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Proof Systems for Super-Strict Implication

Abstract. This paper studies proof systems for the logics of super-strict implication ST2–ST5, which correspond to C.I. Lewis' systems S2–S5 freed of paradoxes of strict implication. First, Hilbert-style axiomatic systems are introduced and shown to be sound and complete by simulating STn in Sn and backsimulating Sn in STn, respectively (for n = 2, . . . , 5). Next, G3-style labelled sequent calculi are investigated. It is shown that these calculi have the good structural properties that are distinctive of G3-style calculi, that they are sound and complete, and it is shown that the proof search for G3.ST2 is terminating and therefore the logic is decidable.

Keywords: Super-strict implication, Axiomatic systems, Labelled sequent calculi, Completeness, Structural proof theory, Decidability, Definable conditionals.

1. Introduction

The goal of this paper is to develop proof systems for normal and non-normal logics of super-strict implication (SSI) $\rhd$. SSI is a refinement of C.I. Lewis' strict implication ($⥽$) that has been studied in [4,5]. The idea behind SSI is to free strict implication from its paradoxes. For this, $A \rhd B$ is defined as true whenever $A ⥽ B$ is true and, moreover, A is possible.[1]

Thus the formulas of *Antilogical Antecedent* (AA) and *Tautological Consequent* (TC)

$$\bot \rhd B \quad \text{(AA)} \qquad\qquad A \rhd \top \quad \text{(TC)}$$

All authors have contributed equally to this work.

[1] The idea of strengthening strict implication in this way goes back to Priest [21]. Constructions similar to $\rhd$ were also studied in [32,33], where $⥽$ is analysed in terms of a dynamic semantics and $\Diamond A$ is treated as a presupposition; and in [22,23,26] (see further references therein) and [7], where $⥽$ is analysed as a variably strict conditional $>$ and $\Diamond A$ as the corresponding outer modality $\neg(A > \bot)$. Compare [27,34] for related accounts.

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are both invalid for $\rhd$. These are also known as the first and second paradox of strict implication.

By avoiding these paradoxes, $\rhd$ provides a better explication of *entailment-related* uses of implication than $\prec$. In this respect, we agree with connexivists [15] and relevantists [1,29], but we disagree with Lewis [11, p. 250], who proposed a formal argument—his so-called *independent proof*—for the validity of $\bot \prec B$. His argument is based on accepting the following inferential principles: $\wedge$-elimination ($\wedge$E), $\vee$-introduction ($\vee$I), and disjunctive syllogism (DS). These principles are all the ingredients needed for *ex falso quodlibet* (i.e., the derivation of an arbitrary formula B from a contradiction $A \wedge \neg A$):

$$\dfrac{\dfrac{\dfrac{A \wedge \neg A}{A}\,\wedge\mathrm{E}}{A \vee B}\,\vee\mathrm{I} \qquad \dfrac{A \wedge \neg A}{\neg A}\,\wedge\mathrm{E}}{B}\,\mathrm{DS}$$

Moreover, if we take $\prec$ to be an object language representation of the underlying derivability relation—i.e., if we assume the deduction theorem for $\prec$—then we must accept the following version of the first paradox of strict implication: $(A \wedge \neg A) \prec B$. Neither connexivists nor relevantists are convinced by this argument: they reject *ex falso quodlibet* and at least one of the principles used in Lewis' proof. In particular, connexivists usually reject $\wedge E$ or correspondingly the object-language law of *Simplification* '$(A \wedge B)$ implies A' (SI)[2]; relevantists usually reject DS.

The rejection of *ex falso quodlibet*, be it based on the rejection of $\wedge E$ or of DS, involves a major departure from classical logic since it means that at least one of $\wedge$, $\vee$, and $\neg$ does not satisfy its Boolean semantics.

We believe this is too high a price to pay. We rather prefer to supplement classical propositional logic with an implication connective $\rhd$ aimed at expressing entailment-related uses of implication, in the same spirit as Lewis' $\prec$. It turns out that for $\rhd$ the paradoxes of $\prec$ become invalid, since the antecedent $\bot$ of the first paradox (AA) is impossible and the antecedent A of the second paradox (TC) is not always possible. What becomes valid, instead, is the negation of the first paradox, $\neg(\bot \rhd B)$ (*No Antilogical Antecedent*, NAA), and its symmetric counterpart, $\neg(B \rhd \bot)$ (*No Antilogical Consequent*, NAC). Furthermore, $\rhd$ allows us to express $\prec$ as well as the

[2] Routley takes the failure of simplification as a distinctive feature of connexive logics, but this is not so since, e.g., Wansing's system C rejects DS but accepts simplification and $\wedge E$, cf. [16].

unary modal operators $\Box$ and $\Diamond$ as follows: $\Box A = (\top \rhd A)$, $\Diamond A = \neg\Box\neg A$, $A \prec B = \Box(A \supset B)$, where $\supset$ stands for the material implication. We show that logics of SSI preserve all validities of a semantics for $\prec$ in the following weak sense: if $A \prec B$ is valid (in a semantics for $\prec$) then, $\Diamond A \supset (A \rhd B)$ is valid (in the corresponding semantics for $\rhd$), see Corollary 7. As a consequence, logics of $\rhd$ can be seen as logics of $\prec$ freed from paradoxical instances.

Moreover, as shown in [4,5], both normal and non-normal logics for $\rhd$ are *Boethian* logics:[3] $\rhd$ is a non-symmetric implication validating both *Aristotle's Thesis* (AT) and *weak Boethius' Thesis* (wBT):

$$\neg(A \rhd \neg A) \qquad\qquad \neg(\neg A \rhd A) \qquad \text{(AT)}$$

$$(A \rhd B) \supset \neg(A \rhd \neg B) \qquad\qquad (A \rhd \neg B) \supset \neg(A \rhd B) \qquad \text{(wBT)}$$

But the logics of SSI are not fully connexive, since strong Boethius' Thesis BT (obtained by replacing $\supset$ with $\rhd$ in wBT), does not hold. In this respect, $\rhd$ is similar to Pizzi's *consequential implication* [20]. As the latter, $\rhd$ is motivated by the rejection of AA, TC, and the acceptance of the Boethian connexive principles AT and wBT. And as for consequential implication, the stronger BT fails for $\rhd$. However, $\rhd$ differs from consequential implication in crucial aspects. First, by the way in which the strict implication is strengthened. Consequential implication, $A \rightarrow B$, strengthens the strict implication by imposing that both $\Box B \supset \Box A$ and $\Diamond B \supset \Diamond A$ are true, whereas super-strict implication only imposes that $\Diamond A$ is true. Second, by the reason for which adding BT is problematic (see Propositions 3.18 and 3.19 from [20] and our Section 5.3).

In previous work, logics for $\rhd$ were studied mostly from a semantical perspective: [4] introduced relational semantics for normal logics of $\rhd$ and [5] for some non-normal ones. The only proof systems introduced were labelled sequent calculi for normal logics of $\rhd$ [4], but no proof system has been introduced yet for non-normal ones and no fully object-language axiomatisation of these logics, be it normal or non-normal has been provided. This paper fills these gaps by introducing a sound and complete Hilbert-style axiomatisation of the logics of $\rhd$ for semantics corresponding to Lewis' systems S2–S5, as well as labelled sequent calculus for each of these logics.[4]

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[3] In [4,5] logics of SSI were called weakly connexive. The more appropriate label *Boethian* was suggested by Claudio Pizzi.

[4] We avoid considering the weaker logic corresponding to Lewis' S1, for the sake of brevity.

The underlying idea is this: We already know that (non-normal) relational models provide a semantics for Lewis's non-normal logic S2 for $\prec$, the logic of all formulas in the language $\mathcal{L}_{\prec}$ that are valid in reflexive non-normal relational models. Furthermore, via the standard modal translation of $\prec$ in terms of $\Box$ (defining $A \prec B$ as $\Box(A \supset B)$), Lemmon [9] provided a modal axiomatisation of S2. In this paper we introduce an axiomatic system for the companion super-strict implication logic ST2 of all formulas in the language $\mathcal{L}_{\triangleright}$ that are valid in reflexive non-normal relational models (now used to interpret $\triangleright$). For the completeness results, we rely on the fact that $\prec$, $\Box$, and $\Diamond$ are expressible in terms of $\triangleright$ and *vice-versa*: this allows us to use the method of *translation and backtranslation* developed and used by one of the authors to axiomatise many different conditional logics [23–25]. That is, we prove the Soundness Theorem of S2 with respect to the class of reflexive non-normal relational models by simulating the system ST2 in the system S2. Vice versa, Completeness is obtained by backsimulating S2 in ST2.

Based on this ground result, we further consider axiomatic extensions of ST2. More precisely, we provide axiomatic systems for the logics ST3, ST4 and ST5, the logics of valid $\mathcal{L}_{\triangleright}$-formulas over the reflexive non-normal relational models which also are, respectively, transitive, transitive and normal, normal and with an equivalence relation as accessibility relation. The corresponding Soundness and Completeness results are proved similarly via simulation and backsimulation methods using respectively the known results for the corresponding systems of strict implication S3, S4 and S5.

Finally, we introduce the labelled calculi G3.STn for the logics STn ($2 \leq n \leq 5$). We prove that such calculi have good structural properties: height-preserving admissibility of the rules of weakening and contraction, syntactic admissiblity of cut, and the height-preserving invertibility of all rules. We also provide corresponding Soundness and Completeness results. We prove Completeness by simulating the axiomatic system for STn within the calculus G3.STn. Finally, we prove the decidability of G3.ST2 by showing the termination of proof-search and, as a consequence, we prove that models for ST2 have the finite model property.

The rest of the paper is organised as follows: Section 2 presents the syntax and semantics for logics of $\triangleright$ and of $\prec$ and shows that they are intertranslatable. In Section 3 we discuss valid and invalid principles for $\triangleright$, and introduce an axiomatisation of ST2, which we then prove sound and complete in Section 4, as explained above. Section 5 does the same for the logics ST3–ST5, and discusses the problem of adding BT to normal systems. Section 6 presents labelled sequent calculi G3.ST2–G3.ST5. Section 7 shows that these calculi have good structural properties and Section 8 proves that they

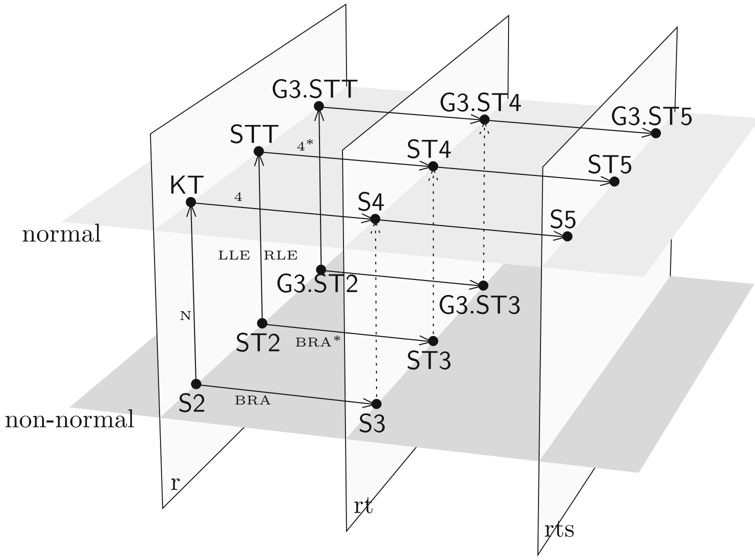

Figure 1. Relations between the systems of strict implication (front plane) super-strict implication (two back planes) and the properties of the accessibility relation (r = reflexive, t = transitive, s = symmetric) in normal (top plane) and non-normal (bottom plane) relational semantics

are sound and complete and that G3.ST2 is decidable. Finally, Appendix A shows that the deduction theorem for material implication holds in ST2 and S2. We shall use the deduction theorem in our proofs throughout the paper.

