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Refutations and Proofs in the Paraconsistent Modal Logics: KN4 and KN4.D

Abstract. Axiomatic proof/refutation systems for the paraconsistent modal logics: **KN4** and **KN4.D** are presented. The completeness proofs boil down to showing that every sequent is either provable or refutable. By constructing finite tree-type countermodels from refutations, the refined characterizations of these logics by classes of finite tree-type frames are established. The axiom systems also provide decision procedures for these logics.

Keywords: Paraconsistent modal logic, Refutation systems, The finite model property, Decidability.

1. Introduction

The paraconsistent modal logic **KN4** is defined in [5] semantically by modifying the models for the basic modal logic **K**. Instead of the classical two values (true, false), the four values—true, false, both, neither—are introduced in order to reject the principle that a contradiction entails everything (*ex contradictione quodlibet*). Syntactic characterizations of this logic were given in [5] (a Hilbert-style system) and in [4] (a sequent system).

In this paper, we offer a refutation system (which is an axiomatic system) for the non-valid sequents of **KN4**, and we slightly modify the proof system presented in [4] by eliminating *Weakening*. As a result, we have a pair of derivation systems: one generating proofs and the other generating refutations. We prove that these axiom systems are sound and complete for **KN4**. What is more, we show that these systems have interesting applications. They enable establishing both the finite model property and decidability of this logic. Also, our completeness proofs are constructive. (The completeness proofs in [4,5] involve all models, and they are not constructive.)

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By simple modifications, we obtain similar axiom systems for **KN4.D** (an important paraconsistent deontic logic).

2. Preliminaries

Let **FOR** be the set of all formulas generated from the set $\mathbf{VAR} = \{p_1, p_2, \ldots\}$ of propositional variables by the connectives: $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (implication), $\Box$ (necessity).

More precisely, every p_i is a formula and if φ, ψ are formulas then so are $\varphi \wedge \psi, \varphi \vee \psi, \varphi \rightarrow \psi, \neg\varphi, \Box\varphi$.

The letters p, q denote members of **VAR**. Also, lowercase Greek letters stand for formulas, and uppercase Greek letters stand for *finite* sets of formulas.

Following the notation in [4], by a *sequent* we mean a pair $\mathsf{s} = \Gamma \rhd \Delta$. (The standard symbols $\vdash, \Rightarrow$ will be used for derivability and strong implication, respectively.)

We sometimes write Γ, Δ instead of $\Gamma \cup \Delta$ and φ, Γ instead of $\{\varphi\} \cup \Gamma$.

A *rule instance* is a pair S/s, where $\mathsf{S} \cup \{\mathsf{s}\}$ is a finite set of sequents. And a *rule* is a set of rule instances.

Let **Q** be a set of rules. We say that a sequent t is **Q**-*derivable* from a set T of sequents (symbolically, $\mathsf{T} \vdash_{\mathbf{Q}} \mathsf{t}$) iff there is a finite sequence $\mathsf{s}_1, \ldots, \mathsf{s}_n$ such that $\mathsf{s}_n = \mathsf{t}$ and for every $1 \leq i \leq n$, eiher $\mathsf{s}_i \in \mathsf{T}$ or s_i is obtained from some preceding sequents by a rule of **Q**.

By a *frame* we mean a pair $\mathcal{W} = (W, R)$, where W is a non-empty set of points (worlds) and R is a binary relation on W. A *model* is a pair $\mathcal{M} = (\mathcal{W}, V)$, where $\mathcal{W}$ is a frame and V is a valuation, that is, a function assigning to a given propositional variable p the set $V(p)$ of points at which p is true.

Given a model $\mathcal{M}$, we write $\mathcal{M}, w \models p$ (and say that p is true at w, or w satisfies p) iff $w \in V(p)$.

The satisfaction relation $\models$ is then extended to complex formulas as follows.

$\mathcal{M}, w \models \varphi \wedge \psi$ iff $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$
$\mathcal{M}, w \models \varphi \vee \psi$ iff $\mathcal{M}, w \models \varphi$ or $\mathcal{M}, w \models \psi$
$\mathcal{M}, w \models \varphi \rightarrow \psi$ iff $\mathcal{M}, w \models \varphi$ implies $\mathcal{M}, w \models \psi$
$\mathcal{M}, w \models \neg\varphi$ iff $\mathcal{M}, w \not\models \varphi$
$\mathcal{M}, w \models \Box\varphi$ iff $\forall x(wRx$ implies $\mathcal{M}, x \models \varphi)$

(Note that every formula is either true or false (that is, not true) at a point.)

We say that a formula $\varphi \in \mathbf{FOR}$ is *valid in* a frame $\mathcal{W}$ iff for every model $\mathcal{M} = (\mathcal{W}, V)$ and $w \in W$, we have: $\mathcal{M}, w \models \varphi$. And a formula is *valid* iff it is valid in all frames. The basic modal logic **K** is the set of all valid formulas.

While **K** is detemined by all frames, other standard modal logics correspond to classes of frames with certain restrictions on the accessibility relation R. In paricular, the deontic logic **KD** is determined by the class of frames where R is serial, that is $\forall x \exists y (xRy)$.

As usual, "iff" stands for "if an only if". Sometimes, we write ...X (X')...Y (Y').. instead of .....X...Y..., and ...X' ...Y'...

3. Refutation Systems

In general, by a logic we mean a set of formulas involving some standard connectives and closed under *substitution*, *modus ponens*, and possibly some other rules (like *necessitation*).

In this paper, by a refutation system we mean an axiom system consisting of refutation axioms (which are formulas) and (pure) refutation rules having the form

$$\frac{\varphi_1 \cdots \varphi_n}{\psi}$$

We say that a refutation rule is *sound* iff it preserves nonvalidity, that is, $\psi \notin \mathbf{L}$ whenever $\varphi_1 \notin \mathbf{L}, \ldots, \varphi_n \notin \mathbf{L}$. A sound refutation rule with the property that $\psi \notin \mathbf{L}$ implies $\varphi_1 \notin \mathbf{L}, \ldots, \varphi_n \notin \mathbf{L}$ is called *invertible*.

We also say that a formula φ is refutable iff φ is derivable from refutaton axioms by refutation rules. Such a refutation system is sound for a logic **L** iff we have:

(*Soundness*) If φ is refutable, then $\varphi \notin \mathbf{L}$.

And it is complete for **L** iff we have:

(*Completeness*) If $\varphi \notin \mathbf{L}$, then φ is refutable.

We can prove that a refutation system is sound for **L** either (1) by showing that the refutation rules are sound and the refutation axioms are not in **L** or (2) by constructing a countermodel from every refutation. In this paper, we choose the latter method because it also establishes the finite model property.

These concepts can be generalized by using sequents rather than formulas.

The idea of refutation goes back to Aristotle, but it was Łukasiewicz [10] who introduced the concept of axiomatic refutation system in modern logic (for more on refutation systems, see e.g. [7,14]). In particular, Łukasiewicz introduced the following refutation rules.

(*Reverse substitution*) $\frac{\varphi}{\psi}$ (where φ is a substitution instance of ψ)

(*Reverse modus ponens* (**L**)) $\frac{\varphi}{\psi}$ (where $\psi \rightarrow \varphi$ is provable in some standard proof system for **L**)

For example, the refutation system consisting of the refutation axiom $\perp$ and the refutation rules: *reverse substitution* and *reverse modus ponens*(**CL**) is sound and complete for Classical Propositional Logic (**CL**). In this paper, we focus on refutation systems that are both *reverse substitution*-free and *reverse modus ponens*-free. We also use sequents rather than formulas.

4. Paraconsistent Modal Logic

As Goble [5] points out, it is worth studying modal extensions of paraconsistent logic because of their applications e.g. in knowledge representation and in deontic logic.

In [5], the paraconsistent modal logic **KN4** is the **K**-like modal extension of the logic **BN4** (studied in [3]), and **BN4** extends the four-valued logic of *First-Degree Entailment* [1,2].

More precisely, our language $\mathbf{FOR}^{\Rightarrow}_{\sim}$ is obtained from **FOR** by replacing $\neg, \rightarrow$ with $\sim$ (strong negation), $\Rightarrow$ (strong implication), respectively. By a *literal* we mean either φ or $\sim\varphi$, where $\varphi \in \mathbf{VAR}$. We define:

$\Box\Phi = \{\Box\varphi : \varphi \in \Phi\}$ and $\sim\Phi = \{\sim\varphi : \varphi \in \Phi\}$.

The *modal degree* of a formula φ ($mdeg(\varphi)$ for short) is defined as follows.

1. $mdeg(\varphi) = 0$ if $\varphi \in \mathbf{VAR}$.

2.1 $mdeg(\sim\varphi) = mdeg(\varphi)$.

2.2 $mdeg(\varphi \otimes \psi) = max(mdeg(\varphi), mdeg(\psi))$, where $\otimes \in \{\wedge, \vee \Rightarrow\}$.

2.3 $mdeg(\Box\varphi) = 1 + mdeg(\varphi)$.

Thus, for example, $mdeg(\Box p \Rightarrow \Box\Box p) = 2$.

If $\Phi = \{\varphi_1, \ldots, \varphi_n\}$ then $mdeg(\Phi) = max(mdeg(\varphi_1), \ldots, mdeg(\varphi_n))$.

The modal degree of a sequent $\Gamma \rhd \Delta$ is $mdeg(\Gamma \cup \Delta)$.

Also, if $\mathsf{S} = \{\mathsf{s}_1, \ldots, \mathsf{s}_n\}$ is a set of sequents, then the modal degree of S is $max(mdeg(\mathsf{s}_1), \ldots, mdeg(\mathsf{s}_n))$.

