Norihiro Kamide

Refutation-Aware Gentzen-Style Calculi for Propositional Until-Free Linear-Time Temporal Logic

Abstract. This study introduces refutation-aware Gentzen-style sequent calculi and Kripke-style semantics for propositional until-free linear-time temporal logic. The sequent calculi and semantics are constructed on the basis of the refutation-aware setting for Nelson's paraconsistent logic. The cut-elimination and completeness theorems for the proposed sequent calculi and semantics are proven.

Keywords: Linear-time temporal logic, Refutation-aware Gentzen-style sequent calculus, Refutation-aware Kripke-style semantics, Completeness theorem, Cut-elimination theorem.

1. Introduction

1.1. Refutation-Aware Calculi and Semantics

First, we roughly explain the informal notion of refutation- or falsification-aware calculi (proof systems) and semantics. The present study is based on this notion and develops some *refutation-aware Gentzen-style sequent calculi* and *Kripke-style semantics* for a standard temporal logic, (propositional until-free) *linear-time temporal logic* (LTL) [52]. This notion is not new and is based on previous studies focused on representing refutation-aware reasoning [25,43,47,59]. Based on these studies, it was informally suggested in [35] that proof systems and/or semantics are said to be *falsification-* or *refutation-aware* if they are capable of providing (or representing) the direct (or explicit) falsifications or refutations of given negated formulas (except for the negated atomic formula). Using some of refutation-aware proof systems and semantics, we can simultaneously handle or represent refutation (or falsification) and verification (or falsification). Using a refutation-aware proof system, we can also directly obtain a disproof of a given negated formula, where a disproof represents a refutation process for the given formula.

Presented by **Heinrich Wansing**; *Received* September 26, 2021

Studia Logica (2023) 111: 979–1014
https://doi.org/10.1007/s11225-023-10052-7

Thus, we can obtain both proof and disproof in the system. For more explanations on refutation-aware proof systems (or inference rules), see Sect. 1.4. In this study, we also call *refutation-aware logics* (or *reasoning*) and *logical methods* for the logics and methods that are constructed based on some refutation-aware calculi and semantics.

Next, we explain traditional motivations for studying refutation-aware logics, calculi, and semantics. The notion of refutation or falsification is critical in the fields of computer science and philosophy. Actually, falsification-aware model checking, which is an automated method for verifying inconsistent concurrent systems, has been studied, for example, in [9,23,37,38]. Adequate representation of refutation-aware reasoning is considered as a major concern in philosophy [25,43,47,59]. Thus, refutation-aware logics are required for these analyses. The typical examples of refutation-aware logics are *Nelson's paraconsistent four-valued logic* N4 [2,40,41,49,69], *Belnap and Dunn's four-valued logic* (*Belnap–Dunn logic*, *Dunn–Belnap logic*, or *first-degree entailment logic*) [4,5,14], *bi-intuitionistic logics* [47,54–56,71], and *dual-intuitionistic logics* (or *falsification logics*) [11,20,59,60,67]. Some of these logics such as Nelson's N4 have refutation-aware semantics that provide explicit interpretations of refutations (i.e., the satisfaction relations of the refutations and verifications in the semantics are clearly divided into $\models^-$ and $\models^+$, respectively). The above-mentioned refutation-aware logics are also regarded as *paraconsistent or inconsistency-tolerant logics* [53], which have no *axiom of explosion* $(\alpha \wedge \neg\alpha) \rightarrow \beta$ and are known to be useful for handling inconsistency-tolerant reasoning.

1.2. The Aims of this Study

Next, we explain the aims of this study. In this study, we aim to obtain philosophically plausible refutation-aware Gentzen-style sequent calculi and Kripke-style semantics that can provide a clear understanding of the refutation-aware temporal reasoning within propositional until-free LTL and its paraconsistent subsystem. Some of these proposed sequent calculi and semantics explicitly distinguish between refutation (or falsification) and verification (or justification). On the one hand, the refutation-aware sequent calculi and semantics for the above-mentioned paraconsistent logics provide a clear understanding of the refutation-aware and inconsistency-tolerant reasoning within the underlying logics. On the other hand, the standard semantics and sequent calculi for (non-paraconsistent) temporal logics are unsuitable for representing refutation-aware and inconsistency-tolerant temporal reasoning within the logics. Thus, we try to construct the above-mentioned refutation-aware framework.

Furthermore, we also aim to obtain a compatible and unified framework for generalizing and integrating paraconsistent and standard temporal logics. Actually, the proposed refutation-aware framework is highly compatible with a paraconsistent temporal logic, *paraconsistent linear-time temporal logic* (PLTL) [39]. A paraconsistent subsystem of PLTL and LTL can be easily obtained in a modular way from the proposed framework by deleting only a few items. Namely, the proposed framework is regarded as a unified generalization or integration of those for PLTL and LTL. Additionally, the proposed framework is also regarded as a generalized extension of that for Nelson's N4. Namely, the proposed refutation-aware Gentzen-style sequent calculi and Kripke-style semantics for propositional until-free LTL are regarded as natural and straightforward extensions of the existing and traditionally studied refutation-aware Gentzen-style sequent calculi and Kripke-style semantics for N4.

1.3. Linear-Time Temporal Logic and Its Paraconsistent Variant

LTL and its applications in computer science have been studied widely [3, 6, 10, 12, 15, 18, 28, 33, 39, 44, 45, 52]. From a purely proof-theoretic point of view, many of cut-free and complete Gentzen-style sequent calculi have been introduced for propositional until-free LTL (i.e., the propositional fragment of LTL without the until operator) and its extensions and modifications [3, 6, 18, 28, 33, 44, 50, 51, 65, 66]. The until operator in LTL entails a certain difficult situation in constructing a simple cut-free two-sided LK-compatible Gentzen-style sequent calculus. A few cut-free and complete sequent calculi extended by adding the until operator were successfully developed in [6, 18]. However, we cannot use these calculi in this study because these are not compatible with the present approach dealing with refutation-aware and inconsistency-tolerant calculi.

Since cut-elimination theorem for Gentzen-style sequent calculus plays a critical role in obtaining some good properties including decidability and subformula property, many of Gentzen-style sequent calculi have been constructed for until-free LTL. For example, in [44], a Gentzen-style sequent calculus LT_ω was introduced for first-order until-free LTL, and the cut-elimination and completeness theorems for this calculus were proven. In [3], a 2-sequent calculus $2S\omega$ was introduced for first-order until-free LTL, along with proofs of the cut-elimination and completeness theorems for this calculus. In [28], an equivalence was shown between the propositional fragments of LT_ω and $2S\omega$, and alternative proofs of the cut-elimination theorems for the propositional fragments of LT_ω and $2S\omega$ were given based on this equivalence. In [33], some embedding-based proofs were presented for the cut-elimination and completeness theorems of LT_ω and its propositional fragment.

In [39], a Gentzen-style sequent calculus PLT_ω was introduced for PLTL extending propositional LT_ω, and the cut-elimination and completeness theorems for PLT_ω were proven using an embedding-based method. PLT_ω was introduced combining LT_ω with a refutation-aware Gentzen-style sequent calculus for Nelson's N4. The logic PLTL, which was formalized as PLT_ω, is regarded as a paraconsistent extension of propositional until-free LTL by adding a paraconsistent negation connective. In this study, we introduce and investigate some alternatives to LT_ω and PLT_ω referred to as NLT_ω, DLT_ω, NLT_ω^-, and DLT_ω^-. Thus, in what follows, we use the simple name LT for propositional until-free LTL.

1.4. The Results of this Study

In this study, we introduce and evaluate refutation-aware Gentzen-style sequent calculi and Kripke-style semantics for LT. The proposed sequent calculi and semantics are extensions or generalizations of the refutation-aware Gentzen-style sequent calculi and normal- and dual-style semantics for classical logic introduced previously, as constructed in [35] based on the idea of the refutation-aware setting for Nelson's N4. To extend or generalize this classical logic framework to LT, we need to solve some technical problems concerned with interactions between the classical negation connective and temporal operators. For example, the interactions between the classical negation connective and next-time operators must be treated carefully in the proofs of the cut-elimination and completeness theorems for the proposed calculi and semantics.

We now present the details of the results of this study. First, we introduce a refutation-aware normal Gentzen-style sequent calculus NLT_ω, which can be roughly considered as a modified subsystem of PLT_ω. Second, we introduce a refutation-aware dual Gentzen-style sequent calculus DLT_ω that is constructed on the basis of the dual sequents $\Gamma \Rightarrow^+ \Delta$ (verification) and $\Gamma \Rightarrow^- \Delta$ (refutation). We then prove the equivalences among LT_ω, NLT_ω and DLT_ω as well as obtain the cut-elimination theorems for NLT_ω and DLT_ω. Third, we introduce refutation-aware normal Kripke-style semantics for LT using the normal satisfaction relation $\models^\star$. Fourth, we introduce refutation-aware dual Kripke-style semantics for LT using the dual satisfaction relations $\models^+$ (verification) and $\models^-$ (refutation). Finally, we prove the equivalences among the normal, refutation-aware normal, and refutation-aware dual Kripke-style semantics for LT and obtain the completeness theorems with respect to the proposed and existing semantics for LT_ω, NLT_ω, and DLT_ω.

Next, we explain the differences between LT_ω and DLT_ω. On the one hand, the treatment of classical negation in LT_ω is not refutation-aware; this is because the classical negation connective $\neg$ in LT_ω is characterized by the following logical inference rules with the i-nested next-time operator X:

$$\frac{\Gamma \Rightarrow \Delta, \mathrm{X}^i\alpha}{\mathrm{X}^i\neg\alpha, \Gamma \Rightarrow \Delta}\ (\neg\text{left}) \quad \frac{\mathrm{X}^i\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \mathrm{X}^i\neg\alpha}\ (\neg\text{right})$$

where ($\neg$left) and ($\neg$right) do not represent explicit (or direct) refutation (or falsification) of α. On the other hand, the treatment of classical negation in DLT_ω is refutation-aware; DLT_ω has no ($\neg$left) and ($\neg$right) but has some refutation-aware dual-logical inference rules based on the refutation-aware negative sequent $\Rightarrow^-$. Examples of the refutation-aware dual-logical inference rules in DLT_ω are of the form

$$\frac{\mathrm{X}^i\alpha, \Gamma \Rightarrow^- \Delta \quad \mathrm{X}^i\beta, \Gamma \Rightarrow^- \Delta}{\mathrm{X}^i(\alpha \wedge \beta), \Gamma \Rightarrow^- \Delta}\ (-\wedge\text{left}) \quad \frac{\Gamma \Rightarrow^- \Delta, \mathrm{X}^i\alpha, \mathrm{X}^i\beta}{\Gamma \Rightarrow^- \Delta, \mathrm{X}^i(\alpha \wedge \beta)}\ (-\wedge\text{right})$$

where ($-\wedge$left) and ($-\wedge$right) represent the explicit refutation (or falsification) of the conjunction $\alpha \wedge \beta$. The proposed falsification-aware dual-system DLT_ω has refutation-aware dual-logical inference rules for all connectives and temporal operators.

The remainder of this paper is structured as follows. In Sect. 2, we introduce LT_ω, NLT_ω, and DLT_ω; prove the equivalences between them; and show the cut-elimination theorems for NLT_ω and DLT_ω. Additionally, we observe that the paraconsistent subsystems NLT^-_ω and DLT^-_ω of NLT_ω and DLT_ω, respectively, can be easily obtained by deleting a few initial sequents. In Sect. 3, we introduce the normal, refutation-aware normal, and refutation-aware dual Kripke-style semantics for LT; prove the equivalences among these semantics; and show the completeness theorems with respect to these semantics for LT_ω, NLT_ω, and DLT_ω. In Sect. 4, we conclude this study, address some remarks, and present some related works.

2. Refutation-Aware Gentzen-Style Sequent Calculi

2.1. Normal Gentzen-Style Sequent Calculus

Formulas of LT are constructed from countably many propositional variables by $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (implication), $\neg$ (negation), G (anytime or globally in the future), F (some-time or eventually in the future), and X (next-time). We consider in this study formulas without until operator and first-order quantifiers. We use lower-case letters $p, q, \ldots$ to denote propositional variables, Greek lower-case letters $\alpha, \beta, \ldots$ to denote formulas, and Greek capital letters $\Gamma, \Delta, \ldots$ to denote finite (possibly empty) sets of formulas. For any $\sharp \in \{\neg, \mathrm{X}, \mathrm{G}, \mathrm{F}\}$, we use an expression $\sharp\Gamma$ to denote the set $\{\sharp\gamma \mid \gamma \in \Gamma\}$. We use the symbol $\equiv$ to denote the equality of symbols, the symbol ω to denote the set of natural numbers, lower-case letters i, j and k to denote any natural numbers, and the symbol $\geq$ or $\leq$ to denote the standard one. We define an expression $\mathrm{X}^i\alpha$ for any $i \in \omega$ inductively by $\mathrm{X}^0\alpha \equiv \alpha$ and $\mathrm{X}^{n+1}\alpha \equiv \mathrm{X}^n\mathrm{X}\alpha$. We call an expression of the form $\Gamma \Rightarrow \Delta$ a *sequent*. We use an expression $L \vdash S$ to denote the fact that a sequent S is provable in a sequent calculus L where L in this expression will occasionally be omitted. We use an expression $\alpha \Leftrightarrow \beta$ to denote the abbreviation of the sequents $\alpha \Rightarrow \beta$ and $\beta \Rightarrow \alpha$. We say that "two sequent calculi L_1 and L_2 are *theorem-equivalent*" if $\{S \mid L_1 \vdash S\} = \{S \mid L_2 \vdash S\}$. We say that "a rule R of inference is *admissible* in a sequent calculus L" if the following condition is satisfied: for any instance

$$\frac{S_1 \quad \cdots \quad S_n}{S}$$

of R, if $L \vdash S_i$ for all i, then $L \vdash S$. Furthermore, we say that "R is *derivable* in L" if there is a derivation from $S_1, \cdots, S_n$ to S in L, where a derivation is a finite tree of sequents followed from some inference rules in L. The notations and notions presented above are used for all sequent calculi and semantics discussed in this paper.

