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Intuitionistic Public Announcement Logic with Distributed Knowledge

Abstract. We develop intuitionistic public announcement logic over intuitionistic **K**, **KT**, **K4**, and **S4** with distributed knowledge. We reveal that a recursion axiom for the distributed knowledge is *not* valid for a frame class discussed in [12] but valid for the restricted frame class introduced in [20,26]. The semantic completeness of the static logics for this restricted frame class is established via the concept of pseudo-model.

Keywords: Intuitionistic epistemic logic, Public announcement logic, Distributed knowledge.

1. Introduction

Public announcement logic [15], by which one can express change of agents' knowledge caused by truthful public announcement of a formula, is an expansion of epistemic logic over classical logic (cf. [5]). It is the simplest dynamic epistemic logic [22]. Since [15], many expansions and variants have been studied over classical logic. Distributed knowledge is one of notions of group knowledge, which represents a kind of aggregation of pieces of knowledge owned by each member of a group. Public announcement logic has been expanded with distributed knowledge in [7,8,24] over classical logic. Especially, [24] can be seen as one of bases of our work. It develops a public announcement logic with distributed knowledge $\mathcal{PAD}$ and a public announcement logic with distributed knowledge and common knowledge $\mathcal{PACD}$, which are based on **S5** epistemic logic, establishes the completeness of the logics, and examines the expressivity and computational complexity of the logics.

Public announcement logic based on intuitionistic logic is also studied, e.g. in [3,10,13]. According to [10], intuitionistic public announcement logic can be useful when we deal with change of *constructive* knowledge. They expand intuitionistic modal logic **IK** [18,19] and **MIPC** [16] with public

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announcement operators. In particular, the syntax of intuitionistic modal logic **IK** [18,19] has both $\Box$ and $\Diamond$, whose semantics are provided as follows:

$$M, w \Vdash \Box\varphi \text{ iff for all } v \in W, \text{ if } (w, v) \in (\leqslant; R) \text{ then } M, v \Vdash \varphi,$$
$$M, w \Vdash \Diamond\varphi \text{ iff for some } v \in W, (w, v) \in R \text{ and } M, v \Vdash \varphi,$$

where $R_1; R_2 := \{(x, z) \mid \text{there exists } y \text{ such that } xR_1y \text{ and } yR_2z\}$ and $M := (W, \leqslant, R, V)$ is an intuitionistic Kripke model $(W, \leqslant, V)$ equipped with a binary relation R on W such that $R; \leqslant \subseteq \leqslant; R$ and $R^{-1}; \leqslant \subseteq \leqslant; R^{-1}$ (R^{-1} is the converse of R). Ma et al. [10] provide an algebraic semantics with intuitionistic public announcement logic based on **IK** [18,19] or **MIPC** [16]. Nomura et al. [13] and Balbiani and Galmiche [3] studied intuitionistic public announcement logic based on **IK** in terms of relational semantics explained as above.

There are also a few intuitionistic epistemic logics with distributed knowledge [9,12,21]. While there is no attempt to study epistemic logic with distributed knowledge based on the above semantics given in [3,10,13], Jäger and Marti [9] develop intuitionistic epistemic logic based on **K** and **KT** with distributed knowledge, and prove its semantic completeness. It is noted that the syntax in [9] expands the syntax of intuitionistic logic with knowledge operators $K_a\varphi$ (where $a \in \mathsf{Agt}$ and Agt is a finite non-empty set of agents) and distributed knowledge operator $D\varphi$ and it does *not* contain the dual (i.e., the corresponding diamond operator) of knowledge operator or distributed knowledge operator. It is also noted that a model in [9] is an intuitionistic Kripke model $(W, \leqslant, V)$ equipped with a family $(R_a)_{a\in\mathsf{Agt}}$ of binary relations R_a such that $\leqslant; R_a \subseteq R_a$, where the semantics of $K_a\varphi$ and $D\varphi$ are defined as follows:

$$M, w \Vdash K_a\varphi \text{ iff for all } v \in W, \text{ if } (w, v) \in R_a \text{ then } M, v \Vdash \varphi,$$
$$M, w \Vdash D\varphi \text{ iff for all } v \in W, \text{ if } (w, v) \in \textstyle\bigcap_{a\in\mathsf{Agt}} R_a \text{ then } M, v \Vdash \varphi.$$

In the semantics, the imposed condition $\leqslant; R_a \subseteq R_a$ assures the heredity or monotonicity condition (once a formula holds at a state w, the formula continues to hold at all $\leqslant$-successor states of w) which characterizes intuitionistic logic. Finally, Jäger and Marti [9] proved the semantic completeness of their logic with distributed knowledge operator D in terms of the notion of *strict extension* [9, Definition 4.4] and did not employ "tree unraveling" technique, which is common for proving semantic completeness of *classical* epistemic logic with distributed knowledge (cf. [2,5,25]).

Murai and Sano [12] generalize the logic in [9], in that distributed knowledge operator is parameterized by a group G, i.e., a *non-empty* subset of whole agents Agt, while [9] deals with only distributed knowledge for the whole agents. Based on the same notion of model as in [9], the semantics of $D_G\varphi$ is defined as follows:

$$M, w \Vdash D_G\varphi \text{ iff for all } v \in W, \text{ if } (w, v) \in \bigcap_{a\in G} R_a \text{ then } M, v \Vdash \varphi.$$

Also, not only the axioms (K) and (T) but also (4) and (D) are introduced. Note that the axiom (D) is restricted to a single agent (i.e., $\neg D_{\{a\}}\bot$), because seriality for each R_a is generally not preserved under taking intersection among a group (refer to [2] over a setting over classical logic), while reflexivity and transitivity are always preserved. In [12], the outline of semantic completeness is provided via the notion of pseudo-model and the "tree unraveling" technique, and the corresponding sequent calculi are investigated.

In this paper, we modify the semantics provided in [12] of intuitionistic epistemic logic with distributed knowledge, and then develop intuitionistic public announcement logic with distributed knowledge, by expanding the logics in [12] except the ones having the axiom (D) with a public announcement operator. Our contribution is twofold. First, we show that condition of "stability" is necessary for our public announcement logic to be sound. We prove that a recursion axiom for the distributed knowledge is *not* sound for the class of all frames satisfying the above condition $\leqslant; R_a \subseteq R_a$. Moreover, we show that the soundness of the intuitionistic public announcement logic holds if the following frame condition is imposed: $\leqslant; R_a; \leqslant \subseteq R_a$ (which is equivalent to $R_a; \leqslant \subseteq R_a$ over the class of all frames). Following [17,20], we call such a frame a *stable* frame. It is remarked that the same condition has been also studied in [26], where such a frame is called a $\Box$-*frame*. The reader may wonder about the meaning of the frame condition $\leqslant; R_a; \leqslant \subseteq R_a$. It is known that binary relations on a set W can be identified with the union-preserving functions on $\mathcal{P}(W)$. When W is equipped with a pre-order $\leqslant$, the set of all upsets on W corresponds to binary relations R on W which satisfies $\leqslant; R; \leqslant \subseteq R$ (cf. [20, p.502]), where X is an *upset* (with respect to $\leqslant$) if $x \in X$ and $x \leqslant y$ imply $y \in X$ for every $x, y \in W$.

Second, we show the strong completeness of the static logic with respect to a class of *stable* frames. To show the strong completeness proof of our public announcement logic with respect to a class of stable frames by the standard method using recursion axioms (described in, e.g., [22]), the strong completeness of the static logic (i.e. the logic presented in [12]) with respect to a class of *stable* frames should be proved, not the strong completeness with respect to a class of *arbitrary* frames, which was proved in [12]. We provide a full-detailed proof of the strong completeness with respect to *stable* models.

The proof differs from the one in [12] in that a pseudo-model obtained by "tree unraveling" is stable in this paper, while it is generally not in [12].

The paper is organized as follows. In Section 2, we introduce syntax and semantics for intuitionistic epistemic logic with distributed knowledge to be made dynamic and the notion of stability of Kripke frame. Section 3 defines Hilbert systems of the logics, and state soundness results. In Section 4, strong completeness of the Hilbert systems of the logics with respect to the suitable classes of stable frames is shown, via a notion of "pseudo-model". In Section 5, we expand the intuitionistic epistemic logic with recursion axioms for a public announcement operator, and prove its completeness by reducing to the completeness of static one via translation from a formula possibly with public announcement operators to a formula without any public announcement operators. Section 6 concludes the paper.

2. Syntax and Semantics of Intuitionistic Epistemic Logics with Distributed Knowledge Operators

We denote a finite set of agents by Agt. We call a *nonempty* subset of Agt a "group" and denote it by G, H, etc. We denote by Grp the set of all groups, i.e., the set of all non-empty subsets of Agt. Let Prop be a countably infinite set of propositional variables and Form be the set of all formulas defined inductively as:

$$\mathsf{Form} \ni \varphi ::= p \mid \bot \mid (\varphi \to \varphi) \mid (\varphi \vee \varphi) \mid (\varphi \wedge \varphi) \mid D_G\varphi,$$

where $p \in \mathsf{Prop}$ and $G \in \mathsf{Grp}$. We will follow the standard rules for elimination of parentheses. We read $D_G\varphi$ as "φ is distributed knowledge among a group G". We define $\neg\varphi$ as $\varphi \to \bot$, $\varphi \leftrightarrow \psi$ as $(\varphi \to \psi) \wedge (\psi \to \varphi)$, and the epistemic operator $K_a\varphi$ (read "agent a knows that φ") as $D_{\{a\}}\varphi$. For a set of formulas Γ, let $D_G^{-1}\Gamma := \{\varphi \in \mathsf{Form} \mid D_G\varphi \in \Gamma\}$. As noted above, an expression of the form $D_\varnothing\varphi$ is *not* a well-formed formula, since $\varnothing$ is *not* a group. It is noted that the syntax given in [11] coincides with the fragment of our syntax with distributed knowledge operators $\{D_{\{a\}} \mid a \in \mathsf{Agt}\} \cup \{D_{\mathsf{Agt}}\}$ alone. Finally, unlike [3,6,19], it is also remarked that this paper will not consider the dual of $D_G\varphi$, i.e., the corresponding diamond operator to D_G.

We introduce Kripke semantics for intuitionistic multi-agent epistemic logic with distributed knowledge, along the lines of [9].

Definition 2.1. (Frame and Model) A tuple $F = (W, \leqslant, (R_a)_{a\in\mathsf{Agt}})$ is a *frame* if: W is a non-empty set of states; $\leqslant$ is a preorder on W; $(R_a)_{a\in\mathsf{Agt}}$ is a family of binary relations on W, indexed by agents; and $\leqslant ; R_a \subseteq R_a$

(for all $a \in \mathsf{Agt}$), where $R_1; R_2 := \{(x, z) \mid \text{there exists } y \text{ such that } xR_1y \text{ and } yR_2z\}$. A pair $M = (F, V)$ is a *model* if F is a frame, and a valuation function $V\colon \mathsf{Prop} \to \mathcal{P}(W)$ satisfies the heredity condition, i.e., if $w \in V(p)$ and $w \leqslant v$, then $v \in V(p)$. We denote an underlying set of states of a frame F or a model M by $|F|$ or $|M|$. For a model $M = (W, \leqslant, (R_a)_{a\in\mathsf{Agt}}, V)$ and a state $w \in W$, a pair (M, w) is called a *pointed model*.

The following notion of stability is needed in intuitionistic public announcement logic with distributed knowledge introduced later.

Definition 2.2. (Stable frame) Let $\leq$ be a preorder on a set X. A relation $R \subseteq X \times X$ is *stable* if $\leqslant; R \subseteq R$ and $R; \leqslant \subseteq R$. A frame $F = (W, \leqslant, (R_a)_{a\in\mathsf{Agt}})$ is *stable* if each $R_a \subseteq W \times W$ is stable. We denote by $\mathbb{ST}$ the class of all stable frames. A model $M = (F, V)$ is *stable* if the underlying frame F is stable.

When $\leqslant$ is the identity relation of W (i.e., $\leqslant := \{(w, w) \mid w \in W\}$), it is easy to see that every binary relation R on W is stable.

Proposition 2.3. *Let $R \subseteq X \times X$ and $\leq$ be a preorder on X. 1. R is stable iff $\leq; R; \leq \subseteq R$. 2. If R is reflexive and transitive, then $\leq; R \subseteq R$ implies $R; \leq \subseteq R$.*

Proof. (item 1) Left-to-right: By the condition $\leq; R \subseteq R$ of stability, we have $(\leq; R); \leq \subseteq R; \leq$. Then, together with the condition $R; \leq \subseteq R$ of stability, we have $\leq; R; \leq \subseteq R$. Right-to-left: We show $\leq; R \subseteq R$ and $R; \leq \subseteq R$. It is evident from reflexivity of $\leq$ that $\leq; R \subseteq \leq; R; \leq \subseteq R$ and that $R; \leq \subseteq \leq; R; \leq \subseteq R$. (item 2) By reflexivity and transitivity of R, we have $R; \leq \subseteq R; \leq; R \subseteq R; R \subseteq R$. ■

Satisfaction relation $M, w \Vdash \varphi$ between pointed models and formulas is defined recursively as follows:

$M, w \Vdash p$	iff $w \in V(p)$,
$M, w \Vdash \bot$	Never,
$M, w \Vdash \varphi \to \psi$	iff for all $v \in W$, if $w \leqslant v$ then $M, v \nVdash \varphi$ or $M, v \Vdash \psi$,
$M, w \Vdash \varphi \wedge \psi$	iff $M, w \Vdash \varphi$ and $M, w \Vdash \psi$,
$M, w \Vdash \varphi \vee \psi$	iff $M, w \Vdash \varphi$ or $M, w \Vdash \psi$,
$M, w \Vdash D_G\varphi$	iff for all $v \in W$, if $(w, v) \in \bigcap_{a\in G} R_a$ then $M, v \Vdash \varphi$.

It is noted from our definition of $K_a\varphi := D_{\{a\}}\varphi$ that the satisfaction of $K_a\varphi$ at a state w of a model M is given as follows:

$$M, w \Vdash K_a\varphi \text{ iff for all } v \in W, \text{ if } (w, v) \in R_a \text{ then } M, v \Vdash \varphi.$$

We have the following heredity property for all formulas.