A model is a triple $\mathcal{M} = (\mathcal{W}, v^+, v^-)$, where $\mathcal{W}$ is a frame, v^+ is a verification valuation (that is, a function assigning to a propositional variable p the set $v^+(p)$ of points at which p is true), and v^- is a falsification valuation (that is, a function assigning to a propositional variable p the set $v^-(p)$ of points at which p is false).

Given a model $\mathcal{M}$, we write $\mathcal{M}, w \models^+ p$ (and say that w verifies p) iff $w \in v^+(p)$; and we write $\mathcal{M}, w \models^- p$ (and say that w falsifies p) iff $w \in v^-(p)$.

The verification and falsification relations ($\models^+$ and $\models^-$) are extende to complex formulas as follows.

$\mathcal{M}, w \models^+ \varphi \wedge \psi$ iff $\mathcal{M}, w \models^+ \varphi$ and $\mathcal{M}, w \models^+ \psi$
$\mathcal{M}, w \models^- \varphi \wedge \psi$ iff $\mathcal{M}, w \models^- \varphi$ or $\mathcal{M}, w \models^- \psi$
$\mathcal{M}, w \models^+ \varphi \vee \psi$ iff $\mathcal{M}, w \models^+ \varphi$ or $\mathcal{M}, w \models^+ \psi$
$\mathcal{M}, w \models^- \varphi \vee \psi$ iff $\mathcal{M}, w \models^- \varphi$ and $\mathcal{M}, w \models^- \psi$
$\mathcal{M}, w \models^+ \varphi \Rightarrow \psi$ iff ($\mathcal{M}, w \models^+ \varphi$ implies $\mathcal{M}, w \models^+ \psi$) and ($\mathcal{M}, w \models^- \psi$ implies $\mathcal{M}, w \models^- \varphi$)
$\mathcal{M}, w \models^- \varphi \Rightarrow \psi$ iff $\mathcal{M}, w \models^+ \varphi$ and $\mathcal{M}, w \models^- \psi$
$\mathcal{M}, w \models^+ \sim \varphi$ iff $\mathcal{M}, w \models^- \varphi$
$\mathcal{M}, w \models^- \sim \varphi$ iff $\mathcal{M}, w \models^+ \varphi$
$\mathcal{M}, w \models^+ \Box\varphi$ iff $\forall x(wRx$ implies $\mathcal{M}, x \models^+ \varphi)$
$\mathcal{M}, w \models^- \Box\varphi$ iff $\exists x(wRx$ and $\mathcal{M}, x \models^- \varphi)$

Note that every formula is either true or false or both or neither at a point. In particular, if p is true at w then $\sim p$ is false at w, if p is false at w then $\sim p$ is true at w, if p is both at w then $\sim p$ is both at w, and if p is neither at w then $\sim p$ is neither at w.

Note also that the verification condition for $\varphi \Rightarrow \psi$ is that for $\rightarrow$ (if φ is true at w then ψ is true at w) plus the condition: if ψ is false at w then φ is false at w.

We say that a formula $\varphi \in \mathbf{FOR}^{\Rightarrow}_{\sim}$ is *valid in* a frame $\mathcal{W}$ iff for every model $\mathcal{M} = (\mathcal{W}, v^+, v^-)$ and $w \in W$, we have: $\mathcal{M}, w \models^+ \varphi$. And a formula is *valid* iff it is valid in all frames. The logic **KN4** is the set of all valid formulas.

Definition 4.1. (i) Let w be a point in a model. $\mathcal{M}$. A sequent $\mathsf{s}(= \Gamma \rhd \Delta)$ is *true at* w iff we have:

If $\mathcal{M}, w \models^+ \gamma$ for all $\gamma \in \Gamma$, then $\mathcal{M}, w \models^+ \delta$ for some $\delta \in \Delta$.

(ii) A sequent s is *valid in* a frame $\mathcal{W}$ iff for every model $\mathcal{M} = (\mathcal{W}, v^+, v^-)$ and $w \in W$, s is *true* at w.

(iii) A sequent s is *valid* iff s is valid in all frames.

By restricting the accessibility relation R, we obtain the paraconsistent counterparts of standard modal logics. All these logics are paraconsistent in the sense that their consequence relations (defined as preserving truth at every point in every model) are not explosive. (A consequence relation is explosive if a contradiction $\varphi, \sim \varphi$ entails any conclusion ψ.) Indeed, to reject *ex contradictione quodlibet*, consider a model in which p is both true and false at w and q is not true at w. Then $\sim p$ is both true and false at w, so the contradiction $p, \sim p$ is true at w but q is not true at w.

The paraconsistent deontic logic **KN4.D** is determined by the class of frames with R serial. By a **KN4.D** *frame* (a **KN4.D** *model*) we mean a frame $\mathcal{W} = (W, R)$ (a model $(\mathcal{W}, v^+, v^-)$) such that R is serial. A sequent s is **KN4.D**-*valid* iff s is valid in all **KN4.D** frames.

5. Proof System

Our proof system for **KN4** consists of

Proof axioms:

$$\varphi, \Gamma \rhd \Delta, \varphi$$

where φ is a literal, and the following set **Prf** of proof rules.

- *Non-modal rules*:

$$\frac{\varphi, \psi, \Gamma \rhd \Delta}{\varphi \wedge \psi, \Gamma \rhd \Delta}(L\wedge) \qquad \frac{\Gamma \rhd \Delta, \varphi \quad \Gamma \rhd \Delta, \psi}{\Gamma \rhd \Delta, \varphi \wedge \psi}(R\wedge)$$

$$\frac{\sim\varphi, \Gamma \rhd \Delta \quad \sim\psi, \Gamma \rhd \Delta}{\sim(\varphi \wedge \psi), \Gamma \rhd \Delta}(L\sim\wedge) \qquad \frac{\Gamma \rhd \Delta, \sim\varphi, \sim\psi}{\Gamma \rhd \Delta, \sim(\varphi \wedge \psi)}(R\sim\wedge)$$

$$\frac{\varphi, \Gamma \rhd \Delta \quad \psi, \Gamma \rhd \Delta}{\varphi \vee \psi, \Gamma \rhd \Delta}(L\vee) \qquad \frac{\Gamma \rhd \Delta, \varphi, \psi}{\Gamma \rhd \Delta, \varphi \vee \psi}(R\vee)$$

$$\frac{\sim\varphi, \sim\psi, \Gamma \rhd \Delta}{\sim(\varphi \vee \psi), \Gamma \rhd \Delta}(L\sim\vee) \qquad \frac{\Gamma \rhd \Delta, \sim\varphi \quad \Gamma \rhd \Delta, \sim\psi}{\Gamma \rhd \Delta, \sim(\varphi \vee \psi)}(R\sim\vee)$$

$$\frac{\Gamma \rhd \Delta, \varphi, \sim\psi \quad \sim\varphi, \Gamma \rhd \Delta, \varphi \quad \psi, \Gamma \rhd \Delta, \sim\psi \quad \sim\varphi, \psi, \Gamma \rhd \Delta}{\varphi \Rightarrow \psi, \Gamma \rhd \Delta}(L\Rightarrow)$$

$$\frac{\varphi, \Gamma \rhd \Delta, \psi \quad \sim\psi, \Gamma \rhd \Delta, \sim\varphi}{\Gamma \rhd \Delta, \varphi \Rightarrow \psi}(R\Rightarrow)$$

$$\frac{\varphi, \Gamma \rhd \Delta}{\sim\sim\varphi, \Gamma \rhd \Delta}(L\sim\sim) \qquad \frac{\Gamma \rhd \Delta, \varphi}{\Gamma \rhd \Delta, \sim\sim\varphi}(R\sim\sim)$$

- *Modal rules*:

$$\frac{\Gamma_1, \sim\varphi \rhd \sim\Gamma_2}{\Box\Gamma_1, \sim\Box\varphi, \sim\Box\Delta_2, \Theta_1 \rhd \Theta_2, \Box\Delta_1, \sim\Box\Gamma_2}(L\Box)$$

$$\frac{\Gamma_1 \rhd \varphi, \sim\Gamma_2}{\Box\Gamma_1, \sim\Box\Delta_2, \Theta_1 \rhd \Theta_2, \Box\varphi, \Box\Delta_1, \sim\Box\Gamma_2}(R\Box)$$

where Θ_1, Θ_2 are finite sets of literals.

We say that a sequent s is *provable* iff **AxRef** $\vdash_{\mathbf{Ref}}$ s, where **AxPrf** is the set of proof axioms. (And we say that a formula φ is provable iff so is the sequent $\emptyset \rhd \varphi$.)

Example 5.1. Here is a proof for the sequent $\emptyset \rhd \Box(p \wedge q) \Rightarrow \Box p$.

1. $p, q \rhd p$ (proof axiom)
2. $p \wedge q \rhd p$ (1, $L\wedge$)
3. $\Box(p \wedge q) \rhd \Box p$ (2, $R\Box$)
4. $\sim p \rhd \sim p, \sim q$ (proof axiom)
5. $\sim p \rhd \sim(p \wedge q)$ (4, $R \sim\wedge$)
6. $\sim\Box p \rhd \sim\Box(p \wedge q)$ (5, $L\Box$)
7. $\emptyset \rhd \Box(p \wedge q) \Rightarrow \Box p$ (3,6, $R \Rightarrow$)

Intuitive Explanations

Proof rules preserve validity, and refutation rules preserve non-validity. In fact, all non-modal proof rules are invertible. These rule correspond to the conditions in the definition of a model.