We define a normal Gentzen-style sequent calculus LT_ω for LT.

Definition 2.1. (LT_ω) The initial sequents of LT_ω are of the form: for any propositional variable p,

$$\mathrm{X}^i p \Rightarrow \mathrm{X}^i p.$$

The structural rules of LT_ω are of the form:

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \alpha, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (\text{cut}) \quad \frac{\Gamma \Rightarrow \Delta}{\alpha, \Gamma \Rightarrow \Delta}\ (\text{we-left}) \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \alpha}\ (\text{we-right}).$$

The logical inference rules of LT_ω are of the form:

$$\frac{\mathrm{X}^i\alpha, \mathrm{X}^i\beta, \Gamma \Rightarrow \Delta}{\mathrm{X}^i(\alpha \wedge \beta), \Gamma \Rightarrow \Delta}\ (\wedge\text{left}) \quad \frac{\Gamma \Rightarrow \Delta, \mathrm{X}^i\alpha \quad \Gamma \Rightarrow \Delta, \mathrm{X}^i\beta}{\Gamma \Rightarrow \Delta, \mathrm{X}^i(\alpha \wedge \beta)}\ (\wedge\text{right})$$

$$\frac{\mathrm{X}^i\alpha, \Gamma \Rightarrow \Delta \quad \mathrm{X}^i\beta, \Gamma \Rightarrow \Delta}{\mathrm{X}^i(\alpha \vee \beta), \Gamma \Rightarrow \Delta}\ (\vee\text{left}) \quad \frac{\Gamma \Rightarrow \Delta, \mathrm{X}^i\alpha, \mathrm{X}^i\beta}{\Gamma \Rightarrow \Delta, \mathrm{X}^i(\alpha \vee \beta)}\ (\vee\text{right})$$

$$\frac{\Gamma \Rightarrow \Sigma, \mathrm{X}^i\alpha \quad \mathrm{X}^i\beta, \Delta \Rightarrow \Pi}{\mathrm{X}^i(\alpha \rightarrow \beta), \Gamma, \Delta \Rightarrow \Sigma, \Pi}\ (\rightarrow\text{left}) \quad \frac{\mathrm{X}^i\alpha, \Gamma \Rightarrow \Delta, \mathrm{X}^i\beta}{\Gamma \Rightarrow \Delta, \mathrm{X}^i(\alpha \rightarrow \beta)}\ (\rightarrow\text{right})$$

$$\frac{\Gamma \Rightarrow \Delta, \mathrm{X}^i\alpha}{\mathrm{X}^i\neg\alpha, \Gamma \Rightarrow \Delta}\ (\neg\text{left}) \quad \frac{\mathrm{X}^i\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \mathrm{X}^i\neg\alpha}\ (\neg\text{right})$$

$$\frac{\mathrm{X}^{i+k}\alpha, \Gamma \Rightarrow \Delta}{\mathrm{X}^i\mathrm{G}\alpha, \Gamma \Rightarrow \Delta}\ (\text{Gleft}) \quad \frac{\{\ \Gamma \Rightarrow \Delta, \mathrm{X}^{i+j}\alpha\ \}_{j\in\omega}}{\Gamma \Rightarrow \Delta, \mathrm{X}^i\mathrm{G}\alpha}\ (\text{Gright})$$

$$\frac{\{\ \mathrm{X}^{i+j}\alpha, \Gamma \Rightarrow \Delta\ \}_{j\in\omega}}{\mathrm{X}^i\mathrm{F}\alpha, \Gamma \Rightarrow \Delta}\ (\text{Fleft}) \quad \frac{\Gamma \Rightarrow \Delta, \mathrm{X}^{i+k}\alpha}{\Gamma \Rightarrow \Delta, \mathrm{X}^i\mathrm{F}\alpha}\ (\text{Fright}).$$

REMARK 2.2. We make the following remarks.

1. The system (first-order) LT_ω (with some slight modifications) was originally introduced by Kawai in [44] for first-order LT. The original system was introduced for a first-order sequent calculus with Barcan formula. In the original system, the next-time operator was not used as a modal operator but used as a special symbol.
2. The cut-elimination theorem holds for LT_ω. Namely, the rule (cut) is admissible in cut-free LT_ω. This cut-elimination theorem for LT_ω was originally proved by Kawai in [44] and was indirectly re-proved by Kamide in [28] via the cut-free equivalence between LT_ω and Baratella-Masini's cut-free 2-sequent calculus $2\mathrm{S}\omega$ [3].
3. Note that the inference rules (Gright) and (Fleft) in LT_ω have an infinite number of premises.
4. The sequents of the form $\mathrm{X}^i\alpha \Rightarrow \mathrm{X}^i\alpha$ for any formula α are provable in cut-free LT_ω. This fact can be proved by induction on the complexity of α. See, e.g., [28].

2.2. Refutation-Aware Normal Gentzen-Style Sequent Calculus

We define a refutation-aware normal Gentzen-style sequent calculus NLT_ω for LT.

DEFINITION 2.3. (NLT_ω) NLT_ω is obtained from LT_ω by replacing ($\neg$left) and ($\neg$right) with the initial sequents of the form: for any propositional variable p,

$$\mathrm{X}^i\neg p \Rightarrow \mathrm{X}^i\neg p \qquad \mathrm{X}^i p, \mathrm{X}^i\neg p \Rightarrow \qquad \Rightarrow \mathrm{X}^i p, \mathrm{X}^i\neg p$$

and the negated logical inference rules of the form:

$$\frac{\mathrm{X}^i\neg\alpha, \Gamma \Rightarrow \Delta \quad \mathrm{X}^i\neg\beta, \Gamma \Rightarrow \Delta}{\mathrm{X}^i\neg(\alpha \wedge \beta), \Gamma \Rightarrow \Delta}\ (\neg\wedge\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \mathrm{X}^i\neg\alpha, \mathrm{X}^i\neg\beta}{\Gamma \Rightarrow \Delta, \mathrm{X}^i\neg(\alpha \wedge \beta)}\ (\neg\wedge\text{right})$$

$$\frac{\mathrm{X}^i\neg\alpha, \mathrm{X}^i\neg\beta, \Gamma \Rightarrow \Delta}{\mathrm{X}^i\neg(\alpha\vee\beta), \Gamma \Rightarrow \Delta}\ (\neg\vee\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \mathrm{X}^i\neg\alpha \quad \Gamma \Rightarrow \Delta, \mathrm{X}^i\neg\beta}{\Gamma \Rightarrow \Delta, \mathrm{X}^i\neg(\alpha\vee\beta)}\ (\neg\vee\text{right})$$

$$\frac{\mathrm{X}^i\alpha, \mathrm{X}^i\neg\beta, \Gamma \Rightarrow \Delta}{\mathrm{X}^i\neg(\alpha\to\beta), \Gamma \Rightarrow \Delta}\ (\neg\to\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \mathrm{X}^i\alpha \quad \Gamma \Rightarrow \Delta, \mathrm{X}^i\neg\beta}{\Gamma \Rightarrow \Delta, \mathrm{X}^i\neg(\alpha\to\beta)}\ (\neg\to\text{right})$$

$$\frac{\mathrm{X}^i\alpha, \Gamma \Rightarrow \Delta}{\mathrm{X}^i\neg\neg\alpha, \Gamma \Rightarrow \Delta}\ (\neg\neg\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \mathrm{X}^i\alpha}{\Gamma \Rightarrow \Delta, \mathrm{X}^i\neg\neg\alpha}\ (\neg\neg\text{right})$$

$$\frac{\mathrm{X}^i\neg\alpha, \Gamma \Rightarrow \Delta}{\neg\mathrm{X}^i\alpha, \Gamma \Rightarrow \Delta}\ (\neg\text{Xleft}) \qquad \frac{\Gamma \Rightarrow \Delta, \mathrm{X}^i\neg\alpha}{\Gamma \Rightarrow \Delta, \neg\mathrm{X}^i\alpha}\ (\neg\text{Xright})$$

$$\frac{\{\ \mathrm{X}^{i+j}\neg\alpha, \Gamma \Rightarrow \Delta\ \}_{j\in\omega}}{\mathrm{X}^i\neg\mathrm{G}\alpha, \Gamma \Rightarrow \Delta}\ (\neg\text{Gleft}) \qquad \frac{\Gamma \Rightarrow \Delta, \mathrm{X}^{i+k}\neg\alpha}{\Gamma \Rightarrow \Delta, \mathrm{X}^i\neg\mathrm{G}\alpha}\ (\neg\text{Gright})$$

$$\frac{\mathrm{X}^{i+k}\neg\alpha, \Gamma \Rightarrow \Delta}{\mathrm{X}^i\neg\mathrm{F}\alpha, \Gamma \Rightarrow \Delta}\ (\neg\text{Fleft}) \qquad \frac{\{\ \Gamma \Rightarrow \Delta, \mathrm{X}^{i+j}\neg\alpha\ \}_{j\in\omega}}{\Gamma \Rightarrow \Delta, \mathrm{X}^i\neg\mathrm{F}\alpha}\ (\neg\text{Fright}).$$

REMARK 2.4. We make the following remarks.

1. The non-negated logical inference rules of NLT_ω represent "verification (or justification)" and the negated logical inference rules of NLT_ω represent "refutation (or falsification)".
2. Let NLT_ω^- be the subsystem that is obtained from NLT_ω by deleting the initial sequents of the form $\mathrm{X}^i p, \mathrm{X}^i\neg p \Rightarrow$ and $\Rightarrow \mathrm{X}^i p, \mathrm{X}^i\neg p$. Then, NLT_ω^- is (a slight non-essential modification of) the classical-negation-free fragment of the Gentzen-style sequent calculus PLT_ω introduced in [39] for *paraconsistent linear-time temporal logic* (PLTL).
3. PLT_ω used both the classical negation connective $\neg$ and the paraconsistent negation connective $\sim$ and was introduced as an extension of LT_ω.
4. The cut-elimination theorem for PLT_ω was proved in [39]. Thus, we can also obtain the cut-elimination theorem for NLT_ω^-.
5. PLT_ω was shown in [39] to be embeddable into LT_ω. Similar to this fact, we can show that NLT_ω^- is embeddable into the classical-negation free fragment of LT_ω.

6. The sequents of the form $\mathrm{X}^i\alpha \Rightarrow \mathrm{X}^i\alpha$ for any formula α are provable in cut-free NLT_ω^-. This fact can be proved by induction on the complexity of α.

PROPOSITION 2.5. *The following sequents are provable in cut-free* NLT_ω*: for any formula* α*,*

1. $\mathrm{X}^i\alpha \Rightarrow \mathrm{X}^i\alpha$,
2. $\mathrm{X}^i\alpha, \mathrm{X}^i\neg\alpha \Rightarrow$,
3. $\Rightarrow \mathrm{X}^i\alpha, \mathrm{X}^i\neg\alpha$.

PROOF. By simultaneous induction on α. The cases for $\mathrm{X}^i\alpha \Rightarrow \mathrm{X}^i\alpha$ can be shown in the same way as those for LT_ω. Thus, in the following, we show only the cases for $\mathrm{X}^i\alpha, \mathrm{X}^i\neg\alpha \Rightarrow$ and $\Rightarrow \mathrm{X}^i\alpha, \mathrm{X}^i\neg\alpha$. We distinguish the cases according to the form of α and show only the following cases.

