let* $A \in \mathsf{RWL}$ *and* X *a non-empty subset of* A. *Then*

$$\mathrm{F}^o(X) = \{a \in A\colon a \geq \Box^n(x_1 \wedge \cdots \wedge x_m) \text{ for some } x_1, \ldots, x_m \in X \text{ and } n, m \in \mathbb{N}\}.$$

In particular, $\mathrm{F}^o(a) = \{b \in A\colon b \geq \Box^n(a) \text{ for some } n \in \mathbb{N}\}$ *for every* $a \in A$.

Proof. It follows from a direct computation. ∎

In the proof of the following proposition, which is a generalization of [6, Theorem 6.17], we will use Theorem 53 and Lemma 55.

Proposition 56. *Let* $A \in \mathsf{RWL}$.

(1) *A is simple if and only if for every $a \in A$ such that $a \neq 1$ there exists $n \in \mathbb{N}$ such that $\Box^n(a) = 0$.*

(2) *If A is a non-trivial algebra, A is subdirectly irreducible if and only if there is $a \neq 1$ such that for every $b \neq 1$ there exists $n \in \mathbb{N}$ such that $\Box^n(b) \leq a$.*

Proof. In order to prove (1), assume that A is simple, so $\mathrm{Fi}_\Box(A) = \{\{1\}, A\}$. Let $a \in A$ such that $a \neq 1$, so $\mathrm{F}^o(1) \neq \{1\}$, i.e., $\mathrm{F}^o(a) = A$. Since $0 \in A = \mathrm{F}^o(a)$, there exists $n \in \mathbb{N}$ such that $\Box^n(a) = 0$. Conversely, suppose that for every $a \in A$ such that $a \neq 1$ there exists $n \in \mathbb{N}$ such that $\Box^n(a) = 0$. We need to show that A is simple, which is equivalent to show that $\mathrm{Fi}_\Box(A) = \{\{1\}, A\}$. Let $F \in \mathrm{Fi}_\Box(A)$ such that $F \neq \{1\}$. Then there exists $a \in A$ such that $a \neq 1$, so it follows from the hypothesis that there exists $n \in \mathbb{N}$ such that $\Box^n(a) = 0$. Since $a \in F$ and F is an open filter we get $0 \in F$. Hence, $F = A$. Therefore, A is simple.

Finally we will show (2). Let A be a non-trivial algebra. Suppose that A is subdirectly irreducible, so there exists $F \in \mathrm{Fi}_\Box(A)$ such that $F \neq \{1\}$ and $F \subseteq H$ for every $H \in \mathrm{Fi}_\Box(A)$ such that $H \neq \{1\}$. Since $F \neq \{1\}$, there exists $a \in F$ such that $a \neq 1$. Let $b \in A$ with $b \neq 1$. Then $\mathrm{F}^o(b) \neq \{1\}$, so $F \subseteq \mathrm{F}^o(b)$. Hence, $a \in \mathrm{F}^o(b)$. Then there exists $n \in \mathbb{N}$ such that $\Box^n(b) \leq a$. Conversely, suppose that there is $a \neq 1$ such that for every $b \neq 1$ there exists $n \in \mathbb{N}$ such that $\Box^n(b) \leq a$. In particular we have that $\mathrm{F}^o(a) \neq \{1\}$. Let $F \in \mathrm{Fi}_\Box(A)$ such that $F \neq \{1\}$. We will see that $\mathrm{F}^o(a) \subseteq F$, which is equivalent to see that $a \in F$. Since $F \neq \{1\}$, there exists $b \neq 1$ such that $b \in F$. Thus, it follows from the hypothesis that there exists $n \in \mathbb{N}$ such that $\Box^n(b) \leq a$. Taking into account that F is an open filter we get $a \in F$, which was our aim. Therefore, A is subdirectly irreducible. ∎

Let $A \in \mathsf{RWL}$ and $a, b \in A$. We write $\theta(a, b)$ for the congruence generated by the pair (a, b). The following result, which give us a description of principal congruences, follows from Lemma 51 and [30, Corollary 12].

PROPOSITION 57. *Let $A \in \mathsf{RWL}$. Then $(c,d) \in \theta(a,b)$ if and only if there exists $n \in \mathbb{N}$ such that $\Box^n(a \Leftrightarrow b) \leq c \Leftrightarrow d$.*

9. Weak Heyting–Lewis Algebras

In this section we will explain how the topological duality developed for WL-lattices can be applied to the case of Heyting algebras with a weak strict implication, also called weak Heyting–Lewis algebras or iA^--algebras in [24, Definition 3.1]. The variety of weak Heyting–Lewis algebras is the algebraic semantic of the logic iP^- [17,18]. Let us note that the results given in this section respond to a question raised in [9] referring to the development of a semantics for the logic iP^-, or iA^- in the notation of [9].