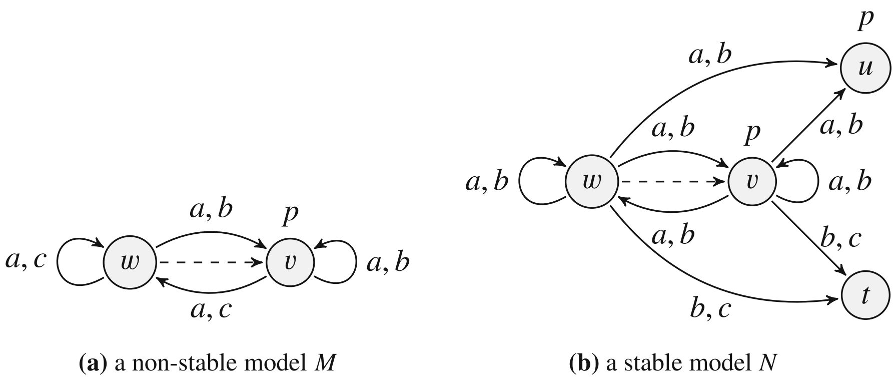

Figure 1. Examples of models

Proposition 2.4. (*Heredity*) *If* $M, w \Vdash \varphi$ *and* $w \leqslant v$*, then* $M, v \Vdash \varphi$*.*

Proof. By induction on φ. For the case where $\varphi := D_G\psi$, it is noted that the condition $\leqslant; R_a \subseteq R_a$ of a frame implies that $\leqslant; \bigcap_{a\in G} R_a \subseteq \bigcap_{a\in G} R_a$. ■

Example 2.5. Fig. 1 a is an example of a non-stable model. The preorder is depicted by a dotted arrow. Note that we omit reflexive arrows for the preorder. It is not stable because $w(\leqslant; R_c; \leqslant)v$ holds but wR_cv fails (if c-arrows were added from w to v and from v to v, then it would be stable). The valuation V of M is defined by $V(p) = \{v\}$ for $\mathsf{Prop} := \{p\}$. It is easy to see that the valuation V satisfies the heredity condition. In this model, it can be seen that different groups have different distributed knowledge even at the same state. Indeed, $D_{\{a,b\}}p$ is true at w, but $D_{\{a,c\}}p$ is false at w. We can see that seriality for each agent's relation is not always preserved under taking intersection among a group. Namely, R_b and R_c are serial but $R_b \cap R_c$ is not in the example. This is why we should restrict (D) axiom to $\neg D_{\{a\}}\bot$, as defined in Table 1 below. Figure 1b is an example of a stable model named N used later in the explanation of PAL extension. This model is stable, because $R_x; \leqslant \subseteq R_x$ holds for all $x \in \mathsf{Agt}$, in particular, it suffices to take care of the non-reflexive $\leqslant$-arrow from w and v and then it is easy to see that $wR_xw \leqslant v$ and wR_xv hold for $x \in \{a, b\}$. From this model too, it can be seen that different groups have different distributed knowledge even at the same state. Indeed, $D_{\{b,c\}}\neg p$ is true at v, but $D_{\{a,b\}}\neg p$ is false at v.

Definition 2.6. (Validity) Given a frame $F = (W, \leqslant, (R_a)_{a\in\mathsf{Agt}})$, we say that a formula φ is *valid* in F (notation: $F \Vdash \varphi$) if $(F, V), w \Vdash \varphi$ for every valuation function V and every $w \in W$. A formula φ is valid in a class $\mathbb{F}$ of frames (notation: $\mathbb{F} \Vdash \varphi$) if $F \Vdash \varphi$ for every $F \in \mathbb{F}$. A formula φ is a *semantic consequence* of Γ in a frame class $\mathbb{F}$ if for all frames $F \in \mathbb{F}$, for all valuations V on F, for all states $w \in |F|$, if $(F, V), w \Vdash \Gamma$, then $(F, V), w \Vdash \varphi$. We write it as "$\Gamma \Vdash_{\mathbb{F}} \varphi$".

3. Hilbert Systems

Hilbert systems for intuitionistic epistemic logics with D_G operators are given in Table 1. A Hilbert system $\mathsf{H}(\mathbf{IntK})$ consists of axioms and rules for intuitionistic logic, axioms (Incl) and (K), and a rule (Nec). Axioms (Incl) and (K) and a rule (Nec) come from Hilbert system for epistemic logic over classical logic with D_G- operators (cf. [5]). Hilbert systems $\mathsf{H}(\mathbf{IntKT})$, $\mathsf{H}(\mathbf{IntKD})$, $\mathsf{H}(\mathbf{IntK4})$, $\mathsf{H}(\mathbf{IntK4D})$, and $\mathsf{H}(\mathbf{IntS4})$ are defined as axiomatic expansions of $\mathsf{H}(\mathbf{IntK})$ with (T), (D), (4), (4) and (D), and (T) and (4), respectively. If we focus on the fragment of our syntax with the following distributed knowledge operators $\{D_{\{a\}} \mid a \in \mathsf{Agt}\} \cup \{D_{\mathsf{Agt}}\}$ alone, the resulting restricted axiomatizations of $\mathsf{H}(\mathbf{IntK})$ and $\mathsf{H}(\mathbf{IntKT})$ coincides with axiomatizations **IDK** and **IDT** given in [9], respectively. As we have seen in Example 2.5, seriality for each agent's relation is not always preserved under taking intersection among a group and so we cannot generalize the axiom (D) $\neg D_{\{a\}}\bot$ to an axiom of the form $\neg D_G\bot$. A similar phenomenon over classical logic is already observed in [25, p.266]. Let **X** be any of **IntK**, **IntKT**, **IntKD**, **IntK4**, **IntK4D**, and **IntS4** in what follows. The notion

Table 1. Axioms and rules for Hilbert systems

Axioms and rules for intuitionistic logic			
$(\mathbf{k})$	$\varphi \to (\psi \to \varphi)$	$(\wedge\mathbf{e}_1)$	$(\varphi \wedge \psi) \to \varphi$
$(\mathbf{s})$	$(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$	$(\wedge\mathbf{e}_2)$	$(\varphi \wedge \psi) \to \psi$
$(\vee\mathbf{i}_1)$	$\varphi \to (\varphi \vee \psi)$	$(\wedge\mathbf{i})$	$\varphi \to (\psi \to (\varphi \wedge \psi))$
$(\vee\mathbf{i}_2)$	$\psi \to (\varphi \vee \psi)$	$(\bot)$	$\bot \to \varphi$
$(\vee\mathbf{e})$	$(\varphi \to \chi) \to ((\psi \to \chi) \to ((\varphi \vee \psi) \to \chi))$	(MP)	From φ and $\varphi \to \psi$, infer ψ
Axioms and rules for $\mathsf{H}(\mathbf{IntK})$			
(Incl)	$D_G\varphi \to D_H\varphi$ $(G \subseteq H)$	(K)	$D_G(\varphi \to \psi) \to (D_G\varphi \to D_G\psi)$
(Nec)	From φ, infer $D_G\varphi$		
Additional axioms for D_G operators			
(T) $D_G\varphi \to \varphi$	(D) $\neg D_{\{a\}}\bot$	(4)	$D_G\varphi \to D_G D_G\varphi$

of provability in each system is defined as usual, and the fact that a formula φ is provable in $\mathsf{H}(\mathbf{X})$ is denoted by "$\vdash_{\mathsf{H}(\mathbf{X})} \varphi$".

Definition 3.1. A formula φ is *derivable* from Γ in a logic $\mathbf{X}$ if $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge \Gamma' \rightarrow \varphi$ for some finite set Γ' which is a subset of Γ. We write it as "$\Gamma \vdash_{\mathsf{H}(\mathbf{X})} \varphi$".

Definition 3.2. The class $\mathbb{F}(\mathbf{X})$ of frames corresponding to $\mathbf{X}$ is defined as follows.

- $\mathbb{F}(\mathbf{IntK})$ is the class of all frames.
- $\mathbb{F}(\mathbf{IntKT})$ is the class of all frames such that R_a is reflexive ($a \in \mathsf{Agt}$).
- $\mathbb{F}(\mathbf{IntKD})$ is the class of all frames such that R_a is serial ($a \in \mathsf{Agt}$).
- $\mathbb{F}(\mathbf{IntK4})$ is the class of all frames such that R_a is transitive ($a \in \mathsf{Agt}$).
- $\mathbb{F}(\mathbf{IntK4D})$ is the class of all frames such that R_a is transitive and serial ($a \in \mathsf{Agt}$).
- $\mathbb{F}(\mathbf{IntS4})$ is the class of all frames such that R_a is reflexive and transitive ($a \in \mathsf{Agt}$).

Here, reflexivity, seriality, and transitivity are defined ordinarily.

Theorem 3.3. *If $\vdash_{\mathsf{H}(\mathbf{X})} \varphi$, then $\mathbb{F}(\mathbf{X}) \Vdash \varphi$. Moreover, $\vdash_{\mathsf{H}(\mathbf{X})} \varphi$ implies $\mathbb{F}(\mathbf{X}) \cap \mathbb{ST} \Vdash \varphi$.*

Proof. The latter is an obvious consequence from the former. We can prove the former by induction on φ. Note that axioms (T) and (4) are valid in reflexive and transitive frames, respectively, because if R_a is reflexive or transitive for any $a \in G$, $\bigcap_{a \in G} R_a$ is also reflexive or transitive, respectively. ■

4. Completeness

In the present section, we provide a proof of strong completeness theorems of our logic with respect to stable models.

Theorem 4.1. *Let $\mathbf{X}$ be any of* **IntK**, **IntKT**, **IntKD**, **IntK4**, **IntK4D**, *and* **IntS4** *and $\Gamma \cup \{\varphi\}$ be a set of formulas. If $\Gamma \Vdash_{\mathbb{F}(\mathbf{X}) \cap \mathbb{ST}} \varphi$, then $\Gamma \vdash_{\mathsf{H}(\mathbf{X})} \varphi$.*

Following the spirit of [4], we show Theorem 4.1 via "pseudo-models". In order to show Theorem 4.1, we first construct a canonical stable pseudo-model (at Section 4.1), and then transform it into a stable pseudo-model with the intersection condition by a "tree unraveling" method (at Subsection 4.2).

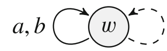

Figure 2. A pseudo-frame

DEFINITION 4.2. A tuple $F = (W, \leqslant, (R_G)_{G \in \mathsf{Grp}})$ is a *pseudo-frame* if: (1)$\leqslant ; R_G \subseteq R_G$ for any $G \in \mathsf{Grp}$; and (2) (the inclusion condition) $R_H \subseteq R_G$ if $G \subseteq H$. A pair $M = (F, V)$ is a *pseudo-model* if F is a pseudo-frame, and a valuation function $V \colon \mathsf{Prop} \to \mathcal{P}(W)$ satisfies the heredity condition, i.e., if $w \in V(p)$ and $w \leqslant v$, then $v \in V(p)$. Let us say that a pseudo-frame $F = (W, \leqslant, (R_G)_{G \in \mathsf{Grp}})$ or -model $M = (W, \leqslant, (R_G)_{G \in \mathsf{Grp}}, V)$ is *stable* if $\leqslant; R_G; \leqslant \subseteq R_G$ holds for any $G \in \mathsf{Grp}$. For a pseudo-model M, a state $w \in |M|$, and a formula φ, a *pseudo-satisfaction relation* $M, w \Vdash^{\mathsf{ps}} \varphi$ is defined the same as the satisfaction relation $\Vdash$, except for the clause for $D_G\varphi$: that is,

$$M, w \Vdash^{\mathsf{ps}} D_G\varphi \text{ iff for all } v \in W, \text{ if } (w, v) \in R_G \text{ then } M, v \Vdash^{\mathsf{ps}} \varphi.$$

Namely, in a pseudo-model, an operator D_G is treated like a primitive box operator, parameterized by a group. Considering the definition of satisfaction relation for $D_G\varphi$, a pseudo-frame can be seen as a frame in the sense of Definition 2.1 if the following "intersection condition" is satisfied.

DEFINITION 4.3. A pseudo-frame $F = (W, \leqslant, (R_G)_{G \in \mathsf{Grp}})$ satisfies the *intersection condition* if $R_G = \bigcap_{a \in G} R_{\{a\}}$ for any group G.

The intersection condition is not always satisfied as in the following.

EXAMPLE 4.4. Figure 2 is an example of a pseudo-frame. We name it F_{ex}. Note that $\{a\}$ is written as "a" and $R_{\{a,b\}}$ is defined as $\emptyset$ here. Since $R_{\{a,b\}} = \emptyset$, the condition of "$R_H \subseteq R_G$ if $G \subseteq H$" is self-evidently satisfied, i.e., $R_{\{a,b\}} \subseteq R_{\{a\}}$ and $R_{\{a,b\}} \subseteq R_{\{b\}}$. Note that the intersection condition is false for a group $\{a, b\}$, because $R_{\{a\}} \cap R_{\{b\}} \not\subseteq R_{\{a,b\}}$. Any frame can be regarded as a pseudo-frame with only relations for singleton groups, as in F_{ex}.

The following provides a characterization of the intersection condition.

PROPOSITION 4.5. *A pseudo-frame $F = (W, \leqslant, (R_G)_{G \in \mathsf{Grp}})$ satisfies the intersection condition iff $R_{G_1} \cap R_{G_2} \subseteq R_{G_1 \cup G_2}$ for all groups G_1, G_2.*

PROOF. Because the left-to-right direction is immediate, we only prove the right-to-left direction. Suppose that $R_{G_1} \cap R_{G_2} \subseteq R_{G_1 \cup G_2}$ for all groups G_1, G_2. By the inclusion condition, it suffices to prove $\bigcap_{a \in G} R_{\{a\}} \subseteq R_G$ for any group G. But, this follows from our supposition by induction on the cardinality of G. ∎

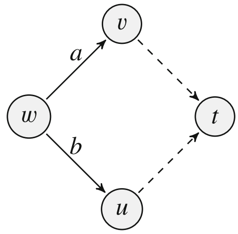

Figure 3. Pseudo-model M

The reader may wonder if we can construct a stable pseudo-frame or -model from a possibly non-stable pseudo-frame or -model (the remaining description of this section is of independent interest and so the reader who is interested in the completeness proof can skip it to move to the next section). The following operation called *stablization* can turn a pseudo-model into a stable one without changing satisfaction on any state.