1. Case $\alpha \equiv \beta{\to}\gamma$: We obtain the required facts:

$$\dfrac{\dfrac{\overset{\vdots\ \textit{Ind. hyp.}}{\mathrm{X}^i\beta \Rightarrow \mathrm{X}^i\beta} \qquad \overset{\vdots\ \textit{Ind. hyp.}}{\mathrm{X}^i\gamma, \mathrm{X}^i\neg\gamma \Rightarrow}}{\mathrm{X}^i(\beta{\to}\gamma), \mathrm{X}^i\beta, \mathrm{X}^i\neg\gamma \Rightarrow}\ (\to\text{left})}{\mathrm{X}^i(\beta{\to}\gamma), \mathrm{X}^i\neg(\beta{\to}\gamma) \Rightarrow}\ (\neg{\to}\text{left})$$

$$\dfrac{\dfrac{\dfrac{\overset{\vdots\ \textit{Ind. hyp.}}{\mathrm{X}^i\beta \Rightarrow \mathrm{X}^i\beta}}{\mathrm{X}^i\beta \Rightarrow \mathrm{X}^i\gamma, \mathrm{X}^i\beta}\ (\text{we-right})}{\Rightarrow \mathrm{X}^i(\beta{\to}\gamma), \mathrm{X}^i\beta}\ (\to\text{right}) \quad \dfrac{\dfrac{\overset{\vdots\ \textit{Ind. hyp.}}{\Rightarrow \mathrm{X}^i\gamma, \mathrm{X}^i\neg\gamma}}{\mathrm{X}^i\beta \Rightarrow \mathrm{X}^i\gamma, \mathrm{X}^i\neg\gamma}\ (\text{we-left})}{\Rightarrow \mathrm{X}^i(\beta{\to}\gamma), \mathrm{X}^i\neg\gamma}\ (\to\text{right})}{\Rightarrow \mathrm{X}^i(\beta{\to}\gamma), \mathrm{X}^i\neg(\beta{\to}\gamma)}\ (\neg{\to}\text{right}).$$

2. Case $\alpha \equiv \mathrm{G}\beta$: We obtain the required facts:

$$\dfrac{\dfrac{\overset{\vdots\ \textit{Ind. hyp.}}{\{\mathrm{X}^{i+j}\beta, \mathrm{X}^{i+j}\neg\beta \Rightarrow\}_{j\in\omega}}}{\{\mathrm{X}^i\mathrm{G}\beta, \mathrm{X}^{i+j}\neg\beta \Rightarrow\}_{j\in\omega}}\ (\text{Gleft})}{\mathrm{X}^i\mathrm{G}\beta, \mathrm{X}^i\neg\mathrm{G}\beta \Rightarrow}\ (\neg\text{Gleft}) \qquad \dfrac{\dfrac{\overset{\vdots\ \textit{Ind. hyp.}}{\{\Rightarrow \mathrm{X}^{i+j}\beta, \mathrm{X}^{i+j}\neg\beta\}_{j\in\omega}}}{\{\Rightarrow \mathrm{X}^{i+j}\beta, \mathrm{X}^i\neg\mathrm{G}\beta\}_{j\in\omega}}\ (\neg\text{Gright})}{\Rightarrow \mathrm{X}^i\mathrm{G}\beta, \mathrm{X}^i\neg\mathrm{G}\beta}\ (\text{Gright}).$$

■

PROPOSITION 2.6. *The following rules are derivable in* NLT_ω *using* (cut)*:*

$$\dfrac{\Gamma \Rightarrow \Delta, \mathrm{X}^i\alpha}{\mathrm{X}^i\neg\alpha, \Gamma \Rightarrow \Delta}\ (\neg\text{left}) \qquad \dfrac{\mathrm{X}^i\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \mathrm{X}^i\neg\alpha}\ (\neg\text{right})$$

$$\frac{\Gamma \Rightarrow \Delta, \mathrm{X}^i\neg\alpha}{\mathrm{X}^i\alpha, \Gamma \Rightarrow \Delta}\ (\neg\text{left}^{-1}) \quad \frac{\mathrm{X}^i\neg\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \mathrm{X}^i\alpha}\ (\neg\text{right}^{-1}).$$

PROOF. We show only the derivability of (¬left) as follows.

$$\dfrac{\Gamma \Rightarrow \Delta, \mathrm{X}^i\alpha \quad \overset{\vdots\ \textit{Prop. } 2.5}{\mathrm{X}^i\alpha, \mathrm{X}^i\neg\alpha \Rightarrow}}{\mathrm{X}^i\neg\alpha, \Gamma \Rightarrow \Delta}\ (\text{cut}).$$

■

PROPOSITION 2.7. *The following rules are admissible in cut-free* NLT_ω*:*

$$\frac{\neg\mathrm{X}^i\alpha, \Gamma \Rightarrow \Delta}{\mathrm{X}^i\neg\alpha, \Gamma \Rightarrow \Delta}\ (\neg\text{Xleft}^{-1}) \quad \frac{\Gamma \Rightarrow \Delta, \neg\mathrm{X}^i\alpha}{\Gamma \Rightarrow \Delta, \mathrm{X}^i\neg\alpha}\ (\neg\text{Xright}^{-1}).$$

PROOF. The case for $(\neg\text{Xleft}^{-1})$ is proved by induction on the proofs P of $\neg\mathrm{X}^i\alpha, \Gamma \Rightarrow \Delta$ in cut-free NLT_ω, and the case for $(\neg\text{Xright}^{-1})$ is proved by induction on the proofs P of $\Gamma \Rightarrow \Delta, \neg\mathrm{X}^i\alpha$ in cut-free NLT_ω. ■

THEOREM 2.8. *(Equivalence between* NLT_ω *and* LT_ω*) The systems* NLT_ω *and* LT_ω *are theorem-equivalent.*

PROOF. Obviously, the negated initial sequents of NLT_ω are provable in LT_ω, and the negated logical inference rules of NLT_ω are derivable in LT_ω. Conversely, (¬left) and (¬right) in LT_ω are derivable in NLT_ω using (cut) by Proposition 2.6. Therefore, NLT_ω and LT_ω are theorem-equivalent. ■

THEOREM 2.9. *(Cut-elimination for* NLT_ω*) The rule* (cut) *is admissible in cut-free* NLT_ω*.*

PROOF. We give a sketch of the proof. As mentioned in Remark 2.4, we have the the cut-elimination theorem for the subsystem NLT_ω^- of NLT_ω obtained by deleting the initial sequents of the form $\mathrm{X}^ip, \mathrm{X}^i\neg p \Rightarrow$ and $\Rightarrow \mathrm{X}^ip, \mathrm{X}^i\neg p$. Thus, it is sufficient to consider the cases of the initial sequents $\mathrm{X}^ip, \mathrm{X}^i\neg p \Rightarrow$ and $\Rightarrow \mathrm{X}^ip, \mathrm{X}^i\neg p$. Since the cases for $\mathrm{X}^ip, \mathrm{X}^i\neg p \Rightarrow$ and the cases for $\Rightarrow \mathrm{X}^ip, \mathrm{X}^i\neg p$ can be similarly considered in a symmetric manner, we here consider only the cases for $\mathrm{X}^ip, \mathrm{X}^i\neg p \Rightarrow$. We now demonstrate some of the cases below.

1. Case when the left upper sequent of the cut is derived from a single premise left logical inference rule R:

$$\dfrac{\dfrac{\overset{\vdots\ P}{\Gamma \Rightarrow \Delta, \mathrm{X}^i\neg p}}{\Gamma^* \Rightarrow \Delta, \mathrm{X}^i\neg p}\ (R) \quad \mathrm{X}^i\neg p, \mathrm{X}^ip \Rightarrow}{\mathrm{X}^ip, \Gamma^* \Rightarrow \Delta}\ (\text{cut})$$

where P is a cut-free proof. In this case, we can transform this proof into the following proof that can eliminate the cut by induction hypothesis:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\, P\\ \Gamma \Rightarrow \Delta, \mathrm{X}^i\neg p\end{array} \quad \mathrm{X}^i\neg p, \mathrm{X}^i p \Rightarrow}{\mathrm{X}^i p, \Gamma \Rightarrow \Delta}\ (\text{cut})}{\mathrm{X}^i p, \Gamma^* \Rightarrow \Delta}\ (R).$$

2. Case when the left upper sequent of the cut is derived from (we-right) where the principal formula of (we-right) is $\mathrm{X}^i\neg p$:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\, P\\ \Gamma \Rightarrow \Delta\end{array}}{\Gamma \Rightarrow \Delta, \mathrm{X}^i\neg p}\ (\text{we-right}) \quad \mathrm{X}^i\neg p, \mathrm{X}^i p \Rightarrow}{\mathrm{X}^i p, \Gamma \Rightarrow \Delta}\ (\text{cut})$$

where P is a cut-free proof. In this case, we can transform this proof into the following cut-free proof:

$$\dfrac{\begin{array}{c}\vdots\, P\\ \Gamma \Rightarrow \Delta\end{array}}{\mathrm{X}^i p, \Gamma \Rightarrow \Delta}\ (\text{we-left}).$$

■

Theorem 2.10. *(Classical-negation-elimination for* NLT_ω*) The rules* ($\neg$left) *and* ($\neg$right) *are admissible in cut-free* NLT_ω*.*

Proof. By Proposition 2.6, ($\neg$left) and ($\neg$right) are derivable in NLT_ω using (cut). Then, by Theorem 2.9, (cut) in the derivations of ($\neg$left) and ($\neg$right) can be eliminated. Thus, ($\neg$left) and ($\neg$right) are admissible in cut-free NLT_ω. ■

Remark 2.11. We make the following remarks.

1. The cut-elimination theorem for NLT_ω can be proved directly as shown by Gentzen for his original sequent calculus LK for classical logic [19].
2. The cut-elimination theorem for NLT_ω^- can be proved in a similar way as in [39], wherein an embedding-based proof of the cut-elimination theorem for PLT_ω was given.
3. However, the embedding-based method presented in [39] for PLT_ω cannot work for NLT_ω.

2.3. Refutation-Aware Dual Gentzen-Style Sequent Calculus

We introduce a refutation-aware dual Gentzen-style sequent calculus DLT_ω for LT. We call an expression of the form $\Gamma \Rightarrow^+ \Delta$ or $\Gamma \Rightarrow^- \Delta$ *positive sequent* or *negative sequent*, respectively. We use the symbol $*$ to denote an arbitrary element of $\{+,-\}$. The intuitive meanings of $\Gamma \Rightarrow^+ \Delta$ and $\Gamma \Rightarrow^- \Delta$ are verification (or justification) and refutation (or falsification), respectively. More precisely, the meaning of $\Gamma \Rightarrow^+ \Delta$ is the same as that of $\Gamma \Rightarrow \Delta$, and the meaning of $\Gamma \Rightarrow^- \Delta$ is the same as that of $\neg\Gamma \Rightarrow \neg\Delta$ (i.e., if we can refute all of the formulas in Γ, then we can refute one of the formulas in Δ). These meanings will be justified in Theorem 2.16.

DEFINITION 2.12. (DLT_ω) The initial sequents of DLT_ω are of the form: for any propositional variable p,

$$\mathrm{X}^i p \Rightarrow^* \mathrm{X}^i p \qquad \mathrm{X}^i p, \mathrm{X}^i \neg p \Rightarrow^* \qquad \Rightarrow^* \mathrm{X}^i p, \mathrm{X}^i \neg p.$$

The structural inference rules of DLT_ω are of the form:

$$\frac{\Gamma \Rightarrow^* \Delta, \alpha \quad \alpha, \Sigma \Rightarrow^* \Pi}{\Gamma, \Sigma \Rightarrow^* \Delta, \Pi}\ (*\text{cut})$$

$$\frac{\Gamma \Rightarrow^* \Delta}{\alpha, \Gamma \Rightarrow^* \Delta}\ (*\text{we-left}) \qquad \frac{\Gamma \Rightarrow^* \Delta}{\Gamma \Rightarrow^* \Delta, \alpha}\ (*\text{we-right}).$$

The conversion inference rules of DLT_ω are of the form:

$$\frac{\neg\Gamma, \Delta \Rightarrow^- \neg\Sigma, \Pi}{\Gamma, \neg\Delta \Rightarrow^+ \Sigma, \neg\Pi}\ (-\text{to}+) \qquad \frac{\neg\Gamma, \Delta \Rightarrow^+ \neg\Sigma, \Pi}{\Gamma, \neg\Delta \Rightarrow^- \Sigma, \neg\Pi}\ (+\text{to}-).$$

The positive logical inference rules of DLT_ω are of the form:

$$\frac{\mathrm{X}^i\alpha, \mathrm{X}^i\beta, \Gamma \Rightarrow^+ \Delta}{\mathrm{X}^i(\alpha \wedge \beta), \Gamma \Rightarrow^+ \Delta}\ (+\wedge\text{left}) \qquad \frac{\Gamma \Rightarrow^+ \Delta, \mathrm{X}^i\alpha \quad \Gamma \Rightarrow^+ \Delta, \mathrm{X}^i\beta}{\Gamma \Rightarrow^+ \Delta, \mathrm{X}^i(\alpha \wedge \beta)}\ (+\wedge\text{right})$$

$$\frac{\mathrm{X}^i\alpha, \Gamma \Rightarrow^+ \Delta \quad \mathrm{X}^i\beta, \Gamma \Rightarrow^+ \Delta}{\mathrm{X}^i(\alpha \vee \beta), \Gamma \Rightarrow^+ \Delta}\ (+\vee\text{left}) \qquad \frac{\Gamma \Rightarrow^+ \Delta, \mathrm{X}^i\alpha, \mathrm{X}^i\beta}{\Gamma \Rightarrow^+ \Delta, \mathrm{X}^i(\alpha \vee \beta)}\ (+\vee\text{right})$$

$$\frac{\Gamma \Rightarrow^+ \Delta, \mathrm{X}^i\alpha \quad \mathrm{X}^i\beta, \Sigma \Rightarrow^+ \Pi}{\mathrm{X}^i(\alpha \rightarrow \beta), \Gamma, \Sigma \Rightarrow^+ \Delta, \Pi}\ (+\rightarrow\text{left}) \qquad \frac{\mathrm{X}^i\alpha, \Gamma \Rightarrow^+ \Delta, \mathrm{X}^i\beta}{\Gamma \Rightarrow^+ \Delta, \mathrm{X}^i(\alpha \rightarrow \beta)}\ (+\rightarrow\text{right})$$

$$\frac{\mathrm{X}^i\neg\alpha, \Gamma \Rightarrow^+ \Delta}{\neg\mathrm{X}^i\alpha, \Gamma \Rightarrow^+ \Delta}\ (+\neg\text{Xleft}) \qquad \frac{\Gamma \Rightarrow^+ \Delta, \mathrm{X}^i\neg\alpha}{\Gamma \Rightarrow^+ \Delta, \neg\mathrm{X}^i\alpha}\ (+\neg\text{Xright})$$

$$\frac{\neg\mathrm{X}^i\alpha, \Gamma \Rightarrow^+ \Delta}{\mathrm{X}^i\neg\alpha, \Gamma \Rightarrow^+ \Delta}\ (+\neg\text{Xleft}^{-1}) \qquad \frac{\Gamma \Rightarrow^+ \Delta, \neg\mathrm{X}^i\alpha}{\Gamma \Rightarrow^+ \Delta, \mathrm{X}^i\neg\alpha}\ (+\neg\text{Xright}^{-1})$$

$$\frac{\mathrm{X}^{i+k}\alpha, \Gamma \Rightarrow^+ \Delta}{\mathrm{X}^i\mathrm{G}\alpha, \Gamma \Rightarrow^+ \Delta}\ (+\text{Gleft}) \qquad \frac{\{\ \Gamma \Rightarrow^+ \Delta, \mathrm{X}^{i+j}\alpha\ \}_{j \in \omega}}{\Gamma \Rightarrow^+ \Delta, \mathrm{X}^i\mathrm{G}\alpha}\ (+\text{Gright})$$

$$\frac{\{\ \mathrm{X}^{i+j}\alpha, \Gamma \Rightarrow^{+} \Delta\ \}_{j\in\omega}}{\mathrm{X}^{i}\mathrm{F}\alpha, \Gamma \Rightarrow^{+} \Delta}\ (+\text{Fleft}) \quad \frac{\Gamma \Rightarrow^{+} \Delta, \mathrm{X}^{i+k}\alpha}{\Gamma \Rightarrow^{+} \Delta, \mathrm{X}^{i}\mathrm{F}\alpha}\ (+\text{Fright}).$$