For example, take the rule $L \sim\vee$. We have: $\sim(\varphi \vee \psi), \Gamma \rhd \Delta$ is non-valid iff ($\sim(\varphi \vee \psi)$ is true, all $\gamma \in \Gamma$ are true and all $\delta \in \Delta$ are not true at some point w in some model) iff ($\varphi \vee \psi$ is false, all $\gamma \in \Gamma$ are true and all $\delta \in \Delta$ are not true at some point w in some model) iff (both φ and ψ are false, all $\gamma \in \Gamma$ are true and all $\delta \in \Delta$ are not true at some point w in some model) iff (both $\sim\varphi$ and $\sim\psi$ are true, all $\gamma \in \Gamma$ are true and all $\delta \in \Delta$ are not true at some point w in some model) iff $\sim\varphi, \sim\psi, \Gamma \rhd \Delta$ is non-valid.

The intuitions for the other rules are similar. Note that the condition for $\mathcal{M}, w \models^+ \varphi \Rightarrow \psi$ means ($\mathcal{M}, w \not\models^+ \varphi$ or $\mathcal{M}, w \models^+ \psi$) and ($\mathcal{M}, w \not\models^- \psi$ or $\mathcal{M}, w \models^- \varphi$), which means ($\mathcal{M}, w \not\models^+ \varphi$ or $\mathcal{M}, w \models^+ \psi$) and ($\mathcal{M}, w \not\models^+ \sim\psi$ or $\mathcal{M}, w \models^+ \sim\varphi$), which is equivalent to ($\mathcal{M}, w \not\models^+ \varphi$ and $\mathcal{M}, w \not\models^+ \sim\psi$) or ($\mathcal{M}, w \models^+ \psi$ and $\mathcal{M}, w \not\models^+ \sim\psi$) or ($\mathcal{M}, w \not\models^+ \varphi$ and $\mathcal{M}, w \models^+ \sim\varphi$) or ($\mathcal{M}, w \models^+ \psi$ and $\mathcal{M}, w \models^+ \sim\varphi$). Hence the form of the rule $L \Rightarrow$.

The modal proof rules are justified in the proof of Proposition 5.5.

Remark 5.2. Our proof system is just like the sequent system in [4] but *Weakening* ($\frac{\Gamma \rhd \Delta}{\Gamma, \Gamma' \rhd \Delta, \Delta'}$) is deleted and the modal rules

$$\frac{\sim\varphi, \Gamma \rhd \sim\Delta}{\sim\Box\varphi, \Box\Gamma \rhd \sim\Box\Delta}(L\Box \sim\Box) \qquad \frac{\Gamma \rhd \varphi, \sim\Delta}{\Box\Gamma \rhd \Box\varphi, \sim\Box\Delta}(R \sim\Box\Box)$$

are replaced with the rules $L\Box, R\Box$, respectively. Also, we use sets rather than multisets.

Example 5.3. Consider a sequent s of the form $\Box\Gamma \rhd \Box\psi$, where

$\Gamma = \{\varphi_1, \varphi_2, \ldots, \varphi_{100}\}$.

There is only one possibility of deriving s in our proof system, namely from $\Gamma \rhd \psi$ by $R\Box$. On the other hand, when we have *Weakening*, we have to consider all possible premises $\Box\Phi \rhd \Box\psi$, where $\Phi \subseteq \Gamma$.

Proof Soundness

PROPOSITION 5.4. *Let* $\frac{\mathsf{S}}{\mathsf{t}}$ *be a non-modal rule.*

(i) *Let* w *be a point in a model. All* $\mathsf{s} \in \mathsf{S}$ *are true at* w *iff so is* t.

(ii) *All* $\mathsf{s} \in \mathsf{S}$ *are valid iff so is* t.

PROOF. We only consider the rule $R\wedge$.

(i) We have: $\Gamma \rhd \Delta, \varphi \wedge \psi$ is not true at w iff all $\gamma \in \Gamma$ are true, all $\delta \in \Delta$ are not true, and $\varphi \wedge \psi$ is not true at w iff (by the verification condition for $\wedge$) all $\gamma \in \Gamma$ are true, all $\delta \in \Delta$ are not true, and (φ is not true or ψ is not true at w) iff (all $\gamma \in \Gamma$, all $\delta \in \Delta$ are not true, and φ is not true at w) or (all $\gamma \in \Gamma$ are true, all $\delta \in \Delta$ are not true, and ψ is not true at w) iff $\Gamma \rhd \Delta, \varphi$ is not true at w or $\Gamma \rhd \Delta, \psi$ is not true at w.

(ii) We have: t is not valid iff t is not true at some w is some model. So, by (i), some $\mathsf{s} \in \mathsf{S}$ is not true at w. Hence, some $\mathsf{s} \in \mathsf{S}$ is not valid, which gives the result. ■

PROPOSITION 5.5. *The modal rules* $L\Box, R\Box$ *preserve validity.*

PROOF. Suppose that the conclusion of $R\Box$ is non-valid. Then, $\Box\varphi$ is not true, every formula in $\sim\Box\Gamma_2$ is not true, and all formulas in $\Box\Gamma_1$ are true at some w in some model. So, there is a point x with wRx such that φ is not true at x. Also, all formulas in Γ_1 are true and all formulas in $\sim\Gamma_2$ are not true at x. Hence, the premise of $R\Box$ is not true at x, so the premise of $R\Box$ is non-valid, which gives the result. (The argument for the rule $L\Box$ is similar.) ■

THEOREM 5.6. *If a sequent* s *is provable, then* s *is valid.*

PROOF. From Propositions 5.4(ii), 5.5 and the fact that every proof axiom is valid. ■

The Logic KN4.D

KN4.D Proof System Our proof system for **KN4.D** is that for **KN4** with the following modification.

Modal rules: $L\Box$, $R\Box$, and

$$\frac{\Gamma_1 \rhd \sim\Gamma_2}{\Box\Gamma_1, \Theta_1 \rhd \Theta_2, \sim\Box\Gamma_2}(\Box\sim\Box)$$

where Θ_1, Θ_2 are finite sets of literals. We say that a sequent s is **KN4.D**-*provable* iff s is derivable in this axiom system.

EXAMPLE 5.7. Here is a **KN4.D** proof for the sequent $\Box p \rhd \sim\Box\sim p$.

1. $p \rhd p$ (proof axiom)
2. $p \rhd \sim\sim p$ (1, $R\sim\sim$)
3. $\Box p \rhd \sim\Box\sim p$ (2, $\Box\sim\Box$)

THEOREM 5.8. *If a sequent* s *is* **KN4.D**-*provable, then* s *is* **KN4.D**-*valid.*

6. Refutation System

Our refutation system for **KN4** consists of

Refutation axioms: Every sequent

$$\Box\Gamma_1, \Theta_1 \rhd \Theta_2, \sim\Box\Gamma_2$$

where Θ_1, Θ_2 are finite sets of literals such that $\Theta_1 \cap \Theta_2 = \emptyset$, and the following set **Ref** of refutation rules.

- *Non-modal rules*:

$$\frac{\varphi, \psi, \Gamma \rhd \Delta}{\varphi \wedge \psi, \Gamma \rhd \Delta}(rL\wedge) \qquad \frac{\Gamma \rhd \Delta, \varphi}{\Gamma \rhd \Delta, \varphi \wedge \psi} \quad \frac{\Gamma \rhd \Delta, \psi}{\Gamma \rhd \Delta, \varphi \wedge \psi}(rR\wedge)$$

$$\frac{\sim\varphi, \Gamma \rhd \Delta}{\sim(\varphi \wedge \psi), \Gamma \rhd \Delta} \quad \frac{\sim\psi, \Gamma \rhd \Delta}{\sim(\varphi \wedge \psi), \Gamma \rhd \Delta}(rL\sim\wedge) \quad \frac{\Gamma \rhd \Delta, \sim\varphi, \sim\psi}{\Gamma \rhd \Delta, \sim(\varphi \wedge \psi)}(rR\sim\wedge)$$

$$\frac{\varphi, \Gamma \rhd \Delta}{\varphi \vee \psi, \Gamma \rhd \Delta} \quad \frac{\psi, \Gamma \rhd \Delta}{\varphi \vee \psi, \Gamma \rhd \Delta}(rL\vee) \qquad \frac{\Gamma \rhd \Delta, \varphi, \psi}{\Gamma \rhd \Delta, \varphi \vee \psi}(rR\vee)$$

$$\frac{\sim\varphi, \sim\psi, \Gamma \rhd \Delta}{\sim(\varphi \vee \psi), \Gamma \rhd \Delta}(rL\sim\vee) \quad \frac{\Gamma \rhd \Delta, \sim\varphi}{\Gamma \rhd \Delta, \sim(\varphi \vee \psi)} \quad \frac{\Gamma \rhd \Delta, \sim\psi}{\Gamma \rhd \Delta, \sim(\varphi \vee \psi)}(rR\sim\vee)$$

$$\frac{\Gamma \rhd \Delta, \varphi, \sim\psi}{\varphi \Rightarrow \psi, \Gamma \rhd \Delta} \quad \frac{\sim\varphi, \Gamma \rhd \Delta, \varphi}{\varphi \Rightarrow \psi, \Gamma \rhd \Delta} \quad \frac{\psi, \Gamma \rhd \Delta, \sim\psi}{\varphi \Rightarrow \psi, \Gamma \rhd \Delta} \quad \frac{\sim\varphi, \psi, \Gamma \rhd \Delta}{\varphi \Rightarrow \psi, \Gamma \rhd \Delta}(rL\Rightarrow)$$

$$\frac{\varphi, \Gamma \rhd \Delta, \psi}{\Gamma \rhd \Delta, \varphi \Rightarrow \psi} \quad \frac{\sim\psi, \Gamma \rhd \Delta, \sim\varphi}{\Gamma \rhd \Delta, \varphi \Rightarrow \psi}(rR\Rightarrow)$$

$$\frac{\varphi, \sim\psi, \Gamma \rhd \Delta}{\sim(\varphi \Rightarrow \psi), \Gamma \rhd \Delta}(rL\sim\Rightarrow) \quad \frac{\Gamma \rhd \Delta, \varphi}{\Gamma \rhd \Delta, \sim(\varphi \Rightarrow \psi)} \quad \frac{\Gamma \rhd \Delta, \sim\psi}{\Gamma \rhd \Delta, \sim(\varphi \Rightarrow \psi)}(rR\sim\Rightarrow)$$

$$\frac{\varphi, \Gamma \rhd \Delta}{\sim\sim\varphi, \Gamma \rhd \Delta}(rL\sim\sim) \qquad \frac{\Gamma \rhd \Delta, \varphi}{\Gamma \rhd \Delta, \sim\sim\varphi}(rR\sim\sim)$$