DEFINITION 58. An algebra $\langle A, \vee, \wedge, \rightarrow, \Rightarrow, 0, 1\rangle$ of type $(2,2,2,2,0,0)$ is a weak Heyting–Lewis algebra, or WHL-algebra for short, if $\langle A, \vee, \wedge, \rightarrow, 0, 1\rangle$ is a Heyting algebra and the following conditions are satisfied for every $a, b, c \in A$:

(1) $a \Rightarrow (b \wedge c) = (a \Rightarrow b) \wedge (a \Rightarrow c)$,

(2) $(a \Rightarrow b) \wedge (b \Rightarrow c) \leq a \Rightarrow c$,

(3) $a \Rightarrow a = 1$.

The variety of WHL-algebra will be denoted by WHL. When no confusion is likely we will write A instead of $\langle A, \vee, \wedge, \rightarrow, \Rightarrow, 0, 1\rangle$.

PROPOSITION 59. *Let $A \in \mathsf{WHL}$. Then $\langle A, \vee, \wedge, \Rightarrow, 0, 1\rangle$ is a WL-lattice.*

PROOF. Let $a, b, c \in A$ and suppose $a \leq b$. Then by Eqs (1) and (3) $a \Rightarrow a = 1 \leq a \Rightarrow b$. As condition $(a \Rightarrow b) \wedge (b \Rightarrow c) \leq a \Rightarrow c$ is equivalent to $b \Rightarrow c \leq (a \Rightarrow b) \rightarrow (a \Rightarrow c)$, we get that $b \Rightarrow c \leq 1 \rightarrow (a \Rightarrow c) = a \Rightarrow c$. Then, by Remark 8, we have $(a \vee b) \Rightarrow c \leq (a \Rightarrow c) \wedge (b \Rightarrow c)$. Thus, $\langle A, \vee, \wedge, \Rightarrow, 0, 1\rangle$ is a WL-lattice. ■

It is clear that if $\mathcal{F} = \langle X, \leq, M\rangle$ is a WL-frame, then $A(\mathcal{F}) = \langle \mathrm{Up}(X), \cup, \cap, \Rightarrow_{\leq}, \Rightarrow_M, \emptyset, X\rangle$ is a WHL-algebra where the intuitionistic implication $\Rightarrow_{\leq}$ is defined as $U \Rightarrow_{\leq} V = \{x \in X \colon [x) \cap U \subseteq V\}$, for all $U, V \in \mathrm{Up}(X)$.

In a similar way to the proof of Theorem 14 and taking into account the representation theory for Heyting algebras [2] we can prove the following representation theorem for WHL-algebras.

THEOREM 60. *Let $A \in \mathsf{WHL}$. Then the map $\varphi \colon A \rightarrow \mathrm{Up}(X(A))$ defined by $\varphi(a) = \{P \in X(A) \colon a \in P\}$ is an embedding of algebras.*

The topological representation for WHL-algebras is a mixer of the topological representation for Heyting algebras by means of Esakia spaces [13] and the topological representation for WL-lattices by means of WL-spaces (16).

We recall that an Esakia space is a Priestley space X such that for each clopen subset C of X the set $(C]$ is also clopen [13].

Definition 61. A WL-space $\langle X, M\rangle$ is a *WEL-space* if X is an Esakia space.

It is clear that if $\langle X, M\rangle$ be a WEL-space, then $\langle D(X), \cup, \cap, \Rightarrow_{\leq}, \Rightarrow_M, \emptyset, X\rangle$ is a WHL-algebra. Conversely, if A is a WHL-algebra, then $\langle X(A), M_A\rangle$ is a WHL-space, where $X(A)$ is the Esakia space associated to the Heyting algebra A.

Let X and Y be Esakia spaces. An Esakia morphism is a continuous and monotone map $f\colon X \to Y$ satisfying the additional condition: for every $x \in X$ and every $y \in Y$, if $f(x) \leq y$ then there exists $z \in X$ such that $x \leq z$ and $f(z) = y$. If $f\colon X \to Y$ is an Esakia morphism then the map $f^*\colon D(Y) \to D(X)$ defined by $f^*(U) = f^{-1}(U)$ is a homomorphism of Heyting algebras. If $h\colon A \to B$ is a homomorphism of Heyting algebras, then the map $h^*\colon X(B) \to X(A)$ defined by $h^*(P) = h^{-1}(P)$ is an Esakia morphism. Moreover, there exists a dual equivalence between the algebraic category of Heyting algebras and the category whose objects are Esakia spaces and whose morphisms are Esakia morphisms.