The negative logical inference rules of DLT_ω are of the form:

$$\frac{\mathrm{X}^{i}\alpha, \Gamma \Rightarrow^{-} \Delta \quad \mathrm{X}^{i}\beta, \Gamma \Rightarrow^{-} \Delta}{\mathrm{X}^{i}(\alpha \wedge \beta), \Gamma \Rightarrow^{-} \Delta}\ (-\wedge\text{left}) \quad \frac{\Gamma \Rightarrow^{-} \Delta, \mathrm{X}^{i}\alpha, \mathrm{X}^{i}\beta}{\Gamma \Rightarrow^{-} \Delta, \mathrm{X}^{i}(\alpha \wedge \beta)}\ (-\wedge\text{right})$$

$$\frac{\mathrm{X}^{i}\alpha, \mathrm{X}^{i}\beta, \Gamma \Rightarrow^{-} \Delta}{\mathrm{X}^{i}(\alpha \vee \beta), \Gamma \Rightarrow^{-} \Delta}\ (-\vee\text{left}) \quad \frac{\Gamma \Rightarrow^{-} \Delta, \mathrm{X}^{i}\alpha \quad \Gamma \Rightarrow^{-} \Delta, \mathrm{X}^{i}\beta}{\Gamma \Rightarrow^{-} \Delta, \mathrm{X}^{i}(\alpha \vee \beta)}\ (-\vee\text{right})$$

$$\frac{\mathrm{X}^{i}\neg\alpha, \mathrm{X}^{i}\beta, \Gamma \Rightarrow^{-} \Delta}{\mathrm{X}^{i}(\alpha \rightarrow \beta), \Gamma \Rightarrow^{-} \Delta}\ (-\rightarrow\text{left}) \quad \frac{\Gamma \Rightarrow^{-} \Delta, \mathrm{X}^{i}\neg\alpha \quad \Gamma \Rightarrow^{-} \Delta, \mathrm{X}^{i}\beta}{\Gamma \Rightarrow^{-} \Delta, \mathrm{X}^{i}(\alpha \rightarrow \beta)}\ (-\rightarrow\text{right})$$

$$\frac{\mathrm{X}^{i}\neg\alpha, \Gamma \Rightarrow^{-} \Delta}{\neg\mathrm{X}^{i}\alpha, \Gamma \Rightarrow^{-} \Delta}\ (-\neg\text{Xleft}) \quad \frac{\Gamma \Rightarrow^{-} \Delta, \mathrm{X}^{i}\neg\alpha}{\Gamma \Rightarrow^{-} \Delta, \neg\mathrm{X}^{i}\alpha}\ (-\neg\text{Xright})$$

$$\frac{\neg\mathrm{X}^{i}\alpha, \Gamma \Rightarrow^{-} \Delta}{\mathrm{X}^{i}\neg\alpha, \Gamma \Rightarrow^{-} \Delta}\ (-\neg\text{Xleft}^{-1}) \quad \frac{\Gamma \Rightarrow^{-} \Delta, \neg\mathrm{X}^{i}\alpha}{\Gamma \Rightarrow^{-} \Delta, \mathrm{X}^{i}\neg\alpha}\ (-\neg\text{Xright}^{-1})$$

$$\frac{\{\ \mathrm{X}^{i+j}\alpha, \Gamma \Rightarrow^{-} \Delta\ \}_{j\in\omega}}{\mathrm{X}^{i}\mathrm{G}\alpha, \Gamma \Rightarrow^{-} \Delta}\ (-\text{Gleft}) \quad \frac{\Gamma \Rightarrow^{-} \Delta, \mathrm{X}^{i+k}\alpha}{\Gamma \Rightarrow^{-} \Delta, \mathrm{X}^{i}\mathrm{G}\alpha}\ (-\text{Gright})$$

$$\frac{\mathrm{X}^{i+k}\alpha, \Gamma \Rightarrow^{-} \Delta}{\mathrm{X}^{i}\mathrm{F}\alpha, \Gamma \Rightarrow^{-} \Delta}\ (-\text{Fleft}) \quad \frac{\{\ \Gamma \Rightarrow^{-} \Delta, \mathrm{X}^{i+j}\alpha\ \}_{j\in\omega}}{\Gamma \Rightarrow^{-} \Delta, \mathrm{X}^{i}\mathrm{F}\alpha}\ (-\text{Fright}).$$

Remark 2.13. We make the following remarks.

1. There is no inital sequents of the form $\mathrm{X}^{i}\neg p \Rightarrow^{*} \mathrm{X}^{i}\neg p$ in DLT_ω, because these sequents are provable in cut-free DLT_ω using the conversion inference rules.
2. A refutation-aware dual Gentzen-style sequent calculus with positive and negative sequents was introduced in [40, 41] for Nelson's paraconsistent logic N4. DLT_ω is regarded as an LT-adapted version of the refutation-aware dual Gentzen-style sequent calculus for N4. Although the refutation-aware normal Gentzen-style sequent calculus PLT_ω for PLTL was introduced in [39], a refutation-aware dual Gentzen-style sequent calculus that is theorem-equivalent to PLT_ω was not introduced in [39].
3. Let DLT^{-}_ω be the paraconsistent subsystem that is obtained from DLT_ω by deleting the initial sequents of the form $\mathrm{X}^{i}p, \mathrm{X}^{i}\neg p \Rightarrow^{*}$ and $\Rightarrow^{*} \mathrm{X}^{i}p, \mathrm{X}^{i}\neg p$. Then, DLT^{-}_ω and NLT^{-}_ω are theorem-equivalent. This fact can be straightforwardly obtained by Theorem 2.16 (with some slight modifications).

Proposition 2.14. *The following sequents are provable in cut-free* DLT_ω*: for any formula* α*,*

1. $\mathrm{X}^i\alpha \Rightarrow^* \mathrm{X}^i\alpha$,
2. $\mathrm{X}^i\alpha, \mathrm{X}^i\neg\alpha \Rightarrow^*$,
3. $\Rightarrow^* \mathrm{X}^i\alpha, \mathrm{X}^i\neg\alpha$.

Proof. By simultaneous induction on α. We distinguish the cases according to the form of α, and show some cases for $\mathrm{X}^i\alpha, \mathrm{X}^i\neg\alpha \Rightarrow^+$.

1. Case $\alpha \equiv \beta{\rightarrow}\gamma$: We obtain the required fact:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\ \textit{Ind. hyp.}\\ \mathrm{X}^i\beta \Rightarrow^+ \mathrm{X}^i\beta\end{array} \qquad \begin{array}{c}\vdots\ \textit{Ind. hyp.}\\ \mathrm{X}^i\neg\gamma, \mathrm{X}^i\gamma \Rightarrow^+\end{array}}{\mathrm{X}^i\beta, \mathrm{X}^i\neg\gamma, \mathrm{X}^i(\beta{\rightarrow}\gamma) \Rightarrow^+}\ (+{\rightarrow}\mathrm{left})}{\mathrm{X}^i\beta, \neg\mathrm{X}^i\gamma, \mathrm{X}^i(\beta{\rightarrow}\gamma) \Rightarrow^+}\ (+\neg\mathrm{Xleft})}{\neg\mathrm{X}^i\beta, \mathrm{X}^i\gamma, \neg\mathrm{X}^i(\beta{\rightarrow}\gamma) \Rightarrow^-}\ (+\mathrm{to}-)}{\mathrm{X}^i\neg\beta, \mathrm{X}^i\gamma, \neg\mathrm{X}^i(\beta{\rightarrow}\gamma) \Rightarrow^-}\ (-\neg\mathrm{Xleft}^{-1})}{\mathrm{X}^i(\beta{\rightarrow}\gamma), \neg\mathrm{X}^i(\beta{\rightarrow}\gamma) \Rightarrow^-}\ (-{\rightarrow}\mathrm{left})}{\neg\mathrm{X}^i(\beta{\rightarrow}\gamma), \mathrm{X}^i(\beta{\rightarrow}\gamma) \Rightarrow^+}\ (-\mathrm{to}+)}{\mathrm{X}^i\neg(\beta{\rightarrow}\gamma), \mathrm{X}^i(\beta{\rightarrow}\gamma) \Rightarrow^+}\ (+\neg\mathrm{Xleft}^{-1}).$$

2. Case $\alpha \equiv \mathrm{G}\beta$: We obtain the required fact:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\ \textit{Ind. hyp.}\\ \{\mathrm{X}^{i+j}\beta, \mathrm{X}^{i+j}\neg\beta \Rightarrow^+\}_{j\in\omega}\end{array}}{\{\mathrm{X}^{i+j}\beta, \neg\mathrm{X}^{i+j}\beta \Rightarrow^+\}_{j\in\omega}}\ (+\neg\mathrm{Xleft})}{\{\mathrm{X}^i\mathrm{G}\beta, \neg\mathrm{X}^{i+j}\beta \Rightarrow^+\}_{j\in\omega}}\ (+\mathrm{Gleft})}{\{\neg\mathrm{X}^i\mathrm{G}\beta, \mathrm{X}^{i+j}\beta \Rightarrow^-\}_{j\in\omega}}\ (+\mathrm{to}-)}{\neg\mathrm{X}^i\mathrm{G}\beta, \mathrm{X}^i\mathrm{G}\beta \Rightarrow^-}\ (-\mathrm{Gleft})}{\mathrm{X}^i\mathrm{G}\beta, \neg\mathrm{X}^i\mathrm{G}\beta \Rightarrow^+}\ (-\mathrm{to}+)}{\mathrm{X}^i\mathrm{G}\beta, \mathrm{X}^i\neg\mathrm{G}\beta \Rightarrow^+}\ (+\neg\mathrm{Xleft}^{-1}).$$

■

Proposition 2.15. *The following rules are derivable in cut-free* DLT_ω*:*

$$\dfrac{\mathrm{X}^i\alpha, \Gamma \Rightarrow^* \Delta}{\mathrm{X}^i\neg\neg\alpha, \Gamma \Rightarrow^* \Delta}\ (*\neg\neg\mathrm{left}) \qquad \dfrac{\Gamma \Rightarrow^* \Delta, \mathrm{X}^i\alpha}{\Gamma \Rightarrow^* \Delta, \mathrm{X}^i\neg\neg\alpha}\ (*\neg\neg\mathrm{right}).$$

PROOF. We show only the case for (−¬¬left) as follows.

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathrm{X}^i\alpha, \Gamma \Rightarrow^- \Delta}{\neg\mathrm{X}^i\alpha, \neg\Gamma \Rightarrow^+ \neg\Delta}\ (-\mathrm{to}+)}{\mathrm{X}^i\neg\alpha, \neg\Gamma \Rightarrow^+ \neg\Delta}\ (+\neg\mathrm{Xleft}^{-1})}{\neg\mathrm{X}^i\neg\alpha, \Gamma \Rightarrow^- \Delta}\ (+\mathrm{to}-)}{\mathrm{X}^i\neg\neg\alpha, \Gamma \Rightarrow^- \Delta}\ (-\neg\mathrm{Xleft}^{-1}).$$

■

THEOREM 2.16. *(Equivalence between* DLT_ω *and* NLT_ω*) Let* Γ *and* Δ *be (possibly empty) sets of formulas.*

1. $\mathrm{DLT}_\omega \vdash \Gamma \Rightarrow^+ \Delta$ *iff* $\mathrm{NLT}_\omega \vdash \Gamma \Rightarrow \Delta$.
2. $\mathrm{DLT}_\omega \vdash \Gamma \Rightarrow^- \Delta$ *iff* $\mathrm{NLT}_\omega \vdash \neg\Gamma \Rightarrow \neg\Delta$.

PROOF. We show 1 and 2 simultaneously as follows.

• ($\Longrightarrow$): By induction on the proofs P of $\Gamma \Rightarrow^* \Delta$ in DLT_ω. We distinguish the cases according to the last inference of P, and show some cases.