Modal rule:

$$\frac{\{\Gamma_1 \rhd \varphi, \sim\Gamma_2 : \varphi \in \Delta_1\} \cup \{\Gamma_1, \sim\varphi \rhd \sim\Gamma_2 : \varphi \in \Delta_2\}}{\Box\Gamma_1, \sim\Box\Delta_2, \Theta_1 \rhd \Theta_2, \Box\Delta_1, \sim\Box\Gamma_2}(rLR)$$

where Θ_1, Θ_2 are finite sets of literals such that $\Theta_1 \cap \Theta_2 = \emptyset$, and $\Delta_1 \cup \Delta_2 \neq \emptyset$.

We say that a sequent s is *refutable* iff $\mathbf{AxRef} \rhd_{\mathbf{Ref}} \mathsf{s}$, where $\mathbf{AxRef}$ is the set of refutation axioms. (And we say that a formula φ is refutable iff so is the sequent $\emptyset \rhd \varphi$.)

EXAMPLE 6.1. Any sequent $\Box\varphi \rhd \sim\Box\psi$, where φ, ψ are arbitrary formulas, is a refutation axiom.

EXAMPLE 6.2. Here is a refutation for the sequent $\Box\Box\Box q, \Box p \rhd \Box\Box p$.

1. $\Box q \rhd p$ (ref. axiom)
2. $\Box\Box q, p \rhd \Box p$ (1, rLR)
3. $\Box\Box\Box q, \Box p \rhd \Box\Box p$ (2, rLR)

Intuitive Explanations

Every refutation axiom $\Box\Gamma_1, \Theta_1 \rhd \Theta_2, \sim\Box\Gamma_2$ is non-valid in the one-point frame $(\{w_0\}, \emptyset)$. Indeed, define a model on this frame by putting: φ is true at w_0 iff $\varphi \in \Theta_1$, and φ is false at w_0 iff $\sim\varphi \in \Theta_1$ ($\varphi \in \mathbf{VAR}$). Then, every member of Θ_1 (Θ_2) is true (not true) at w_0 (because $\Theta_1 \cap \Theta_2 = \emptyset$). Also, every member of $\Box\Gamma_1$ ($\sim\Box\Gamma_2$) is true (not true) at w_0 (because there is no point accessible from w_0).

The non-modal refutation rules are simply reversals of the non-modal proof rules. For example, consider the rule $rL\vee$. By Proposition 5.4(ii), we have: $\varphi \vee \psi, \Gamma \rhd \Delta$ is non-valid iff either $\varphi, \Gamma \rhd \Delta$ is non-valid or $\psi, \Gamma \rhd \Delta$ is non-valid. So, if $\varphi, \Gamma \rhd \Delta$ is non-valid then so is $\varphi \vee \psi, \Gamma \rhd \Delta$, and if $\psi, \Gamma \rhd \Delta$ is non-valid then so is $\varphi \vee \psi, \Gamma \rhd \Delta$.

In order to justify the modal refutation rule, we use the amalgamation construction (see e.g. [9]). To simplify our description, we assume that $\Delta_2 = \Gamma_2 = \emptyset$, so the conclusion of the rule rLR is $\Box\Gamma_1, \Theta_1 \rhd \Theta_2, \Box\Delta_1$. (In the general case Δ_2 (Γ_2) behaves like Δ_1 (Γ_1).) Let $\{\varphi_1, \ldots, \varphi_m\} = \Delta_1$. Suppose that all premises $\Gamma_1 \rhd \varphi_1, \ldots, \Gamma_1 \rhd \varphi_m$ are non-valid. Then, there are models

$$(\mathcal{W}_1, v_1^+, v_1^-) \cdots (\mathcal{W}_m, v_m^+, v_m^-)$$

(with $\mathcal{W}_i = (W_i, R_i)$) and points $w_1 \in W_1, \ldots, w_m \in W_m$ such that all members of Γ_1 are true at w_i and φ_i is not true at w_i ($1 \leq i \leq m$). Construct a new model $((W, R), v^+, v^-)$, where W is the union of $W_1, \ldots, W_m$ together with a new point w_0, the relation R is the union of $R_1, \ldots, R_m$ together with $\{(w_0, w_i) : 1 \leq i \leq m\}$ (so $w_1, \ldots, w_m$ are all points accessible from w_0), and v^+ (v^-) is a valuation preserving $v_1^+, \ldots, v_m^+$ ($v_1^-, \ldots, v_m^-$) and such that: φ is true at w_0 iff $\varphi \in \Theta_1$, and φ is false at w_0 iff $\sim\varphi \in \Theta_1$ ($\varphi \in \mathbf{VAR}$). Then, all members of $\Box\Gamma_1$ are true at w_0 (because all members of Γ_1 are true at each w_i), all members of $\Box\Delta_1$ are not true at w_0 (because every φ_i is not true at w_i), and each member of Θ_1 (Θ_2) is true (not true) at w_0. Hence, the conclusion $\Box\Gamma_1, \Theta_1 \rhd \Theta_2, \Box\Delta_1$ is not true at w_0. So, it is non-valid.

Remark 6.3. The rule rLR is invertible because (by Proposition 5.5) we also have: If the conclusion is non-valid, then so is every premise.

The Logic KN4.D

KN4.D Refutation System Our refutation system for **KN4.D** is that for **KN4** with the following modifications.

Refutation axioms: Every sequent $\Theta_1 \rhd \Theta_2$, where Θ_1, Θ_2 are finite sets of literals such that $\Theta_1 \cap \Theta_2 = \emptyset$.

Refutation modal rule:

$$\frac{\{\Gamma_1 \rhd \varphi, {\sim}\Gamma_2 : \varphi \in \Delta_1\} \cup \{\Gamma_1, {\sim}\varphi \rhd {\sim}\Gamma_2 : \varphi \in \Delta_2\}}{\Box\Gamma_1, {\sim}\Box\Delta_2, \Theta_1 \rhd \Theta_2, \Box\Delta_1, {\sim}\Box\Gamma_2}(rLR_{\mathbf{D}})$$

where Θ_1, Θ_2 are finite sets of literals such that $\Theta_1 \cap \Theta_2 = \emptyset$. If $\Delta_1 \cup \Delta_2 = \emptyset$ then $rLR_{\mathbf{D}}$ is the rule $\Box \sim \Box$ ($\frac{\Gamma_1 \rhd {\sim}\Gamma_2}{\Box\Gamma_1, \Theta_1 \rhd \Theta_2, {\sim}\Box\Gamma_2}$). We say that a sequent s is **KN4.D**-*refutable* iff s is derivable in this axiom system.

Example 6.4. Here is a **KN4.D** refutation for the sequent $\Box p \rhd {\sim}\Box{\sim}q$.

1. $p \rhd q$ (ref. axiom)
2. $p \rhd {\sim\sim}q$ (1, $rR{\sim\sim}$)
3. $\Box p \rhd {\sim}\Box{\sim}q$ (2, $rLR_{\mathbf{D}}$)

Remark 6.5. The rule $rLR_{\mathbf{D}}$ is invertible.

Normal Sequents

Definition 6.6. A *normal sequent* is a sequent

$$\Box\Gamma_1, {\sim}\Box\Delta_2, \Theta_1 \rhd \Theta_2, \Box\Delta_1, {\sim}\Box\Gamma_2$$

where Θ_1, Θ_2 are finite sets of literals. We say that a normal sequent is *special* iff $\Theta_1 \cap \Theta_2 = \emptyset$.

Thus, the conclusion of the rule rLR (as well as any refutation axiom) is a special normal sequent.

7. Refutation Soundness

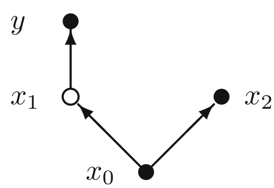
Refutation Trees

In order to establish refutation soundness, we first represent derivations in our refutation system as refutation trees. By transforming every refutation tree into a finite tree-type countermodel, we also establish the refined characterizations of **KN4** and **KN4.D** analogous to the characterizations of standard modal logics by classes of finite tree-type frames, see e.g. [8]. (The logic **K** is characterized by the class of finite, irreflexive, intransistive trees; and the logic **KD** is characterized by the class of finite, intransitive trees with reflexive end nodes and all other nodes irreflexive.)

By a *refutation tree* for a sequent s we mean a finite (immediate) successor-tree $\mathcal{RT}$ whose nodes are labelled with sequents and which satisfies the following conditions. (The label of a node x in $\mathcal{RT}$ is denoted by $s(x)$.)

- The origin x_0 is labelled with s.
- If x is an end node, then $s(x)$ is a refutation axiom.
- If $x_1, \ldots, x_k$ are the immediate successors of a node x, then $s(x)$ is obtained from $s(x_1), \ldots, s(x_k)$ by a refutation rule.