DEFINITION 4.6. Let $M = (W, \leqslant, (R_G)_{G\in\mathsf{Grp}}, V)$ be a pseudo-model. We say that a stable pseudo-model $M^{\mathsf{st}} := (W, \leqslant, (R_G; \leqslant)_{G\in\mathsf{Grp}}, V)$ is *the stabilization of* M.

PROPOSITION 4.7. *Let* $M = (W, \leqslant, (R_G)_{G\in\mathsf{Grp}}, V)$ *be a pseudo-model. Then,* M^{st} is *a stable pseudo-model. Moreover,* $M, w \Vdash \varphi$ *iff* $M^{\mathsf{st}}, w \Vdash \varphi$ *for any* $\varphi \in \mathsf{Form}$ *and* $w \in W$.

PROOF. First, we show that M^{st} is a stable pseudo-model. We show $\leqslant ; (R_G; \leqslant); \leqslant \subseteq R_G; \leqslant$. Using $\leqslant; R_G \subseteq R_G$ and the transitivity of $\leqslant$, we have $\leqslant; R_G; \leqslant; \leqslant \subseteq R_G; \leqslant; \leqslant \subseteq R_G; \leqslant$. Further, if $G \subseteq H$, we have $R_H; \leqslant \subseteq R_G; \leqslant$ by $R_H \subseteq R_G$.

Next, we show the latter by induction on φ. Only the case $\varphi = D_G\psi$ is treated. (left-to-right) Assume that $M, w \Vdash D_G\psi$. We show that $M^{\mathsf{st}}, w \Vdash D_G\psi$. Fix v such that $w(R_G; \leqslant)v$. Then, there is u such that wR_Gu and $u \leqslant v$. From the former and the assumption, we have $M, u \Vdash \psi$. Then, by the latter and the heredity, $M, v \Vdash \psi$. (right-to-left) Obvious by $R_G \subseteq R_G; \leqslant$. ■

However, in general, the intersection condition $\left(\bigcap_{a\in G} R_{\{a\}} = R_G\right)$ is not preserved under the stabilization. That is, there is a pseudo-model such that it satisfies $\bigcap_{a\in G} R_{\{a\}} = R_G$ for any $G \in \mathsf{Grp}$ but it does not satisfy the following: $\bigcap_{a\in G}(R_{\{a\}}; \leqslant) = R_G; \leqslant$ for any $G \in \mathsf{Grp}$. Let $\mathsf{Agt} = \{a, b\}$. The pseudo-model M depicted in Figure 3 is such a model. The pseudo-model M is defined as $(W, \leqslant, R_{\{a\}}, R_{\{b\}}, R_{\{a,b\}}, V)$, where $W = \{w, v, u, t\}$, $\leqslant = \{(v, t), (u, t)\} \cup \{(x, x) \mid x \in W\}$, $R_{\{a\}} = \{(w, v)\}$, $R_{\{b\}} = \{(w, u)\}$, and $R_{\{a,b\}} = \emptyset$. The valuation V can be any. The solid line stands for the relations for groups and the dotted arrow stands for the preorder. Reflexive arrows for the preorder is omitted in the figure. We can easily see that $\bigcap_{a\in G} R_{\{a\}} = R_G$ for any $G \in \mathsf{Grp}$. However, $\bigcap_{c\in G}(R_{\{c\}};\leqslant) = R_G;\leqslant$ is not true when $G = \{a,b\}$, because $(w,t) \in \big(R_{\{a\}};\leqslant\big) \cap \big(R_{\{b\}};\leqslant\big)$ but $(w,t) \notin R_{\{a,b\}};\leqslant = \emptyset$.

As can be guessed from the counterexample, "$(R_{G_1};\leqslant) \cap (R_{G_2};\leqslant) \subseteq R_{G_1\cup G_2};\leqslant$" is a sufficient (and necessary) condition for the preservation of the intersection condition. The following is an immediate consequence of Proposition 4.5.

PROPOSITION 4.8. *Let M be a pseudo-model with $(R_{G_1};\leqslant) \cap (R_{G_2};\leqslant) \subseteq R_{G_1\cup G_2};\leqslant$ for any $G_1, G_2 \in \mathsf{Grp}$. Then, M^{st} enjoys the intersection condition.*

4.1. Canonical Pseudo-model

This section defines a canonical pseudo-model of our logics. Since D_G operators are interpreted as primitive box-like operators indexed by a group in a pseudo-model, a canonical pseudo-model defined here is almost the same as the canonical model of intuitionistic epistemic logics without distributed knowledge, which is described in detail, e.g., in [11, Chapter 1]. Unlike [11, Chapter 1], we assure that our canonical pseudo-model is *stable*, too. In what follows, let $\mathbf{X}$ be any of **IntK**, **IntKT**, **IntKD**, **IntK4**, **IntK4D**, and **IntS4**.

DEFINITION 4.9. A set Γ of formulas is $\mathbf{X}$-*consistent* if $\Gamma \nvdash_{\mathsf{H}(\mathbf{X})} \bot$. We say that the set Γ is *prime* if $\varphi_1 \vee \varphi_2 \in \Gamma$ implies $\varphi_1 \in \Gamma$ or $\varphi_2 \in \Gamma$. We also say that Γ is an $\mathbf{X}$-*theory* if $\Gamma \vdash_{\mathsf{H}(\mathbf{X})} \varphi$ implies $\varphi \in \Gamma$.

The following are useful properties of a consistent and prime theory.

LEMMA 4.10. *Let a set Γ of formulas be an $\mathbf{X}$-consistent and prime $\mathbf{X}$-theory.*

1. $\Gamma \vdash_{\mathsf{H}(\mathbf{X})} \varphi$ *iff* $\varphi \in \Gamma$.
2. *If* $\{\varphi, \varphi \to \psi\} \subseteq \Gamma$, *then* $\psi \in \Gamma$.
3. $\bot \notin \Gamma$.
4. $\varphi \wedge \psi \in \Gamma$ *iff* $\varphi \in \Gamma$ *and* $\psi \in \Gamma$.
5. $\varphi \vee \psi \in \Gamma$ *iff* $\varphi \in \Gamma$ *or* $\psi \in \Gamma$.
6. *If* $\varphi \to \psi \notin \Gamma$, *then* $\Gamma \cup \{\varphi\} \nvdash_{\mathsf{H}(\mathbf{X})} \psi$.
7. *If* $D_G\psi \notin \Gamma$, *then* $D_G^{-1}\Gamma \nvdash_{\mathsf{H}(\mathbf{X})} \psi$.

PROOF.

(1) The left-to-right is evident since Γ is $\mathbf{X}$-theory. We show the right-to-left. Suppose $\varphi \in \Gamma$. Then $\Gamma \vdash_{\mathsf{H}(\mathbf{X})} \varphi$ is the case because $\vdash_{\mathsf{H}(\mathbf{X})} \varphi \to \varphi$.

(2) Suppose $\{\varphi, \varphi \to \psi\} \subseteq \Gamma$. Then, by item 1., we have $\Gamma \vdash_{\mathsf{H}(\mathbf{X})} \varphi$ and $\Gamma \vdash_{\mathsf{H}(\mathbf{X})} \varphi \to \psi$, that is, $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge \Gamma_1 \to \varphi$ and $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge \Gamma_2 \to (\varphi \to \psi)$ for some finite sets $\Gamma_1, \Gamma_2 \subseteq \Gamma$. Then we have $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge(\Gamma_1 \cup \Gamma_2) \to \varphi$ and $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge(\Gamma_1 \cup \Gamma_2) \to (\varphi \to \psi)$, which jointly entail $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge(\Gamma_1 \cup \Gamma_2) \to (\varphi \wedge (\varphi \to \psi))$. Since we have $\vdash_{\mathsf{H}(\mathbf{X})} (\varphi \wedge (\varphi \to \psi)) \to \psi$ as an intuitionistic theorem, we obtain $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge(\Gamma_1 \cup \Gamma_2) \to \psi$, which means that $\Gamma \vdash_{\mathsf{H}(\mathbf{X})} \psi$. Since Γ is $\mathbf{X}$-theory, $\psi \in \Gamma$.

(3) Suppose $\bot \in \Gamma$ for contradiction. Then, by item 1., $\Gamma \vdash_{\mathsf{H}(\mathbf{X})} \bot$. However, this contradicts with $\mathbf{X}$-consistency of Γ.

(4) First, we show the left-to-right. Suppose $\varphi \wedge \psi \in \Gamma$. By item 1., it suffices to show $\Gamma \vdash_{\mathsf{H}(\mathbf{X})} \varphi$ and $\Gamma \vdash_{\mathsf{H}(\mathbf{X})} \psi$. It is the case that $\Gamma \vdash_{\mathsf{H}(\mathbf{X})} \varphi$, because we have $\vdash_{\mathsf{H}(\mathbf{X})} \varphi \wedge \psi \to \varphi$ as an axiom. $\Gamma \vdash_{\mathsf{H}(\mathbf{X})} \psi$ can be similarly shown. Next, we show the right-to-left. Suppose $\varphi \in \Gamma$ and $\psi \in \Gamma$. By item 1., it suffices to show $\Gamma \vdash_{\mathsf{H}(\mathbf{X})} \varphi \wedge \psi$, which is the case because we have $\vdash_{\mathsf{H}(\mathbf{X})} \varphi \wedge \psi \to \varphi \wedge \psi$ as an intuitionistic theorem.

(5) The left-to-right is the case by primeness of Γ. We show the right-to-left. Suppose $\varphi \in \Gamma$ or $\psi \in \Gamma$. By item 1., it suffices to show $\Gamma \vdash_{\mathsf{H}(\mathbf{X})} \varphi \vee \psi$. First, assume $\varphi \in \Gamma$. Then $\Gamma \vdash_{\mathsf{H}(\mathbf{X})} \varphi \vee \psi$ is the case because we have $\vdash_{\mathsf{H}(\mathbf{X})} \varphi \to \varphi \vee \psi$ as an axiom. Similarly, $\Gamma \vdash_{\mathsf{H}(\mathbf{X})} \varphi \vee \psi$ is the case by $\vdash_{\mathsf{H}(\mathbf{X})} \psi \to \varphi \vee \psi$ when assuming $\psi \in \Gamma$.

(6) We show the contraposition. Suppose $\Gamma \cup \{\varphi\} \vdash_{\mathsf{H}(\mathbf{X})} \psi$. Then, there exists a finite subset Γ' of Γ such that $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge \Gamma' \to \psi$. First, assume $\varphi \notin \Gamma'$. Then, it turns out that $\Gamma \vdash_{\mathsf{H}(\mathbf{X})} \psi$. Since we also have $\Gamma \vdash_{\mathsf{H}(\mathbf{X})} \psi \to (\varphi \to \psi)$, $\Gamma \vdash_{\mathsf{H}(\mathbf{X})} \varphi \to \psi$ is obtained. Then, $\varphi \to \psi \in \Gamma$ because Γ is an $\mathbf{X}$-theory. Next, assume $\varphi \in \Gamma'$. Put $\Delta := \Gamma' - \{\varphi\}$. Since $\vdash_{\mathsf{H}(\mathbf{X})} (\bigwedge \Gamma' \to \psi) \leftrightarrow (\bigwedge \Delta \to \varphi \to \psi)$, we have $\Gamma \vdash_{\mathsf{H}(\mathbf{X})} \varphi \to \psi$. Then, $\varphi \to \psi \in \Gamma$ because Γ is an $\mathbf{X}$-theory.

(7) We show the contraposition. $D_G^{-1}\Gamma \vdash_{\mathsf{H}(\mathbf{X})} \psi$. Then, there exists a finite subset Δ of $D_G^{-1}\Gamma$ such that $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge \Delta \to \psi$. By axiom (K), we have $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge D_G\Delta \to D_G\psi$, which means that $\Gamma \vdash_{\mathsf{H}(\mathbf{X})} D_G\psi$. Then, $D_G\psi \in \Gamma$ because Γ is an $\mathbf{X}$-theory.

■

Lemma 4.11. (*Lindenbaum, [11, Lemma 1.16]*) *Let* $\Gamma \cup \{\varphi\}$ *be a set of formulas. If* $\Gamma \nvdash_{\mathsf{H}(\mathbf{X})} \varphi$*, then there is an* $\mathbf{X}$*-consistent and prime* $\mathbf{X}$*-theory* Γ^+ *such that* $\Gamma \subseteq \Gamma^+$ *and* $\Gamma^+ \nvdash_{\mathsf{H}(\mathbf{X})} \varphi$*.*

Definition 4.12. A *canonical pseudo-model* $M^{\mathbf{X}} = (W^{\mathbf{X}}, \leqslant^{\mathbf{X}}, (R^{\mathbf{X}}_G)_{G \in \mathsf{Grp}}, V^{\mathbf{X}})$ is defined as follows:

- $W^{\mathbf{X}} := \{\Gamma \in \mathcal{P}(\mathsf{Form}) \mid \Gamma$ is an $\mathbf{X}$-consistent and prime $\mathbf{X}$-theory$\}$.
- $\Gamma \leqslant^{\mathbf{X}} \Delta$ iff $\Gamma \subseteq \Delta$.
- $\Gamma R^{\mathbf{X}}_G \Delta$ iff $D^{-1}_G \Gamma \subseteq \Delta$.
- $V^{\mathbf{X}}(p) := \{\Gamma \in W^{\mathbf{X}} \mid p \in \Gamma\}$.