Remark 62. If $f\colon X \to Y$ is a morphism between Esakia spaces, then the image of every closed upset is a closed upset. Indeed, let $Z \in \mathrm{CUp}(X)$. We known that $f[Z]$ is a closed of Y. We prove that is an upset. Let $x \leq y$ and $x \in f[Z]$. Then there exists $z \in Z$ such that $x = f(z)$. So, $f(z) \leq y$. As f is a morphism between Esakia spaces, there exists $w \in X$ such that $z \leq w$ and $f(w) = y$. So, $y \in f[Z]$. Therefore, $f[Z]$ is an upset.

Let $\langle X_1, M_1\rangle$ and $\langle X_2, M_2\rangle$ be two WHL-spaces. A map $f\colon X_1 \to X_2$ is a WHL-morphism if is an Esakia morphism and a WL-morphism. We write SWHL for the category whose objects are WHL-spaces and whose morphisms are WHL-morphisms.

We abuse notation and also write WHL for the algebraic category whose members are WHL-algebras.

Theorem 63. *The assignment $A \mapsto \langle X(A), M_A\rangle$ and $h \mapsto h^*$ defines a functor* $\mathrm{X}\colon \mathsf{WHL} \to \mathsf{SWHL}$, *and the assignment $\langle X, M\rangle \mapsto \langle D(X), \cup, \cap, \Rightarrow_{\leq}, \Rightarrow_M, \emptyset, X\rangle$ and $f \mapsto f^*$ defines a functor* $\mathrm{D}\colon \mathsf{SWHL} \to \mathsf{WHL}$. *Moreover, the* *functors* X *and* D *establish a dual equivalence between the categories* WHL *and* SWHL*.*

Proof. It follows from Esakia duality, Theorem 14, Propositions 19, 21 and 22, and Corollary 26. ■

We finish this section by describing the congruences in the variety of the WHL-algebras. We recall that the congruences of a Heyting A algebra are characterized by filters, or using Esakia duality, the congruences of A are characterized by closed upsets of the Esakia space of A. Taking this into account we can give the following characterization of congruences in WHL-algebras. The proof is exactly the same as the one given in Lemma 49.

Theorem 64. *Let* $A \in \mathsf{WHL}$*. Let* Y *be a closed subset of* $X(A)$*. Then the following conditions are equivalent:*

(1) $\theta(Y)$ *is a congruence of* A*.*

(2) Y *is a* M_A*-closed upset of* $X(A)$*.*

10. Conclusions

In this paper we have presented a topological duality for the variety WL of weak Lewis distributive lattices, which are the $\{\vee, \wedge, \Rightarrow, \bot, \top\}$-subreducts of the class of algebras $\{\langle \mathcal{P}(X), \Rightarrow_M \rangle : \langle X, M \rangle$ is a neighbourhood frame$\}$, where the implication $\Rightarrow_M$ is defined by $U \Rightarrow_M V = \{x \in X : \forall Y \in M(x)$ $(Y \subseteq U$ implies $Y \subseteq V)\}$, for all $U, V \in \mathcal{P}(X)$. This duality generalizes the duality given in [6] for the variety of weak Heyting algebras. We have also give a topological representation for some extensions of WL, as well as a topological characterization of the the congruences in the special case of the variety of weak Lewis distributive lattices satisfying the additional condition $a \wedge (a \Rightarrow b) \leq b$. Moreover, we have studied the connection between a particular class of WL-frames and the class of intuitionistic neighbourhood frames defined in [26, Definition 2.1].

The weak Heyting–Lewis algebras were first defined in [24] as the algebraic semantics of the arithmetical base preservativity logic iP^-. The topological duality for WL-lattices is extended to a duality for weak Heyting–Lewis algebras. The results presented in this paper are the first steps concerning to the study of the algebraic and relational semantics of the arithmetical base preservativity logic iP^-.

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References

[1] Ardeshir, M., and W. Ruitenburg, Basic propositional calculus I, *Mathematical Logic Quarterly* 44:317–343, 1998.

[2] Balbes, R., and P. Dwinger, *Distributive Lattices*, University of Missouri Press, Columbia, 1974.