The paper thus investigates two kinds of proof systems for super-strict implication: the Hilbert-style systems ST2–ST5 (Sections 2–5) and the labelled sequent calculi G3.ST2–G3.ST5 (Sections 6–8). These are independent of each other, and a reader interested in only one of them can read the relevant part without reference to the other part. Nonetheless, we will show some interesting interrelations between the two kinds of proof systems: the Hilbert-style systems will be used in one proof of the completeness of the labelled calculi (Theorem 39) and the latter will be used in one proof of the soundness of the former (Corrolary 40) (Figure 1).

2. Syntax and Semantics

This Section introduces the syntax and semantics for super-strict implication and states the inter-translatability with strict implication.

2.1. Syntax

The language $\mathcal{L}_\Box$ of modal logics and the language $\mathcal{L}_\rhd$ for logics of super-strict implication are generated by the following grammars (where p is in some non-empty set of sentential variables $\mathcal{P}$, which for simplicity we assume at most countable):

$$A{:=}\ p \mid \neg A \mid A \wedge A \mid A \vee A \mid A \supset A \mid \Box A \qquad (\mathcal{L}_\Box)$$
$$A{:=}p \mid \neg A \mid A \wedge A \mid A \vee A \mid A \supset A \mid A \rhd A \qquad (\mathcal{L}_\rhd)$$

Parentheses follow the usual conventions and the modal operators $\Box$ and $\rhd$ bind lighter than all other operators. Capital roman letters will be used as meta-variables for formulas (of the appropriate language). The weight of a formula, $\mathtt{w}(A)$, is given by the number of binary operators occurring therein. For both languages, we denote by $\top$ an arbitrary propositional tautology (for example $p \vee \neg p$); $\bot = \neg\top$; and $(A \equiv B) = (A \supset B) \wedge (B \supset A)$. In the language $\mathcal{L}_\Box$, we define $\Diamond A = \neg \Box \neg A$. In the language $\mathcal{L}_\rhd$, we use the following abbreviation for the *inner modality* $\boxdot A = (\top \rhd A)$, the dual being $\diamond\!\!\!\cdot = \neg \boxdot \neg$. Strict implication ($⥽$) is defined in $\mathcal{L}_\Box$ as $A ⥽ B = \Box(A \supset B)$ and in $\mathcal{L}_\rhd$ as $A ⥽ B = \top \rhd (A \supset B) = \boxdot(A \supset B)$.[5] The formulas that $\mathcal{L}_\Box$ and $\mathcal{L}_\rhd$ have in common constitute the classical propositional language $\mathcal{L}$ (over $\mathcal{P}$). A formula from $\mathcal{L}$ will be called *classical*.

2.2. Semantics

We here present Kripke's semantics for non-normal logics. This semantics is based on the fact that non-normal systems, such as S2 and S3, can be consistently extended with the axiom $\Diamond \Diamond A$: we can have models with accessible points where every formula, $\bot$ included, is possible. Kripke used this fact to extend the well-known relational semantics for normal modalities with so-called *non-normal points* where every formula is possible and no formula is necessary, see [2,8,31]. We use $\wp(W)$ to designate the powerset of W and the long arrow notation $f : X \longrightarrow Y$ to designate that f is a total function from X to Y.

Definition 1. (Models) A *relational model* $\mathfrak{M}$ is a tuple $\langle W, R, N, V \rangle$, where $W \neq \emptyset$, R is a reflexive relation over W, $N \subseteq W$, and $V : \mathcal{P} \longrightarrow \wp(W)$.

W are the *worlds*, to be denoted by $w, v, u, \ldots$. R is the *accessibility relation*, and we write wRv for v being accessible from w. Reflexivity means that

[5] Another option is the more complicated $A ⥽ B = \neg(\neg(A \supset B) \rhd \neg(A \supset B))$.

all worlds are accessible to themselves (wRw). N are the *normal worlds* with respect to which truth in a model and validity will be defined. V is a usual *valuation function*, assigning to each propositional variable p a set of worlds $V(p)$—the worlds where p is true (also called *p-worlds*, for short). Our relational models are sometimes also called *reflexive non-normal relational models.*

The models for $\mathcal{L}_\Box$ and $\mathcal{L}_\triangleright$ are in fact the same. However, we need to distinguish the model relation for $\mathcal{L}_\Box$, designated by $\vDash$, from the model relation for $\mathcal{L}_\triangleright$, designated by $\vDash_\triangleright$ (when necessary).

DEFINITION 2. Truth in a world in a model for $\mathcal{L}_\Box$ is defined standardly for sentential variables and for the classical connectives $\neg$, $\wedge$, $\vee$, and $\supset$; for $\Box$ we have:

$$w \vDash \Box A \quad \text{iff} \quad w \in N \text{ and } \forall v \in W \ (wRv \text{ implies } v \vDash A)$$

DEFINITION 3. Truth in a world in a model for $\mathcal{L}_\triangleright$ is defined as in a model for $\mathcal{L}_\Box$ for classical connectives and it is defined as follows for $\triangleright$:

$$w \vDash_\triangleright A \triangleright B \quad \text{iff} \quad \begin{array}{ll} \text{(i)} & w \in N \text{ and} \\ \text{(ii)} & \exists v \in W (wRv \text{ and } v \vDash_\triangleright A) \text{ and} \\ \text{(iii)} & \forall v \in W \ (wRv \text{ implies } v \vDash_\triangleright A \supset B \ ) \end{array}$$

A formula is *true* in a model if it is true in all normal points of that model; it is *valid* in a class of models whenever it is true in all models in that class. A *logic* (over $\mathcal{L}_\Box$ or $\mathcal{L}_\triangleright$) is the set of all formulas that are valid in some class of models. In particular, we consider the following logics over $\mathcal{L}_\Box$ ($\mathcal{L}_\triangleright$):

- S2 (ST2) is the set of formulas that are valid in the class of all models;
- S3 (ST3) is the set of formulas that are valid in the class of all transitive models, where a model is *transitive* if $\forall w, v, u \in W(wRv \ \& \ vRu \supset wRu)$
- KT (STT) is the set of formulas that are valid in the class of all normal models, where a model is *normal* if $N = W$;
- S4 (ST4) is the set of formulas that are valid in the class of all normal transitive models;
- S5 (ST5) is the set of formulas that are valid in the class of all normal, transitive and symmetric models, where a model is *symmetric* if $\forall w, v \in W(wRv \text{ implies } vRw)$.

2.3. Intertranslatability

We are now going to show that logics over $\mathcal{L}_\Box$ are intertranslatable with logics over $\mathcal{L}_\rhd$ by means of two truth-preserving translations $\circ$ and $\bullet$.

Definition 4. The *translation* $\circ : \mathcal{L}_\rhd \longrightarrow \mathcal{L}_\Box$ is defined by

1. $p^\circ = p$
2. $(\neg A)^\circ = \neg A^\circ$
3. $(A \sharp B)^\circ = (A^\circ \sharp B^\circ)$ for $\sharp \in \{\wedge, \vee, \supset\}$
4. $(A \rhd B)^\circ = \Box(A^\circ \supset B^\circ) \wedge \Diamond A^\circ$

We call A° the translate of A. When A is classical, we have $A^\circ = A$. In particular $\top^\circ = \top$ and $\bot^\circ = \bot$.

Definition 5. The *backtranslation* $\bullet : \mathcal{L}_\Box \longrightarrow \mathcal{L}_\rhd$ is defined by

1. $p^\bullet = p$
2. $(\neg A)^\bullet = \neg A^\bullet$
3. $(A \sharp B)^\bullet = (A^\bullet \sharp B^\bullet)$ for $\sharp \in \{\wedge, \vee, \supset\}$
4. $(\Box A)^\bullet = (\top \rhd A^\bullet)$

We call $A^\bullet$ the backtranslate of A. The backtranslation uses the fact that the models are reflexive and hence serial. $\top \rhd A$ (our $\boxdot A$) expresses not only $\Box A$ but also $\Diamond \top$.[6] As before, when A is classical, we have $A^\bullet = A$. In particular $\top^\bullet = \top$ and $\bot^\bullet = \bot$.

The following shows that the translation is semantically well behaved. That is, C and C° express the same proposition:

Lemma 6. *For all $C \in \mathcal{L}_\rhd$, we have $w \vDash_\rhd C$ iff $w \vDash C^\circ$.*

Proof. By induction on the complexity of the formula. It suffices to verify $C = (A \rhd B)$, assuming the property holds for A, B (IH).

$$
\begin{array}{lll}
w \vDash_\rhd (A \rhd B) & \text{iff } w \in N \ \&\ \exists v \in W(wRv \ \&\ v \vDash_\rhd A) & \\
 & \quad \&\ \forall v \in W(wRv \text{ implies } v \vDash_\rhd A \supset B) & \vDash_\rhd \\
 & \text{iff } w \in N \ \&\ \neg\forall v \in W(wRv \text{ implies } v \nvDash_\rhd A) & \\
 & \quad \&\ w \in N \ \&\ \forall v \in W(wRv \text{ implies } v \nvDash_\rhd A \text{ or } v \vDash_\rhd B) & \\
 & \text{iff } w \in N \ \&\ \neg\forall v \in W(wRv \text{ implies } v \nvDash A^\circ) & \\
 & \quad \&\ w \in N \ \&\ \forall v \in W(wRv \text{ implies } v \nvDash A^\circ \text{ or } v \vDash B^\circ) & \text{IH} \\
 & \text{iff } w \vDash \Box(A^\circ \supset B^\circ) \wedge \neg\Box\neg A^\circ & \vDash \\
 & \text{iff } w \vDash (A \rhd B)^\circ & \circ
\end{array}
$$

[6] For non-serial models, we would need $(\Box A)^\bullet = \neg(\neg A^\bullet \rhd \neg A^\bullet)$.

A similar fact can be proven for $\bullet$, but we won't need it. ■

From the previous Lemma, we obtain our weak validity preservation result:

Corollary 7. *If* $\models A^\circ \strictif B^\circ$ *then* $\models_{\rhd} \Diamonddot A \supset (A \rhd B)$.

Proof. Suppose $\models A^\circ \strictif B^\circ$. That is $\models \Box(A^\circ \supset B^\circ)$ by definition of $\strictif$ in $\mathcal{L}_\Box$. Consider an arbitrary normal world w of some reflexive non-normal relational model, and assume $w \models_{\rhd} \Diamonddot A$. That is, in w and according to $\models_{\rhd}$, we have $\neg(\top \rhd \neg A)$ by definition of $\Diamonddot$. Thus, in w but according to $\models$, we obtain $\neg(\Diamond \top \wedge \Box(\top \supset \neg A^\circ))$ by Lemma 6. Hence $\Box \bot \vee \neg \Box \neg A^\circ$. And thus (since w is normal) $\neg \Box \neg A^\circ$, that is $\Diamond A^\circ$. Thus overall $w \models \Diamond A^\circ \wedge \Box(A^\circ \supset B^\circ)$, so that $w \models (A \rhd B)^\circ$. Hence by Lemma 6 again, $w \models_{\rhd} A \rhd B$. Since we proved $\Diamonddot A \supset (A \rhd B)$ for an aribtrary normal world w, we can conclude $\models_{\rhd} \Diamonddot A \supset (A \rhd B)$. ■

3. Axiomatic Systems for S2 and ST2

We are now going to introduce axiomatic systems for the logics S2 and ST2. As usual, an axiomatic *derivation* from the set of assumptions Γ is a sequence of formulas such that each formula is an axiom or a member of Γ or it follows from formulas preceding it by a rule of the system. We use $\Gamma \vdash_{\mathsf{S}} A$ to say that the formula A is derivable from the set of assumptions Γ in the system S; we omit the subscript whenever the system is clear from the context. A *theorem* is a formula derivable from the empty set of assumptions; we write $\vdash_{\mathsf{S}} A$ when A is a theorem of the system S.