Proposition 4.13. *The canonical model* $M^{\mathbf{X}}$ *of Definition 4.12* is *a stable pseudo-model.*

Proof. First, we show that $\leqslant^{\mathbf{X}}; R^{\mathbf{X}}_G; \leqslant^{\mathbf{X}} \subseteq R^{\mathbf{X}}_G$. Suppose that $\Gamma(\leqslant^{\mathbf{X}}; R^{\mathbf{X}}_G; \leqslant^{\mathbf{X}})\Delta$. Then, there are $\Theta_1, \Theta_2 \in W^{\mathbf{X}}$ such that $\Gamma \subseteq \Theta_1 R^{\mathbf{X}}_G \Theta_2 \subseteq \Delta$. By $\Gamma \subseteq \Theta_1$, we get $D^{-1}_G \Gamma \subseteq D^{-1}_G \Theta_1$. It follows from $D^{-1}_G \Theta_1 \subseteq \Theta_2$ that $D^{-1}_G \Gamma \subseteq \Theta_2$. Finally, it follows from $\Theta_2 \subseteq \Delta$ that $D^{-1}_G \Gamma \subseteq \Delta$, as required. Next, we show that $R^{\mathbf{X}}_H \subseteq R^{\mathbf{X}}_G$ if $G \subseteq H$. Assume that $\Gamma R^{\mathbf{X}}_H \Delta$ and $\varphi \in D^{-1}_G \Gamma$, i.e., $D_G \varphi \in \Gamma$. We show that $\varphi \in \Delta$. Since $D_G \varphi \to D_H \varphi$ is an axiom in any $\mathsf{H}(\mathbf{X})$ and hence $D_G \varphi \to D_H \varphi \in \Gamma$, we have $D_H \varphi \in \Gamma$ by item 2. of Lemma 4.10. Then, by $\Gamma R^{\mathbf{X}}_H \Delta$, $\varphi \in \Delta$. Finally, it is obvious that $V^{\mathbf{X}}$ satisfies the heredity condition. ∎

Lemma 4.14. (*Truth Lemma*) *For all formulas* φ *and* $\mathbf{X}$*-consistent and prime* $\mathbf{X}$*-theories* Γ*,* $\varphi \in \Gamma$ *if and only if* $M^{\mathbf{X}}, \Gamma \Vdash^{\mathsf{ps}} \varphi$*.*

Proof. By induction on φ. We show the case $\varphi = D_G \psi$. First, we show the left-to-right. Assume $D_G \psi \in \Gamma$ and fix any $\Delta \in W^{\mathbf{X}}$ such that $\Gamma R^{\mathbf{X}}_G \Delta$, i.e., $D^{-1}_G \Gamma \subseteq \Delta$. Clearly, $\psi \in \Delta$, and by the induction hypothesis, we have $M^{\mathbf{X}}, \Delta \Vdash^{\mathsf{ps}} \psi$. Next, We show the contraposition of the right-to-left. Assume $D_G \psi \notin \Gamma$. By item 7. of Lemma 4.10 and Lemma 4.11, there is an $\mathbf{X}$-consistent and prime $\mathbf{X}$-theory Δ such that $D^{-1}_G \Gamma \subseteq \Delta$ and $\Delta \nvdash_{\mathsf{H}(\mathbf{X})} \psi$. By item 1. of Lemma 4.10 and induction hypothesis, we have $M^{\mathbf{X}}, \Delta \nVdash^{\mathsf{ps}} \psi$, which shows $M^{\mathbf{X}}, \Gamma \nVdash^{\mathsf{ps}} D_G \psi$. ∎

For the axioms (T), (D), and (4), the canonical pseudo-model satisfies the corresponding property on relations for D_G.

Proposition 4.15. *1. If* $\mathbf{X}$ *has the axiom* (T)*,* $R^{\mathbf{X}}_G$ *is reflexive in* $M^{\mathbf{X}}$*. 2. If* $\mathbf{X}$ *has the axiom* (D)*,* $R^{\mathbf{X}}_{\{a\}}$ *is serial in* $M^{\mathbf{X}}$*. 3. If* $\mathbf{X}$ *has the axiom* (4)*,* $R^{\mathbf{X}}_G$ *is transitive in* $M^{\mathbf{X}}$*.*

PROOF. We only show item 2. Fix any $\mathbf{X}$-consistent and prime $\mathbf{X}$-theory Γ. The aim is to find an $\mathbf{X}$-consistent and prime $\mathbf{X}$-theory Δ such that $D^{-1}_{\{a\}}\Gamma \subseteq \Delta$. By Lemma 4.11, it suffices to show $D^{-1}_{\{a\}}\Gamma \nvdash_{\mathsf{H}(\mathbf{X})} \bot$. Assuming the contrary, we have $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge_{i=1}^{n}\varphi_i \to \bot$ for some $\varphi_1 \dots \varphi_n \in D^{-1}_{\{a\}}\Gamma$. By (Nec), (K), and intuitionistic propositional tautologies, $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge_{i=1}^{n} D_{\{a\}}\varphi_i \to D_{\{a\}}\bot$. Since $D_{\{a\}}\varphi_i \in \Gamma$, it means $\Gamma \vdash_{\mathsf{H}(\mathbf{X})} D_{\{a\}}\bot$. However, we also have $\Gamma \vdash_{\mathsf{H}(\mathbf{X})} \neg D_{\{a\}}\bot$ by the assumption, which leads to contradiction by items from 1. to 3. of Lemma 4.10. ■

4.2. Tree Unraveling

This section introduces a method called "tree unraveling", which transforms a stable pseudo-model into another stable pseudo-model satisfying the intersection condition $\bigcap_{a\in G} R_{\{a\}} = R_G$ (i.e., a model in the sense of Definition 2.1). Our definitions below are intuitionistic generalizations of definitions proposed in [4] over classical logic.

DEFINITION 4.16. Let $M = (W, \leqslant, (R_G)_{G\in\mathsf{Grp}}, V)$ be a pseudo-model. A pseudo-model $M' = (W', \leqslant \cap(W' \times W'), (R_G \cap (W' \times W'))_{G\in\mathsf{Grp}}, V')$ is a *generated submodel* of M if: $W' \subseteq W$; If $w \in W'$ and $w \leqslant w'$ then $w' \in W'$; If $w \in W'$ and wR_Gw' then $w' \in W'$; and $V'(p) = V(p) \cap W'$ for any $p \in \mathsf{Prop}$. For $X \subseteq |M|$, we define M_X as the smallest generated submodel containing X. If $M = M_X$, we say that M is generated by X.

PROPOSITION 4.17. *Let M be a pseudo-model and M' be a generated submodel of M. Then, for any formula φ and $w \in |M'|$, $M', w \Vdash^{\mathsf{ps}} \varphi$ iff $M, w \Vdash^{\mathsf{ps}} \varphi$.*

DEFINITION 4.18. Let $M = (F, V)$ be a pseudo-model generated by $w \in W$, where $F = (W, \leqslant, (R_G)_{G\in\mathsf{Grp}})$.

- We put $w_0 := w$ and define Finpath(F, w) as

$$\{\langle w_0, L_1, w_1, L_2, \cdots, L_n, w_n\rangle \mid n \geq 0, L_i \in \{\leqslant, R_G\}_{G\in\mathsf{Grp}}, w_{i-1}L_iw_i \text{ for all } 1 \leq i \leq n\}.$$

  An element of Finpath(F, w) is *a path (from w)* and denote it by $\overrightarrow{u}$, $\overrightarrow{v}$, etc.

- For $\overrightarrow{u} = \langle w_0, L_1, w_1, L_2, \cdots, L_{n-1}, w_{n-1}, L_n, w_n\rangle \in$ Finpath(F, w),

$$\mathsf{body}(\overrightarrow{u}) := \langle w_0, L_1, w_1, L_2, \cdots, L_{n-1}, w_{n-1}\rangle \text{ and } \mathsf{tail}(\overrightarrow{u}) := w_n.$$

- We say that paths $\overrightarrow{u}, \overrightarrow{v} \in$ Finpath(F, w) satisfy a relation $\overrightarrow{u} \preccurlyeq \overrightarrow{v}$ if and only if $\overrightarrow{v} = \overrightarrow{u}^\frown\langle\leqslant, w'\rangle$, where $^\frown$ is concatenation of two tuples.

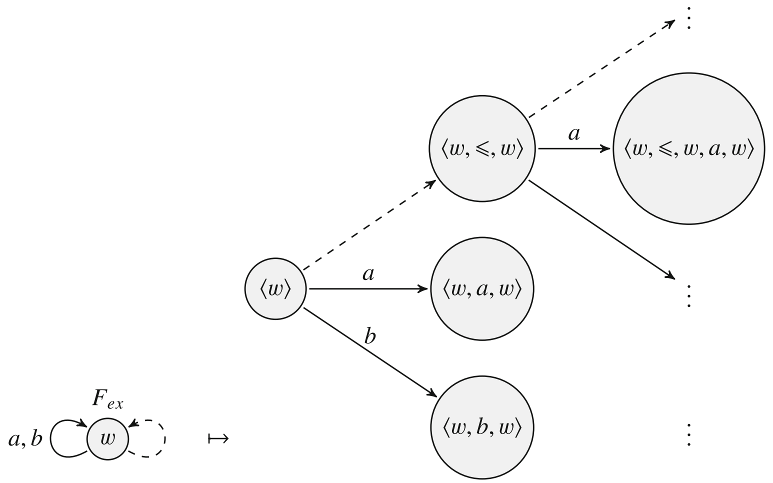

Figure 4. Tree unraveling

- For paths $\overrightarrow{u}, \overrightarrow{v} \in \text{Finpath}(F, w)$, $\overrightarrow{u}\mathcal{R}_G\overrightarrow{v}$ iff $\overrightarrow{v} = \overrightarrow{u}^\frown\langle R_H, w'\rangle$ and $G \subseteq H$.
- A *valuation function* $\mathcal{V}\colon \mathsf{Prop} \to \mathcal{P}(\text{Finpath}(F, w))$ is defined by:

$$\mathcal{V}(p) = \{\overrightarrow{u} \in \text{Finpath}(F, w) \mid \mathsf{tail}(\overrightarrow{u}) \in V(p)\}.$$

Take F_{ex} in Figure 2 (recall Example 4.4 and note that a and b in the figure denote $\{a\}$ and $\{b\}$ respectively). The set $\text{Finpath}(F_{ex}, w)$ of paths on F_{ex} and $\preccurlyeq$ and $\mathcal{R}_G$ on this set are drawn in Figure 4. The point is that the a-arrow and b-arrow on w in F_{ex} are transformed into two arrows with different destinations, so that the condition "$R_{\{a\}} \cap R_{\{b\}} = R_{\{a,b\}}$" is not satisfied in F_{ex} (as we saw in Example 4.4) but becomes satisfied in $\text{Finpath}(F_{ex}, w)$. However, as it is, $(\text{Finpath}(F_{ex}, w), \preccurlyeq, (\mathcal{R}_G)_{G\in\mathsf{Grp}})$ is not a pseudo-frame, since $\preccurlyeq$ itself is not a preorder and the condition "$\leqslant; R_G \subseteq R_G$" is not satisfied because, for example, there is no a-arrow from $\langle w\rangle$ to $\langle w, \leqslant, w, a, w\rangle$. Therefore, a preorder and relations for D_G on $\text{Finpath}(F, w)$ in general should be defined as follows.

Definition 4.19. (Tree Unraveling) Let $M = (F, V)$ be a pseudo-model generated by $w \in W$, where $F = (W, \leqslant, (R_G)_{G\in\mathsf{Grp}})$. *Tree unravelings* of a pointed pseudo-model (M, w) are defined as follows[1]

[1] If we replace all occurrences of "$(\preccurlyeq^*; \mathcal{R}_G; \preccurlyeq^*)$" with "$(\preccurlyeq :^*; \mathcal{R}_G)$" then those provides simpler definitions for possibly non-stable models than those given in [12, Theorem 4.1].

1. $\mathrm{Tree}(M, w) := (\mathrm{Finpath}(F, w), \preccurlyeq^*, (\preccurlyeq^*; \mathcal{R}_G; \preccurlyeq^*)_{G \in \mathsf{Grp}}, \mathcal{V})$,
2. $\mathrm{Tree}^\circ(M, w) := \Big(\mathrm{Finpath}(F, w), \preccurlyeq^*, ((\preccurlyeq^*; \mathcal{R}_G; \preccurlyeq^*) \cup \preccurlyeq^*)_{G \in \mathsf{Grp}}, \mathcal{V}\Big)$,
3. $\mathrm{Tree}^+(M, w) := \left(\mathrm{Finpath}(F, w), \preccurlyeq^*, \left((\preccurlyeq^*; \mathcal{R}_G; \preccurlyeq^*)^+\right)_{G \in \mathsf{Grp}}, \mathcal{V}\right)$,
4. $\mathrm{Tree}^*(M, w) := \left(\mathrm{Finpath}(F, w), \preccurlyeq^*, \left(((\preccurlyeq^*; \mathcal{R}_G; \preccurlyeq^*) \cup \preccurlyeq^*)^+\right)_{G \in \mathsf{Grp}}, \mathcal{V}\right)$,

where R^+ is defined as the transitive closure of R and R^* as the reflexive transitive closure of R.

It is easy to see that $(\preccurlyeq^*; \mathcal{R}_G; \preccurlyeq^*) \cup \preccurlyeq^*$ is reflexive, since $\preccurlyeq^*$ is reflexive. Therefore, $((\preccurlyeq^*; \mathcal{R}_G; \preccurlyeq^*) \cup \preccurlyeq^*)^+$ is also reflexive. Well-definedness of these models are verified in Proposition 4.22 below. Tree unravelings $\mathrm{Tree}(M, w)$, $\mathrm{Tree}^\circ(M, w)$, $\mathrm{Tree}^+(M, w)$ and $\mathrm{Tree}^*(M, w)$ will be used if all the underlying binary relations R_G on W are arbitrary, reflexive, transitive, and both reflexive and transitive, respectively. We also establish that all tree unravelings of Definition 4.19 are stable (see Proposition 4.22).

The following two propositions are useful for the purpose.