Example 7.1. Here is a refutation tree for $\mathsf{s} = \Box q, {\sim}\Box p \rhd \Box(p \wedge \Box q)$.



We have: $s(x_0) = \mathsf{s}$, $s(x_1) = q \rhd p \wedge \Box q$, $s(x_2) = q, {\sim}p \rhd \emptyset$, $s(y) = q \rhd p$. Both $s(y)$ and $s(x_2)$ are refutation axioms. The sequent $s(x_1)$ is obtained from $s(y)$ by the non-modal rule $rR\wedge$, and $s(x_0)$ is obtained from $s(x_1), s(x_2)$ by the modal rule. The black nodes are the nodes whose labels are special normal sequents.

Refined Countermodels

By modifying some results in [13], we now define the finite irreflexive intransitive tree $\mathcal{T}$ corresponding to a refutation tree $\mathcal{RT}$. It will be obtained from $\mathcal{RT}$ by removing the nodes whose labels are obtained by non-modal rules (so the remaining nodes are the nodes whose labels are either refutation axioms or have been obtained by the modal rule). First, for every node x in $\mathcal{RT}$, we define its corresponding world $x^\star$.

Definition 7.2. Let x be a node in a refutation tree $\mathcal{RT}$.

- If $s(x)$ is a refutation axiom, then $x^\star = x$.
- If $s(x)$ is obtained from $s(x_1)$ by a non-modal rule, then $x^\star = x_1^\star$.

- If $s(x)$ is obtained from $s(x_1), \ldots, s(x_k)$ by the modal rule, then $x^\star = x$.

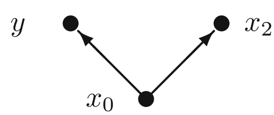
Definition 7.3. Let $\mathcal{RT}$ be a refutation tree. Then its corresponding finite successor-tree $\mathcal{T} = (W, \prec)$, where $W = \{x^\star : x$ is a node in $\mathcal{RT}\}$, and the relation $\prec$ on W is defined as follows.

$x \prec y$ iff $s(x)$ is obtained from (its immediate successors) $s(x_1), \ldots, s(x_k)$ by the modal rule and $y = x_i^\star$ for some $1 \leq i \leq k$. (Note that $\mathcal{T}$ is a finite, irreflexive, intransitive tree.)

Remark 7.4. If w is a node in $\mathcal{T}$, then either $s(w)$ is a refutation axiom or $s(w)$ is obtained by the modal rule, so $s(w)$ is a special normal sequent.

Our model corresponding to $\mathcal{RT}$ is $(\mathcal{T}, v^+, v^-)$, where v^+, v^- are defined as follows. Let w be a node in $\mathcal{T}$. Then $s(w)$ is a special normal sequent $\Box\Gamma_1, {\sim}\Box\Delta_2, \Theta_1 \rhd \Theta_2, \Box\Delta_1, {\sim}\Box\Gamma_2$ where Θ_1, Θ_2 are finite sets of literals such that $\Theta_1 \cap \Theta_2 = \emptyset$. We define v^+, v^- as valuations satisfying the following conditions. $x \in v^+(p)$ iff $p \in \Theta_1$, and $x \in v^-(p)$ iff ${\sim}p \in \Theta_1$.

Example 7.5. Let $\mathcal{RT}$ be the refutation tree for the sequent s defined in Example 7.1. The frame $\mathcal{T}$ is the following irreflexive intransitive tree.



We have: $y^\star = y$, $x_2^\star = x_2$, $x_1^\star = y$, and $x_0^\star = x_0$. And v^+, v^- are valuations such that $y \in v^+(q)$, $x_2 \in v^+(q)$, $x_2 \in v^-(p)$. (Note that $\Theta_1 = \Theta_2 = \emptyset$ in $s(x_0)$.)

Theorem 7.6. *Let x be a node in a refutation tree $\mathcal{RT}$. Then the sequent $s(x)$ is not true at $x^\star$.*

Proof. (by induction on the number n_x of nodes in the subtree of $\mathcal{RT}$ generated by x).

(1) $n_x = 1$. Then $s(x)$ is a refutation axiom $\Box\Gamma_1, \Theta_1 \rhd \Theta_2, {\sim}\Box\Gamma_2$ (and $\Theta_1 \cap \Theta_2 = \emptyset$). Since $x^\star = x$, we have: $s(x^\star) = s(x)$. By definition, every formula in Θ_1 is true and every formula in Θ_2 is not true at $x^\star$. Also, $\Box\Gamma_1$ is true and every $\psi \in {\sim}\Box\Gamma_2$ is not true at $x^\star$ (because $x \prec y$ for no y). Hence $s(x)$ is not true at $x^\star$.

(2) $n_x > 1$ and we assume that the theorem holds for the subtrees with fewer elements than n_x.

(2.1) $s(x)$ is obtained from $s(x_1)$ by a non-modal rule. Then $x^\star = x_1^\star$. Since $n_{x_1} < n_x$, by the induction hypothesis, $s(x_1)$ is not true at $x_1^\star$. Hence $s(x_1)$ is not true at $x^\star$. So, by Proposition 5.4(i), $s(x)$ is not true at $x^\star$.

(2.2) $s(x)$ is obtained from $s(x_1), \ldots, s(x_{m+n})$ by rLR. Then

$s(x) = \Box\Gamma_1, \sim\Box\Delta_2, \Theta_1 \rhd \Theta_2, \Box\Delta_1, \sim\Box\Gamma_2$ and $\Theta_1 \cap \Theta_2 = \emptyset$
$s(x_i) = \Gamma_1 \rhd \varphi_i, \sim\Gamma_2$ $(1 \leq i \leq m)$
$s(x_i) = \Gamma_1, \sim\varphi_i \rhd \sim\Gamma_2$ $(m+1 \leq i \leq m+n)$

where $\{\varphi_1, \ldots, \varphi_m\} = \Delta_1$, $\{\varphi_{m+1}, \ldots, \varphi_{m+n}\} = \Delta_2$.

Also, $x^\star = x$ (so $s(x^\star) = s(x)$) and $n_{x_i} < n_x$ for all i. By the induction hypothesis, $s(x_i)$ is not true at $x_i^\star$ $(1 \leq i \leq m+n)$, so

every formula in Γ_1 is true at $x_i^\star$ for all $1 \leq i \leq m+n$,
every formula in Γ_2 is not false at $x_i^\star$ for all $1 \leq i \leq m+n$,
φ_i is not true at $x_i^\star$ $(1 \leq i \leq m)$ and φ_i is false at $x_i^\star$ $(m+1 \leq i \leq m+n)$.

Hence, each $\Box\varphi_i$ $(1 \leq i \leq m)$ is not true at $x^\star$ and every $\sim\Box\varphi_i$ $(m+1 \leq i \leq m+n)$ is true at $x^\star$ (because $x^\star \prec x_i^\star$ for all i). Also, every formula in $\Box\Gamma_1$ is true and every formula in $\sim\Box\Gamma_2$ is not true at $x^\star$. By definition, each formula in Θ_1 is true and each formula in Θ_2 is not true at $x^\star$.

Therefore $s(x)$ is not true at $x^\star$, as required. ∎

Corollary 7.7. *Every refutable sequent is non-valid in some finite, irreflexive, intransitive tree.*

Proof. Assume that a sequent s is refutable. Then s has a refutation tree $\mathcal{RT}$ with root x, and $s(x) = s$. By Theorem 7.6, s is not true at $x^\star$, which gives the result. ∎

Refutation Soundness

Theorem 7.8. *If a sequent* s *is refutable, then* s *is non-valid.*

Proof. From Corollary 7.7. ∎

The Logic KN4.D

1. Definition 7.3 is modified as follows.

   The end nodes in $\mathcal{T}$ are now reflexive.

2. The proof of Theorem 7.6 is modified as follows.

   (1) $n_x = 1$. Then $s(x)$ is a refutation axiom $\Theta_1 \rhd \Theta_2$ and $\Theta_1 \cap \Theta_2 = \emptyset$.

   (2.2) $s(x)$ is obtained by $rLR_{\mathbf{D}}$. If $\Delta_1 \cup \Delta_2 = \emptyset$ (so $s(x)$ is obtained from $s(y)$ by $\Box \sim\Box$), then

   $s(x) = \Box\Gamma_1, \Theta_1 \rhd \Theta_2, \sim\Box\Gamma_2$ and $\Theta_1 \cap \Theta_2 = \emptyset$;

$s(y) = \Gamma_1 \rhd \sim\Gamma_2$

By the induction hypothesis, $s(y)$ is not true at $y^\star$, so every formula in Γ_1 is true at $y^\star$ and every formula in Γ_2 is not false at $y^\star$. Hence, $\Box\Gamma_1$ is true and $\sim\Box\Gamma_2$ is not true at $x^\star$ (because $x^\star \prec y^\star$). By definition, each formula in Θ_1 is true and each formula in Θ_2 is not true at $x^\star$.

THEOREM 7.9. *If a sequent* s *is* **KN4.D**-*refutable, then* s *is not* **KN4.D**-*valid.*

8. Reductions to Normal Sequents

Intutive Explanations

Our completeness proof, in fact, describes the following normal form procedure.

1. Reduce a given sequent s to some normal sequents $\mathsf{s}_1, \ldots, \mathsf{s}_n$ such that:

   $\mathsf{s} \in$ **KN4** iff each $\mathsf{s}_i \in$ **KN4**.

   (Thus, if all s_i are valid, so is s; and if some s_i is non-valid, so is s.)

2. Pick some s_i. We may assume that it is special. (Otherwise it is a proof axiom.) Consider the premises of the rule rLR with s_i as the conclusion. They are of modal degree lower than that of s_i. Now, if all the premises are refutable, then so is s_i (by rLR); and if some premise is provable, then so is s_i (by $L\Box$ or $R\Box$).