[3] Celani, S., Concurrent algebras: an algebraic study of a fragment of concurrent propositional dynamic logic, *Algebra Universalis* 66:183–204, 2011.

[4] Celani, S., Topological duality for Boolean algebras with a normal n-ary monotonic operator, *Order* 26:49–67, 2009.

[5] Celani, S., and R. Jansana, A closer look at some subintuitionistic logics, *Notre Dame Journal of Formal Logic* 42:225–255, 2003.

[6] Celani, S., and R. Jansana, Bounded distributive lattices with strict implication, *Mathematical Logic Quarterly* 51:219–246, 2005.

[7] Corsi, G., Weak logics with strict implication, *Mathematical Logic Quarterly* 33:389–406, 1987.

[8] Davey, B., and H. A. Priestley, *Introduction to Lattices and Order*. Cambridge University Press, 1994.

[9] de Groot, J., T. Litak, and D. Pattinson, *Gödel-McKinsey-Tarski and Blok-Esakia for Heyting-Lewis Implication*, arXiv:2105.01873

[10] de Groot, J., and D. Pattinson, Monotone subintuitionistic logic: duality and transfer results, *Notre Dame Journal of Formal Logic* 63:213–242, 2022.

[11] Došen, K., Modal translation in K and D, *Diamonds and Defaults, The Series Synthese Library* 229:103–127, 1994.

[12] Epstein, G., and A. Horn, Logics which are characterized by subresiduated lattices, *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik* 22:199–210, 1976.

[13] Esakia, L., *Heyting algebras. Duality theory*, G. Bezhanishvili, and W. Holliday, (eds.), translated from the Russian by A. Evseev, vol. 50 of *Trends in Logic*, Springer, 2019.

[14] Gehrke, M., and S. van Gool, *Topological Duality for Distributive Lattices: Theory and Applications*, arXiv:2203.03286, 2023.

[15] Goldblatt, R., Parallel action: concurrent dynamic logic with independent modalities, *Studia Logica* 51:551–578, 1992.

[16] Hansen, H. H., *Monotonic Modal Logic*, Master's Thesis, University of Amsterdam, 2003.

[17] Iemhoff, R., *Provability Logic and Admissible Rules*, Ph.D. Thesis, University of Amsterdam, 2001.

[18] Iemhoff, R., Preservativity logic: an analogue of interpretability logic for constructive theories, *Mathematical Logic Quarterly* 49:230–249, 2003.

[19] Iemhoff, R., D. De Jongh, and C. Zhou, Properties of intuitionistic provability and preservativity logics, *Logic Journal of the IGPL* 13:615–636, 2005.

[20] Lewis, C. I., The matrix algebra for implications, *The Journal of Philosophy, Psychology and Scientific Methods* 11:589–600, 1920.

[21] Lewis, C. I., *A Survey of Symbolic Logic*, University of California Press, 1918.

[22] Lewis, C. I., and C. H. Langford, *Symbolic Logic*, in *The Century philosophical Series*, Century Company, New York-London, 1932.

[23] Litak, T., and A. Visser, Lewis meets Brouwer: constructive strict implication, *Indagationes Mathematicae* 29:36–90, 2018.

[24] Litak, T., and A. Visser, *Lewisian Fixed Points I: Two Incomparable Constructions*, arXiv:1905.09450.

[25] Maleki, F. S., and D. de Jongh, Weak subintuitionistic logics, *Logic Journal of the IGPL* 25:214–231, 2017.

[26] Moniri, M., and F. Shirmohammadzadeh Maleki, Another neighbourhood semantics for intuitionistic logic, *Logic Journal of the IGPL* https://doi.org/10.1093/jigpal/jzac069.2022.

[27] Pacuit, E., *Neighborhood Semantics for Modal Logic*, Springer, 2017.

[28] Priestley, H. A., Representation of bounded distributive lattice by means of orderer Stone spaces, *Bulletin of the London Mathematical Society* 2:186–190, 1970.

[29] Restall G., Subintuitionistic logics, *Notre Dame Journal of Formal Logic* 35:116–129, 1994.

[30] San Martín, H. J., On congruences in weak implicative semi-lattices, *Soft Computing* 21:3167–3176, 2017.

[31] Visser, A., A propositional logic with explicit fixed points, *Studia Logica* 40:155–175, 1981.

[32] Visser, A., Substitutions of $\sum_1^0$-sentences: Explorations between intuitionistic propositional logic and intuitionistic arithmetic, *Annals of Pure and Applied Logic* 114:227–271, 2002.

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