Proposition 4.20. *If $G_1 \subseteq G_2$ then $\mathcal{R}_{G_2} \subseteq \mathcal{R}_{G_1}$.*

Proof. Suppose $\overrightarrow{u} \mathcal{R}_{G_2} \overrightarrow{v}$, i.e., $\overrightarrow{v}$ is of the form $\overrightarrow{u} {}^\frown \langle R_H, w' \rangle$ and $G_2 \subseteq H$. We thus have $G_1 \subseteq H$ by the assumption, and hence $\overrightarrow{u} \mathcal{R}_{G_1} \overrightarrow{v}$. ■

Proposition 4.21. *Let $R, R_1, \cdots, R_n$ be binary relations on a set X.*

1. *If $R_1 \subseteq R_2$, then $R; R_1 \subseteq R; R_2$.*
2. *If $R_1 \subseteq R_2$, then $R_1; R \subseteq R_2; R$.*
3. *If $R_1 \subseteq R_2$, then $R_1^n \subseteq R_2^n$ for all $n \in \mathbb{N}$. In particular, $R_1^+ \subseteq R_2^+$ and $R_1^* \subseteq R_2^*$.*
4. *$R; \bigcap_{i \in I} R_i \subseteq \bigcap_{i \in I}(R; R_i)$.*
5. *$(\bigcap_{i \in I} R_i); R \subseteq \bigcap_{i \in I}(R_i; R)$.*

Proposition 4.22. *All the tree unravelings of a pointed pseudo-model (M, w) defined in Definition 4.19 are stable pseudo-models.*

Proof. The condition 1 "$\leqslant; R_G \subseteq R_G$ for any G" of pseudo-frame is obvious by the transitivity of $\preccurlyeq^*$. The condition for stability is similarly verified also by the transitivity of $\preccurlyeq^*$. The inclusion condition holds by items 1 and 3 of Proposition 4.21 and Proposition 4.20. We show that $\mathcal{V}$ satisfies heredity. Take any $p \in \mathsf{Prop}$ and suppose $\overrightarrow{u} \in \mathcal{V}(p)$ and $\overrightarrow{u} \preccurlyeq^* \overrightarrow{v}$. By the former, we have $\mathsf{tail}(\overrightarrow{u}) \in V(p)$. By the latter, it is easily seen that $\mathsf{tail}(\overrightarrow{u}) \leqslant \mathsf{tail}(\overrightarrow{v})$. Since V satisfies heredity, it turns out that $\mathsf{tail}(\overrightarrow{v}) \in V(p)$, which means $\overrightarrow{v} \in \mathcal{V}(p)$. ■

It is noted that $\mathrm{Tree}^{\mathtt{x}}(M, w)$ is: a reflexive pseudo-model if $\mathtt{x} = \circ$; a transitive pseudo-model if $\mathtt{x} = +$; and a reflexive and transitive pseudo-model if $\mathtt{x} = *$, which is easily seen from the definition of the relation for D_G in each tree unraveling. Moreover, seriality is inherited from an original pseudo-model:

Proposition 4.23. *If $R_{\{a\}}$ is serial, then $\preccurlyeq^*;\mathcal{R}_{\{a\}}$ and $\left(\preccurlyeq^*;\mathcal{R}_{\{a\}}\right)^+$ are serial.*

Proof. Assume that $R_{\{a\}}$ is serial. It suffices to show that $\mathcal{R}_{\{a\}}$ is serial. Take $\overrightarrow{u} \in \mathrm{Finpath}(F, w)$. Since $R_{\{a\}}$ is serial, $\mathsf{tail}(\overrightarrow{u})R_{\{a\}}x$ for some $x \in W$. Therefore, we conclude that $\left(\overrightarrow{u}, \overrightarrow{u}^\frown\langle R_{\{a\}}, x\rangle\right) \in \mathcal{R}_{\{a\}}$. ■

Proposition 4.24. *Let $F = (W, \leqslant, (R_G)_{G\in\mathsf{Grp}})$ be a pseudo-frame and $w \in W$. Let us denote $\mathcal{R}_G$ or $\preccurlyeq$ on $\mathrm{Finpath}(F, w)$ by $\mathcal{R}, \mathcal{S}, \mathcal{R}_1, \mathcal{R}_2$, etc. The following hold.*

1. *If $\overrightarrow{u}\mathcal{R}\overrightarrow{v}$, then $\overrightarrow{u} = \mathsf{body}(\overrightarrow{v})$.*
2. *If $(\overrightarrow{u}, \overrightarrow{v}) \in (\mathcal{R}_1; \cdots; \mathcal{R}_m); (\mathcal{S}_1; \cdots; \mathcal{S}_n)$ $(m \geq 0, n \geq 1)$, then there exists the unique $\overrightarrow{t}$ such that $(\overrightarrow{u}, \overrightarrow{t}) \in (\mathcal{R}_1; \cdots; \mathcal{R}_m)$ and $(\overrightarrow{t}, \overrightarrow{v}) \in (\mathcal{S}_1; \cdots; \mathcal{S}_n)$. In particular, such unique $\overrightarrow{t}$ is defined as $\mathsf{body}^n(\overrightarrow{v}) := \underbrace{\mathsf{body}(\cdots(\mathsf{body}}_{n \text{ times}}(\overrightarrow{v}))\cdots)$.*
3. *If $(\mathcal{R}_1; \cdots; \mathcal{R}_m) \cap (\mathcal{S}_1; \cdots; \mathcal{S}_n) \neq \emptyset$, then $n = m$.*
4. $\mathcal{R}_1; \cdots; \mathcal{R}_m; \bigcap_{i\in I}(\mathcal{S}_{(i,1)}; \cdots; \mathcal{S}_{(i,n_i)}) = \bigcap_{i\in I}\left(\mathcal{R}_1; \cdots; \mathcal{R}_m; \mathcal{S}_{(i,1)}; \cdots; \mathcal{S}_{(i,n_i)}\right)$.
5. $\bigcap_{i\in I}\left(\mathcal{S}_{(i,1)}; \cdots; \mathcal{S}_{(i,n_i)}\right); \mathcal{R}_1; \cdots; \mathcal{R}_m = \bigcap_{i\in I}\left(\mathcal{S}_{(i,1)}; \cdots; \mathcal{S}_{(i,n_i)}; \mathcal{R}_1; \cdots; \mathcal{R}_m\right)$.

Proof. Item 1 is obvious by the definition of $\mathcal{R}_G$ and $\preccurlyeq$.

Item 5. is shown similarly to item 4.. We show the remaining items below.

- (item 2.) By induction on n. If $n = 1$, there exists $\overrightarrow{t}$ such that $\overrightarrow{u}(\mathcal{R}_1; \cdots; \mathcal{R}_n)\overrightarrow{t}$ and $\overrightarrow{t}\mathcal{S}_1\overrightarrow{v}$. By item 1., $\overrightarrow{t} = \mathsf{body}(\overrightarrow{v})$. Since $(\mathcal{R}_1; \cdots; \mathcal{R}_m); (\mathcal{S}_1; \cdots; \mathcal{S}_n) = (\mathcal{R}_1; \cdots; \mathcal{R}_m; \mathcal{S}_1); (\mathcal{S}_2; \cdots; \mathcal{S}_n)$, we obtain $\overrightarrow{u}(\mathcal{R}_1; \cdots; \mathcal{R}_m; \mathcal{S}_1)\mathsf{body}^{n-1}(\overrightarrow{v})$ and $\mathsf{body}^{n-1}(\overrightarrow{v})(\mathcal{S}_2; \cdots; \mathcal{S}_n)\overrightarrow{v}$ hold by induction hypothesis. Then, the former implies that $\overrightarrow{u}(\mathcal{R}_1; \cdots; \mathcal{R}_m)\mathsf{body}^n(\overrightarrow{v})$.

- (item 3.) Fix some $(\overrightarrow{u}, \overrightarrow{v}) \in (\mathcal{R}_1; \cdots; \mathcal{R}_m) \cap (\mathcal{S}_1; \cdots; \mathcal{S}_n)$. If $m = 0$, we have $\overrightarrow{u} = \overrightarrow{v}$. Then n should clearly be 0. If $n = 0$, then $m = 0$ holds by the same argument. Assume that $n, m > 0$. Applying item 2. with $m = 0$, we obtain $\overrightarrow{u} = \mathsf{body}^m(\overrightarrow{v})$ and $\overrightarrow{u} = \mathsf{body}^n(\overrightarrow{v})$. Therefore, $n = m$.
- (item 4.) By item 3., if there are $i, j \in I$ such that $n_i \neq n_j$, the equation is equivalent to $\emptyset = \emptyset$. So, we assume $n_i = n_j$ for all $i, j \in I$,and denote it by n. The left-to-right is obvious from item 4 of Proposition 4.21. We show the converse. Assume that $(\overrightarrow{u}, \overrightarrow{v}) \in (\mathcal{R}_1; \cdots; \mathcal{R}_m; \mathcal{S}_{(i,1)}; \cdots; \mathcal{S}_{(i,n)})$ for all i. By item 2., $(\overrightarrow{u}, \mathsf{body}^n(\overrightarrow{v})) \in (\mathcal{R}_1; \cdots; \mathcal{R}_m)$ and $(\mathsf{body}^n(\overrightarrow{v}), \overrightarrow{v}) \in (\mathcal{S}_{(i,1)}; \cdots; \mathcal{S}_{(i,n)})$ for all i. Then, we have $(\overrightarrow{u}, \overrightarrow{v}) \in \mathcal{R}_1; \cdots; \mathcal{R}_m; \bigcap_{i \in I}(\mathcal{S}_{(i,1)}; \cdots; \mathcal{S}_{(i,n)})$, as desired.

■

Proposition 4.25. *All the tree unravelings of a pointed pseudo-model*(M, w) *satisfy the intersection condition. That is:*

1. $\bigcap_{a \in G}(\preccurlyeq^*; \mathcal{R}_{\{a\}}; \preccurlyeq^*) = \preccurlyeq^*; \mathcal{R}_G; \preccurlyeq^*$ *holds in* $\mathrm{Tree}(M, w)$.
2. $\bigcap_{a \in G}((\preccurlyeq^*; \mathcal{R}_{\{a\}}; \preccurlyeq^*) \cup \preccurlyeq^*) = (\preccurlyeq^*; \mathcal{R}_G; \preccurlyeq^*) \cup \preccurlyeq^*$ *holds in* $\mathrm{Tree}^\circ(M, w)$.
3. $\bigcap_{a \in G}\left(\preccurlyeq^*; \mathcal{R}_{\{a\}}; \preccurlyeq^*\right)^+ = (\preccurlyeq^*; \mathcal{R}_G; \preccurlyeq^*)^+$ *holds in* $\mathrm{Tree}^+(M, w)$.
4. $\bigcap_{a \in G}\left((\preccurlyeq^*; \mathcal{R}_{\{a\}}; \preccurlyeq^*) \cup \preccurlyeq^*\right)^+ = ((\preccurlyeq^*; \mathcal{R}_G; \preccurlyeq^*) \cup \preccurlyeq^*)^+$ *holds in* Tree^* (M, w),.

Proof. Item 2. and item 4. are easy consequences of item 1. and item 3., respectively. Thus, we show item 1. and item 3. below with the help of Proposition 4.5.

- (item 1.) It suffices to show that $(\preccurlyeq^*; \mathcal{R}_{G_1}; \preccurlyeq^*) \cap (\preccurlyeq^*; \mathcal{R}_{G_2}; \preccurlyeq^*) \subseteq \preccurlyeq^*; \mathcal{R}_{G_1 \cup G_2}; \preccurlyeq^*$ by Proposition 4.5. Fix any $(\overrightarrow{u}, \overrightarrow{v}) \in (\preccurlyeq^*; \mathcal{R}_{G_1}; \preccurlyeq^*) \cap (\preccurlyeq^*; \mathcal{R}_{G_2}; \preccurlyeq^*)$. We show that $\overrightarrow{u}(\preccurlyeq^*; \mathcal{R}_{G_1 \cup G_2}; \preccurlyeq^*)\overrightarrow{v}$. By the definitions of $\preccurlyeq$ and $\mathcal{R}_G$, items from 1. to 3. of Proposition 4.24 tell us that

  $$\overrightarrow{v} = \overrightarrow{u}^\frown\langle \leqslant, w_1, \cdots, w_{n-1}, \leqslant, w_n, R_H, u_0, \leqslant, u_1, \cdots, u_{m-1}, \leqslant, u_m\rangle,$$

  where $n, m \geq 0$ and $G_i \subseteq H$ $(i = 1, 2)$. Since $G_1 \cup G_2 \subseteq H$, the goal is immediate.
- (item 3.) By Proposition 4.5, it suffices to show that

  $$(\preccurlyeq^*; \mathcal{R}_{G_1}; \preccurlyeq^*)^+ \cap (\preccurlyeq^*; \mathcal{R}_{G_2}; \preccurlyeq^*)^+ \subseteq (\preccurlyeq^*; \mathcal{R}_{G_1 \cup G_2}; \preccurlyeq^*)^+ .$$

  Let us assume $(\overrightarrow{u}, \overrightarrow{v}) \in (\preccurlyeq^*; \mathcal{R}_{G_1}; \preccurlyeq^*)^+ \cap (\preccurlyeq^*; \mathcal{R}_{G_2}; \preccurlyeq^*)^+$. Our goal is to show that $\overrightarrow{u}\,(\preccurlyeq^*; \mathcal{R}_{G_1 \cup G_2}; \preccurlyeq^*)^+\,\overrightarrow{v}$. For each $i \in \{1, 2\}$, there exist natural numbers n_i, $k_{(i,1)}$, ..., $k_{(i,n_i)}$, $l_{(i,1)}$, ..., $l_{(i,n_i)}$ such that

$$\overrightarrow{u}(\preccurlyeq^{k_{(i,1)}};\mathcal{R}_{G_i};\preccurlyeq^{l_{(i,1)}});\cdots;(\preccurlyeq^{k_{(i,n_i)}};\mathcal{R}_{G_i};\preccurlyeq^{l_{(i,n_i)}})\overrightarrow{v}.$$

By items from 1. to 3. of Proposition 4.24, we have $n_1 = n_2$, $k_{(1,1)} = k_{(2,1)}$, ..., $k_{(1,n_1)} = k_{(2,n_2)}$, $l_{(1,1)} = l_{(2,1)}$, ..., $l_{(1,n_1)} = l_{(2,n_2)}$. So, we drop the subscript i to simply write n, k_1, $\dots k_n$, l_1, ..., l_n. Therefore, by a similar argument for Tree(M, w) (item 1.), we obtain:

$$\overrightarrow{u}(\preccurlyeq^{k_1};\mathcal{R}_{G_1\cup G_2};\preccurlyeq^{l_1});\cdots;(\preccurlyeq^{k_n};\mathcal{R}_{G_1\cup G_2};\preccurlyeq^{l_n})\overrightarrow{v},$$

which implies our goal.