3.1. Lemmon's Axiomatisation

The following Hilbert-style axiomatisation of S2 was given by Lemmon [9]:

Definition 8. (Lemmon's axiomatisation of S2)

	Axioms of S2
PT	Classical propositional tautologies in the language $\mathcal{L}_\Box$
K	$\Box(A \supset B) \supset (\Box A \supset \Box B)$
T	$\Box A \supset A$
	Rules of S2
MP	If $\Gamma \vdash A$ and $\Gamma \vdash A \supset B$ then $\Gamma \vdash B$
rN	If A is an instance of one of the axioms, then $\vdash \Box A$
BR	If $\vdash \Box(A \supset B)$ then $\vdash \Box(\Box A \supset \Box B)$

K is also known as the *Distribution axiom*, T as the *Truth axiom*, MP is *Modus Ponens* for $\supset$, rN stands for *restricted Necessitation*, BR is *Becker's rule*. $\mathsf{S2}$ was shown to be sound and complete for relational models in $\mathcal{L}_{\Box}$ [8].

3.2. Super-Strict Axiomatisation

By T* and K* we designate the axioms T and K written for $\boxdot$.

We can similarly ask for the logic of super-strict implication in relational models (the super-strict companion $\mathsf{ST2}$ to $\mathsf{S2}$). We use a Hilbert-style axiomatisation:

Definition 9. (axiomatisation of $\mathsf{ST2}$)

	Axioms of $\mathsf{ST2}$	
PT	Classical propositional tautologies in $\mathcal{L}_{\rhd}$	
AT	$\neg(A \rhd \neg A)$	Aristotle's Thesis
PA	$(A \rhd B) \supset \dot{\Diamond} A$	Possible Antecedent
INC	$(A \rhd B) \supset \boxdot(A \supset B)$	Inclusion
AND	$(A \rhd B) \wedge (A \rhd C) \supset (A \rhd B \wedge C)$	Conjunction of Consequents
TID	$\top \rhd \top$	Tautological ID
SPRES	$\boxdot(A \supset B) \wedge \dot{\Diamond} A \supset (A \rhd B)$	Strong Preservation
T*	$\boxdot A \supset A$	Truth axiom for $\boxdot$
	Rules of $\mathsf{ST2}$	
MP	If $\Gamma \vdash A$ and $\Gamma \vdash A \supset B$, then $\Gamma \vdash B$	Modus Ponens
rLLE	If $a \equiv B$ is an instance of PT, then $\vdash (A \rhd C) \supset (B \rhd C)$	Restricted Left Logical Equivalence
rRW	If $A \supset B$ is an instance of PT, then $\vdash (C \rhd A) \supset (C \rhd B)$	Restricted Right Weakening
BR*	If $\vdash \boxdot(A \supset B)$ then $\vdash \boxdot(\boxdot A \supset \boxdot B)$	Becker's rule
rN*	If A is an instance of T*, K*, PA, INC or SPRES, then $\vdash \boxdot A$	restricted Necessitation

AT is *Aristotle's Thesis.* PA expresses that the conditional implies the possibility of the antecedent. INC has the object language form of a postulate known as *Inclusion* in belief revision theory and expresses that the conditional implies the strict conditional. SPRES is a form of Preservation called *Strong Preservation* (in belief revision theory). Together with PA and INC, this expresses the definition of $\rhd$ in terms of $\boxdot$. AND is a standard axiom of conditional logic which expresses that the consequent of a conditional is closed under conjunction. TID is the identity for $\top$, but more importantly it expresses that $\top$ is possible (in the sense of $\Diamond$). T* expresses that the inner modality implies truth. The rules rLLE and rRW are restricted versions of *Left Logical Equivalence* and *Right Weakening*, which are normal rules of conditional logic. BR* can be seen as the translate of the rule BR.[7] The rule rN* is again restricted, to the instances of T*, K*, PA, INC or SPRES. The restriction does not mean that the formulas to which rN* can be applied need to be axioms, they only need to be of a certain form.[8] The notion of a derivation is the same here as before.

3.3. Principles of Super-Strict Implication

LEMMA 10. *The following principles are invalid:*

ID	$A \rhd A$	Identity Conditional
C	$(A \rhd B) \supset (\neg B \rhd \neg A)$	Contraposition
SA	$(A \rhd C) \supset (A \wedge B \rhd C)$	Strengthening the Antecedent
TC	$A \rhd \top$	Tautological Consequent
AA	$\bot \rhd B$	Antilogical Antecedent
SI	$A \wedge B \rhd A$	Simplification
CEM	$(A \rhd B) \vee (A \rhd \neg B)$	Conditional Excluded Middle
BT	$(A \rhd B) \rhd \neg(A \rhd \neg B)$	(strong) Boethius Thesis
S	$(A \rhd B) \supset (B \rhd A)$	Symmetry

PROOF. To disprove the first eight principles, we use a frame (i.e., a model without valuation) with a normal world $w \in N$ such that $R(w) = \{w\}$, changing only the valuation for each principle.
ID. Let $w \vDash_{\rhd} \neg p$. Then $p \rhd p$ is false in w, since there is no accessible p-world.
C. Assume $w \vDash_{\rhd} p \wedge q$. Then $p \rhd q$ is true in w, but $\neg q \rhd \neg p$ is false in w, since there is no accessible $\neg q$-world.

[7] In non-serial frames we would need to use a modified version.

[8] Note that K* follows from rN* and T* (see Lemma 11). Observe also that we could have restricted necessitation to all axioms of ST2 (and K*) except for TID, instead of restricting it to T*, K*, PA, INC, and SPRES, cf. Lemma 34. We have chosen a more minimal axiomatisation that is sufficient to prove completeness. Two other options are: (1) to drop rN* and extend rRW for $C = \top$ to formulas $\top \supset X$ where X is an instance of T*, K*, PA, INC and SPRES, or (2) drop rN* as well as PA, INC, and SPRES, and adopt instead as additional axioms the boxed versions $\boxdot Y$ where Y = T*, K*, PA, INC, SPRES.

SA. Assume $w \vDash_{\triangleright} p \wedge \neg q \wedge r$. Then $p \triangleright r$ is true in w, but $p \wedge q \triangleright r$ is false in w, since there is no accessible world where $p \wedge q$ is true.
TC. $\bot \triangleright \top$ is false in w since there is no accessible $\bot$-world.
AA. See TC.
SI. Let $w \vDash p \wedge \neg q$. Then $p \wedge q \triangleright p$ is false in w, since there is no accessible $p \wedge q$-world.
CEM. Let $w \vDash \neg p \wedge q$. Then $p \triangleright q$ is false since there is no accessible p-world, and $p \triangleright \neg q$ is false for the same reason.
BT. As in ID—$p \triangleright p$ is false in w, hence there is no accessible world where $p \triangleright p$ is true. Hence BT is false in w.
S. We extend the previous frame to $R(w) = \{w, v\}$. Let the valuation be such that $w \vDash_{\triangleright} p \wedge q$ and $v \vDash_{\triangleright} \neg p \wedge q$. Then we have $p \triangleright q$ in w. But we do not have $q \triangleright p$ in w, since there is an accessible q-world which is not a p-world. ■

ID is a standard principle of conditional logic—the axiom of *Identity*. C is *Contraposition*, and SA is the often criticized *Strengthening of the Antecedent*. TC and AA are the paradoxes of strict implication *Tautological Consequent* and *Antilogical Antecedent*, SI is *Simplification*, CEM is *Conditional Excluded Middle* and equivalent to the converse of wBT (CwBT), BT is *strong Boethius Thesis*, and S is *Symmetry*.

Apart from CEM, BT and S, all these principles are valid for strict implication (replacing $\triangleright$ by $\prec$). Since $\triangleright$ invalidates TC and AA, it avoids the paradoxes of strict implication. Since $\triangleright$ invalidates SI, it shares the connexivists' doubts about C. I. Lewis' independent proof. Since $\triangleright$ invalidates S, it can be thought of as a directional connective, as any implicative connective should be. As David Lewis' counterfactual, $\triangleright$ invalidates CEM, and thus differs essentially from Stalnaker's conditional. The invalidity of BT is largely due to the fact that BT would imply the possibility of any $A \triangleright B$, and hence in particular of $A \triangleright \bot$, which is always false (see below). The remaining invalidities may be motivated as follows: The reaction of the non-monotonic reasoning tradition to strict implication was not that much to criticise TC and AA, but rather SA. Indeed, SA was rejected by the early proponents of this tradition—Stalnaker [30] and Lewis [12]. These authors also endorsed unrestricted Right Weakening (RW) as well as Identity (ID), and thus also Simplification (SI). As a consequence, they needed to reject Contraposition (since C + RW implies SA). Super-strict implication shares rejection of SA and C with these accounts.[9] Super-strict implication however differs from

[9] C holds for strict implication, as well as for the so-called evidential conditional [3,28].

the non-monotonic reasoning tradition and variably strict conditionals in that ID is invalid and that RW is sensibly restricted, so that the invalidity of SI may equally be explained from the invalidity of ID or the restricted use of RW.

LEMMA 11. *The following principles are derivable:*

K*	$\boxdot(A \supset B) \supset (\boxdot A \supset \boxdot B)$	Distribution for $\boxdot$
MI	$(A \rhd B) \supset (A \supset B)$	implied Material Implication
MP($\rhd$)	$A, A \rhd B \vdash B$	Modus Ponens for $\rhd$
NAA	$\neg(\bot \rhd B)$	No Antilogical Antecedent
NAC	$\neg(A \rhd \bot)$	No Antilogical consequent
wBT	$(A \rhd B) \supset \neg(A \rhd \neg B)$	weak Boethius Thesis
C*	$\diamonddot \neg B \wedge (A \rhd B) \supset (\neg B \rhd \neg A)$	Restricted Contraposition
$\diamonddot$ID	$\diamonddot A \supset (A \rhd A)$	Possibility to ID
ID$\diamonddot$	$(A \rhd A) \supset \diamonddot A$	ID to Possibility
DW	$(A \rhd B) \supset (A \rhd B \vee C)$	Disjunctive weakening
OR	$(A \rhd C) \wedge (B \rhd C) \supset (A \vee B \rhd C)$	Disjunction of Antecedents
RM	$(A \rhd C) \wedge \neg(A \rhd \neg B) \supset (A \wedge B \rhd C)$	Rational Monotonicity
CM	$(A \rhd C) \wedge (A \rhd B) \supset (A \wedge B \rhd C)$	Cautious Monotonicity
wTR	$(A \rhd B) \wedge (B \rhd C) \supset (A \rhd C)$	weak Transitivity
CTR	$(A \rhd B) \wedge (A \wedge B \rhd C) \supset (A \rhd C)$	Cumulative Transitivity

PROOF. We use the deduction theorem for $\supset$ in ST2 (Appendix, Lemma 52), MP and other classical reasoning, often without saying (e.g. $\supset$ contraposes and is transitive).

K*. Applying rN* to a formula of form K*, we obtain $\boxdot(\boxdot(A \supset B) \supset (\boxdot A \supset \boxdot B))$. By T* we thus get $\boxdot(A \supset B) \supset (\boxdot A \supset \boxdot B)$. This is K*.

MI. Follows from INC and T*.

MP($\rhd$). Follows from MI.

NAA. $\boxdot \top$ by TID. That is $\neg \diamonddot \bot$. Therefore $\neg(\bot \rhd B)$ by contraposing PA.

NAC. $\bot \supset \neg A$ is PT. Thus $(A \rhd \bot) \supset (A \rhd \neg A)$ by rRW. Since $\supset$ contraposes, we obtain $\neg(A \rhd \neg A) \supset \neg(A \rhd \bot)$. Yet $\neg(A \rhd \neg A)$ by AT. Hence $\neg(A \rhd \bot)$.

wBT. Suppose for reductio that $A \rhd B$ and $A \rhd \neg B$. Hence $A \rhd \bot$ by AND. This contradicts NAC.

C*. Suppose $\diamonddot \neg B$ and $A \rhd B$. The second implies $\boxdot(A \supset B)$ by INC. Thus $\boxdot(\neg B \supset \neg A)$ by rRW. Together with $\diamonddot \neg B$ this yields $\neg B \rhd \neg A$ by SPRES.