1. Case (+to−): The last inference of P is of the form:

$$\dfrac{\neg\Gamma, \Delta \Rightarrow^+ \neg\Sigma, \Pi}{\Gamma, \neg\Delta \Rightarrow^- \Sigma, \neg\Pi}\ (+\mathrm{to}-).$$

   By induction hypothesis for 1, we have: $\mathrm{NLT}_\omega \vdash \neg\Gamma, \Delta \Rightarrow \neg\Sigma, \Pi$. Then, we obtain the required fact:

$$\begin{array}{c} \vdots \\ \neg\Gamma, \Delta \Rightarrow \neg\Sigma, \Pi \\ \vdots\ (\neg\neg\mathrm{left}), (\neg\neg\mathrm{right}). \\ \neg\Gamma, \neg\neg\Delta \Rightarrow \neg\Sigma, \neg\neg\Pi \end{array}$$

2. Case (−→left): The last inference of P is of the form:

$$\dfrac{\mathrm{X}^i\neg\alpha, \mathrm{X}^i\beta, \Gamma \Rightarrow^- \Delta}{\mathrm{X}^i(\alpha\rightarrow\beta), \Gamma \Rightarrow^- \Delta}\ (-\rightarrow\mathrm{left}).$$

By induction hypothesis for 2, we have $\mathrm{NLT}_\omega \vdash \neg\mathrm{X}^i\neg\alpha, \neg\mathrm{X}^i\beta, \neg\Gamma \Rightarrow \neg\Delta$. Then, we obtain the required fact:

$$
\dfrac{\dfrac{\dfrac{\vdots\ \mathit{Prop.}\ 2.5}{\mathrm{X}^i\alpha \Rightarrow \mathrm{X}^i\alpha}}{\mathrm{X}^i\alpha \Rightarrow \mathrm{X}^i\neg\neg\alpha}\,(\neg\neg\mathrm{right}) \qquad \dfrac{\dfrac{\vdots}{\neg\mathrm{X}^i\neg\alpha, \neg\mathrm{X}^i\beta, \neg\Gamma \Rightarrow \neg\Delta}}{\mathrm{X}^i\neg\neg\alpha, \mathrm{X}^i\neg\beta, \neg\Gamma \Rightarrow \neg\Delta}\,\vdots\,(\neg\mathrm{Xleft}^{-1})\ \mathit{Prop.}\ 2.7}{\dfrac{\mathrm{X}^i\alpha, \mathrm{X}^i\neg\beta, \neg\Gamma \Rightarrow \neg\Delta}{\mathrm{X}^i\neg(\alpha\rightarrow\beta), \neg\Gamma \Rightarrow \neg\Delta}\,(\neg{\rightarrow}\mathrm{left}).}\,(\mathrm{cut})
$$

3. Case $(-\neg\mathrm{Xleft})$: The last inference of P is of the form:

$$
\dfrac{\mathrm{X}^i\neg\alpha, \Gamma \Rightarrow^- \Delta}{\neg\mathrm{X}^i\alpha, \Gamma \Rightarrow^- \Delta}\,(-\neg\mathrm{Xleft}).
$$

By induction hypothesis for 2, we have: $\mathrm{NLT}_\omega \vdash \neg\mathrm{X}^i\neg\alpha, \neg\Gamma \Rightarrow \neg\Delta$. Then, we obtain the required fact:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\vdots\ \mathit{Prop.}\ 2.5}{\mathrm{X}^i\alpha \Rightarrow \mathrm{X}^i\alpha}}{\neg\neg\mathrm{X}^i\alpha \Rightarrow \mathrm{X}^i\alpha}\,(\neg\neg\mathrm{left})}{\neg\neg\mathrm{X}^i\alpha \Rightarrow \mathrm{X}^i\neg\neg\alpha}\,(\neg\neg\mathrm{right}) \qquad \dfrac{\dfrac{\vdots}{\neg\mathrm{X}^i\neg\alpha, \neg\Gamma \Rightarrow \neg\Delta}}{\mathrm{X}^i\neg\neg\alpha, \neg\Gamma \Rightarrow \neg\Delta}\,(\neg\mathrm{Xleft}^{-1})\ \mathit{Prop.}\ 2.7}{\neg\neg\mathrm{X}^i\alpha, \neg\Gamma \Rightarrow \neg\Delta}\,(\mathrm{cut}).
$$

4. Case $(-\neg\mathrm{Xleft}^{-1})$: The last inference of P is of the form:

$$
\dfrac{\neg\mathrm{X}^i\alpha, \Gamma \Rightarrow^- \Delta}{\mathrm{X}^i\neg\alpha, \Gamma \Rightarrow^- \Delta}\,(-\neg\mathrm{Xleft}^{-1}).
$$

By induction hypothesis for 2, we have: $\mathrm{NLT}_\omega \vdash \neg\neg\mathrm{X}^i\alpha, \neg\Gamma \Rightarrow \neg\Delta$. Then, we obtain the required fact:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots\ \mathit{Prop.}\ 2.5}{\mathrm{X}^i\alpha \Rightarrow \mathrm{X}^i\alpha}}{\mathrm{X}^i\neg\neg\alpha \Rightarrow \mathrm{X}^i\alpha}\,(\neg\neg\mathrm{left})}{\neg\mathrm{X}^i\neg\alpha \Rightarrow \mathrm{X}^i\alpha}\,(\neg\mathrm{Xleft})}{\neg\mathrm{X}^i\neg\alpha \Rightarrow \neg\neg\mathrm{X}^i\alpha}\,(\neg\neg\mathrm{right}) \qquad \dfrac{\vdots}{\neg\neg\mathrm{X}^i\alpha, \neg\Gamma \Rightarrow \Delta}}{\neg\mathrm{X}^i\neg\alpha, \neg\Gamma \Rightarrow \neg\Delta}\,(\mathrm{cut}).
$$

5. Case $(-\mathrm{Gleft})$: The last inference of P is of the form:

$$
\dfrac{\{\mathrm{X}^{i+j}\alpha, \Gamma \Rightarrow^- \Delta\}_{j\in\omega}}{\mathrm{X}^i\mathrm{G}\alpha, \Gamma \Rightarrow^- \Delta}\,(-\mathrm{Gleft}).
$$

By induction hypothesis for 2, we have: $\mathrm{NLT}_\omega \vdash \neg \mathrm{X}^{i+j}\alpha, \neg\Gamma \Rightarrow \neg\Delta$ for any $j \in \omega$. Then, we obtain the required fact:

$$\dfrac{\dfrac{\dfrac{\vdots}{\{\neg \mathrm{X}^{i+j}\alpha, \neg\Gamma \Rightarrow \neg\Delta\}_{j\in\omega}}}{\{\mathrm{X}^{i+j}\neg\alpha, \neg\Gamma \Rightarrow \neg\Delta\}_{j\in\omega}}\ (\neg\mathrm{Xleft}^{-1})\ \textit{Prop. } 2.7}{\dfrac{\mathrm{X}^{i}\neg\mathrm{G}\alpha, \neg\Gamma \Rightarrow \neg\Delta}{\neg\mathrm{X}^{i}\mathrm{G}\alpha, \neg\Gamma \Rightarrow \neg\Delta}\ (\neg\mathrm{Xleft}).}\ (\neg\mathrm{Gleft})$$

• ($\Longleftarrow$): By induction on the proofs P of $\Gamma \Rightarrow \Delta$ or $\neg\Gamma \Rightarrow \neg\Delta$ in NLT_ω. We distinguish the cases according to the last inference of P, and show some cases.

1. Case ($\neg{\rightarrow}$left) for 1: The last inference of P is of the form:

$$\dfrac{\mathrm{X}^{i}\alpha, \mathrm{X}^{i}\neg\beta, \Gamma \Rightarrow \Delta}{\mathrm{X}^{i}\neg(\alpha{\rightarrow}\beta), \Gamma \Rightarrow \Delta}\ (\neg{\rightarrow}\mathrm{left}).$$

By induction hypothesis for 1, we have: $\mathrm{DLT}_\omega \vdash \mathrm{X}^{i}\alpha, \mathrm{X}^{i}\neg\beta, \Gamma \Rightarrow^{+} \Delta$. Then, we obtain the required fact:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\mathrm{X}^{i}\alpha, \mathrm{X}^{i}\neg\beta, \Gamma \Rightarrow^{+} \Delta}}{\mathrm{X}^{i}\alpha, \neg\mathrm{X}^{i}\beta, \Gamma \Rightarrow^{+} \Delta}\ (+\neg\mathrm{Xleft})}{\neg\mathrm{X}^{i}\alpha, \mathrm{X}^{i}\beta, \neg\Gamma \Rightarrow^{-} \neg\Delta}\ (+\mathrm{to}-)}{\mathrm{X}^{i}\neg\alpha, \mathrm{X}^{i}\beta, \neg\Gamma \Rightarrow^{-} \neg\Delta}\ (-\neg\mathrm{Xleft}^{-1})}{\mathrm{X}^{i}(\alpha{\rightarrow}\beta), \neg\Gamma \Rightarrow^{-} \neg\Delta}\ (-{\rightarrow}\mathrm{left})}{\neg\mathrm{X}^{i}(\alpha{\rightarrow}\beta), \Gamma \Rightarrow^{+} \Delta}\ (-\mathrm{to}+)}{\mathrm{X}^{i}\neg(\alpha{\rightarrow}\beta), \Gamma \Rightarrow^{+} \Delta}\ (+\neg\mathrm{Xleft}^{-1}).$$

2. Case ($\neg{\rightarrow}$left) for 2: The last inference of P is of the form:

$$\dfrac{\alpha, \neg\beta, \neg\Gamma \Rightarrow \neg\Delta}{\neg(\alpha{\rightarrow}\beta), \neg\Gamma \Rightarrow \neg\Delta}\ (\neg{\rightarrow}\mathrm{left}).$$

By induction hypothesis for 1, we have: $\mathrm{DLT}_\omega \vdash \alpha, \neg\beta, \neg\Gamma \Rightarrow^{+} \neg\Delta$. Then, we obtain the required fact:

$$\dfrac{\dfrac{\dfrac{\vdots}{\alpha, \neg\beta, \neg\Gamma \Rightarrow^{+} \neg\Delta}}{\neg\alpha, \beta, \Gamma \Rightarrow^{-} \Delta}\ (+\mathrm{to}-)}{\alpha{\rightarrow}\beta, \Gamma \Rightarrow^{-} \Delta}\ (-{\rightarrow}\mathrm{left}).$$

3. Case (¬¬left) for 1: The last inference of P is of the form:

$$\frac{X^i\alpha, \Gamma \Rightarrow \Delta}{X^i\neg\neg\alpha, \Gamma \Rightarrow \Delta}\ (\neg\neg\text{left}).$$

   By induction hypothesis for 1, we have: $\text{DLT}_\omega \vdash X^i\alpha, \Gamma \Rightarrow^+ \Delta$. Then, we obtain the required fact:

$$\frac{\begin{array}{c}\vdots\\ X^i\alpha, \Gamma \Rightarrow^+ \Delta\end{array}}{X^i\neg\neg\alpha, \Gamma \Rightarrow^+ \Delta}\ (+\neg\neg\text{left})\ \textit{Prop. } 2.15.$$

4. Case (¬¬left) for 2: The last inference of P is of the form:

$$\frac{\alpha, \neg\Gamma \Rightarrow \neg\Delta}{\neg\neg\alpha, \neg\Gamma \Rightarrow \neg\Delta}\ (\neg\neg\text{left}).$$

   By induction hypothesis for 1, we have: $\text{DLT}_\omega \vdash \alpha, \neg\Gamma \Rightarrow^+ \neg\Delta$. Then, we obtain the required fact:

$$\frac{\begin{array}{c}\vdots\\ \alpha, \neg\Gamma \Rightarrow^+ \neg\Delta\end{array}}{\neg\alpha, \Gamma \Rightarrow^- \Delta}\ (+\text{to}-).$$

5. Case (¬Xleft) for 1: The last inference of P is of the form:

$$\frac{X^i\neg\alpha, \Gamma \Rightarrow \Delta}{\neg X^i\alpha, \Gamma \Rightarrow \Delta}\ (\neg\text{Xleft}).$$

   By induction hypothesis for 1, we have: $\text{DLT}_\omega \vdash X^i\neg\alpha, \Gamma \Rightarrow^+ \Delta$. Then, we obtain the required fact:

$$\frac{\begin{array}{c}\vdots\\ X^i\neg\alpha, \Gamma \Rightarrow^+ \Delta\end{array}}{\neg X^i\alpha, \Gamma \Rightarrow^+ \Delta}\ (+\neg\text{Xleft}).$$

6. Case (¬Xleft) for 2: The last inference of P is of the form:

$$\frac{X^i\neg\alpha, \neg\Gamma \Rightarrow \neg\Delta}{\neg X^i\alpha, \neg\Gamma \Rightarrow \neg\Delta}\ (\neg\text{Xleft}).$$

By induction hypothesis for 1, we have: $\text{DLT}_\omega \vdash \text{X}^i\neg\alpha, \neg\Gamma \Rightarrow^+ \neg\Delta$. Then, we obtain the required fact:

$$\dfrac{\dfrac{\dfrac{\vdots}{\text{X}^i\neg\alpha, \neg\Gamma \Rightarrow^+ \neg\Delta}}{\neg\text{X}^i\alpha, \neg\Gamma \Rightarrow^+ \neg\Delta}\ (+\neg\text{Xleft})}{\text{X}^i\alpha, \Gamma \Rightarrow^- \Delta}\ (+\text{to}-).$$

7. Case ($\neg$Gleft) for 1: The last inference of P is of the form:

$$\dfrac{\{\text{X}^{i+j}\neg\alpha, \Gamma \Rightarrow \Delta\}_{j\in\omega}}{\text{X}^i\neg\text{G}\alpha, \Gamma \Rightarrow \Delta}\ (\neg\text{Gleft}).$$