■

Definition 4.26. (bounded morphism) Let $M = (W, \leqslant, (R_G)_{G\in\mathsf{Grp}}, V)$ and $M' = (W', \leqslant', (R'_G)_{G\in\mathsf{Grp}}, V')$ be pseudo-models. A function $f\colon W \to W'$ is called a bounded morphism from M to M' if and only if any of the following is satisfied:

($\leqslant$-forth) if $w \leqslant v$ then $f(w) \leqslant' f(v)$,

($\leqslant$-back) if $f(w) \leqslant' v'$ then there is $v \in W$ such that $w \leqslant v$ and $v' = f(v)$,

(R_G-forth) if wR_Gv then $f(w)R'_Gf(v)$,

(R_G-back) if $f(w)R'_Gv'$ then there is $v \in W$ such that wR_Gv and $v' = f(v)$,

(Atom) $w \in V(p)$ iff $f(w) \in V'(p)$.

If there is a surjective bounded morphism from M to M', we denote it by $M \twoheadrightarrow M'$.

Proposition 4.27. *Let f be a bounded morphism from a pseudo-model M to a pseudo-model M'. For any formula φ and $w \in W$, $M, w \Vdash^{\mathsf{ps}} \varphi$ iff $M', f(w) \Vdash^{\mathsf{ps}} \varphi$.*

Theorem 4.28. *Let $M = (F, V)$ be a stable pseudo-model generated by w.*

1. Tree$(M, w) \twoheadrightarrow M$.
2. Tree$^\circ(M, w) \twoheadrightarrow M$, *if all R_Gs are reflexive.*
3. Tree$^+(M, w) \twoheadrightarrow M$, *if all R_Gs are transitive.*
4. Tree$^*(M, w) \twoheadrightarrow M$, *if all R_Gs are reflexive and transitive.*

Proof. In any item, we define a function f from Finpath(F, w) to $|M|$ as one which maps $\overrightarrow{u} \in$ Finpath(F, w) to $\mathsf{tail}(\overrightarrow{u})$. Surjectivity is evident from the definition of Finpath(F, w) and the assumption that M is generated by w. We show that f is a bounded morphism. The condition for a valuation is obviously satisfied by the definition of $\mathcal{V}$. We check the condition for preorder. Suppose $\overrightarrow{u} \preccurlyeq^* \overrightarrow{v}$. Then, it is obvious by the definition of $\preccurlyeq$ and the transitivity of $\leqslant$ that $f(\overrightarrow{u}) \leqslant f(\overrightarrow{v})$. Suppose $f(\overrightarrow{u}) \leqslant v$. Put $\overrightarrow{v} := \overrightarrow{u}^\frown\langle\leqslant, v\rangle$, which clearly gives $\overrightarrow{u} \preccurlyeq \overrightarrow{v}$ and $f(\overrightarrow{v}) = v$. We check the condition for R_G relation. The back condition is easier. Suppose $f(\overrightarrow{u})R_G v$. Put $\overrightarrow{v} := \overrightarrow{u}^\frown\langle R_G, v\rangle$, which clearly gives $\overrightarrow{u}\mathcal{R}_G\overrightarrow{v}$ and $v = f(\overrightarrow{v})$. Since $\mathcal{R}_G \subseteq (\preccurlyeq^*;\mathcal{R}_G;\preccurlyeq^*), ((\preccurlyeq^*;\mathcal{R}_G;\preccurlyeq^*)\cup \preccurlyeq^*), (\preccurlyeq^*;\mathcal{R}_G;\preccurlyeq^*)^+, ((\preccurlyeq^*;\mathcal{R}_G;\preccurlyeq^*)\cup \preccurlyeq^*)^+$, the back condition turns out to be satisfied in all items by $\overrightarrow{v}$. We check the forth condition for each item below.

- (1.) Suppose $\overrightarrow{u}(\preccurlyeq^*;\mathcal{R}_G;\preccurlyeq^*)\overrightarrow{v}$. By definition,

  $$\overrightarrow{v} = \overrightarrow{u}^\frown\langle\leqslant, w_1, \cdots, w_{n-1}, \leqslant, w_n, R_H, w_n, \leqslant, w_{n+1}, \cdots, \leqslant w_m\rangle$$

  where $G \subseteq H$. Therefore, $f(\overrightarrow{u})(\leqslant;R_H;\leqslant)f(\overrightarrow{v})$, which entails $f(\overrightarrow{u})R_H f(\overrightarrow{v})$ since M is a stable pseudo-model. By $R_H \subseteq R_G$, we have $f(\overrightarrow{u})R_G f(\overrightarrow{v})$.

- (2.) Suppose $\overrightarrow{u}((\preccurlyeq^*;\mathcal{R}_G;\leqslant)\cup \preccurlyeq^*)\overrightarrow{v}$, i.e. $\overrightarrow{u}(\preccurlyeq^*;\mathcal{R}_G;\leqslant)\overrightarrow{v}$ or $\overrightarrow{u} \preccurlyeq^* \overrightarrow{v}$. In the former case, the same argument as item 1. can be applied. In the latter case, we have $f(\overrightarrow{u}) \leqslant f(\overrightarrow{v})$. Since R_G is reflexive, we get $f(\overrightarrow{u})(\leqslant;R_G;\leqslant)f(\overrightarrow{v})$ hence $f(\overrightarrow{u})R_G f(\overrightarrow{v})$ since M is a stable pseudo-model.

- (3.) Suppose $\overrightarrow{u}\ (\preccurlyeq^*;\mathcal{R}_G;\leqslant)^+\ \overrightarrow{v}$. Thus, $\overrightarrow{u}\ (\preccurlyeq^*;\mathcal{R}_G;\leqslant)^m\ \overrightarrow{v}$ for some natural number $m \geqslant 1$. By repeating a similar argument for item 1., we can obtain $f(\overrightarrow{u})(\leqslant;R_G;\leqslant)^m f(\overrightarrow{v})$. Since R_G is transitive and stable, $(\leqslant;R_G;\leqslant)^m \subseteq R_G^m \subseteq R_G$. Thus, $f(\overrightarrow{u})R_G f(\overrightarrow{v})$.

- (4.) This is shown similarly to item 2. with the help of item 3..

■

Now we are ready to prove Theorem 4.1: Given any **X** from **IntK**, **IntKT**, **IntKD**, **IntK4**, **IntK4D**, and **IntS4**, if $\Gamma \Vdash_{\mathbb{F}(\mathbf{X})\cap\mathbb{ST}} \varphi$ then $\Gamma \vdash_{\mathsf{H}(\mathbf{X})} \varphi$.

Proof. We show the contraposition. Assume $\Gamma \nvdash_{\mathsf{H}(\mathbf{X})} \varphi$. By Lemma 4.11, We can find an **X**-consistent and prime **X**-theory Γ^+ such that $\Gamma \subseteq \Gamma^+$ and $\Gamma^+ \nvdash_{\mathsf{H}(\mathbf{X})} \varphi$. Since $\Gamma \subseteq \Gamma^+$, $M^{\mathbf{X}}, \Gamma^+ \Vdash^{\mathsf{ps}} \Gamma$ by the left-to-right direction of Lemma 4.14. On the other hand, $M^{\mathbf{X}}, \Gamma^+ \nVdash^{\mathsf{ps}} \varphi$ by the right-to-left direction of Lemma 4.14 and item 1. of Lemma 4.10. We take an appropriate tree unraveling depending on **X**.

- ($\mathbf{X} = \mathbf{IntK}, \mathbf{IntKD}$) We can take Tree $\left(M^{\mathbf{X}}_{\Gamma^+}, \Gamma^+\right)$, because $M^{\mathbf{X}}_{\Gamma^+}$ is a pseudo-model generated by Γ^+ by Proposition 4.13. Since Tree $\left(M^{\mathbf{X}}_{\Gamma^+}, \Gamma^+\right)$

can be seen as a model in the sense of Definition 2.1 by Proposition 4.25, it suffices to show that $(M^{\mathbf{X}}, \Gamma^{+})$ satisfies exactly the same formulas as $\left(\text{Tree}\left(M^{\mathbf{X}}_{\Gamma^{+}}, \Gamma^{+}\right), \langle\Gamma^{+}\rangle\right)$ in the case of **IntK**. First, $(M^{\mathbf{X}}, \Gamma^{+})$ satisfies exactly the same formulas as $(M^{\mathbf{X}}_{\Gamma^{+}}, \Gamma^{+})$ by Proposition 4.17. Next, $(M^{\mathbf{X}}_{\Gamma^{+}}, \Gamma^{+})$ satisfies exactly the same formulas as $\left(\text{Tree}\left(M^{\mathbf{X}}_{\Gamma^{+}}, \Gamma^{+}\right), \langle\Gamma^{+}\rangle\right)$ by Proposition 4.27, because $f(\langle\Gamma^{+}\rangle) = \Gamma^{+}$ and f is a bounded morphism, which is shown to exist in Theorem 4.28. It follows that $\text{Tree}\left(M^{\mathbf{X}}_{\Gamma^{+}}, \Gamma^{+}\right), \langle\Gamma^{+}\rangle \Vdash \Gamma$ but $\text{Tree}\left(M^{\mathbf{X}}_{\Gamma^{+}}, \Gamma^{+}\right), \langle\Gamma^{+}\rangle \nVdash \varphi$. By Proposition 4.22, $\text{Tree}\left(M^{\mathbf{X}}_{\Gamma^{+}}, \Gamma^{+}\right) \in \mathbb{F}(\mathbf{IntK}) \cap \mathbb{ST}$ hence $\Gamma \nVdash_{\mathbb{F}(\mathbf{IntK})\cap\mathbb{ST}} \varphi$, as desired. In the case of **IntKD**, we need to additionally show that $\text{Tree}\left(M^{\mathbf{X}}_{\Gamma^{+}}, \Gamma^{+}\right)$ is serial. This is true, because of item 2 of Proposition 4.15, the obvious fact that seriality is preserved under taking generated submodel, and Proposition 4.23.

- ($\mathbf{X} = \mathbf{IntKT}$) Take $\text{Tree}^{\circ}\left(M^{\mathbf{X}}_{\Gamma^{+}}, \Gamma^{+}\right)$. We can show that $\left(\text{Tree}^{\circ}\left(M^{\mathbf{X}}_{\Gamma^{+}}, \Gamma^{+}\right), \langle\Gamma^{+}\rangle\right)$, which can be seen as a model by Proposition 4.25, satisfies exactly the same formulas as $(M^{\mathbf{X}}, \Gamma^{+})$ by the same argument as the case of **IntK**. Note that $\text{Tree}^{\circ}\left(M^{\mathbf{X}}_{\Gamma^{+}}, \Gamma^{+}\right)$ is reflexive.
- ($\mathbf{X} = \mathbf{IntK4}, \mathbf{IntK4D}$) Take $\text{Tree}^{+}\left(M^{\mathbf{X}}_{\Gamma^{+}}, \Gamma^{+}\right)$. We can show $\left(\text{Tree}^{+}\left(M^{\mathbf{X}}_{\Gamma^{+}}, \Gamma^{+}\right), \langle\Gamma^{+}\rangle\right)$, which can be seen as a model by Proposition 4.25, satisfies exactly the same formulas as $(M^{\mathbf{X}}, \Gamma^{+})$ by the same argument as the case of **IntK**. Note that $\text{Tree}^{+}\left(M^{\mathbf{X}}_{\Gamma^{+}}, \Gamma^{+}\right)$ is transitive. For the case of **IntK4D**, we can additionally show that $\text{Tree}^{+}\left(M^{\mathbf{X}}_{\Gamma^{+}}, \Gamma^{+}\right)$ is serial by the same argument as the case of **IntKD**.
- ($\mathbf{X} = \mathbf{IntS4}$) Take $\text{Tree}^{*}\left(M^{\mathbf{X}}_{\Gamma^{+}}, \Gamma^{+}\right)$. We can show $\left(\text{Tree}^{*}\left(M^{\mathbf{X}}_{\Gamma^{+}}, \Gamma^{+}\right), \langle\Gamma^{+}\rangle\right)$, which can be seen as a model by Proposition 4.25, satisfies exactly the same formulas as $(M^{\mathbf{X}}, \Gamma^{+})$ by the same argument as the case of **IntK**. Note that $\text{Tree}^{*}\left(M^{\mathbf{X}}_{\Gamma^{+}}, \Gamma^{+}\right)$ is reflexive and transitive.

■

5. Public Announcement Expansions

This section adds the public announcement operator $[\varphi]$ (read: "after a truthful announcement of φ") to our static syntax of intuitionistic epistemic logic with distributed knowledge. Recall that our underlying syntax and semantics are different from those of [3,6,10,13] as we have seen in the introduction. We will not consider a dual diamond-version $\langle\varphi\rangle$ of the public announcement operator $[\varphi]$ in the public announcement expansion, either. This simplification enables us to apply the outside-in strategy, which rewrites one of the *outermost* occurrences of the public announcement operators in a formula, of proving the semantic completeness (in Section 5.2). It is noted that the inside-out strategy, which rewrites one of the *innermost* occurrences of the public announcement operators in a formula, of proving the semantic completeness were employed in [3,10]. A subtlety between outside-in and inside-out strategies, the reader is referred to, e.g., [23].

To apply the outside-in strategy for the semantic completeness of our public announcement expansion, we should have axioms (called *recursion axioms*) which state how the public announcement operators commutes with propositional variables and each of the all logical connectives. Section 5.1 shows that the natural recursion axiom $[\varphi]D_G\psi \leftrightarrow (\varphi \rightarrow D_G(\varphi \rightarrow [\varphi]\psi))$[2] for the distributed knowledge operator is not valid on all models (Proposition 5.8) but valid on all stable models (Proposition 5.9). We also show that the recursion axiom $[\varphi][\psi]\chi \leftrightarrow [\varphi \wedge [\varphi]\psi]\chi$, which is a key axiom for the outside-in strategy, is valid on all models (Proposition 5.7).