$\diamonddot$ID. Suppose $\diamonddot A$. $\top \rhd \top$ by TID. But $\top \supset (A \supset A)$ is PT. Hence $\top \rhd (A \supset A)$ by rRW. That is $\boxdot(A \supset A)$. Together with $\diamonddot A$, this yields $A \rhd A$ by SPRES.
ID$\diamonddot$. By PA.
DW. $B \supset B \vee C$ is PT. Thus $(A \rhd B) \supset (A \rhd B \vee C)$ by rRW.
OR. Suppose $A \rhd C$ and $B \rhd C$. From the first we get $\boxdot(A \supset C)$ by INC, and $\diamonddot A$ by PA, thus $\diamonddot(A \vee B)$: By rRW $(\top \rhd \neg(A \vee B)) \supset (\top \rhd \neg A)$. Since $\diamonddot A$, i.e., $\neg(\top \rhd \neg A)$, we obtain $\neg(\top \rhd \neg(A \vee B))$. That is $\diamonddot(A \vee B)$. Similarly from $B \rhd C$ we obtain $\boxdot(B \supset C)$. This together with $\boxdot(A \supset C)$ yields $\boxdot(A \vee B \supset C)$ by AND and rRW. Together with $\diamonddot(A \vee B)$ we obtain $A \vee B \rhd C$ by SPRES.
RM. Suppose $A \rhd C$ and $\neg(A \rhd \neg B)$. From the first we obtain $\diamonddot A$ and $\boxdot(A \supset C)$ (cf. the proof for OR). From $\boxdot(A \supset C)$ we obtain $\boxdot(A \wedge B \supset C)$ by rRW. Contraposing SPRES, from $\neg(A \rhd \neg B)$ we obtain $\neg \diamonddot A$ or $\neg \boxdot(A \supset \neg B)$. Since $\diamonddot A$, we must have $\neg \boxdot(A \supset \neg B)$. That is $\diamonddot(A \wedge B)$. This together with $\boxdot(A \wedge B \supset C)$ delivers $A \wedge B \rhd C$ by SPRES.
CM. Suppose $A \rhd B$ and $A \rhd C$. From the first, we get $\neg(A \rhd \neg B)$ by wBT. Together with $A \rhd C$ this yields $A \wedge B \rhd C$ by RM.
wTR. Suppose $A \rhd B$ and $B \rhd C$. Thus $\diamonddot A$, $\boxdot(A \supset B)$ and $\boxdot(B \supset C)$. The two last yield $\boxdot(A \supset C)$ by AND and rRW. Together with $\diamonddot A$ this yields $A \rhd C$.
CTR. Assume $A \rhd B$ and $A \wedge B \rhd C$. Thus $\boxdot(A \supset B)$, and $\boxdot(A \wedge B \supset C)$ by INC. Hence $\boxdot(A \supset C)$ by AND and rRW. From $A \rhd B$ we also obtain $\diamonddot A$. Together with $\boxdot(A \supset C)$, this yields $A \rhd C$ by SPRES. ■

Due to the soundness result proven below, these derivabilities are validities, and a standard semantic proof is possible as well (left to the reader).[10]

K* is K for $\boxdot$, MI stands for the fact that the conditional (here $\rhd$) implies *Material Implication* and MP($\rhd$) stands for Modus Ponens for $\rhd$. NAA stands for *No Antilogical Antecedent*, NAC is the reverse and stands for *No Antilogical Consequent*, wBT is *weak Boethius Thesis*, C* is a restricted form of Contraposition (C), $\diamonddot$*ID* restricts Identity (ID), together with *ID*$\diamonddot$ this says that the identity conditional expresses the inner possibility. OR, RM and CM are standard principles of non-monotonic or variably strict conditionals, where OR is also known as *Disjunction in the Antecedent*, RM stands for *Rational Monotonicity*, and CM for *Cautious Monotonicitiy*.

[10] For example for MI: Suppose $A \rhd B$ is true in w. Thus by the defining clause (iii) for all $v \in W$ such that wRv we have that $A \supset B$ is true in v. By reflexivity we obtain that $A \supset B$ is true in w.

wTR is *weak Transitivity*, and CTR is known as *Cumulative Transitivity*.[11] wTR is not to be confused with (strong) *Transitivity* (TR): $(A \rhd B) \wedge (B \rhd C) \rhd (A \rhd C)$.

By adopting NAA, the super-strict implication rejects the first paradox of strict implication (AA) by accepting the negation,[12] NAC also expresses that there can be no impossible antecedent conditional. Indeed, $A \rhd \bot$ usually expresses that A is impossible in terms of the outer modality. So that by NAC, $\rhd$ has no outer modality. More importantly, adopting NAC—the hallmark of non-vacuism [23]—comes down to treating all impossible antecedent conditionals as false. By wBT combined with AT, the logic of super-strict implication is a Boethian logic, although it is not fully connexive. Interestingly and contrary to strict implication, super-strict implication validates a restricted version of RW (DW) without validating SA, since it invalidates C. Finally, strong Transitivity (TR) is invalid for superstrict implication but derivable for strict implication $\rhd$ in $\mathsf{S3}$.

Due to OR, RM and CM, super-strict implication shares important similarities with the non-monotonic reasoning tradition. Indeed, starting with Stalnaker [30] and D. Lewis [12], this tradition adopted the latter two principles as admissible restrictions of the problematic principle SA. The similarity is best seen by considering an equivalent axiomatisation of $\mathsf{ST2}$—call it $\mathsf{ST2}'$—where we replace SPRES by RM and $\odot$ID.[13] $\mathsf{ST2}'$ can be compared to one of the most famous systems of (non-monotonic) conditional logic – the system VW of D. Lewis [12]. This system can be axiomatised by MP, the unrestricted versions LLE, RW, and the axioms PT, ID, OR, AND, CM, RM, INC, T*.[14] Thus the only difference is that $\mathsf{ST2}$ has the restricted versions rLLE, rRW, rN*, the restricted ID in the form of $\odot$ID and TID, and that additionally PA and AT hold, and that OR and CM need not be stated explicitly.

[11] CTR is sometimes denoted CUT.

[12] Interestingly the negation of the second paradox (TC): $\neg(A \rhd \top)$ (*No Tautological Consequent* NTC) is invalid due to TID.

[13] SPRES and $\odot$ID are derivable in $\mathsf{ST2}$. Conversely, SPRES is derivable by RM and $\odot$ID and the remaining axioms: From RM we get $\boxdot(A \supset B) \wedge \odot A \supset (A \rhd (A \supset B))$, using LLE applied to the PT equivalence $A \equiv (A \wedge \top)$. But $(A \rhd (A \supset B)) \supset \odot A$ by PA and $\odot A \supset (A \rhd A)$ by $\odot$ID. Furthermore $(A \rhd (A \supset B)) \wedge (A \rhd A) \supset (A \rhd B)$ by AND and rRW. Chaining the results, we obtain $\boxdot(A \supset B) \wedge \odot A \supset (A \rhd B)$.

[14] INC is redundant, as it follows from ID, RW, OR. Readers might be more acquainted with the axiomatisation for VW where instead of INC and T* we have MI. The two axiomatisations are equivalent: INC and T* imply MI; MI implies T*. We use the axiomatisation with INC and T* for comparative reasons.

The essential difference, however, to the non-monotonic reasoning tradition is this: Non-monotonic reasoning does not only reject C to avoid SA all by maintaining RW, it also needs to reject wTR to avoid SA all by maintaining RW and ID (since wTR + ID + RW also implies SA). Super-strict implication takes a different route by maintaining wTR but allowing only restricted forms of ID (TID, $\Diamond$ID). Furthermore, although our ground logic for (non-normal) super-strict implication does not adopt unrestricted RW as an axiom, we will shortly see that we can consistently extend the logic of super-strict implication to systems where unrestricted RW holds.

In the following, we establish completeness of the axiomatisation of super-strict implication, using a method developed by Raidl [23,24]. The 'neutral conditional' examined in [22,23] is a similar strengthening of an underlying conditional $>$, namely $(A \rhd' B) = (A > B) \wedge \neg(A > \bot)$. The similarity lies in the fact that our super-strict conditional can equivalently be conceived as $(A \rhd B) = \Box(A \supset B) \wedge \neg\Box(A \supset \bot)$. The difference is that Raidl uses a semantics where the underlying $>$ satisfies the unrestricted LLE and the unrestricted RW, so that $\rhd'$ satisfies them as well. These however do not hold unrestrictedly for our (non-normal) super-strict implication. For this reason our Lemma 16, required to prove our completeness result here, needs to be obtained by an intermediary step.

4. Soundness and Completeness of ST2

We here prove soundness and completeness of the system ST2 in our (reflexive non-normal) relational models used to interpret the language $\mathcal{L}_{\rhd}$.[15]

4.1. Soundness

LEMMA 12. (Simulation) *For all* $\chi \in \mathcal{L}_{\rhd}$, *if* $\vdash_{\mathsf{ST2}} \chi$ *then* $\vdash_{\mathsf{S2}} \chi^\circ$.

PROOF. By induction on the length of the derivation. *Derivations of length 1.* Then χ is an axiom of ST2. Eight cases are possible: either χ is an instance of PT, AT, PA, INC, AND, TID, SPRES, or T*.
PT. Suppose χ is an instance of PT in the language $\mathcal{L}_{\rhd}$. Thus there is a classical formula $\varphi[p_1, \ldots, p_n]$ with variables in $p_1, \ldots, p_n$ and there are formulas $\psi_1, \ldots, \psi_n$ such that $\chi = \varphi[\psi_1/p_1, \ldots, \psi_n/p_n]$. It is then provable by induction on the complexity of φ that $\chi^\circ = (\varphi[\psi_1/p_1, \ldots, \psi_n/p_n])^\circ =$

[15] Appendix A shows that the deduction theorem for $\supset$ holds in ST2. Hence most proofs here and in the next Section could have been shortened. Nevertheless we have opted for direct proofs.

$\varphi[\psi_1^\circ/p_1, \ldots, \psi_n^\circ/p_n]$ (compare Lemma 1 from [23]). But the latter formula is an instance of PT in $\mathcal{L}_\Box$. Hence it is derivable in $\mathsf{S2}$.

AT. Suppose χ is an instance of AT. The translate is of the form $\neg(\Box(A \supset \neg A) \wedge \neg\Box\neg A)$. $\neg\Box\neg A \vee \Box\neg A$ is PT in $\mathcal{L}_\Box$. $(A \supset \neg A) \supset \neg A$ is PT. Hence $\Box((A \supset \neg A) \supset \neg A)$ by rN. Therefore $\Box(A \supset \neg A) \supset \Box\neg A$ by K. Since $\supset$ contraposes, we get $\neg\Box\neg A \supset \neg\Box(A \supset \neg A)$. By PT, we obtain the translate of AT.

PA. The translate of PA is of the form $\Box(A \supset B) \wedge \Diamond A \supset \neg(\Box(\top \supset \neg A) \wedge \Diamond\top)$. Assume $\Box(A \supset B)$, $\Diamond A$. Therefore $\neg\Box\neg A$. $(\top \supset \neg A) \supset \neg A$ is PT. Hence $\Box(\top \supset \neg A) \supset \Box\neg A$ by rN, K and MP. Thus $\neg\Box\neg A \supset \neg\Box(\top \supset \neg A)$. Since $\neg\Box\neg A$, we obtain $\neg\Box(\top \supset \neg A)$. Therefore also $\neg\Box(\top \supset \neg A) \vee \neg\Diamond\top$, that is $\neg(\Box(\top \supset \neg A) \wedge \Diamond\top)$.

INC. The translate of INC is of the form $\Box(A \supset B) \wedge \Diamond A \supset \Box(\top \supset (A \supset B)) \wedge \Diamond\top$. Assume $\Box(A \supset B)$, $\Diamond A$. From the first we obtain $\Box(\top \supset (A \supset B))$ by PT, rN, K and MP. And $\Diamond\top$ holds by T.