PR-Reductions

We now describe the reductions to normal forms in the following syntactic way. Let **P** (**R**) denote the set of the non-modal proof rules (refutation rules).

DEFINITION 8.1. A sequent s is **PR**-*reducible* to a finite set T of sequents iff both (i) $\mathsf{T} \vdash_{\mathbf{P}} \mathsf{s}$ and (ii) $\mathsf{t} \vdash_{\mathbf{R}} \mathsf{s}$ for some $\mathsf{t} \in \mathsf{T}$.

PROPOSITION 8.2. *If a sequent* s *is* **PR**-*reducible to a finite set* T *of sequents, then we have:* s *is valid iff so is every* $\mathsf{t} \in \mathsf{T}$.

PROOF. From Proposition 5.4(ii). ■

The *length* $l(\varphi)$ of a formula φ is defined as follows.

DEFINITION 8.3. (1) If φ is a literal or a (negated) $\Box$ formula, then $l(\varphi) = 0$.

(2.1) If $\varphi =\sim \psi$ (and ψ is neither a variable nor a $\Box$ formula), then $l(\varphi) = l(\psi) + 1$.

(2.2) If $\varphi = \psi_1 \otimes \psi_2$, where $\otimes \in \{\wedge, \vee, \Rightarrow\}$, then $l(\varphi) = l(\psi_1) + l(\psi_2) + 1$.

The length $l(\Gamma)$ of a set $\Gamma = \{\varphi_1, \ldots, \varphi_n\}$ is $l(\varphi_1) + \cdots + l(\varphi_n)$. And the length $l(s)$ of a sequent $\mathsf{s} = \Gamma \rhd \Delta$ is $l(\Gamma \cup \Delta)$.

LEMMA 8.4. *Every sequent* $\mathsf{s}(= \Gamma \rhd \Delta)$ *of modal degree* k *is* **PR**-*reducible to some normal sequents* $\mathsf{s}_1, \ldots, \mathsf{s}_n$ *of modal degree* $\leq k$.

PROOF. (by induction on the length $l(s)$ of a sequent $\mathsf{s} = \Gamma \rhd \Delta$ of modal degree k).

(1) $l(\mathsf{s}) = 0$. Then every $\varphi \in \Gamma \cup \Delta$ is of length 0, so s is normal.

(2) $l(\mathsf{s}) > 0$ and we assume that the lemma holds for sequents of length $< l(\mathsf{s})$. Then some $\varphi \in \Gamma \cup \Delta$ is of length > 0, say $\varphi \in \Delta$. We only consider the case where $\varphi = \psi_1 \wedge \psi_2$. Thus, s has the form

$$\Gamma \rhd \Delta', \psi_1 \wedge \psi_2$$

Consider the sequents

$$\mathsf{s}_1 = \Gamma \rhd \Delta', \psi_1 \qquad \mathsf{s}_2 = \Gamma \rhd \Delta', \psi_2$$

Both s_1 and s_2 are of length $< l(\mathsf{s})$, and they are of modal degree $\leq k$. Hence, by the induction hypothesis, each s_i $(i \in \{1, 2\})$ is **PR**-reducible to some normal forms

$$\mathsf{s}_i^1, \ldots, \mathsf{s}_i^{n_i}$$

of modal degree $\leq k$. Also, by the rules $(R\wedge)$ and $(rR\wedge)$, s is **PR**-reducible to $\mathsf{s}_1, \mathsf{s}_2$. Therefore, s is **PR**-reducible to

$$\mathsf{s}_1^1, \ldots, \mathsf{s}_1^{n_1}, \mathsf{s}_2^1, \ldots, \mathsf{s}_2^{n_2}$$

which was to be shown. ■

9. Completeness

THEOREM 9.1. *Every sequent* s *is either provable or refutable.*

PROOF. (by induction on the modal degree k of s).

(1) $k = 0$. Then s is $\Box$-free. By Lemma 8.4, s is **PR**-reducible to some finite set T of $\Box$-free normal sequents. Thus, every $\mathsf{t} \in \mathsf{T}$ has the form $\Theta_1 \rhd \Theta_2$, where Θ_1, Θ_2 are finite sets of literals. If $\Theta_1 \cap \Theta_2 \neq \emptyset$ then t is provable (because t is a proof axiom), and if $\Theta_1 \cap \Theta_2 = \emptyset$ then t is refutable (because t is a refutation axiom). Hence, each $\mathsf{t} \in \mathsf{T}$ is either provable or refutable. If all $t \in \mathsf{T}$ are provable, then so is s; and if some $\mathsf{t} \in \mathsf{T}$ is refutable, then so is s (by Definition 8.1). Therefore, either s is provable or s is refutable.

(2) $k > 0$ and we assume that the theorem holds for sequents of modal degree $< k$. By Lemma 8.4, s is **PR**-reducible to some finite set T of normal sequents of modal degree $\leq k$. Take any sequent $\mathsf{t} \in \mathsf{T}(= \Box\Gamma_1, \sim\Box\Delta_2, \Theta_1 \rhd \Theta_2, \Box\Delta_1, \sim\Box\Gamma_2)$. If $\Theta_1 \cap \Theta_2 \neq \emptyset$ then t is provable, so we assume that $\Theta_1 \cap \Theta_2 = \emptyset$. Also, if $\Delta_1 \cup \Delta_2 = \emptyset$ then t is refutable, so we assume that $\Delta_1 \cup \Delta_2 \neq \emptyset$.

Let $\{\varphi_1, \ldots, \varphi_m\} = \Delta_1$, $\{\varphi_{m+1}, \ldots, \varphi_{m+n}\} = \Delta_2$. Consider the sequents

$$\mathsf{t}_i = \Gamma_1 \rhd \varphi_i, \sim\Gamma_2 \qquad (1 \leq i \leq m)$$
$$\mathsf{t}_i = \Gamma_1, \sim\varphi_i \rhd \sim\Gamma_2 \qquad (m+1 \leq i \leq m+n)$$

Each t_i is of modal degree $< k$, so by the induction hypothesis, every t_i is either provable or refutable $(1 \leq i \leq m+n)$.

(Case 1) For some $1 \leq i \leq m+n$, t_i is provable. Then t is provable by $L\Box$ or by $R\Box$.

(Case 2) For all $1 \leq i \leq m+n$, t_i is refutable. Then t by rLR.

Hence, t is provable or t is refutable for any $\mathsf{t} \in \mathsf{T}$. If each $\mathsf{t} \in \mathsf{T}$ is provable, then so is s. And if some $\mathsf{t} \in \mathsf{T}$ is refutable, then so is s. Therefore, either s is provable or s is refutable. ■

Proof Completeness

THEOREM 9.2. *If a sequent* s *is valid, then* s *is provable.*

PROOF. Assume that a sequent s is valid. Then s is not refutable (by Theorem 7.8). Hence, by Theorem 9.1, s is provable. ■

Refutation Completeness

THEOREM 9.3. *If a sequent* s *is non-valid, then* s *is refutable.*

PROOF. Assume that a sequent s is non-valid. Then s is not provable (by Theorem 5.6). Hence, by Theorem 9.1, s is refutable. ■

The Finite Model Property

THEOREM 9.4. *The logic* **KN4** *is characterized by the class of finite, irreflexive, intransitive trees.*

PROOF. Assume that $\varphi \notin$ **KN4**. Then the sequent $\emptyset \rhd \varphi$ is non-valid. So, by Theorem 9.3, it is refutable. Hence, by Corollary 7.7, the sequent $\emptyset \rhd \varphi$ is non-valid in some finite irreflexive intransitive tree. Thus, φ is not valid in that tree either. ■

The Logic KN4.D

The proof of Theorem 9.1 is modified as follows.

If $\Delta_1 \cup \Delta_2 = \emptyset$ then we consider the sequent $t_0 = \Gamma_1 \rhd \sim\Gamma_2$. It is of modal degree $< k$, so by the induction hypothesis, it is either provable or refutable. Now, if t_0 is provable, then so is t by $\Box \sim\Box$; and if t_0 is refutable, then so is t by $rLR_{\mathbf{D}}$.

THEOREM 9.5. *If a sequent* s *is* **KN4.D**-*valid, then* s *is* **KN4.D**-*provable.*

THEOREM 9.6. *If a sequent* s *is not* **KN4.D**-*valid, then* s *is* **KN4.D**-*refutable.*

THEOREM 9.7. *The logic* **KN4.D** *is characterized by the class of finite, intransitive trees with reflexive end nodes and all other nodes irreflexive.*

10. Reduction Procedures

Our axiom systems provide decision procedures called reduction procedures (see [14]).

By a *reduction rule* we mean a set of pairs $\frac{\mathsf{S}}{\{\mathsf{S}_1,\ldots,\mathsf{S}_n\}}$, where $\mathsf{S}, \mathsf{S}_1, \ldots, \mathsf{S}_n$ are finite sets of sequents. We also write $\frac{\mathsf{S}}{\mathsf{S}_1|\ldots|\mathsf{S}_n}$ instead of $\frac{\mathsf{S}}{\{\mathsf{S}_1,\ldots,\,\mathsf{S}_n\}}$ and $\frac{\mathsf{S}}{\mathsf{S}_1}$ instead of $\frac{\mathsf{S}}{\{\mathsf{S}_1\}}$. Let $VAL(\mathsf{L})$ denote the set of sequents valid in a logic $\mathbf{L}$. We say that a reduction rule is *sound for* a logic $\mathbf{L}$ iff we have: $\mathsf{S} \subseteq VAL(\mathbf{L})$ iff some $\mathsf{S}_i \subseteq VAL(\mathbf{L})$.