5.1. Public Announcement Expansion Over Stable Models

We expand our syntax with the public announcement operator and define the set of all formulas of the expanded syntax as:

$$\mathsf{Form}^+ \ni \varphi ::= p \mid \bot \mid (\varphi \rightarrow \varphi) \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid D_G\varphi \mid [\varphi]\varphi,$$

where $p \in \mathsf{Prop}$ and $G \in \mathsf{Grp}$. We follow the standard rules for elimination of the parentheses

Definition 5.1. Let $M = (W, \leqslant, (R_a)_{a\in\mathsf{Agt}}, V)$ and $\varphi, \psi \in \mathsf{Form}^+$. The satisfaction relation $M, w \Vdash$ is defined as before except:

$M, w \Vdash [\varphi]\psi$ iff for all $v \in W, w \leqslant v$ and $M, v \Vdash \varphi$ jointly imply $M^\varphi, v \Vdash \psi$,

where $M^\varphi := ([\![\varphi]\!]_M, \leqslant^\varphi, (R^\varphi_a)_{a\in\mathsf{Agt}}, V^\varphi)$ is defined as follows:

- $[\![\varphi]\!]_M := \{w \in W \mid M, w \Vdash \varphi\}$,
- $\leqslant^\varphi := \leqslant \cap([\![\varphi]\!]_M \times [\![\varphi]\!]_M)$,
- $R^\varphi_a := R_a \cap ([\![\varphi]\!]_M \times [\![\varphi]\!]_M)$,
- $V^\varphi(p) := V(p) \cap [\![\varphi]\!]_M$.

[2] The axiom of the same shape is included in Hilbert system of the public announcement logic over *classical logic* with distributed knowledge in [24]. When $G = \{a\}$ and so $D_{\{a\}}$ is K_a, the axiom is included in Hilbert systems in [3,10,13] of intuitionistic public announcement logic and it is valid in all intended models in [3,13].

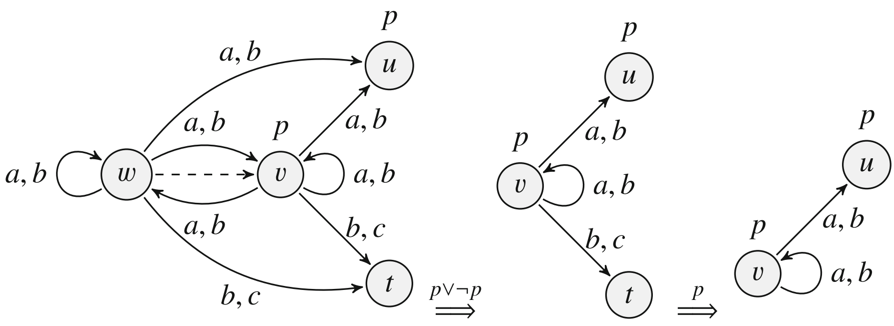

Figure 5. Models N, $N^{p\vee\neg p}$, and $(N^{p\vee\neg p})^p$

Under these definitions, the heredity condition still holds.

Proposition 5.2. *If $M, w \Vdash \varphi$ and $w \leqslant v$, then $M, v \Vdash \varphi$.*

It is noted that this would not hold if we were to adopt "$M, w \Vdash \varphi$ implies $M^{\varphi}, w \Vdash \psi$" as the definition of $M, w \Vdash [\varphi]\psi$. Moreover, it is easy to show the following.

Proposition 5.3. *Let $M = (W, \leqslant, (R_a)_{a\in \mathsf{Agt}}, V)$ and $\varphi \in \mathsf{Form}^+$. Suppose $[\![\varphi]\!] \neq \emptyset$. 1. M^{φ} satisfies all the conditions of a model. 2. If R_a is reflexive (or transitive), then so is R_a^{φ}. 3. If M is stable, then so is M^{φ}.*

Remark 5.4. Seriality is not preserved under taking submodels. A counterexample is: $M := (\{w, v\}, \leqslant, R_a, V)$, where $\mathsf{Agt} = \{a\}$, $\mathsf{Prop} = \{p\}$, $\leqslant := \{(w, w), (v, v)\}$, $R_a := \{(w, v), (v, v)\}$, and $V(p) = \{w\}$. The model M is serial, but M^p is not because $R_a^p = \emptyset$. Hence, the corresponding axiom (D) is not in consideration below.

Example 5.5. In Figure 5, we consider updates of the model N discussed in Example 2.5 (recall Fig. 1 b). If we update the first (leftmost) model by $p \vee \neg p$, we obtain the second model, $N^{p\vee\neg p}$. Note that this updated model can be seen as a classical Kripke model since no proper pair is ordered by $\leqslant^{p\vee\neg p}$. By this update, p becomes a distributed knowledge at v of a group $\{a, b\}$, that is, $N, v \Vdash \neg D_{\{a,b\}}p$ but $N^{p\vee\neg p}, v \Vdash D_{\{a,b\}}p$. Next, we update $N^{p\vee\neg p}$ by p to obtain the model $(N^{p\vee\neg p})^p$. It is easy to see that $(N^{p\vee\neg p})^p$ and N^p are the same. By this update, p becomes a knowledge at v of an agent b, that is, $N^{p\vee\neg p}, v \Vdash \neg D_{\{b\}}p$ but $(N^{p\vee\neg p})^p, v \Vdash D_{\{b\}}p$.

The axioms in Table 2 are for the PAL extension. We call the axiom system expanded from $\mathsf{H}(\mathbf{X})$ by all the axioms in Table 2, $\mathsf{H}(\mathbf{X})^+$, where $\mathbf{X} = \mathbf{IntK}$, $\mathbf{IntKT}$, $\mathbf{IntK4}$, and $\mathbf{IntS4}$. The notion of semantic consequence

Table 2. Additional axioms for public announcement operator

$([]p)$	$[\varphi]p \leftrightarrow (\varphi \to p)$	$([]\bot)$	$[\varphi]\bot \leftrightarrow (\varphi \to \bot)$
$([]\to)$	$[\varphi](\psi \to \chi) \leftrightarrow ([\varphi]\psi \to [\varphi]\chi)$	$([]\wedge)$	$[\varphi](\psi \wedge \chi) \leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi)$
$([]\vee)$	$[\varphi](\psi \vee \chi) \leftrightarrow (\varphi \to [\varphi]\psi \vee [\varphi]\chi)$	$([]D)$	$[\varphi]D_G\psi \leftrightarrow (\varphi \to D_G[\varphi]\psi)$
$([][])$	$[\varphi][\psi]\chi \leftrightarrow [\varphi \wedge [\varphi]\psi]\chi[0.]$		

and derivability is defined in the same way as for $\mathsf{H}(\mathbf{X})$. In what follows, we establish that all the axioms in Table 2 are valid with respect to the class of all *stable* frames. First, we deal with composition of two announcements. Recall from Example 5.5 that $(N^{p\vee\neg p})^p$ and N^p of Fig. 5 are the same. This can be understood as an example of the following lemma, because $(p \vee \neg p) \wedge [p \vee \neg p]p$ and p are equivalent.

Lemma 5.6. *Let M be a model and suppose $(M^\varphi)^\psi$ and $M^{\varphi\wedge[\varphi]\psi}$ are well-defined models. Then, $(M^\varphi)^\psi = M^{\varphi\wedge[\varphi]\psi}$.*

Proof. It suffices to show that $|(M^\varphi)^\psi| = |M^{\varphi\wedge[\varphi]\psi}|$. Assume $w \in |M^{\varphi\wedge[\varphi]\psi}|$, which is equivalent to $M, w \Vdash \varphi \wedge [\varphi]\psi$. This is equivalent to $M, w \Vdash \varphi$ and "$M, v \Vdash \varphi$ implies $M^\varphi, v \Vdash \psi$ for any v such that $w \leqslant v$". By instantiating v with w, we can infer "$M, w \Vdash \varphi$ implies $M^\varphi, w \Vdash \psi$" from the latter. Then, we have $M^\varphi, w \Vdash \psi$ by modus ponens, which means $w \in |(M^\varphi)^\psi|$. For the left-to-right, let us assume $M^\varphi, w \Vdash \psi$. We have to show $M, w \Vdash \varphi$ and "$M, v \Vdash \varphi$ implies $M^\varphi, v \Vdash \psi$ for any v such that $w \leqslant v$". The assumption presupposes $w \in |M^\varphi|$, which means $M, w \Vdash \varphi$, the former goal. For the latter implication, fix a v satisfying $w \leqslant v$. Then, by the heredity, $M, v \Vdash \varphi$, so $w \leqslant^\varphi v$. Then, again by the heredity, $M^\varphi, v \Vdash \psi$. ∎

Proposition 5.7. *The axioms in Table 2 except $([]D)$ are valid in the class of all frames.*

Proof. We show the validity of $([]\vee)$ and $([][])$ alone.

First, we show the validity of $([]\vee)$. For any model M and $w \in |M|$, we show $M, w \Vdash [\varphi](\psi_1 \vee \psi_2) \to (\varphi \to ([\varphi]\psi_1 \vee [\varphi]\psi_2))$. Fix any v such that $w \leqslant v$ and $M, v \Vdash [\varphi](\psi_1 \vee \psi_2)$. To show $M, v \Vdash \varphi \to ([\varphi]\psi_1 \vee [\varphi]\psi_2)$, fix u such that $v \leqslant u$ and $M, u \Vdash \varphi$. We show $M, u \Vdash [\varphi]\psi_1 \vee [\varphi]\psi_2$. From $M, v \Vdash [\varphi](\psi_1 \vee \psi_2)$, $v \leqslant u$, and $M, u \Vdash \varphi$, we have $M^\varphi, u \Vdash \psi_1 \vee \psi_2$. Then, it suffices to show that $M^\varphi, u \Vdash \psi_i$ implies $M, u \Vdash [\varphi]\psi_i$ for $i = 1, 2$. Assume $M^\varphi, u \Vdash \psi_i$ and fix any t such that $u \leqslant t$ and $M, t \Vdash \varphi$. Obviously, we have $u \leqslant^\varphi t$. Then, by the assumption and the heredity, $M^\varphi, t \Vdash \psi_i$. Next, we show $M, w \Vdash (\varphi \to ([\varphi]\psi_1 \vee [\varphi]\psi_2)) \to [\varphi](\psi_1 \vee \psi_2)$. Fix any v such that $w \leqslant v$ and $M, v \Vdash \varphi \to ([\varphi]\psi_1 \vee [\varphi]\psi_2)$. To show $M, v \Vdash [\varphi](\psi_1 \vee \psi_2)$, fix

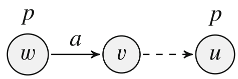

Figure 6. Model falsifying ($[]D$)

u such that $v \leqslant u$ and $M, u \Vdash \varphi$. We show $M^{\varphi}, u \Vdash \psi_1 \vee \psi_2$. Since $v \leqslant u$ and $M, u \Vdash \varphi$, we have $M, u \Vdash [\varphi]\psi_1 \vee [\varphi]\psi_2$. Then, it suffices to show that $M, u \Vdash [\varphi]\psi_i$ implies $M^{\varphi}, u \Vdash \psi_i$ for $i = 1, 2$. This is obviously true because $M, u \Vdash \varphi$.

Second, we show the validity of ($[][]$). First, for any model M and $w \in |M|$, we show $M, w \Vdash [\varphi][\psi]\chi \to [\varphi \wedge [\varphi]\psi]\chi$. Fix any v such that $w \leqslant v$ and $M, v \Vdash [\varphi][\psi]\chi$. To show $M, v \Vdash [\varphi \wedge [\varphi]\psi]\chi$, fix u such that $v \leqslant u$ and $M, u \Vdash \varphi \wedge [\varphi]\psi$. Since $M, u \Vdash \varphi$ and $M, v \Vdash [\varphi][\psi]\chi$, we have $M^{\varphi}, u \Vdash [\psi]\chi$. Also, by $M, u \Vdash [\varphi]\psi$, we have $M^{\varphi}, u \Vdash \psi$. Then, by these two, we obtain $(M^{\varphi})^{\psi}, u \Vdash \chi$, which is equivalent to $M^{\varphi \wedge [\varphi]\psi}, u \Vdash \chi$ by Lemma 5.6. To show the right-to-left, Fix any v such that $w \leqslant v$ and $M, v \Vdash [\varphi \wedge [\varphi]\psi]\chi$. Fix any u such that $v \leqslant u$ and $M, u \Vdash \varphi$. We show $M^{\varphi}, u \Vdash [\psi]\chi$. Fix any t such that $u \leqslant^{\varphi} t$ and $M^{\varphi}, t \Vdash \psi$. Instead of $(M^{\varphi})^{\psi}, t \Vdash \chi$, we use Lemma 5.6 to show $M^{\varphi \wedge [\varphi]\psi}, t \Vdash \chi$. To show this, we show $v \leqslant t$ and $M, t \Vdash \varphi \wedge [\varphi]\psi$. The former is obvious. Also, $M, t \Vdash \varphi$ since $t \in |M^{\varphi}|$. Further, we have $M, t \Vdash [\varphi]\psi$. For this, take s such that $t \leqslant s$ and $M, s \Vdash \varphi$. Then, $M^{\varphi}, s \Vdash \psi$, since $M^{\varphi}, t \Vdash \psi$ and $t \leqslant^{\varphi} s$. ■

The model M depicted in Figure 6 is a counterexample of ($[]D$). Let $\mathsf{Agt} = \{a\}$. Define the model $M = (\{w, v, u\}, \leqslant, R_a, V)$, where $\leqslant = \{(w, w), (v, v), (v, u), (u, u)\}$, $R_a = \{(w, v)\}$, and $V(p) := \{w, u\}$ and $V(q) := \emptyset$. The solid line stands for the relations for agents and the dotted arrow stands for the preorder. Reflexive arrows for the preorder is omitted in the figure. The condition "$\leqslant; R_a \subseteq R_a$" is easily checked. Also, the condition "$R_a; \leqslant \subseteq R_a$" is easily seen not to be satisfied, since $w(R_a; \leqslant)u$ holds but wR_au fails. Therefore, M is not a stable model.