AND. The translate of AND is of the form $\Box(A \supset B) \wedge \Diamond A \wedge \Box(A \supset C) \wedge \Diamond A \supset \Box(A \supset B \wedge C) \wedge \Diamond A$. Assume $\Box(A \supset B)$, $\Diamond A$, $\Box(A \supset C)$. It suffices to prove $\Box(A \supset B \wedge C)$. We have $(A \supset B) \supset ((A \supset C) \supset (A \supset B \wedge C))$ by PT. Thus $\Box(A \supset B) \supset \Box((A \supset C) \supset (A \supset B \wedge C))$ by rN, K and MP. Hence $\Box(A \supset B) \supset (\Box(A \supset C) \supset \Box(A \supset B \wedge C))$ by K and transitivity of $\supset$. Therefore $\Box(A \supset B \wedge C)$, applying MP twice.

TID. The translate of TID is of the form $\Box(\top \supset \top) \wedge \Diamond\top$. We have $\Diamond\top$ due to T and $\Box(\top \supset \top)$ due to PT and rN.

SPRES. The translate of SPRES is of the form $\Box(\top \supset (A \supset B)) \wedge \Diamond\top \wedge \neg(\Box(\top \supset \neg A) \wedge \Diamond\top) \supset \Box(A \supset B) \wedge \Diamond A$. Assume $\Box(\top \supset (A \supset B))$, $\Diamond\top$, $\neg(\Box(\top \supset \neg A) \wedge \Diamond\top)$. From $\Box(\top \supset (A \supset B))$ we obtain $\Box(A \supset B)$ by PT, rN, K, MP. From $\neg(\Box(\top \supset \neg A) \wedge \Diamond\top)$ we obtain $\neg\Box(\top \supset \neg A) \vee \neg\Diamond\top$. But since $\Diamond\top$, we have $\neg\Box(\top \supset \neg A)$. This implies $\Diamond A$: $\neg A \supset (\top \supset \neg A)$ by PT, and hence $\Box\neg A \supset \Box(\top \supset \neg A)$ by rN, K, MP. Thus $\neg\Box(\top \supset \neg A) \supset \Diamond A$.

T*. The translate of T* is $\Box(\top \supset A) \wedge \Diamond\top \supset A$. From $\Box(\top \supset A)$ we obtain $\top \supset A$ by T, and hence A.

Derivations of length $n+1$. Suppose the property (if $\vdash_{\mathsf{ST2}} \chi$ then $\vdash_{\mathsf{S2}} \chi^\circ$) holds for all derivations of length n or smaller, our induction hypothesis (IH), and assume the derivation of χ is of length $n+1$. Then there are two possibilities. Either χ is one of the axioms of $\mathsf{ST2}$, or it is obtained by one of the rules of $\mathsf{ST2}$ (MP, rLLE, rRW, BR*, or rN*). In the first case, the reasoning is as above. Thus let us consider the second case.

MP. Suppose χ is obtained by MP. Then there are previous formulas α and $\alpha \supset \chi$ in the derivation, such that $\vdash_{\mathsf{ST2}} \alpha$ and $\vdash_{\mathsf{ST2}} \alpha \supset \chi$. By IH, $\vdash_{\mathsf{S2}} \alpha^\circ$ and $\vdash_{\mathsf{S2}} (\alpha \supset \chi)^\circ$, thus $\vdash_{\mathsf{S2}} \alpha^\circ \supset \chi^\circ$. Therefore $\vdash_{\mathsf{S2}} \chi^\circ$ by MP.
rLLE. Suppose χ is obtained by rLLE. Then there are formulas A, B, C, such that $\chi = (A \rhd C) \supset (B \rhd C)$ and $A \equiv B$ is PT in $\mathcal{L}_\rhd$. Thus $(A \equiv B)^\circ$ is PT in $\mathcal{L}_\Box$ (same argument as for PT). Hence $A^\circ \equiv B^\circ$ is PT. Therefore $(A^\circ \supset C^\circ) \supset (B^\circ \supset C^\circ)$ is also PT (PT is closed under substitution of equivalents). Thus $\Box(A^\circ \supset C^\circ) \supset \Box(B^\circ \supset C^\circ)$ by rN, K and MP. By a similar argument, we obtain $\Diamond A^\circ \supset \Diamond B^\circ$. Hence $\Box(A^\circ \supset C^\circ) \wedge \Diamond A^\circ \supset \Box(B^\circ \supset C^\circ) \wedge \Diamond B^\circ$. Therefore $\vdash_{\mathsf{S2}} ((A \rhd C) \supset (B \rhd C))^\circ$.
rRW. Similar reasoning.
BR*. Suppose χ is obtained by BR*. Then there is a previous formula in the derivation whose translate is $\Box(\top \supset (A \supset B)) \wedge \Diamond \top$. By IH we have $\Box(\top \supset (A \supset B))$ in $\mathsf{S2}$. We have $\Box(\top \supset (A \supset B)) \supset \Box(A \supset B)$ by PT, rN, K and MP. Since $\Box(\top \supset (A \supset B))$, we obtain $\Box(A \supset B)$. We also have $\Box(A \supset B) \supset \Box((\top \supset A) \supset (\top \supset B))$ by PT, rN, K and MP. Since $\Box(A \supset B)$, MP delivers $\Box((\top \supset A) \supset (\top \supset B))$. Thus $\Box(\Box(\top \supset A) \supset \Box(\top \supset B))$ by BR. Therefore $\Box(\top \supset (\Box(\top \supset A) \supset \Box(\top \supset B)))$ by PT, rN, K, MP. Since we have already shown that we have $\Diamond \top$ in $\mathsf{S2}$, we obtain $\Box(\top \supset (\Box(\top \supset A) \wedge \Diamond \top \supset \Box(\top \supset B) \wedge \Diamond \top)) \wedge \Diamond \top$, using PT and MP. But this is the translate of χ.
rN*. Suppose χ is obtained by rN*. Thus, there is a previous formula θ in the derivation, such that $\chi = \boxdot \theta$, and θ is either an instance of T*, K*, PA, INC or SPRES.
Suppose θ is an instance of T*. We need to find a derivation of $\chi^\circ = (\boxdot \theta)^\circ$, which is of the form $\Box(\top \supset (\Box(\top \supset A) \wedge \Diamond \top \supset A)) \wedge \Diamond \top$. We have $\Diamond \top$ by T. Thus it suffices to prove the first conjunct. We also have $\Box(\top \supset A) \supset (\top \supset A)$ by T. Thus $\Box(\Box(\top \supset A) \supset (\top \supset A))$ by rN applied to T. By PT, K, and MP we obtain $\Box(\Box(\top \supset A) \supset A)$. By a similar reasoning we obtain $\Box(\Box(\top \supset A) \wedge \Diamond \top \supset A)$, and ultimately χ°.
Suppose θ is an instance of K*. We need to find a derivation of $\chi^\circ = (\boxdot \theta)^\circ$, which is of the form $\Box(\top \supset (\Box(\top \supset (A \supset B)) \wedge \Diamond \top \supset (\Box(\top \supset A) \wedge \Diamond \top \supset \Box(\top \supset B) \wedge \Diamond \top))) \wedge \Diamond \top$. We get $\Diamond \top$ from T. Thus it suffices to establish the first conjunct. We get $\Box(\Box(\top \supset (A \supset B)) \supset \Box((\top \supset A) \supset (\top \supset B)))$ by PT, rN, BR. And we get $\Box(\Box((\top \supset A) \supset (\top \supset B)) \supset (\Box(\top \supset A) \supset \Box(\top \supset B)))$ by applying rN to K. Thus by PT, rN and two instances of K, $\Box(\Box(\top \supset (A \supset B)) \supset (\Box(\top \supset A) \supset \Box(\top \supset B)))$. Using PT, rN and K again we get the result, and ultimately χ°.
Suppose θ is an instance of PA. We need to find a derivation of $\chi^\circ = (\boxdot \theta)^\circ$, which is of the form $\Box(\top \supset (\Box(A \supset B) \wedge \Diamond A \supset \neg(\Box(\top \supset \neg A) \wedge \Diamond \top))) \wedge$

$\Diamond\top$. $\Diamond\top$ holds by T. We have $\Box((\top \supset \neg A) \supset \neg A)$. Thus by BR $\Box(\Box(\top \supset \neg A) \supset \Box\neg A)$. Since $\supset$ contraposes, we have $(\Box(\top \supset \neg A) \supset \Box\neg A) \supset (\Diamond A \supset \neg\Box(\top \supset \neg A))$ as PT. Applying rN, K, the previous result and MP, we obtain $\Box(\Diamond A \supset \neg\Box(\top \supset \neg A))$. But $(\Diamond A \supset \neg\Box(\top \supset \neg A)) \supset (\top \supset (\Box(A \supset B) \wedge \Diamond A \supset \neg(\Box(\top \supset \neg A) \wedge \Diamond\top)))$ is PT. Hence rN, K, the previous result and MP deliver $\Box(\top \supset (\Box(A \supset B) \wedge \Diamond A \supset \neg(\Box(\top \supset \neg A) \wedge \Diamond\top)))$.
Suppose θ is an instance of INC. We need to find a derivation of $\chi^\circ = (\boxdot\,\theta)^\circ$, which is of the form $\Box(\top \supset (\Box(A \supset B) \wedge \Diamond A \supset \Box(\top \supset (A \supset B)) \wedge \Diamond\top)) \wedge \Diamond\top$. This works by a similar reasoning as for PA. We have $\Box((A \supset B) \supset (\top \supset (A \supset B)))$ by PT, rN. Thus $\Box(\Box(A \supset B) \supset \Box(\top \supset (A \supset B)))$ by BR. Also $\Box(\top \supset \Diamond\top)$ by applying rN to T (and then contraposing with PT, using rN K). Using this result, PT, K and MP, we obtain $\Box(\top \supset (\Box(A \supset B) \wedge \Diamond A \supset \Box(\top \supset (A \supset B)) \wedge \Diamond\top))$.
Suppose θ is an instance of SPRES. We need to find a derivation of $\chi^\circ = (\boxdot\,\theta)^\circ$, which is of the form $\Box(\top \supset (\Box(\top \supset (A \supset B)) \wedge \Diamond\top \wedge \neg(\Box(\top \supset \neg A) \wedge \Diamond\top) \supset \Box(A \supset B) \wedge \Diamond A)) \wedge \Diamond\top$. This works by a similar reasoning as for the previous ones. By rN applied to T, we obtain $\Box(\top \supset \Diamond\top)$. By PT, rN, and BR, we obtain $\Box(\Box\neg A \supset \Box(\top \supset \neg A))$. Thus by PT, rN, K and MP, we also obtain $\Box(\Box\neg A \supset (\Box(\top \supset \neg A) \wedge \Diamond\top))$. Since $\supset$ contraposes, and by rN, K and MP we get $\Box(\neg(\Box(\top \supset \neg A) \wedge \Diamond\top) \supset \Diamond A)$. By PT, rN, and BR we obtain $\Box(\Box(\top \supset (A \supset B)) \supset \Box(A \supset B))$. Again using PT, rN, K and MP we get $\Box(\Box(\top \supset (A \supset B)) \wedge \neg(\Box(\top \supset \neg A) \wedge \Diamond\top) \supset \Box(A \supset B) \wedge \Diamond A)$. One further application of PT, rN, K and MP allows us to get $\Box(\top \supset (\Box(\top \supset (A \supset B)) \wedge \Diamond\top \wedge \neg(\Box(\top \supset \neg A) \wedge \Diamond\top) \supset \Box(A \supset B) \wedge \Diamond A))$. ■

THEOREM 13. ST2 *is sound for reflexive non-normal relational models in* $\mathcal{L}_\triangleright$.