By induction hypothesis for 1, we have: $\text{DLT}_\omega \vdash \text{X}^{i+j}\neg\alpha, \Gamma \Rightarrow^+ \Delta$ for any $j \in \omega$. Then, we obtain the required fact:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\{\text{X}^{i+j}\neg\alpha, \Gamma \Rightarrow^+ \Delta\}_{j\in\omega}}}{\{\neg\text{X}^{i+j}\alpha, \Gamma \Rightarrow^+ \Delta\}_{j\in\omega}}\ (+\neg\text{Xleft})}{\{\text{X}^{i+j}\alpha, \neg\Gamma \Rightarrow^- \neg\Delta\}_{j\in\omega}}\ (+\text{to}-)}{\text{X}^i\text{G}\alpha, \neg\Gamma \Rightarrow^- \neg\Delta}\ (-\text{Gleft})}{\neg\text{X}^i\text{G}\alpha, \Gamma \Rightarrow^+ \Delta}\ (-\text{to}+)}{\text{X}^i\neg\text{G}\alpha, \Gamma \Rightarrow^+ \Delta}\ (+\neg\text{Xleft}^{-1}).$$

8. Case ($\neg$Gleft) for 2: The last inference of P is of the form:

$$\dfrac{\{\text{X}^{j}\neg\alpha, \neg\Gamma \Rightarrow \neg\Delta\}_{j\in\omega}}{\neg\text{G}\alpha, \neg\Gamma \Rightarrow \neg\Delta}\ (\neg\text{Gleft}).$$

By induction hypothesis for 1, we have: $\text{DLT}_\omega \vdash \text{X}^{j}\neg\alpha, \neg\Gamma \Rightarrow^+ \neg\Delta$ for any $j \in \omega$. Then, we obtain the required fact:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\{\text{X}^{j}\neg\alpha, \neg\Gamma \Rightarrow^+ \neg\Delta\}_{j\in\omega}}}{\{\neg\text{X}^{j}\alpha, \neg\Gamma \Rightarrow^+ \neg\Delta\}_{j\in\omega}}\ (\neg\text{Xleft})}{\{\text{X}^{j}\alpha, \Gamma \Rightarrow^- \Delta\}_{j\in\omega}}\ (+\text{to}-)}{\text{G}\alpha, \Gamma \Rightarrow^- \Delta}\ (-\text{Gleft}).$$

■

Prior to proving the cut-elimination theorem for DLT_ω, we need the following lemma, which is a slight modification of Theorem 2.16.

LEMMA 2.17. *Let* Γ *and* Δ *be sets of formulas.*

1. *If* $\mathrm{DLT}_\omega \vdash \Gamma \Rightarrow^* \Delta$, *then* $\mathrm{NLT}_\omega \vdash \Gamma \Rightarrow \Delta$ *if* $* = +$, *or* $\mathrm{NLT}_\omega \vdash \neg\Gamma \Rightarrow \neg\Delta$ *if* $* = -$.
2. *If* $\mathrm{NLT}_\omega - (\mathrm{cut}) \vdash \Gamma \Rightarrow \Delta$, *then* $\mathrm{DLT}_\omega - \{(-\mathrm{cut}), (+\mathrm{cut})\} \vdash \Gamma \Rightarrow^+ \Delta$.

PROOF. Similar to the proof of Theorem 2.16. The statement 1 is proved by induction on the proofs P of $\Gamma \Rightarrow^* \Delta$ in DLT_ω, and the statement 2 is proved by induction on the cut-free proofs P of $\Gamma \Rightarrow \Delta$ in NLT_ω. ■

THEOREM 2.18. *(Cut-elimination for* LT_ω*) The rule* (cut) *is admissible in cut-free* DLT_ω.

PROOF. Suppose $\mathrm{DLT}_\omega \vdash \Gamma \Rightarrow^* \Delta$. Then, by Lemma 2.17 (1), we have $\mathrm{NLT}_\omega \vdash \Gamma \Rightarrow \Delta$ if $* \equiv +$, or $\mathrm{NLT}_\omega \vdash \neg\Gamma \Rightarrow \neg\Delta$ if $* \equiv -$. Therefore, by Theorem 2.9, we have $\mathrm{NLT}_\omega - (\mathrm{cut}) \vdash \Gamma \Rightarrow \Delta$ or $\mathrm{NLT}_\omega - (\mathrm{cut}) \vdash \neg\Gamma \Rightarrow \neg\Delta$. If $\mathrm{NLT}_\omega - (\mathrm{cut}) \vdash \Gamma \Rightarrow \Delta$, then $\mathrm{DLT}_\omega - (\mathrm{cut}) \vdash \Gamma \Rightarrow^+ \Delta$ by Lemma 2.17 (2). If $\mathrm{NLT}_\omega - (\mathrm{cut}) \vdash \neg\Gamma \Rightarrow \neg\Delta$, then $\mathrm{DLT}_\omega - (\mathrm{cut}) \vdash \neg\Gamma \Rightarrow^+ \neg\Delta$ by Lemma 2.17 (2). Therefore, we obtain $\mathrm{DLT}_\omega - (\mathrm{cut}) \vdash \Gamma \Rightarrow^- \Delta$ by using (+to−). ■

3. Refutation-Aware Kripke-Style Semantics

3.1. Normal Kripke-Style Semantics

We now define a normal Kripke-style semantics for LT. Prior to defining this semantics, we need to introduce some notations. Let Γ be a set $\{\alpha_1, \ldots, \alpha_m\}$ $(m \geq 0)$ of formulas and p be a fixed propositional variable. Then, we use the notation Γ^* to denote $\alpha_1 \vee \cdots \vee \alpha_m$ if $m \geq 1$, and otherwise $\neg(p \rightarrow p)$. We also use the notation Γ_* to denote $\alpha_1 \wedge \cdots \wedge \alpha_m$ if $m \geq 1$, and otherwise $p \rightarrow p$.

DEFINITION 3.1. (*Normal semantics for* LT) Let S be a non-empty set of states and Φ be a set of propositional variables. A structure $M := (\sigma, I)$ is called a *model* if

1. σ is an infinite sequence $s_0, s_1, s_2, \ldots$ of states in S,
2. I is a mapping from Φ to the power set of S.

A satisfaction relation $(M, i) \models \alpha$ for any formula α, where M is a model (σ, I) and i $(\in \omega)$ is some position within σ, is defined inductively by the following clauses:

1. for any $p \in \Phi$, $(M, i) \models p$ iff $s_i \in I(p)$,

2. $(M,i) \models \alpha \wedge \beta$ iff $(M,i) \models \alpha$ and $(M,i) \models \beta$,
3. $(M,i) \models \alpha \vee \beta$ iff $(M,i) \models \alpha$ or $(M,i) \models \beta$,
4. $(M,i) \models \alpha \rightarrow \beta$ iff $(M,i) \not\models \alpha$ or $(M,i) \models \beta$,
5. $(M,i) \models \neg\alpha$ iff $(M,i) \not\models \alpha$,
6. $(M,i) \models \mathrm{X}\alpha$ iff $(M,i+1) \models \alpha$,
7. $(M,i) \models \mathrm{G}\alpha$ iff $(M,j) \models \alpha$ for any $j \geq i$,
8. $(M,i) \models \mathrm{F}\alpha$ iff $(M,j) \models \alpha$ for some $j \geq i$.

A formula α is called *valid* if $(M,0) \models \alpha$ for any model $M := (\sigma, I)$ and the satisfaction relation $\models$ on M. A sequent $\Gamma \Rightarrow \Delta$ is called valid if the formula $\Gamma_* \rightarrow \Delta^*$ is valid.

The following completeness theorem holds [3, 33, 44].

THEOREM 3.2. *(Completeness for* LT_ω *with respect to normal semantics) For any sequent* S*, we have:* $\mathrm{LT}_\omega \vdash S$ *iff* S *is valid.*

REMARK 3.3. For more information on the completeness theorem for LTL, see e.g., [15, 45].

3.2. Refutation-Aware Normal Kripke-Style Semantics

We define a refutation-aware normal Kripke-style semantics for LT.

DEFINITION 3.4. (*Refutation-aware normal semantics for* LT) Let S be a non-empty set of states, Φ be a set of propositional variables, and $\Phi^{\neg}$ be the set $\{\neg p \mid p \in \Phi\}$ of negated propositional variables.

A structure $M := (\sigma, I^\star)$ is called a *refutation-aware normal model* if

1. σ is an infinite sequence $s_0, s_1, s_2, \ldots$ of states in S,
2. $I^\star$ is a mapping from $\Phi \cup \Phi^{\neg}$ to the power set of S such that for any $p \in \Phi$ and any $s \in S$, $s \in I^\star(p)$ iff $s \notin I^\star(\neg p)$.

A refutation-aware normal satisfaction relation $(M,i) \models^\star \alpha$ for any formula α, where M is a refutation-aware normal model $(\sigma, I^\star)$ and i $(\in \omega)$ is some position within σ, is defined inductively by the following clauses:

1. for any $p \in \Phi$, $(M,i) \models^\star p$ iff $s_i \in I^\star(p)$,
2. $(M,i) \models^\star \alpha \wedge \beta$ iff $(M,i) \models^\star \alpha$ and $(M,i) \models^\star \beta$,
3. $(M,i) \models^\star \alpha \vee \beta$ iff $(M,i) \models^\star \alpha$ or $(M,i) \models^\star \beta$,
4. $(M,i) \models^\star \alpha \rightarrow \beta$ iff $(M,i) \not\models^\star \alpha$ or $(M,i) \models^\star \beta$,
5. $(M,i) \models^\star \mathrm{X}\alpha$ iff $(M,i+1) \models^\star \alpha$,

6. $(M, i) \models^{\star} \mathrm{G}\alpha$ iff $(M, j) \models^{\star} \alpha$ for any $j \geq i$,
7. $(M, i) \models^{\star} \mathrm{F}\alpha$ iff $(M, j) \models^{\star} \alpha$ for some $j \geq i$,
8. for any $p \in \Phi$, $(M, i) \models^{\star} \neg p$ iff $s_i \in I^{\star}(\neg p)$,
9. $(M, i) \models^{\star} \neg(\alpha \wedge \beta)$ iff $(M, i) \models^{\star} \neg\alpha$ or $(M, i) \models^{\star} \neg\beta$,
10. $(M, i) \models^{\star} \neg(\alpha \vee \beta)$ iff $(M, i) \models^{\star} \neg\alpha$ and $(M, i) \models^{\star} \neg\beta$,
11. $(M, i) \models^{\star} \neg(\alpha \rightarrow \beta)$ iff $(M, i) \models^{\star} \alpha$ and $(M, i) \models^{\star} \neg\beta$,
12. $(M, i) \models^{\star} \neg\neg\alpha$ iff $(M, i) \models^{\star} \alpha$,
13. $(M, i) \models^{\star} \neg\mathrm{X}\alpha$ iff $(M, i+1) \models^{\star} \neg\alpha$,
14. $(M, i) \models^{\star} \neg\mathrm{G}\alpha$ iff $(M, j) \models^{\star} \neg\alpha$ for some $j \geq i$,
15. $(M, i) \models^{\star} \neg\mathrm{F}\alpha$ iff $(M, j) \models^{\star} \neg\alpha$ for any $j \geq i$.

A formula α is called *n-valid* if $(M, 0) \models^{\star} \alpha$ for any refutation-aware normal model $M := (\sigma, I^{\star})$ and the refutation-aware normal satisfaction relation $\models^{\star}$ on M. A sequent $\Gamma \Rightarrow \Delta$ is called n-valid if the formula $\Gamma_{*} \rightarrow \Delta^{*}$ is n-valid.

Remark 3.5. By deleting the mapping condition "$s \in I^{\star}(p)$ iff $s \notin I^{\star}(\neg p)$" in Definition 3.4, we can obtain a refutation-aware normal Kripke-style semantics for a paraconsistent subsystem of PLTL and LT. This type of refutation-aware normal Kripke-style semantics was not considered in [39].

Proposition 3.6. *For any refutation-aware normal model $M := (\sigma, I^{\star})$, the refutation-aware normal satisfaction relation $\models^{\star}$ on M, any formula α, and any $i \in \omega$, we have: $(M, i) \models^{\star} \alpha$ iff $(M, i) \not\models^{\star} \neg\alpha$.*

Proof. By induction on α. We show some cases.

1. Case $\alpha \equiv p \in \Phi$: $(M, i) \models^{\star} p$ iff $s_i \in I^{\star}(p)$ iff $s_i \notin I^{\star}(\neg p)$ iff $(M, i) \not\models^{\star} \neg p$.
2. Case $\alpha \equiv \beta \wedge \gamma$: $(M, i) \models^{\star} \beta \wedge \gamma$ iff $(M, i) \models^{\star} \beta$ and $(M, i) \models^{\star} \gamma$ iff $(M, i) \not\models^{\star} \neg\beta$ and $(M, i) \not\models^{\star} \neg\gamma$ (by induction hypothesis) iff $(M, i) \not\models^{\star} \neg(\beta \wedge \gamma)$.
3. Case $\alpha \equiv \beta \rightarrow \gamma$: $(M, i) \models^{\star} \beta \rightarrow \gamma$ iff $(M, i) \not\models^{\star} \beta$ or $(M, i) \models^{\star} \gamma$ iff $(M, i) \not\models^{\star} \beta$ or $(M, i) \not\models^{\star} \neg\gamma$ (by induction hypothesis) iff $(M, i) \not\models^{\star} \neg(\beta \rightarrow \gamma)$.
4. Case $\alpha \equiv \neg\beta$: $(M, i) \models^{\star} \neg\beta$ iff $(M, i) \not\models^{\star} \beta$ (by induction hypothesis with contraposition) iff $(M, i) \not\models^{\star} \neg\neg\beta$.
5. Case $\alpha \equiv \mathrm{X}\beta$: $(M, i) \models^{\star} \mathrm{X}\beta$ iff $(M, i+1) \models^{\star} \beta$ iff $(M, i+1) \not\models^{\star} \neg\beta$ (by induction hypothesis) iff $(M, i) \not\models^{\star} \neg\mathrm{X}\beta$.