A *reduction system* $\mathcal{H}$ consists of a set of sequents (called *simple sequents*) whose validity is easy to check and a set of reduction rules. We say that $\mathcal{H}$ is *sound for* a logic $\mathbf{L}$ iff so is each of its reduction rules.

Let $\mathcal{H}$ be a reduction system. An $\mathcal{H}$ reduction tree for a finite set T of sequents is a finite successor tree with nodes as finite sets of sequents satisfying the following conditions.

1. The origin is T.
2. If $\mathsf{S}_1, \ldots, \mathsf{S}_n$ are the immediate successors of a node S, then $\{\mathsf{S}_1, \ldots, \mathsf{S}_n\}$ is obtained from S by a reduction rule of $\mathcal{H}$.
3. The end nodes are finite sets of simple sequents.

We say that T is $\mathcal{H}$-*reducible* iff there is an $\mathcal{H}$ reduction tree for T. And we say that $\mathcal{H}$ is *complete* for $\mathbf{L}$ iff every finite set of sequents is $\mathcal{H}$-reducible.

Reduction System $\mathcal{H}_{\mathbf{KN4}}$

Our reduction system for **KN4** is defined as follows. (We sometimes write $\mathsf{S};\mathsf{T}$ for $\mathsf{S} \cup \mathsf{T}$, and $\mathsf{s};\mathsf{T}$ for $\{\mathsf{s}\} \cup \mathsf{T}$.)

- *Simple sequents*: Every proof axiom (see Sect. 5) and every refutation axiom (see Sect. 6).
- *Reduction rules*:

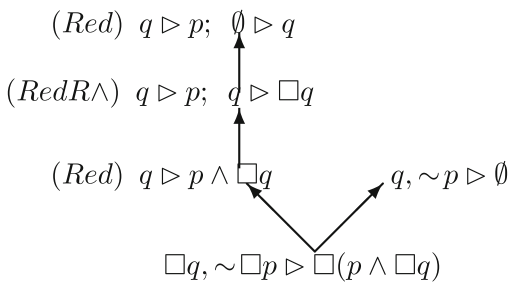
  Non-modal rules: $\frac{\mathsf{s};\mathsf{T}}{\mathsf{S}\cup\mathsf{T}}(RedX)$, where $\frac{\mathsf{S}}{\mathsf{s}}(X)$ is a non-modal proof rule.

  Modal rule: $\frac{\mathsf{s};\mathsf{S}}{\mathsf{s}_1;\mathsf{S}|\ldots|\mathsf{s}_{m+n};\mathsf{S}}(Red)$ where $\frac{\mathsf{s}_1\ldots\mathsf{s}_{m+n}}{\mathsf{s}}$ is the modal refutation rule. (This rule looks like this. $\frac{\Box\Gamma_1,\sim\Box\Delta_2,\Theta_1\rhd\Theta_2,\Box\Delta_1,\sim\Box\Gamma_2;\mathsf{S}}{\{\{\Gamma_1\rhd\varphi,\sim\Gamma_2;\mathsf{S}:\varphi\in\Delta_1\}\cup\{\Gamma_1,\sim\varphi\rhd\sim\Gamma_2;\mathsf{S}:\varphi\in\Delta_2\}\}}$)

Example 10.1. The reduction rule corresponding to the proof rule $R\wedge$.

$$\frac{\Gamma\rhd\Delta,\varphi\wedge\psi;\ \ \mathsf{T}}{\Gamma\rhd\Delta,\varphi;\ \ \Gamma\rhd\Delta,\psi;\ \ \mathsf{T}}(RedR\wedge)$$

Example 10.2. Here is a reduction tree for the sequent $\Box q,\sim\Box p\rhd\Box(p\wedge\Box q)$.



Lemma 10.3. *(i) Let $\frac{\mathsf{S}}{\mathsf{s}}$ be an invertible proof rule for* **L** *(that is,* $\mathsf{S}\subseteq VAL(\mathbf{L})$ *iff* $\mathsf{s}\in VAL(\mathbf{L})$*). Then,* $\frac{\mathsf{s};\mathsf{T}}{\mathsf{S}\cup\mathsf{T}}$ *is a sound reduction rule for* **L**.

(ii) Let Let $\frac{s_1\ldots s_n}{s}$ *be an invertible refutation rule for* **L** *(that is, each* $\mathsf{s}_i\notin VAL(\mathbf{L})$ *iff* $\mathsf{s}\notin VAL(\mathbf{L})$*). Then,* $\frac{\mathsf{s};\mathsf{S}}{\mathsf{s}_1;\mathsf{S}|\ldots|\mathsf{s}_n;\mathsf{S}}$ *is a sound reduction rule for* **L**.

Proof. We only prove (i). Assume that $\frac{\mathsf{S}}{\mathsf{s}}$ is invertible and $\mathsf{S}\cup\mathsf{T}\not\subseteq VAL(\mathbf{L})$. Then there is $\mathsf{t}\in\mathsf{S}\cup\mathsf{T}$ such that $\mathsf{t}\notin VAL(\mathbf{L})$. We may assume that $\mathsf{t}\in\mathsf{S}$. (If $\mathsf{t}\in\mathsf{T}$ then $\mathsf{s};\mathsf{T}\not\subseteq VAL(\mathbf{L})$.) So, $\mathsf{s}\notin VAL(\mathbf{L})$ and $\{\mathsf{s}\}\cup\mathsf{T}\not\subseteq VAL(\mathbf{L})$. ■

Corollary 10.4. *$\mathcal{H}_{\mathbf{KN4}}$ is sound for* **KN4**.

Proof. From Lemma 10.3, Proposition 5.4, and Remark 6.3. ■

Theorem 10.5. *Every finite set* T *of sequents is* $\mathcal{H}_{\mathbf{KN4}}$*-reducible.*

Proof. (The proof is by induction on the modal degree of T, and it is based on the proof of Theorem 9.1.)

(1) $mdeg(\mathsf{T}) = 0$. By Lemma 8.4, every member of T is **PR**-reducible to some finite set of $\Box$-free normal sequents, so there is a sequence $\mathsf{T}_1, \ldots, \mathsf{T}_l$ such that $\mathsf{T}_1 = \mathsf{T}$, T_l is a finite set of simple sequents, and every T_i $(1 < i \leq l)$ is obtained from the preceding set by a non-modal reduction rule. So, this sequence is an $\mathcal{H}_{\mathbf{KN4}}$ reduction tree for T.

(2) $mdeg(\mathsf{T}) = k > 0$. First, reduce T to some finite set T' of normal sequents of modal degree $\leq k$ (see (1)). Suppose $\mathsf{t} \in \mathsf{T}'$ is of modal degree k. For simplicity, we assume that T' contains just one sequent of modal degree k. (Otherwise, we reduce the other ones in the same way by applying *Red*.) Apply the rule *Red* to $\mathsf{T}' = \{\mathsf{t}\} \cup \mathsf{S}$, obtaining the immediate successors

$$\{\mathsf{t}_1\} \cup \mathsf{S} \cdots \{\mathsf{t}_{m+n}\} \cup \mathsf{S}$$

(see the proof of Theorem 9.1). Each of these sets is of modal degree $< k$, so by the induction hypothesis, each of them has an $\mathcal{H}_{\mathbf{KN4}}$ reduction tree. Hence, the tree

$$\begin{array}{c}
\vdots \qquad\qquad\qquad \vdots \\
\{\mathsf{t}_1\} \cup \mathsf{S} \cdots \{\mathsf{t}_{m+n}\} \cup \mathsf{S} \\
\nwarrow \quad \nearrow \\
\{\mathsf{t}\} \cup \mathsf{S} \\
\vdots \\
\mathsf{T}
\end{array}$$

is an $\mathcal{H}_{\mathbf{KN4}}$ reduction tree for T. ■

COROLLARY 10.6. **KN4** *is decidable.*

PROOF. Take any formula φ. Construct an $\mathcal{H}_{\mathbf{KN4}}$ reduction tree for $\{\emptyset \rhd \varphi\}$ (existing by Theorem 10.5). By Corollary 10.4, if all members of some end node are valid, then so is $\{\emptyset \rhd \varphi\}$ (and $\varphi \in \mathbf{KN4}$); and if some member in each end node is non-valid, then so is $\{\emptyset \rhd \varphi\}$ (and $\varphi \notin \mathbf{KN4}$). ■

REMARK 10.7. Similar results can be obtained for the logic **KN4.D**.

11. Hybrid Deduction–Refutation Systems

In this section, we briefly discuss a related method, namely that of hybrid deduction–refutation systems. Hybrid deduction-refutation systems were introduced by Goranko [6]. In [12], such systems for certain FDE-based logics are presented.

Given a set **FORM** of formulas and a logic $\mathbf{L} \subseteq \mathbf{FORM}$, we define the entailment relation $\models_{\mathbf{L}}$ as follows. $\Phi \models_{\mathbf{L}} \varphi$ iff $\bigwedge \Phi \to \varphi \in \mathbf{L}$. A *deduction* (*refutation*) sequent is an object $\Phi \vdash \varphi$ ($\Phi \dashv \varphi$), where $\Phi \cup \{\varphi\} \subseteq \mathbf{FORM}$. A *hybrid sequent* is a deduction sequent or a refutation sequent. A *hybrid rule instance* is a pair S/s, where $\mathsf{S} \cup \{\mathsf{s}\}$ is a finite set of hybrid sequents. A *rule* is a set of hybrid rule instances. And a hybrid system is a set of hybrid rules. A *purely deductive* (*refutational*) system is a hybrid system that contains no refutation (deduction) sequents in its rules.