PROOF. Suppose $\vdash_{\mathsf{ST2}} A$. Thus $\vdash_{\mathsf{S2}} A^\circ$ by Lemma 12. Hence $\vDash A^\circ$ in reflexive non-normal relational models by the known soundness of S2 for reflexive non-normal relational semantics [8]. Hence $\vDash_\triangleright A$ by Lemma 6.

A direct semantic proof of soundness is possible as well. Since this is a standard procedure, we leave it to the reader. Let us illustrate nonetheless with AND: Suppose $w \vDash_\triangleright A \triangleright B$ and $w \vDash_\triangleright A \triangleright C$. Thus (i) $w \in N$, (ii) $\exists v \in W$ such that wRv and $v \vDash_\triangleright A$ and (iii) $\forall v \in W$ if wRv then $v \vDash_\triangleright A \supset B$ and $v \vDash_\triangleright A \supset C$. From the last two, we obtain that $\forall v \in W$ if wRv then $v \vDash_\triangleright A \supset B \wedge C$. Together with (i) and (ii) we can thus conclude that $w \vDash_\triangleright A \triangleright B \wedge C$. ■

4.2. Completeness

LEMMA 14. (Backsimulation) *For all* $\chi \in \mathcal{L}_\Box$, *if* $\vdash_{\mathsf{S2}} \chi$ *then* $\vdash_{\mathsf{ST2}} \chi^\bullet$.

Proof. By induction on the length of the derivation. *Derivations of length 1.* Then χ is an axiom of $\mathsf{S2}$. Three cases are possible: either χ is an instance of PT or an instance of K or an instance of T.
PT. Similar argument as for PT in Lemma 12.
K. The backtranslate of K is of the form $(\top \rhd (A \supset B)) \supset ((\top \rhd A) \supset (\top \rhd B))$. Assume $\top \rhd (A \supset B)$ and $\top \rhd A$. By AND we obtain $\top \rhd (A \supset B) \wedge A$. Hence $\top \rhd B$ by rRW.
T. The backtranslate of T is T*.
Derivations of length $n+1$. Suppose the derivation of χ is of length $n+1$, and the property holds for all shorter derivations. Then there are two possibilities. Either χ is an axiom of $\mathsf{S2}$, or χ is obtained by one of the rules of $\mathsf{S2}$ (MP, rN or BR). In the first case, we reason as above. Thus let us consider the second case.
MP. Same reasoning as for MP in Lemma 12.
rN. Suppose χ is obtained by rN. Then there is a formula A such that $\chi = \Box A$ and A is PT in $\mathcal{L}_{\Box}$ or it is one of the axioms K or T:
Assume that A is PT. By the same argument as for PT, we have that $A^{\bullet}$ is PT in $\mathcal{L}_{\rhd}$. Thus $\top \supset A^{\bullet}$ is PT. Hence $\vdash_{\mathsf{ST2}} (\top \rhd \top) \supset (\top \rhd A^{\bullet})$ by rRW. Yet $\top \rhd \top$ by TID. Therefore $\top \rhd A^{\bullet}$. This is $\chi^{\bullet}$. Therefore $\vdash_{\mathsf{ST2}} (\Box A)^{\bullet}$.
Assume that A is an instance of T. Thus it is of the form $\Box B \supset B$. But $\boxdot B^{\bullet} \supset B$ holds by T*. Applying rN* to T*, we get $\boxdot(\boxdot B^{\bullet} \supset B)$. This is $\chi^{\bullet}$.
Assume that A is an instance of K. Thus it is of the form $\Box(B \supset C) \supset (\Box B \supset \Box C)$. Applying rN* to the following instance of K* $\boxdot(B^{\bullet} \supset C^{\bullet}) \supset (\boxdot^{\bullet} B \supset \boxdot C^{\bullet})$ gives us $\boxdot(\boxdot(B^{\bullet} \supset C^{\bullet}) \supset (\boxdot^{\bullet} B \supset \boxdot C^{\bullet}))$. This is $\chi^{\bullet}$.
BR. Suppose χ is obtained by BR. Then $\chi = \Box(\Box A \supset \Box B)$ and there is a previous formula $\Box(A \supset B)$ in the derivation. By IH there is a derivation of $\top \rhd (A^{\bullet} \supset B^{\bullet})$. Thus by BR* we obtain $\top \rhd ((\top \rhd A^{\bullet}) \supset (\top \rhd B^{\bullet}))$. This is $\chi^{\bullet}$. ■

We call $\chi^{\circ\bullet}$ the *twin* of χ. To prove completeness, we ultimately need to establish that any formula of $\mathcal{L}_{\rhd}$ is provably equivalent to its twin. We procede by an intermediary step, proving first that the strict equivalence between χ and its twin is derivable.

Lemma 15. (Strict Twin Equivalence) *For all* $\chi \in \mathcal{L}_{\rhd}$, *we have* $\vdash_{\mathsf{ST2}} \boxdot(\chi \equiv \chi^{\circ\bullet})$.

Proof. By induction on the complexity of the formula.
Let $\chi = p$. Then $p^{\circ\bullet} = p$. Thus $\top \supset (p \equiv p^{\circ\bullet})$ is PT, and hence $\boxdot(p \equiv p^{\circ\bullet})$ by rRW and TID.

Let $\chi = \neg A$, and suppose as IH that $\vdash_{\mathsf{ST2}} \boxdot(A \equiv A^{\circ\bullet})$. $(A \equiv A^{\circ\bullet}) \supset (\neg A \equiv \neg A^{\circ\bullet})$ is PT. But $(\neg A)^{\circ\bullet} = \neg A^{\circ\bullet}$. Thus $(A \equiv A^{\circ\bullet}) \supset (\neg A \equiv (\neg A)^{\circ\bullet})$ is PT. Since by IH $\boxdot(A \equiv A^{\circ\bullet})$, we obtain $\boxdot(\neg A \equiv (\neg A)^{\circ\bullet})$ by rRW.
Let $\chi = (A \wedge B)$ and suppose as IH that $\vdash_{\mathsf{ST2}} \boxdot(A \equiv A^{\circ\bullet})$ and $\vdash_{\mathsf{ST2}} \boxdot(B \equiv B^{\circ\bullet})$. Thus we obtain $\boxdot((A \equiv A^{\circ\bullet}) \wedge (B \equiv B^{\circ\bullet}))$ by AND. Hence $\boxdot((A \wedge B) \equiv (A^{\circ\bullet} \wedge B^{\circ\bullet}))$ by rRW. But $(A^{\circ\bullet} \wedge B^{\circ\bullet}) = (A \wedge B)^{\circ\bullet}$. Thus $\boxdot((A \wedge B) \equiv (A \wedge B)^{\circ\bullet})$.
The cases for $\chi = (A \vee B)$ and $\chi = (A \supset B)$ work similarly.
Let $\chi = (A \rhd B)$ and suppose as IH that $\vdash_{\mathsf{ST2}} \boxdot(A \equiv A^{\circ\bullet})$ and $\vdash_{\mathsf{ST2}} \boxdot(B \equiv B^{\circ\bullet})$. We obtain $\vdash_{\mathsf{ST2}} \boxdot((A \supset B)^{\circ\bullet} \equiv (A \supset B))$ by a similar reasoning as for $\wedge$. Thus by BR*, we obtain $\boxdot(\boxdot(A \supset B)^{\circ\bullet} \equiv \boxdot(A \supset B))$. We also get $\boxdot(\neg A \equiv \neg A^{\circ\bullet})$, as above. Thus by BR* $\boxdot(\boxdot\neg A \equiv \boxdot\neg A^{\circ\bullet})$. Hence by rRW $\boxdot(\neg\boxdot\neg A \equiv \neg\boxdot\neg A^{\circ\bullet})$. That is $\boxdot(\diamonddot A \equiv \diamonddot A^{\circ\bullet})$. Thus by AND and rRW again, we get $\boxdot(\boxdot(A \supset B) \wedge \diamonddot A \equiv \boxdot(A \supset B)^{\circ\bullet} \wedge \diamonddot A^{\circ\bullet})$. But by applying rN* to PA, INC, and SPRES, we respectively obtain $\boxdot((A \rhd B) \supset \diamonddot A)$, $\boxdot((A \rhd B) \supset \boxdot(A \supset B))$, and $\boxdot((\boxdot(A \supset B) \wedge \diamonddot A) \supset (A \rhd B))$. By AND and rRW, we conclude $\boxdot(A \rhd B \equiv \boxdot(A \supset B) \wedge \diamonddot A)$. By using our previous result, AND and rRW deliver $\boxdot(A \rhd B \equiv \boxdot(A \supset B)^{\circ\bullet} \wedge \diamonddot A^{\circ\bullet})$. Yet, $\boxdot(A \supset B)^{\circ\bullet} \wedge \diamonddot A^{\circ\bullet} = (A \rhd B)^{\circ\bullet}$. Thus we have arrived at $\boxdot(A \rhd B \equiv (A \rhd B)^{\circ\bullet})$. ■

By T*, we can now conclude that χ is provably equivalent to it's twin:

Lemma 16. (Twin Equivalence) *For all* $\chi \in \mathcal{L}_{\rhd}$, *we have* $\vdash_{\mathsf{ST2}} \chi \equiv \chi^{\circ\bullet}$.

Proof. By Lemma 15, T* and MP. ■

This Lemma was also used in [23–25] and can be obtained directly when LLE and RW are unrestricted in the logic. It is because these principles are restricted here, that we needed to pass through first proving that any formula is strictly equivalent to its twin.

From what we have shown, we obtain our desired completeness result:

Theorem 17. ST2 *is complete for reflexive non-normal relational models in* $\mathcal{L}_{\rhd}$.

Proof. Suppose $\vDash_{\rhd} A$. Thus $\vDash A^{\circ}$ by Lemma 6. Hence $\vdash_{\mathsf{S2}} A^{\circ}$ by the known completeness of S2 for relational models in $\mathcal{L}_{\Box}$ [8]. Therefore $\vdash_{\mathsf{ST2}} A^{\circ\bullet}$ by Lemma 14. Hence $\vdash_{\mathsf{ST2}} A$ by Lemma 16. ■

The curious reader might wonder about the following: (1) Is a direct completeness proof possible? (2) Given the known decidability of S2 [18], is ST2 also decidable?[16] The answer to both questions is yes.

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[16]We thank an anonymous referee for raising both questions.

For (1) we should note that a direct completeness proof would roughly amount to using similar translation mechanisms but now on the semantic side. Overall, the canonical model construction would be messier,[17] and it would only be instructive, if we had no completeness proof for S2.

For (2) one can simply use the translation and simulation results. If there is a procedure that decides whether or not $\vdash_{\mathsf{S2}} A$ in finitely many steps, then there is such a procedure that decides whether or not $\vdash_{\mathsf{S2}} B^\circ$ and hence whether or not $\vdash_{\mathsf{ST2}} B$, since we have established that $\vdash_{\mathsf{ST2}} B$ iff $\vdash_{\mathsf{S2}} B^\circ$.

5. Stronger Systems

As one can extend the system S2, we can similarly extend the system ST2 all by keeping the tight relation between the two logics in the sense that they are logics for the same semantics but once interpreted in terms of the strict implication and once interpreted in terms of the super-strict implication.