6. Case $\alpha \equiv \mathrm{G}\beta$: $(M,i) \models^\star \mathrm{G}\beta$ iff $(M,j) \models^\star \beta$ for any $j \geq i$ iff $(M,j) \not\models^\star \neg\beta$ for any $j \geq i$ (by induction hypothesis) iff not-[$(M,j) \models^\star \neg\beta$ for some $j \geq i$] iff not-[$(M,i) \models^\star \neg\mathrm{G}\beta$] iff $(M,i) \not\models^\star \neg\mathrm{G}\beta$.

■

REMARK 3.7. We make the following remarks.

1. Proposition 3.6 shows that the mapping condition "$s \in I^\star(p)$ iff $s \notin I^\star(\neg p)$" in Definition 3.4 can be extended to the refutation-aware normal satisfaction relation for any formula α.
2. By the contraposition of the statement of Proposition 3.6, we can obtain the following classical-negation condition for refutation-aware normal satisfaction relation: $(M,i) \models^\star \neg\alpha$ iff $(M,i) \not\models^\star \alpha$.

Prior to proving the equivalence between the refutation-aware normal and normal Kripke-style semantics (i.e., the equivalence between the n-validity and the validity), we need to show the following two lemmas.

LEMMA 3.8. *For any model $M := (\sigma, I)$, we can construct a refutation-aware normal model $N := (\sigma, I^\star)$ such that for any formula α and any $i \in \omega$, $(M,i) \models \alpha$ iff $(N,i) \models^\star \alpha$.*

PROOF. Let S be a non-empty set of states, σ be an infinite sequence of states in S, M be a model (σ, I), and $\models$ be the satisfaction relation on M. Then, we define a refutation-aware normal model $N := (\sigma, I^\star)$ such that for any $s_i \in S$ and any $p \in \Phi$,

1. $s_i \in I^\star(p)$ iff $s_i \in I(p)$,
2. $s_i \in I^\star(\neg p)$ iff $s_i \notin I(p)$.

Then, we can obtain the mapping condition "$s \in I^\star(p)$ iff $s \notin I^\star(\neg p)$" in Definition 3.4, because we have: $s \in I^\star(p)$ iff $s \in I(p)$ iff $s \notin I^\star(\neg p)$.

We now prove this lemma by induction on α. We show some cases.

1. Case $\alpha \equiv p \in \Phi$: $(M,i) \models p$ iff $s_i \in I(p)$ iff $s_i \in I^\star(p)$ iff $(N,i) \models^\star p$.
2. Case $\alpha \equiv \beta \wedge \gamma$: $(M,i) \models \beta \wedge \gamma$ iff $(M,i) \models \beta$ and $(M,i) \models \gamma$ iff $(N,i) \models^\star \beta$ and $(N,i) \models^\star \gamma$ (by induction hypothesis) iff $(N,i) \models^\star \beta \wedge \gamma$.
3. Case $\alpha \equiv \beta \rightarrow \gamma$: $(M,i) \models \beta \rightarrow \gamma$ iff $(M,i) \not\models \beta$ or $(M,i) \models \gamma$ iff $(N,i) \not\models^\star \beta$ or $(N,i) \models^\star \gamma$ (by induction hypothesis) iff $(N,i) \models^\star \beta \rightarrow \gamma$.
4. Case $\alpha \equiv \neg\beta$: $(M,i) \models \neg\beta$ iff $(M,i) \not\models \beta$ iff $(N,i) \not\models^\star \beta$ (by induction hypothesis) iff $(N,i) \models^\star \neg\beta$ (by Proposition 3.6).

5. Case $\alpha \equiv \mathrm{X}\beta$: $(M, i) \models \mathrm{X}\beta$ iff $(M, i+1) \models \beta$ iff $(N, i+1) \models^{\star} \beta$ (by induction hypothesis) iff $(N, i) \models^{\star} \mathrm{X}\beta$.
6. Case $\alpha \equiv \mathrm{G}\beta$: $(M, i) \models \mathrm{G}\beta$ iff $(M, j) \models \beta$ for any $j \geq i$ iff $(N, j) \models^{\star} \beta$ for any $j \geq i$ (by induction hypothesis) iff $(N, i) \models^{\star} \mathrm{G}\beta$.

■

LEMMA 3.9. *For any refutation-aware normal model $M := (\sigma, I^{\star})$, we can construct a model $N := (\sigma, I)$ such that for any formula α and any $i \in \omega$, $(M, i) \models^{\star} \alpha$ iff $(N, i) \models \alpha$.*

PROOF. Similar to the proof of Lemma 3.8. ■

THEOREM 3.10. *(Equivalence between n-validity and validity) For any sequent S, we have: S is n-valid iff S is valid.*

PROOF. By Lemmas 3.8 and 3.9. ■

3.3. Refutation-Aware Dual Kripke-Style Semantics

We define a refutation-aware dual Kripke-style semantics for LT by using two kinds of satisfaction relations $\models^{+}$ and $\models^{-}$. The intuitive meanings of $\models^{+}$ and $\models^{-}$ are "verification" (or "justification") and "refutation" (or "falsification"), respectively.

DEFINITION 3.11. (*Refutation-aware dual semantics for* LT) Let S be a non-empty set of states and Φ be a set of propositional variables. A structure $M := (\sigma, I^{+}, I^{-})$ is called a *refutation-aware dual model* if

1. σ is an infinite sequence $s_0, s_1, s_2, \ldots$ of states in S,
2. I^{+} and I^{-} are mappings from Φ to the power set of S such that for any $p \in \Phi$ and any $s \in S$, $s \in I^{+}(p)$ iff $s \notin I^{-}(p)$.

Refutation-aware dual satisfaction relations $(M, i) \models^{+} \alpha$ and $(M, i) \models^{-} \alpha$ for any formula α, where M is a refutation-aware dual model (σ, I^{+}, I^{-}) and i ($\in \omega$) is some position within σ, are defined inductively by the following clauses:

1. for any $p \in \Phi$, $(M, i) \models^{+} p$ iff $s_i \in I^{+}(p)$,
2. $(M, i) \models^{+} \alpha \wedge \beta$ iff $(M, i) \models^{+} \alpha$ and $(M, i) \models^{+} \beta$,
3. $(M, i) \models^{+} \alpha \vee \beta$ iff $(M, i) \models^{+} \alpha$ or $(M, i) \models^{+} \beta$,
4. $(M, i) \models^{+} \alpha \rightarrow \beta$ iff $(M, i) \not\models^{+} \alpha$ or $(M, i) \models^{+} \beta$,
5. $(M, i) \models^{+} \neg\alpha$ iff $(M, i) \models^{-} \alpha$,

6. $(M, i) \models^+ \mathrm{X}\alpha$ iff $(M, i+1) \models^+ \alpha$,
7. $(M, i) \models^+ \mathrm{G}\alpha$ iff $(M, j) \models^+ \alpha$ for any $j \geq i$,
8. $(M, i) \models^+ \mathrm{F}\alpha$ iff $(M, j) \models^+ \alpha$ for some $j \geq i$,
9. for any $p \in \Phi$, $(M, i) \models^- p$ iff $s_i \in I^-(p)$,
10. $(M, i) \models^- \alpha \wedge \beta$ iff $(M, i) \models^- \alpha$ or $(M, i) \models^- \beta$,
11. $(M, i) \models^- \alpha \vee \beta$ iff $(M, i) \models^- \alpha$ and $(M, i) \models^- \beta$,
12. $(M, i) \models^- \alpha \rightarrow \beta$ iff $(M, i) \models^+ \alpha$ and $(M, i) \models^- \beta$,
13. $(M, i) \models^- \neg\alpha$ iff $(M, i) \models^+ \alpha$,
14. $(M, i) \models^- \mathrm{X}\alpha$ iff $(M, i+1) \models^- \alpha$,
15. $(M, i) \models^- \mathrm{G}\alpha$ iff $(M, j) \models^- \alpha$ for some $j \geq i$,
16. $(M, i) \models^- \mathrm{F}\alpha$ iff $(M, j) \models^- \alpha$ for any $j \geq i$.

A formula α is called *d-valid* if $(M, 0) \models^+ \alpha$ for any refutation-aware dual model $M := (\sigma, I^+, I^-)$ and the refutation-aware dual satisfaction relations $\models^+$ and $\models^-$ on M. A sequent $\Gamma \Rightarrow \Delta$ is called d-valid if the formula $\Gamma_* \rightarrow \Delta^*$ is d-valid.

Remark 3.12. We make the following remarks.

1. By deleting the mapping condition "$s \in I^+(p)$ iff $s \notin I^-(p)$" in Definition 3.11, we can obtain a refutation-aware dual Kripke-style semantics for a paraconsistent subsystem of PLTL and LT. This type of refutation-aware dual Kripke-style semantics was originally introduced in [39] for PLT_ω.
2. The completeness theorem with respect to a refutation-aware dual Kripke-style semantics for PLT_ω was proved in [39] using an embedding-based method. The dual semantics introduced in [39] for PLT_ω has no mapping condition "$s \in I^+(p)$ iff $s \notin I^-(p)$" and has the standard classical negation condition for the satisfaction relation.

Lemma 3.13. *For any model $M := (\sigma, I)$, we can construct a refutation-aware dual model $N := (\sigma, I^+, I^-)$ such that for any formula α and any $i \in \omega$,*

1. *$(M, i) \models \alpha$ iff $(N, i) \models^+ \alpha$,*
2. *$(M, i) \models \neg\alpha$ iff $(N, i) \models^- \alpha$.*

Proof. Let S be a non empty set of states, σ be an infinite sequence of states in S, M be a model (σ, I), and $\models$ be the satisfaction relation on M.

Then, we define a refutation-aware dual model $N := (\sigma, I^+, I^-)$ such that for any $s_i \in S$ and any $p \in \Phi$,

1. $s_i \in I^+(p)$ iff $s_i \in I(p)$,
2. $s_i \in I^-(p)$ iff $s_i \notin I(p)$.

Then, we can obtain the mapping condition "$s \in I^+(p)$ iff $s \notin I^-(p)$" in Definition 3.11, because we have: $s \in I^+(p)$ iff $s \in I(p)$ iff $s \notin I^-(p)$.

We now prove this lemma by simultaneous induction on α. We show some cases.

1. Case $\alpha \equiv p \in \Phi$: For 1, we obtain: $(M, i) \models p$ iff $s_i \in I(p)$ iff $s_i \in I^+(p)$ iff $(N, i) \models^+ p$. For 2, we obtain: $(M, i) \models \neg p$ iff $s_i \notin I(p)$ iff $s_i \in I^-(p)$ iff $(N, i) \models^- p$.
2. Case $\alpha \equiv \beta \wedge \gamma$: For 1, we obtain: $(M, i) \models \beta \wedge \gamma$ iff $(M, i) \models \beta$ and $(M, i) \models \gamma$ iff $(N, i) \models^+ \beta$ and $(N, i) \models^+ \gamma$ (by induction hypothesis for 1) iff $(N, i) \models^+ \beta \wedge \gamma$. For 2, we obtain: $(M, i) \models \neg(\beta \wedge \gamma)$ iff $(M, i) \not\models \beta \wedge \gamma$ iff $(M, i) \not\models \beta$ or $(M, i) \not\models \gamma$ iff $(M, i) \models \neg\beta$ or $(M, i) \models \neg\gamma$ iff $(N, i) \models^- \beta$ or $(N, i) \models^- \gamma$ (by induction hypothesis for 2) iff $(N, i) \models^- \beta \wedge \gamma$.
3. Case $\alpha \equiv \beta \rightarrow \gamma$: For 1, we obtain: $(M, i) \models \beta \rightarrow \gamma$ iff $(M, i) \not\models \beta$ or $(M, i) \models \gamma$ iff $(N, i) \not\models^+ \beta$ or $(N, i) \models^+ \gamma$ (by induction hypothesis for 1) iff $(N, i) \models^+ \beta \rightarrow \gamma$. For 2, we obtain: $(M, i) \models \neg(\beta \rightarrow \gamma)$ iff $(M, i) \not\models \beta \rightarrow \gamma$ iff $(M, i) \models \beta$ and $(M, i) \not\models \gamma$ iff $(M, i) \models \beta$ and $(M, i) \models \neg\gamma$ iff $(N, i) \models^+ \beta$ and $(N, i) \models^- \gamma$ (by induction hypotheses for 1 and 2) iff $(N, i) \models^- \beta \rightarrow \gamma$.
4. Case $\alpha \equiv \neg\beta$: For 1, we obtain: $(M, i) \models \neg\beta$ iff $(N, i) \models^- \beta$ (by induction hypothesis for 2) iff $(N, i) \models^+ \neg\beta$. For 2, we obtain: $(M, i) \models \neg\neg\beta$ iff $(M, i) \models \beta$ iff $(N, i) \models^+ \beta$ (by induction hypothesis for 1) iff $(N, i) \models^- \neg\beta$.
5. Case $\alpha \equiv \mathrm{X}\beta$: For 1, we obtain: $(M, i) \models \mathrm{X}\beta$ iff $(M, i+1) \models \beta$ iff $(N, i+1) \models^+ \beta$ (by induction hypothesis for 1) iff $(N, i) \models^+ \mathrm{X}\beta$. For 2, we obtain: $(M, i) \models \neg\mathrm{X}\beta$ iff $(M, i) \not\models \mathrm{X}\beta$ iff $(M, i+1) \not\models \beta$ iff $(M, i+1) \models \neg\beta$ iff $(N, i+1) \models^- \beta$ (by induction hypothesis for 2) iff $(N, i) \models^- \mathrm{X}\beta$.
6. Case $\alpha \equiv \mathrm{G}\beta$: For 1, we obtain: $(M, i) \models \mathrm{G}\beta$ iff $(M, j) \models \beta$ for any $j \geq i$ iff $(N, j) \models^+ \beta$ for any $j \geq i$ (by induction hypothesis) iff $(N, i) \models^+ \mathrm{G}\beta$. For 2, we obtain: $(M, i) \models \neg\mathrm{G}\beta$ iff $(M, i) \not\models \mathrm{G}\beta$ iff $(M, j) \not\models \beta$ for some $j \geq i$ iff $(M, j) \models \neg\beta$ for some $j \geq i$ iff $(N, j) \models^- \beta$ for some $j \geq i$ (by induction hypothesis for 2) iff $(N, i) \models^- \mathrm{G}\beta$.