To show that the axiom ($[]D$) is not valid with respect to the class of all frames, we show $[p]K_aq \to (p \to K_a[p]q)$ is not satisfied at the state w in M. First, $M, w \Vdash [p]K_aq$, because $M^p, w \Vdash K_aq$ is vacuously true by the model update which eliminates v, and w is the only world accessible from w by $\leqslant$. Second, however, $M, w \nVdash p \to K_a[p]q$. The antecedent is obviously true at w. To reject the consequent, we focus on v, the only world accessible from w by R_a, to show $M, v \nVdash [p]q$. This is true because we can find u, which is ahead of v, satisfies p, but does not q. To sum up, we have shown the following.

Proposition 5.8. *The axiom* $([]D)$ *is not valid with respect to the class of all frames.*

By focusing on the class of all *stable* frames, the validity of $([]D)$ is recovered.

Proposition 5.9. *The axiom* $([]D)$ *is valid in the class* $\mathbb{ST}$ *of all stable frames.*

Proof. For any stable model M and $w \in |M|$, we show $M, w \Vdash [\varphi]D_G\psi \to (\varphi \to D_G[\varphi]\psi)$. Fix any v such that $w \leqslant v$ and $M, v \Vdash [\varphi]D_G\psi$. We show $M, v \Vdash \varphi \to D_G[\varphi]\psi$. Fix any u such that $v \leqslant u$ and $M, u \Vdash \varphi$. To show $M, u \Vdash D_G[\varphi]\psi$, fix any t such that $(u, t) \in \bigcap_{a \in G} R_a$. We show $M, t \Vdash [\varphi]\psi$. Fix any s such that $t \leqslant s$ and $M, s \Vdash \varphi$. We show $M^{\varphi}, s \Vdash \psi$. By the stability of the underlying frame, we have $\bigcap_{a \in G} R_a; \leqslant = \bigcap_{a \in G} R_a$. Hence, $(u, s) \in \bigcap_{a \in G} R_a$. Then, we have $M^{\varphi}, s \Vdash \psi$, because $M^{\varphi}, u \Vdash D_G\psi$ holds by $M, v \Vdash [\varphi]D_G\psi$, $v \leqslant u$, and $M, u \Vdash \varphi$ and we have $s \in |M^{\varphi}|$. Next, we show $M, w \Vdash (\varphi \to D_G[\varphi]\psi) \to [\varphi]D_G\psi$. Fix any v such that $w \leqslant v$ and $M, v \Vdash \varphi \to D_G[\varphi]\psi$. To show $M, v \Vdash [\varphi]D_G\psi$, fix any u such that $v \leqslant u$ and $M, u \Vdash \varphi$. Take any t such that $(u, t) \in \bigcap_{a \in G} R_a^{\varphi}$. We show $M^{\varphi}, t \Vdash \psi$. By $M, v \Vdash \varphi \to D_G[\varphi]\psi$, $v \leqslant u$, and $M, u \Vdash \varphi$, we have $M, u \Vdash D_G[\varphi]\psi$. Then, $M, t \Vdash [\varphi]\psi$. Since $M, t \Vdash \varphi$, we obtain $M^{\varphi}, t \Vdash \psi$. ■

Therefore, $\mathsf{H}(\mathbf{X})^+$ is sound for the class of all corresponding *stable* frames.

Theorem 5.10. *Let* $\varphi \in \mathsf{Form}^+$*. If* $\vdash_{\mathsf{H}(\mathbf{X})^+} \varphi$*, then* $\Vdash_{\mathbb{F}(\mathbf{X}) \cap \mathbb{ST}} \varphi$*.*

Proof. Theorem 3.3 and Propositions 5.9 and 5.7. ■

5.2. Semantic Completeness of Public Announcement Expansions

Let $\mathbf{X} = \mathbf{IntK}$, $\mathbf{IntKT}$, $\mathbf{IntK4}$, or $\mathbf{IntS4}$. With the help of a translation from Form^+ to Form as discussed in [22], we show the strong completeness of $\mathsf{H}(\mathbf{X})^+$ with respect to the corresponding class of stable frames, i.e., if $\Gamma \Vdash_{\mathbb{F}(\mathbf{X}) \cap \mathbb{ST}} \varphi$ then $\Gamma \vdash_{\mathsf{H}(\mathbf{X})^+} \varphi$, for every $\Gamma \cup \{\varphi\} \subseteq \mathsf{Form}^+$.

Definition 5.11. The *translation* $t : \mathsf{Form}^+ \to \mathsf{Form}$ is defined as follows:

- $t(p) = p$,
- $t(\bot) = \bot$,
- $t(\varphi \Box \psi) = t(\varphi) \Box t(\psi)$,
- $t(D_G\varphi) = D_G t(\varphi)$,
- $t([\varphi]p) = t(\varphi \to p)$,

- $t([\varphi]\bot) = t(\varphi \to \bot)$,
- $t([\varphi](\psi \star \chi)) = t([\varphi]\psi \star [\varphi]\chi)$,
- $t([\varphi](\psi \vee \chi)) = t(\varphi \to [\varphi]\psi \vee [\varphi]\chi)$,
- $t([\varphi]D_G\psi) = t(\varphi \to D_G[\varphi]\psi)$,
- $t([\varphi][\psi]\chi) = t([\varphi \wedge [\varphi]\psi]\chi)$,

where $\Box \in \{\to, \vee, \wedge\}$ and $\star \in \{\to, \wedge\}$.

The following notion of *complexity* of a formula (cf. [22, Definition 7.21]) plays a key role in our strong completeness proof.

DEFINITION 5.12. The *complexity function* $c\colon \mathsf{Form}^+ \to \mathbb{N}$ is defined as follows:

$$\begin{aligned} c(p) &= 1 \\ c(\bot) &= 1 \\ c(\varphi\Box\psi) &= 1 + \max(c(\varphi), c(\psi)) \quad (\Box \in \{\to, \vee, \wedge\}) \\ c(D_G\varphi) &= 1 + c(\varphi) \\ c([\varphi]\psi) &= (4 + c(\varphi)) \cdot c(\psi). \end{aligned}$$

It is easy to see that $c(\varphi) \geq 1$ for any formula φ.

LEMMA 5.13.

1. $c(\psi) > c(\varphi)$ *if* φ *is a proper subformula of* ψ.
2. $c([\varphi]p) > c(\varphi \to p)$.
3. $c([\varphi]\bot) > c(\varphi \to \bot)$.
4. $c([\varphi](\psi \star \chi)) > c([\varphi]\psi \star [\varphi]\chi)$ *where* $\star \in \{\to, \wedge\}$.
5. $c([\varphi](\psi \vee \chi)) > c(\varphi \to [\varphi]\psi \vee [\varphi]\chi)$.
6. $c([\varphi]D_G\psi) > c(\varphi \to D_G[\varphi]\psi)$.
7. $c([\varphi][\psi]\chi) > c([\varphi \wedge [\varphi]\psi]\chi)$.

PROOF. Since the items except item 5. are proved exactly in the same way as in the proof of [22, Lemma 7.22], we focus on item 5. here. We may assume that $c(\psi) \geq c(\chi)$ without loss of generality. Then, $c([\varphi](\psi \vee \chi)) = (4 + c(\varphi)) \cdot (1 + c(\psi))$. On the other hand, $c(\varphi \to [\varphi]\psi \vee [\varphi]\chi) = 1 + \max(c(\varphi), 1 + c([\varphi]\psi)) \overset{\text{(item 1.)}}{=} 1 + 1 + c([\varphi]\psi) = 2 + (4 + c(\varphi)) \cdot c(\psi)$. Then, $c([\varphi](\psi \vee \chi)) - c(\varphi \to [\varphi]\psi \vee [\varphi]\chi) = 2 + c(\varphi) > 0$. ∎

LEMMA 5.14. *The translation* t *is* Form*-valued, i.e.,* $t(\varphi) \in \mathsf{Form}$.

Proof. By induction on $c(\varphi)$. It is obvious if $\varphi = p$ or $\bot$. If $\varphi = \psi_1 \Box \psi_2$ where $\Box \in \{\to, \wedge, \vee\}$, $c(\varphi) > c(\psi_i)$ by item 1. of Lemma 5.13. Hence, by induction hypothesis, $t(\psi_1 \Box \psi_2) = t(\psi_1) \Box t(\psi_2) \in \mathsf{Form}$. The same argument applies to the case where $\varphi = D_G \psi$. Suppose that $\varphi = [\psi]\chi$. Depending on the form of χ, the corresponding item among items 2. to 7. of Lemma 5.13 assures that $t(\varphi) \in \mathsf{Form}$. ■

Lemma 5.15. *Let* $\mathbf{X} = \mathbf{IntK}$, $\mathbf{IntKT}$, $\mathbf{IntK4}$, *or* $\mathbf{IntS4}$. *For all* $\varphi \in \mathsf{Form}^+$, $\vdash_{\mathsf{H}(\mathbf{X})^+} \varphi \leftrightarrow t(\varphi)$.

Proof. By induction on $c(\varphi)$. In what follows, we write $\vdash$ instead of $\vdash_{\mathsf{H}(\mathbf{X})^+}$. If $\varphi = p$ or $\bot$, it is obvious. Suppose $\varphi = \psi_1 \Box \psi_2$ where $\Box \in \{\to, \wedge, \vee\}$. By induction hypothesis, we get $\vdash \psi_i \leftrightarrow t(\psi_i)$. Then, by intuitionistic tautologies, we have $\vdash \psi_1 \Box \psi_2 \leftrightarrow t(\psi_1) \Box t(\psi_2)$, which implies our goal. Suppose $\varphi = D_G \psi$. By induction hypothesis and the axiom (K), $\vdash D_G \psi \leftrightarrow D_G t(\psi)$ holds. Suppose that $\varphi = [\psi]\chi$. Depending on the form of χ, the corresponding item among items 2. to 7. of Lemma 5.13 assures that $\vdash \varphi \leftrightarrow t(\varphi)$ with the help of additional axioms of Table 2. For example, when χ is of the form $\psi_1 \vee \psi_2$, we proceed as follows. By the axiom ($[]\vee$) of Table 2, $\vdash [\varphi](\psi_1 \vee \psi_2) \leftrightarrow (\varphi \to ([\varphi]\psi_1 \vee [\varphi]\psi_2))$. By item 5. of Lemma 5.13 and induction hypothesis, we get $\vdash (\varphi \to ([\varphi]\psi_1 \vee [\varphi]\psi_2)) \leftrightarrow t(\varphi \to ([\varphi]\psi_1 \vee [\varphi]\psi_2))$. By definition of t, we can conclude $\vdash [\varphi](\psi_1 \vee \psi_2) \leftrightarrow t([\varphi](\psi_1 \vee \psi_2))$. ■

Theorem 5.16. (*strong completeness*) *Let* $\mathbf{X} = \mathbf{IntK}$, $\mathbf{IntKT}$, $\mathbf{IntK4}$, *or* $\mathbf{IntS4}$ *and* $\Gamma \cup \{\varphi\} \subseteq \mathsf{Form}^+$. *If* $\Gamma \Vdash_{\mathbb{F}(\mathbf{X}) \cap \mathbb{ST}} \varphi$, *then* $\Gamma \vdash_{\mathsf{H}(\mathbf{X})^+} \varphi$.

Proof. Assume that $\Gamma \Vdash_{\mathbb{F}(\mathbf{X}) \cap \mathbb{ST}} \varphi$. By Theorem 5.10 and Lemma 5.15, we have $t[\Gamma] \Vdash_{\mathbb{F}(\mathbf{X}) \cap \mathbb{ST}} t(\varphi)$, where $t[\Delta] := \{t(\psi) \mid \psi \in \Delta\}$. By Lemma 5.14, $t[\Gamma \cup \{\varphi\}] \subseteq \mathsf{Form}$. It follows from Theorem 4.1 that $t[\Gamma] \vdash_{\mathsf{H}(\mathbf{X})} t(\varphi)$. For some finite set $\Gamma' \subseteq \Gamma$, $\vdash_{\mathsf{H}(\mathbf{X})} \bigwedge t[\Gamma'] \to t(\varphi)$, which implies $\vdash_{\mathsf{H}(\mathbf{X})^+} \bigwedge t[\Gamma'] \to t(\varphi)$ since $\mathsf{H}(\mathbf{X})^+$ extends $\mathsf{H}(\mathbf{X})$. By Lemma 5.15, we get $\vdash_{\mathsf{H}(\mathbf{X})^+} \bigwedge \Gamma' \to \varphi$ hence $\Gamma \vdash_{\mathsf{H}(\mathbf{X})^+} \varphi$. ■

6. Concluding Remark

We comment on possible further directions of research. The first direction is to consider a possibility of adding the diamond version of D_G and/or $[\varphi]$ to this study. Alternatively, we may add distributed knowledge operators and their dual to intuitionistic public announcement logic studied in [3,10,13]. For this direction, [20, Lemma 18] might be useful, since it states when the semantics based on stable models of this paper and the semantics based on intended models in [3,6,10,13] become equivalent, provided there is no distributed knowledge operators. The second direction is to simplify our semantic completeness argument of $\mathsf{H}(\mathbf{X})$ via a similar method given in [25] for classical epistemic logic with distributed knowledge. One of the merits of the method is that the notion of pseudo- (or pre-) model is not necessary. The third direction is to add **S5**-type axioms to our intuitionistic epistemic logic with distributed knowledge. Since Ono [14] showed that there are at least four distinct **S5**-type axioms over the intuitionistic modal logic **S4**, it would be interesting to study the corresponding **S5**-type axioms in our setting. The fourth direction is to expand our syntax with the common knowledge operator (cf. [22]). This amounts to investigating the intuitionistic counterpart of [25]. The final direction is to consider another dynamic expansions of our syntax. In order to formalize changes of agents' constructive knowledge caused by communication among a group, we may add resolution operators [1].

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