5.1. Extensions of S2

Consider the following rule and axioms

If $\vdash A$ then $\vdash \Box A$ — N

$\Box(A \supset B) \supset \Box(\Box A \supset \Box B)$ — BRA

$\Box A \supset \Box\Box A$ — 4

$\Diamond A \supset \Box\Diamond A$ — 5

S3 is obtained from S2 by removing K and replacing the rule BR by the corresponding axiom BRA (so that rN now applies to the axioms PT, T and BRA). KT is obtained from S2 by removing the rule BR and replacing the rule rN by the unrestricted version N. S4 is obtained from KT by adding

[17]For example, for the normal case, instead of constructing an accessibility relation out of formulas in maximal consistent theories (aka canonical worlds) by wRv iff $\{A : \Box A \in w\} \subseteq v$, one would need to construct the relation through wRv iff $\{B : \neg(\neg B \rhd \neg B) \in w\} \subseteq v$, see footnote 6, and then prove analogues to the truth lemma, and the determination lemma, which involves proving several analogue preparatory results as for the canonical Kripke model construction.

the axiom 4, and S5 is obtained from S4 by replacing 4 by the axiom 5.[18] Note that S5 extends S4 which extends S3 which extends S2. (see Figure 1)[19]

5.2. Extensions of ST2

Consider the following rules and axioms:

If $\vdash A \equiv B$ then $\vdash (A \triangleright C) \supset (B \triangleright C)$ LLE

If $\vdash A \supset B$ then $\vdash (C \triangleright A) \supset (C \triangleright B)$ RW

$\boxdot(A \supset B) \supset \boxdot(\boxdot A \supset \boxdot B)$ BRA*

$\boxdot A \supset \boxdot\boxdot A$ 4*

$\diamonddot A \supset \boxdot \diamonddot A$ 5*

The system ST3 is obtained from ST2 by removing AND, and replacing the rule BR* by the axiom BRA*, and now rN* applies to BRA* instead of applying to K*. STT is obtained from ST2 by removing the rules BR* and rN*, and replacing the restricted rules rLLE and rRW by the unrestricted versions LLE and RW. ST4 is obtained from STT by adding 4*, and ST5 is obtained from ST4 by replacing 4* by 5*. Note that ST5 extends ST4 which extends ST3 which extends ST2.[20]

THEOREM 18. STT *is sound and complete for reflexive normal Kripke models in* $\mathcal{L}_{\triangleright}$.

For another equivalent axiomatization of STT, see [26, theorem 3].

PROOF. *Simulation*: Given Lemma 12, we only need to simulate the unrestricted rules LLE, RW. We use the fact that in KT we have the deduction theorem for $\supset$ and substitution of provable equivalents.
LLE. Suppose χ is obtained by LLE. Then there are formulas A, B, C, such that $\chi = (A \triangleright C) \supset (B \triangleright C)$ and $\vdash_{\mathsf{STT}} A \equiv B$. Thus by IH, $\vdash_{\mathsf{KT}} (A \equiv B)^\circ$.

[18] Lemmon's axiomatisation of S4 (replacing rN by N in S3) is equivalent to our S4: From BRA and T we obtain K, from BRA, K and N we also obtain 4; conversely, K and 4 imply BRA.

[19] In S3, BR can be simulated (by BRA, MP), K is derivable (by BRA, T, PT, K), and rN applied to K can also be simulated. In KT, BR can be simulated by K, N, MP.

[20] In ST3, BR* can be simulated (by BRA*, MP) and K* is derivable (by BRA*, T*). Thus AND is also derivable (using K*, INC, PA, SPRES), and rN* applied to K* can also be simulated. In STT as in the stronger systems ST4 and ST5, BR* and rN* can be simulated. In ST5, 4* is derivable. Note that ST4 could equivalently have been obtained from ST3 by extending rN* to N. Furthermore, in ST2 and all stronger systems, the deduction theorem for $\supset$ holds (the proof is similar as for ST2—see appendix A).

Hence $\vdash_{\mathsf{KT}} A^\circ \equiv B^\circ$. We can thus prove $\Box(A^\circ \supset C^\circ) \supset \Box(B^\circ \supset C^\circ)$ by substitution of provable equivalents, as well as $\Diamond A^\circ \supset \Diamond B^\circ$. Hence $\Box(A^\circ \supset C^\circ) \wedge \Diamond A^\circ \supset \Box(B^\circ \supset C^\circ) \wedge \Diamond B^\circ$. But this is the translate of χ.
RW: Similar reasoning.
Backsimulation: From Lemma 14, we know how to backsimulate PT, T, K, MP. It remains to backsimulate N.
N: Suppose χ is obtained by N. Then there is a formula A such that $\chi = \Box A$ and $\vdash_{\mathsf{KT}} A$. By IH, $\vdash_{\mathsf{STT}} A^\bullet$. Thus $\vdash_{\mathsf{STT}} \top \supset A^\bullet$. Therefore $(\top \rhd \top) \supset (\top \rhd A^\bullet)$ by RW. Since $\top \rhd \top$ by TID, we obtain $\top \rhd A^\bullet$. Thus $(\Box A)^\bullet$.
The proofs for transferring soundness and completeness are then the same as in Theorem 13-17. (By RW, we could now prove Lemma 16 directly.) ■

Let S be a modal logic sound and complete for the class of models M in $\mathcal{L}_\Box$. We call ST its *super-strict companion*, when the following hold: ST is sound and complete for M in $\mathcal{L}_\rhd$, the Twin Equivalence Lemma holds in ST, the Translation and Backtranslation Lemma hold for M. Then, we obtain the following extension strategy[21]

Theorem 19. *Let* S *be sound and complete for* M*,* ST *be the super-strict companion to* S*, and* $\mathsf{S}+\mathsf{X}$ *be sound and complete for* $N \subset M$*. Then* $\mathsf{ST}+\mathsf{X}^\bullet$ *is a super-strict companion to* $\mathsf{S}+\mathsf{X}$*.*

Proof. *Simulation*: it suffices to simulate the new axiom $\mathsf{X}^\bullet$, and for this, we establish $\vdash_{\mathsf{S}} A \equiv A^{\bullet\circ}$: We have $\vdash_{\mathsf{ST}} A^\bullet \equiv A^{\bullet\circ\bullet}$ by twin equivalence. Thus $M \vDash_\rhd (A \equiv A^{\bullet\circ})^\bullet$ by soundness of ST and $\bullet$. Hence $M \vDash A \equiv A^{\bullet\circ}$ by the Backtranslation Lemma. Hence $\vdash_{\mathsf{S}} A \equiv A^{\bullet\circ}$ by completeness of S for M. Thus $\mathsf{X}^\bullet$ can be simulated by X. *Backsimulation*: it suffices to backsimulate X, which follows since the backtranslate of X is $\mathsf{X}^\bullet$.
The proofs for transferring soundness and completeness are then the same as in Theorem 13-17, using the soundness and completeness of $\mathcal{S} + \mathsf{X}$ (see [8] or [2]), the Translation Lemma in M, and the twin equivalence in ST. ■

Corollary 20. *1.* $\mathsf{ST3}$ *is sound and complete w.r.t. (reflexive and) transitive models.*

2. $\mathsf{ST4}$ *is sound and complete w.r.t. (reflexive and) transitive normal models.*

3. $\mathsf{ST5}$ *is sound and complete w.r.t. normal models where* R *is an equivalence relation.*

[21] A similar result was proven in [23, Theorem 5]: for the definable companions to variably strict conditional logics.

Proof. For $\mathsf{ST3}$: $\mathsf{ST2}$ is the super-strict companion to $\mathsf{S2}$ (Theorem 13, 17, and the soundness and completeness of $\mathsf{S2}$ for reflexive non-normal relational models). The result then follows by Theorem 19, since BRA$^* =$ BRA$^\bullet$, and because $\mathsf{S3}$ is sound and complete for transitive reflexive non-normal relation models in $\mathcal{L}_\Box$.
Same reasoning for $\mathsf{ST4}$ and $\mathsf{ST5}$, starting from the previous fact that STT is the super-strict companion to KT, and noting that $4^* = 4^\bullet$ and $5^*=5^\bullet$. ■

5.3. What About Strong Boethius Thesis?

We already proved that strong Boethius Thesis (BT) is invalid (Lemma 10). One might thus ask, whether the addition of BT to a system of super-strict implication is at all possible.

The question may at first seem parallel to the one raised by Pizzi and Williamson [20] for their consequential implication $\rightarrow$. These authors show that in normal systems of consequential implication with Strong Boethius Thesis (BT), the addition of MP for $\rightarrow$ makes the following derivable (see their Proposition 3.18):

1. $(A \rightarrow B) \equiv (B \rightarrow A)$ (S)
2. $(A \rightarrow B) \equiv \neg(A \rightarrow \neg B)$ (wBT & CwBT)

And, if in normal systems with BT, we add the principle MI for $\rightarrow$ $((A \rightarrow B) \supset (A \supset B))$, then the following is also derivable (their Proposition 3.19):

3. $(A \rightarrow B) \equiv (A \equiv B)$ (Collapse to $\equiv$)

Based on these results, Pizzi and Williamson argue that it is difficult to interpret BT in normal systems of consequential implication.

The situation is different for super-strict implication. First, note that in normal systems of super-strict implication, the addition of MP for $\rhd$ (MP($\rhd$)) causes no problem. MP($\rhd$) is already derivable in $\mathsf{ST2}$ (Lemma 11) and thus remains derivable in normal systems. Second, (1) for $\rhd$, which is just Symmetry (S), is invalid in non-normal semantics of $\rhd$, and remains invalid in normal semantics.[22] Third, (2) for $\rhd$ is invalid in non-normal semantics of $\rhd$, since although wBT is valid (Lemma 11), the converse $\neg(A \rhd \neg B) \supset (A \rhd B)$ (CwBT) is invalid, since the equivalent CEM is invalid (Lemma 10). Again CEM, and thus (2), remains invalid in normal semantics.[23] Fourth, (3) for $\rhd$ is invalid in non-normal semantics, since $\rhd$ is

[22] The proof of the invalidity of S (Lemma 10) carries over to normal semantics.

[23] Again, the proof of invalidity of CEM (Lemma 10) carries over to normal semantics.

clearly distinct from classical equivalence $\equiv$, due to the modal assumption, and this remains so in normal semantics. So no problems such as (1), (2) or (3) appear in normal systems of $\rhd$, as long as we do not add BT.

The reason to reject BT for $\rhd$ is not that much that it creates a collapse similar to (1)–(3) above, but that the acceptance of BT would amount to accepting that any super-strict implication $A \rhd B$ is possible, simply because $C \rhd D$ implies $\diamond C$ (due to PA), so that $(A \rhd B) \rhd \neg(A \rhd \neg B)$ would imply $\diamond(A \rhd B)$. This creates a contradiction, not a collapse.

Lemma 21. *Adding* BT *to* STT *is inconsistent.*

Proof. Adding BT to a system of strict implication means to add the validity $\diamond(A \rhd B)$ for any A, B (by PA). In particular $\diamond(\bot \rhd \bot)$. This creates a contradiction. Indeed, $\neg(\bot \rhd \bot)$ by NAA. Thus in normal systems $\boxdot \neg(\bot \rhd \bot)$ due to N (or RW), so that $\neg \diamond(\bot \rhd \bot)$ by duality. ■

This is the real reason why adding BT makes no sense.

However, we may consider the following sensible weakening of BT which we call *possibilistic Boethius Thesis*:

$$\diamond(A \rhd B) \supset ((A \rhd B) \rhd \neg(A \rhd \neg B)) \qquad \text{pBT}$$

This is valid in our normal semantics:

Lemma 22. pBT *is valid in reflexive normal relational models.*

Proof. wBT is valid in non-normal semantics, according to $\vDash_{\rhd}$ (Lemma 11 and Theorem 17). Hence wBT is valid in normal semantics. Hence the translate wBT$^\circ$ of wBT—$\Box(A^\circ \supset B^\circ) \wedge \Diamond A^\circ \supset \neg(\Box(A^\circ \supset \neg B^\circ) \wedge \Diamond A^\circ)$— is valid in normal semantics, according to $\vDash$ (Lemma 6). Thus in normal semantics (by N), $\Box\,$wBT$^\circ$— $\Box(\Box(A^\circ \supset B^\circ) \wedge \Diamond A^\circ \supset \neg(\Box(A^\circ \supset \neg B^\circ) \wedge \Diamond A^\circ))$ – is valid, according to $\vDash$.