■

Lemma 3.14. *For any refutation-aware dual model $M := (\sigma, I^+, I^-)$, we can construct a model $N := (\sigma, I)$ such that for any formula α and any $i \in \omega$,*

1. *$(M, i) \models^+ \alpha$ iff $(N, i) \models \alpha$,*
2. *$(M, i) \models^- \alpha$ iff $(N, i) \models \neg\alpha$.*

Proof. Similar to the proof of Lemma 3.13. ■

Theorem 3.15. *(Equivalence between d-validity and validity) For any sequent S, we have: S is d-valid iff S is valid.*

Proof. By Lemmas 3.13 and 3.14. ■

We obtain the following theorem.

Theorem 3.16. *(Completeness for* LT_ω, NLT_ω, *and* DLT_ω*) Let L be* LT_ω, NLT_ω, *or* DLT_ω. *For any sequent S, we have:*

1. *$L \vdash S$ iff S is valid,*
2. *$L \vdash S$ iff S is n-valid,*
3. *$L \vdash S$ iff S is d-valid.*

Proof. By Theorems 2.8, 2.16, 3.2, 3.10, and 3.15. ■

4. Conclusions, Remarks, and Related Works

4.1. Conclusions

In this study, we introduced the refutation-aware Gentzen-style sequent calculi NLT_ω, and DLT_ω for LT (i.e., propositional until-free linear-time temporal logic); proved the equivalences between them and the standard Gentzen-style sequent calculus LT_ω [44]; and proved the cut-elimination theorems for NLT_ω and DLT_ω. Furthermore, we introduced the refutation-aware normal and dual Kripke-style semantics for LT; proved the equivalences among these semantics and the standard semantics for LT; and proved the completeness theorems with respect to these Kripke-style semantics for LT_ω, NLT_ω, and DLT_ω. Additionally, we observed that the paraconsistent subsystems NLT^-_ω and DLT^-_ω of NLT_ω, and DLT_ω, respectively, can be easily obtained in a modular way. These subsystems are also regarded as subsystems and alternatives of the previously introduced Gentzen-style sequent calculus PLT_ω [39] for PLTL (i.e., paraconsistent linear-time temporal logic). It was thus shown in this study that the proposed refutation-aware framework for Gentzen-style sequent calculi and Kripke-style semantics was regarded as a framework compatible with the existing refutation-aware and inconsistency-tolerant framework of PLTL.

We now summarize the merits of the proposed calculi NLT_ω and DLT_ω as follows.

1. Similar to the existing refutation-aware Gentzen-style sequent calculi for Nelson's paraconsistent logic N4 and classical logic, we can obtain a clear understanding of the refutation-aware reasoning within NLT_ω and DLT_ω. Namely, we can directly obtain a disproof of a given negated formula using these calculi, where a disproof represents a refutation process for the given formula. Thus, we can obtain both proof and disproof in the calculi.
2. NLT_ω and DLT_ω are highly compatible with inconsistency-tolerant reasoning. Namely, these calculi are compatible with the paraconsistent variant PLTL of LT, which is useful for appropriately handling inconsistency-tolerant temporal reasoning [39]. In other words, the framework of these calculi is regarded as a generalization of the existing PLTL- and N4-based frameworks. Actually, we can simply define the paraconsistent subsystems NLT_ω^- and DLT_ω^- of NLT_ω and DLT_ω, respectively, by deleting only a few initial sequents from NLT_ω and DLT_ω.

In addition to the above-mentioned merits, we have the following merits for NLT_ω and DLT_ω.

1. Similar to LT_ω, we can obtain some useful results for NLT_ω and DLT_ω by using the infinite premises rules. Using the infinite premises rules, we can obtain a theorem for embedding LT into propositional infinitary logic. By using this embedding theorem, we can obtain the completeness and cut-elimination theorems for NLT_ω and DLT_ω. For the proofs of the embedding, completeness, and cut-elimination theorems for LT_ω, see [30,33,34].
2. Similar to LT_ω, we can obtain some useful variants of NLT_ω and DLT_ω by modifying the infinite premises rules. Replacing the infinite premises rules in NLT_ω and DLT_ω with the finite premises versions of the rules, we can easily define some useful time-bounded fragments of the calculi. Such time-bounded fragments of LTL and LT were studied, for example, in [31–33]. These time-bounded fragments are embeddable into propositional classical logic; have some efficient decision procedures; and have some practical applications [31–33].

As a future work, we intend to develop a sequent-calculus-based *uniform-proof theoretic logic programming* (or *automated theorem proving*) framework [48] using NLT_ω, DLT_ω, NLT^-_ω, and DLT^-_ω. Concerned with this direction, a uniform-proof theoretic *paraconsistent logic programming*, which was based on a refutation-aware Gentzen-style sequent calculus for Nelson's N4, was developed in [29]. We also intend to develop a refutation-aware model checking framework extending the proposed refutation-aware Kripke-style semantics for LT and its paraconsistent subsystem. A work of this direction is in [37]. We also intend to develop a *refutation-aware Gentzen-style natural deduction systems* for LT. Concerned with this direction, some refutation-aware Gentzen-style natural deduction systems for Nelson's N4 were developed in [27].

4.2. Remarks on Until Operator

We have not yet obtained a cut-free and complete refutation-aware normal, dual, or split-context Gentzen-style sequent calculus that is theorem-equivalent to an extended LT_ω with the until operator U. A reason why we cannot obtain such a calculus is that we have not yet obtained a cut-free and complete extension of LT_ω with the addition of U. An extension called $\mathrm{LT}^{\mathrm{U}}_\omega$ with U was considered in [34], although it is unknown whether the cut-elimination and completeness theorems for $\mathrm{LT}^{\mathrm{U}}_\omega$ holds or not. In what follows, we explain this system.

The language of $\mathrm{LT}^{\mathrm{U}}_\omega$ is obtained from the language of LT_ω by adding a binary temporal operator U (until). An expression $\{\mathrm{X}^k\alpha\}_{i\leq k<j}$ for any natural numbers i, j and k is used to represent an arbitrary formula $\mathrm{X}^k\alpha$ with $i \leq k < j$ if there is k with $i \leq k < j$, or the formula α if there is no such k. A sequent expression $\{\Gamma \Rightarrow \Delta, \mathrm{X}^k\alpha\}_{i\leq k<j}$ for any natural numbers i, j and k is used to represent the set $\{\Gamma \Rightarrow \Delta, \mathrm{X}^k\alpha \mid i \leq k < j\}$ if there is k with $i \leq k < j$, or the sequent $\Gamma \Rightarrow \Delta, \alpha$ if there is no such k.

Then, $\mathrm{LT}^{\mathrm{U}}_\omega$ is defined as follows.

Definition 4.1. ($\mathrm{LT}^{\mathrm{U}}_\omega$) $\mathrm{LT}^{\mathrm{U}}_\omega$ is obtained from LT_ω by adding the logical inference rules of the form:

$$\frac{\{\ \mathrm{X}^{i+j}\beta, \{\mathrm{X}^k\alpha\}_{i\leq k<i+j}, \Gamma \Rightarrow \Delta\ \}_{j\in\omega}}{\mathrm{X}^i(\alpha\mathrm{U}\beta), \Gamma \Rightarrow \Delta}\ (\mathrm{Uleft})$$

$$\frac{\Gamma \Rightarrow \Delta, \mathrm{X}^{i+j}\beta \quad \{\Gamma \Rightarrow \Delta, \mathrm{X}^k\alpha\}_{i\leq k<i+j}}{\Gamma \Rightarrow \Delta, \mathrm{X}^i(\alpha\mathrm{U}\beta)}\ (\mathrm{Uright}).$$

Remark 4.2. We make the following remarks.

1. (Uleft) and (Uright) are formulated based on the following informal axiom scheme:

$$\mathrm{X}^i(\alpha \mathrm{U} \beta) \leftrightarrow \bigvee_{j \in \omega} (\mathrm{X}^{i+j}\beta \wedge \bigwedge_{i \le k < i+j} \mathrm{X}^k \alpha).$$

2. The above informal axiom scheme is closely related to the following semantic clause for U:

$$(M, i) \models \alpha \mathrm{U} \beta \text{ iff } \exists j \geq i \ [(M, j) \models \beta \text{ and } \forall i \leq k < j \ (M, k) \models \alpha].$$

3. A weak theorem for one-directionally embedding $\mathrm{LT}^{\mathrm{U}}_{\omega}$ into a cut-free Gentzen-style sequent calculus LK_{ω} for infinitary logic was proved in [34].
4. If we consider to develop falsification-aware variants of $\mathrm{LT}^{\mathrm{U}}_{\omega}$, we need to extend it by adding the release operator R, which is the dual counterpart of U and has the following clause:

$$(M, i) \models \alpha \mathrm{R} \beta \text{ iff } \forall j \geq i \ [(M, j) \models \beta \text{ or } \exists i \leq k < j \ (M, k) \models \alpha].$$

4.3. Related Works

In *model checking* [8,10], which is well-known to be a software verification technique, refutation or falsification plays a critical role in obtaining the counterexample traces for the underlying object specifications. A counterexample-guided abstraction and refinement technique [9] for model checking can be considered an example of a useful refutation- or falsification-aware technique in model checking. *Yet Another Software Model Checker* (YASM) [22,23] is regarded as a refutation-aware model-checker because it explicitly divides and simultaneously performs falsification and verification. YASM could prove and disprove properties with equal effectiveness and was constructed based on Belnap and Dunn's four-valued logic [4,5,14], which is regarded as a subsystem of Nelson's paraconsistent logic N4 [2,49].

Falsification-aware model checking has recently been proposed and studied in [37,38]. This technique is regarded as a generalization of *inconsistency-tolerant (or paraconsistent) model checking* [36,42], which is a variant of model checking. This falsification-aware model checking paradigm was roughly defined based on refutation-aware Kripke-style semantics for some temporal logics. In [38], falsification-aware normal and dual Kripke-style semantics for *computation tree logic* (CTL) [8] were introduced to obtain the theoretical foundation of falsification-aware CTL-model checking. In [37], falsification-aware normal and dual Kripke-style semantics for a wide range of temporal logics including LTL with the until and release operators were also considered for extending the framework of falsification-aware CTL-model checking.

In falsification-aware model checking, we can simultaneously verify and falsify formulas using two falsification-aware dual satisfaction relations $\models^+$ and $\models^-$ in the falsification-aware dual Kripke-style semantics for the underlying logics. Suppose that CTL is considered the basic logic in the following explanation. We can then formally consider a falsification-aware model-checking problem as follows. Suppose that M is a falsification-aware dual CTL model (S, S_0, R, L^+, L^-), where S is a set of states, S_0 is a set of initial states, R is a binary relation on S, and L^+ and L^- are positive and negative labeling functions, respectively. In addition, suppose that $\models^+$ and $\models^-$ are falsification-aware dual CTL-satisfaction relations on M. Then, the *falsification-aware model checking problem* for CTL is defined as follows. For any formula α, find the *verification set* $\{s \in S \mid (M, s) \models^+ \alpha\}$ and the *falsification set* $\{s \in S \mid (M, s) \models^- \alpha\}$. These sets can be simultaneously found. Thus, we can simultaneously perform verification and falsification.

Other traditional refutation-aware systems so called the *Łukasiewicz-style refutation systems* have been proposed and studied, for example, in [21,46,61–64], although a Łukasiewicz-style refutation system for LTL or LT has not yet been developed. For example, some Łukasiewicz-style refutation systems [46] for modal logics including S4 were introduced and studied in [61,62]. An Łukasiewicz-style refutation system for *Wansing's nonmonotonic logic* W [70] was introduced in [64], wherein the decidability and finite model property were proved for W using the refutation system. The logic W is regarded as an extension of Nelson's N4.

Another famous refutation-centric formal system is *Robinson's resolution calculus or principle* [57]. Resolution calculi for LTL and its variants have been studied in developing an efficient method for mechanical theorem proving [1,7,12,13,16,17,26,32,58,68]. Resolution calculi for decidable fragments of first-order LTL were of the utmost interest in this area of research. Most of the *monodic fragments* of first-order LTL were shown to be decidable [24]. The corresponding resolution calculi, which were called *monodic temporal resolution*, have been studied [12]. The method of the monodic temporal resolution was based on the technique of *clausal temporal resolution* [17].

Acknowledgements. We would like to thank the anonymous referees for their valuable comments and suggestions. This research was supported by JSPS KAKENHI Grant Number JP23K10990.

Declarations

Conflict of interest The author has no conflict of interest.

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N. Kamide
Department of Data Science, School of Data Science
Nagoya City University
Takiko Campus, Yamanohata 1, Mizuho-cho, Mizuho-ku
Nagoya Aichi467-8501
Japan
drnkamide08@kpd.biglobe.ne.jp