A (deduction) rule of the form

$$\frac{\Gamma_1 \vdash \varphi_1, \ldots, \Gamma_m \vdash \varphi_m, \Delta_1 \dashv \psi_1, \ldots, \Delta_n \dashv \psi_n}{\Gamma \vdash \alpha}$$

is *sound with respect to* $\mathbf{L}$ iff $\Gamma \models_{\mathbf{L}} \alpha$ whenever

$\Gamma_1 \models_{\mathbf{L}} \varphi_1, \ldots, \Gamma_m \models_{\mathbf{L}} \varphi_m, \Delta_1 \not\models_{\mathbf{L}} \psi_1, \ldots, \Delta_n \not\models_{\mathbf{L}} \psi_n$.

A (refutation) rule of the form

$$\frac{\Gamma_1 \vdash \varphi_1, \ldots, \Gamma_m \vdash \varphi_m, \Delta_1 \dashv \psi_1, \ldots, \Delta_n \dashv \psi_n}{\Gamma \dashv \alpha}$$

is *sound with respect to* $\mathbf{L}$ iff $\Gamma \not\models_{\mathbf{L}} \alpha$ whenever

$\Gamma_1 \models_{\mathbf{L}} \varphi_1, \ldots, \Gamma_m \models_{\mathbf{L}} \varphi_m, \Delta_1 \not\models_{\mathbf{L}} \psi_1, \ldots, \Delta_n \not\models_{\mathbf{L}} \psi_n$.

Let $\mathbf{H}$ be a hybrid system. We say that a hybrid sequent t is $\mathbf{H}$-*dervable* from a set T of hybrid sequents (symbolically, $\mathsf{T} \vdash_{\mathbf{H}} \mathsf{t}$) iff there is a finite sequence $\mathsf{s}_1, \ldots, \mathsf{s}_n$ such that $\mathsf{s}_n = \mathsf{t}$ and for every $1 \leq i \leq n$, eiher $\mathsf{s}_i \in \mathsf{T}$ or s_i is obtained from some preceding hybrid sequents by a rule of $\mathbf{H}$.

$\mathbf{H}$ is *deductively* (*refutationally*) *sound for* $\mathbf{L}$ iff

$\vdash_{\mathbf{H}} \Phi \vdash \varphi$ implies $\Phi \models_{\mathbf{L}} \varphi$ ($\vdash_{\mathbf{H}} \Phi \dashv \varphi$ implies $\Phi \not\models_{\mathbf{L}} \varphi$).

And $\mathbf{H}$ is *deductively* (*refutationally*) *complete for* $\mathbf{L}$ iff

$\Phi \models_{\mathbf{L}} \varphi$ implies $\vdash_{\mathbf{H}} \Phi \vdash \varphi$ ($\Phi \not\models_{\mathbf{L}} \varphi$ implies $\vdash_{\mathbf{H}} \Phi \dashv \varphi$).

Given a hybrid rule, we can form a *derivative rule* by replacing one of the premises with the conclusion and changing the direction of the turnstyle in those sequents. For example, the deduction rule $\frac{\Gamma \vdash \varphi, \Gamma \vdash \varphi \to \psi}{\Gamma \vdash \psi}$ produces the following derivative rules.

$$\frac{\Gamma \dashv \psi, \Gamma \vdash \varphi \to \psi}{\Gamma \dashv \varphi} \qquad \frac{\Gamma \vdash \varphi, \Gamma \dashv \psi}{\Gamma \dashv \varphi \to \psi}$$

In [6,12], it is by forming derivative rules that hybrid systems for the specific logics are obtained.

This paper is not about hybrid systems. It is about pure proof/refutation systems. However, we show that it is possible to produce a hybrid system for a logic $\mathbf{L}$ from a purely deductive system $\mathbf{Pr_L}$ (deductively sound and complete for $\mathbf{L}$) and a purely refutational system $\mathbf{Rf_L}$ (refutationally sound and complete for $\mathbf{L}$). Indeed, let $\mathbf{H} = \mathbf{Pr_L} \cup \mathbf{Rf_L}$.

PROPOSITION 11.1. **H** *is both deductively and refutationally sound and complete for* **L**.

PROOF. 1. Assume that $\vdash_{\mathbf{H}} \Phi \vdash \varphi$ ($\vdash_{\mathbf{H}} \Phi \dashv \varphi$). Then, $\Phi \vdash \varphi$ can be derived only by $\mathbf{Pr_L}$ ($\mathbf{Rf_L}$). So, $\Phi \models_{\mathbf{L}} \varphi$ ($\Phi \not\models_{\mathbf{L}} \varphi$) by $\mathbf{Pr_L}$ ($\mathbf{Rf_L}$) soundness.

2. Assume that $\Phi \models_{\mathbf{L}} \varphi$ ($\Phi \not\models_{\mathbf{L}} \varphi$). Then, $\vdash_{\mathbf{Pr_L}} \Phi \vdash \varphi$ ($\vdash_{\mathbf{Rf_L}} \Phi \dashv \varphi$) by $\mathbf{Pr_L}$ ($\mathbf{Rf_L}$) completeness. So, $\vdash_{\mathbf{H}} \Phi \vdash \varphi$ ($\vdash_{\mathbf{H}} \Phi \dashv \varphi$). ∎

REMARK 11.2. Let **PRF** (**REF**) be the purely deductive (refutational) system consisting of all deduction rules (refutation rules) $\frac{\emptyset}{\mathsf{s}}$, where $\mathsf{s} \in \mathbf{AxPrf}$ ($\mathsf{s} \in \mathbf{AxRef}$) and all deduction (refutation) rules in **Prf** (**Ref**). Also $\rhd$ is replaced with $\vdash$ ($\dashv$). Then **PRF** (**REF**) is deductively (refutationally) sound and complete for **KN4**. Hence, $\mathbf{PRF} \cup \mathbf{REF}$ is both deductively and refutationally sound and complete for **KN4**.

12. Further Research

12.1. Intransitive Paraconsistent Modal Logics

Our procedure for **KN4.D** results from that for **KN4** by simple modifications. However, it is not clear how to extend our results to other logics of this kind. The rule rLR has to be modified somehow, but in a specific logic we would have a specific modification. (Note that the rule rLR is not sound in any proper extension of **KN4** because it enables refuting every sequent nonvalid in **KN4**.) Since the general form of the modal refutation rule is not known, it is also hard to generalize the construction of countermodels from refutation trees. I leave such procedures for intransitive paraconsistent modal logics as an open problem.

12.2. Transitive Paraconsistent Modal Logics

Refutation calculi that are both *reverse substitution*-free and *reverse modus ponens*-free for standard transitive modal logics are known in the literature (see e.g. [13,15,17]). In such a logic **L**, for every formula φ, its normal form $N(\varphi)$ (based on the Mints normal form [11]) can be constructed with the property that: $\varphi \to N(\varphi) \in \mathbf{L}$, and if $N(\varphi) \in \mathbf{L}$ then $\varphi \in \mathbf{L}$. The refutation calculus is for normal forms of this kind, and it provides both a decision procedure and the characterization of **L** by the class of tree-type frames. It seems posssible to obtain similar results for transitive paraconsistent modal logics by first modifying the normal forms, and second by modifying the refutation rules (and the modifications are non-trivial). This topic is studied in [16].

References

[1] Anderson, A., and N. Belnap, *Entailment: The Logic of Relevance and Entailment*, vol. I, Princeton, 1975.

[2] Anderson, A., N. Belnap, and M. Dunn, *Entailment: The Logic of Relevance and Entailment*, vol. II, Princeton, 1992.

[3] Brady, R., Completeness proofs for the systems RM3 and BN4, *Logique et Analyse* 25: 9–32, 1982.

[4] Drobyshevich, S., and H. Wansing, Proof systems for various FDE-based modal logics, *The Review of Symbolic Logic* 13: 729–747, 2020.

[5] Goble, L., Paraconsistent modal logic, *Logique et Analyse* 49: 3–29, 2006.

[6] Goranko, V., Hybrid deduction–refutation systems, *Axioms* 2019. https://doi.org/10.3390/axioms8040118.

[7] Goranko, V., G. Pulcini, and T. Skura, Refutation systems: an overview and some applications to philosophical logics, in F. Liu, H. Ono, and J. Yu, (eds.), *Knowledge, Proofs and Dynamics*, Springer, 2020, pp. 173–197.

[8] Goré, R., Tableau methods for modal and temporal logics, in M. D. Agostino, D. Gabbay, R. Hähnle, and J. Posega, (eds.), *Handbook of Tableau Methods*, Kluwer, 1999, pp. 297–396.

[9] Hughes, G. G., and M. J. Cresswell, *A Companion to Modal Logic*, Methuen, London, 1984.

[10] Łukasiewicz, J., *Aristotle's Syllogistic from the Standpoint of Modern Formal Logic*, Clarendon, Oxford, 1951.

[11] Mints, G., Gentzen-type systems and resolution rules, *Lecture Notes in Computer Science* 417: 198–231, 1990.

[12] Moore, E., Hybrid deduction–refutation systems for FDE-based logics. *Australian Journal of Logic* 18: 599–615, 2021.

[13] Skura, T., Refutations, proofs, and models in the modal logic K4, *Studia Logica* 70: 193–204, 2002.

[14] Skura, T., Refutation systems in propositional logic, in D. M. Gabbay, and F. Guenthner, (eds.), *Handbook of Philosophical Logic*, vol. 16, Springer, 2011, pp. 115–157.

[15] Skura, T., *Refutation Methods in Modal Propositional Logic*, Semper, Warszawa, 2013.

[16] Skura, T., Refutation systems for transitive paraconsistent modal logics, *In preparation* 2024.

[17] Skura, T., and R. Goré, Invertible refutation rules for S4 and extensions. *In preparation* 2024.

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