Now consider a world w in a reflexive normal model where $\diamond(A \rhd B)$, according to $\vDash_{\rhd}$. Thus $\Diamond(A \rhd B)^\circ$ in w, according to $\vDash$ (Lemma 6). But $(\Diamond(A \rhd B)^\circ \wedge \Box\,\text{wBT}^\circ) = \text{pBT}^\circ$. Hence, overall, $w \vDash \text{pBT}^\circ$. Therefore $w \vDash_{\rhd} \text{pBT}$ (Lemma 6). Since w was an arbitrary world in a reflexive normal model, we can conclude that pBT is valid in reflexive normal relational models. ■

We conclude: although it makes no sense to add BT to a normal system of super-strict implication, due to inconsistency, we can add the sensible weakening pBT. Especially to reflexive normal systems, where that addition is in fact redundant, since pBT is already ensured there.

6. Labelled Sequent Calculi

Starting from the labelled calculi for normal logics of SSI introduced in [4] and from the calculi for Lewis' systems introduced in [19] we are now going to introduce labelled sequent calculi for the logics $\mathsf{ST2} - \mathsf{ST5}$. First of all we extend the language with a denumerable set $\mathbb{W}$ of so-called *world-labels*, to be denoted by $w, v, u, \dots$, and with the two predicates $\mathcal{N}$ and $\mathcal{R}$ of arity 1 and 2, respectively. The labelled language $\mathcal{L}_{\rhd}^{\mathbb{W}}$ is the smallest set such that:

1. If $w \in \mathbb{W}$ and $A \in \mathcal{L}_{\rhd}$ then the *labelled formula* $w : A$ is in $\mathcal{L}_{\rhd}^{\mathbb{W}}$;
2. If $w \in \mathbb{W}$ then the *normality atom* $\mathcal{N}w$ is in $\mathcal{L}_{\rhd}^{\mathbb{W}}$;
3. If $w, v \in \mathbb{W}$ then the *relational atom* $w\mathcal{R}v$ is in $\mathcal{L}_{\rhd}^{\mathbb{W}}$.

The $\mathcal{L}_{\rhd}^{\mathbb{W}}$-formula $E[v/u]$ is obtained by replacing in E each occurrence of the label u with an occurrence of the label v. The weight of an $\mathcal{L}_{\rhd}^{\mathbb{W}}$-formula E, $\mathtt{w}(E)$, is given by positing that, if E is $w : A$ for some $A \in \mathcal{L}_{\rhd}$, then it is equal to that of A; else it is 0.

A sequent is an expression of the shape

$$\Gamma \Rightarrow \Delta$$

where Γ is a finite multiset composed of labelled formulas, normality atoms, and relational atoms; Δ, instead, is a finite multiset composed only of labelled formulas. Substitution is extended to multisets and sequents by applying it componentwise.

The rules for $\rhd$ in labelled calculi can be obtained as meaning explanations from the truth conditions for $\rhd$, cf. [14]. In particular, the right-to left clause

if $w \in N$ and $\exists v \in W(wRv$ and $v \models_{\rhd} A)$ and $\forall u \in W(wRu$ implies $u \models_{\rhd} A \supset B)$, then $w \models_{\rhd} A \rhd B$

is expressed by the following right rule:

$$\frac{w\mathcal{R}u, u : A, \mathcal{N}w, w\mathcal{R}v, \Gamma \Rightarrow \Delta, u : B \qquad \mathcal{N}w, w\mathcal{R}v, \Gamma \Rightarrow \Delta, w : A \rhd B, v : A}{\mathcal{N}w, w\mathcal{R}v, \Gamma \Rightarrow \Delta, w : A \rhd B}\ {\scriptstyle R\rhd,\, u \text{ fresh}}$$

where the variable condition on u encodes the existential quantification of the semantic clause. Analogously, the left-to-right truth clause is expressed

Table 1. Rules of the calculi G3.ST2 – G3.ST5

Initial sequents: $w : p, \Gamma \Rightarrow \Delta, w : p$, with p atomic

Logical rules

$$\frac{\Gamma \Rightarrow \Delta, w : A}{w : \neg A, \Gamma \Rightarrow \Delta}\ L\neg \qquad \frac{w : A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, w : \neg A}\ R\neg$$

$$\frac{w : A, w : B, \Gamma \Rightarrow \Delta}{w : A \wedge B, \Gamma \Rightarrow \Delta}\ L\wedge \qquad \frac{\Gamma \Rightarrow \Delta, w : A \qquad \Gamma \Rightarrow \Delta, w : B}{\Gamma \Rightarrow \Delta, w : A \wedge B}\ R\wedge$$

$$\frac{w : A, \Gamma \Rightarrow \Delta \qquad w : B, \Gamma \Rightarrow \Delta}{w : A \vee B, \Gamma \Rightarrow \Delta}\ L\vee \qquad \frac{\Gamma \Rightarrow \Delta, w : A, w : B}{\Gamma \Rightarrow \Delta, w : A \vee B}\ R\vee$$

$$\frac{\Gamma \Rightarrow \Delta, w : A \qquad w : B, \Gamma \Rightarrow \Delta}{w : A \supset B, \Gamma \Rightarrow \Delta}\ L\supset \qquad \frac{w : A, \Gamma \Rightarrow \Delta, w : B}{\Gamma \Rightarrow \Delta, w : A \supset B}\ R\supset$$

$$\frac{\mathcal{N}w, w\mathcal{R}u, u : A, w\mathcal{R}v, w : A \rhd B, \Gamma \Rightarrow \Delta, v : A \qquad \mathcal{N}w, w\mathcal{R}u, u : A, v : B, w\mathcal{R}v, w : A \rhd B, \Gamma \Rightarrow \Delta}{w\mathcal{R}v, w : A \rhd B, \Gamma \Rightarrow \Delta}\ L\rhd, u \text{ fresh}$$

$$\frac{w\mathcal{R}u, u : A, \mathcal{N}w, w\mathcal{R}v, \Gamma \Rightarrow \Delta, u : B \qquad \mathcal{N}w, w\mathcal{R}v, \Gamma \Rightarrow \Delta, w : A \rhd B, v : A}{\mathcal{N}w, w\mathcal{R}v, \Gamma \Rightarrow \Delta, w : A \rhd B}\ R\rhd, u \text{ fresh}$$

Non-logical rules:

$$\frac{w\mathcal{R}w, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ Ref \qquad \frac{w\mathcal{R}u, w\mathcal{R}v, v\mathcal{R}u, \Gamma \Rightarrow \Delta}{w\mathcal{R}v, v\mathcal{R}u, \Gamma \Rightarrow \Delta}\ Trans \qquad \frac{\mathcal{N}w, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ Norm \qquad \frac{v\mathcal{R}w, w\mathcal{R}v, \Gamma \Rightarrow \Delta}{w\mathcal{R}v, \Gamma \Rightarrow \Delta}\ Sym$$

by the following left rule[24]:

$$\dfrac{\mathcal{N}w, w\mathcal{R}u, u : A, w\mathcal{R}v, w : A \rhd B, \Gamma \Rightarrow \Delta, v : A \qquad \mathcal{N}w, w\mathcal{R}u, u : A, v : B, w\mathcal{R}v, w : A \rhd B, \Gamma \Rightarrow \Delta}{w\mathcal{R}v, w : A \rhd B, \Gamma \Rightarrow \Delta} \; L\rhd, u \text{ fresh}$$

The rules of the labelled calculi G3.ST2−G3.ST5 are presented in Table 1. In particular G3.ST2 contains the initial sequents, all logical rules, and *Ref* as its only non-logical rule. G3.ST3 is obtained by adding the non-logical rule *Trans* to G3.ST2. G3.ST4 is obtained by adding the non-logical rule *Norm* to G3.ST3. Finally, G3.ST5 is obtained by adding the non-logical rule *Sym* to G3.ST4. We use G3.STn to denote an arbitrary calculus among them. The calculi G3.ST4 (G3.ST5) can be obtained by simply dropping normality atoms from rules $L/R\rhd$, thus obtaining the calculus G3SS.S4 (G3SS.S5) from [4].

A G3.STn-*derivation* of $\Gamma \Rightarrow \Delta$ is a finite tree of sequents whose root is $\Gamma \Rightarrow \Delta$, whose leaves are initial sequents, and which grows according to the rules of G3.STn. The *height* of a derivation is the number of nodes of one of its longest branches. The expression G3.STn $\vdash^{(n)} \Gamma \Rightarrow \Delta$ means that there is a G3.STn-derivation of $\Gamma \Rightarrow \Delta$ (of height n). A rule is G3.STn-*admissible* if its conclusion is G3.STn-derivable whenever its premisses are G3.STn-derivable, and it is *hp*-G3.STn-*admissible* if it is admissible and its conclusion has a derivation bounded by the height of the derivations of its premisses. In the rules in Table 1 Γ and Δ are called *contexts*; the formulas displayed in the conclusion (premisses only) are called *principal* (*active*).

7. Structural Properties of G3.STn

Lemma 23. G3.STn $\vdash w : A, \Gamma \Rightarrow \Delta, w : A$ *for all formulas* A.

Proof. We just have to apply bottom-up the two rules for $w : A$ and then the lemma follows by the induction hypothesis (IH, for short). ■

Lemma 24. (Substitution) *The following rule of substitution is hp-*G3.STn*-admissible:*

$$\dfrac{\Gamma \Rightarrow \Delta}{\Gamma[v/u] \Rightarrow \Delta[v/u]} \; S$$

Proof. By induction on the height of the derivation $\mathcal{D}$ of $\Gamma \Rightarrow \Delta$. We apply IH to the premiss(es) of the last rule instance in $\mathcal{D}$ (twice if it has a variable

[24]This rule, like all left rules in sequent calculi, has to be read bottom-up since it represents an elimination rule in natural deduction.

condition) and then we conclude $\Gamma[v/u] \Rightarrow \Delta[v/u]$ by an instance of the same rule. ■

THEOREM 25. (Weakening) *The rule of weakening is hp-*G3.STn*-admissible:*

$$\frac{\Gamma \Rightarrow \Delta}{\Pi, \Gamma \Rightarrow \Delta, \Sigma}\ W$$

PROOF. By induction on the height of the derivation $\mathcal{D}$ of $\Gamma \Rightarrow \Delta$. We start by applying Lemma 24 to the premiss(es) of the last rule instance in $\mathcal{D}$ if it is by a rule with a variable condition clashing with some label in Π, Σ; then we apply IH to the (substituted) premiss(es). Finally, we obtain $\Pi, \Gamma \Rightarrow \Delta, \Sigma$ by an instance of the last rule applied in $\mathcal{D}$. ■

COROLLARY 26. *The following rules are admissible in* G3.STn*:*

$$\frac{v : A, \mathcal{N}w, w\mathcal{R}v, w : \boxdot A, \Gamma \Rightarrow \Delta}{w\mathcal{R}v, w : \boxdot A, \Gamma \Rightarrow \Delta}\ L\boxdot \qquad \frac{w\mathcal{R}u, \mathcal{N}w, \Gamma \Rightarrow \Delta, u : A}{\mathcal{N}w, \Gamma \Rightarrow \Delta, w : \boxdot A}\ R\boxdot,\ u \text{ fresh}$$

$$\frac{u : A, w\mathcal{R}u, \mathcal{N}w, \Gamma \Rightarrow \Delta}{\mathcal{N}w, w : \diamonddot A, \Gamma \Rightarrow \Delta}\ L\diamonddot,\ u \text{ fresh} \qquad \frac{\mathcal{N}w, w\mathcal{R}v, \Gamma \Rightarrow \Delta, w : \diamonddot A, v : A}{w\mathcal{R}v, \Gamma \Rightarrow \Delta, w : \diamonddot A}\ R\diamonddot$$

PROOF. The admissiblity of rules $L/R\boxdot$ is shown by the following derivations, where $\boxdot A$ is expressed by $\top \rhd A$, and where we used the known fact that sequents of shape $\Gamma \Rightarrow \Delta, w' : \top$ are derivable ($\top$ was defined as $p \vee \neg p$)

$$\dfrac{\dfrac{}{w\mathcal{R}u, \mathcal{N}w, u : \top, \Gamma' \Rightarrow \Delta, v : \top} \quad \dfrac{v : A, \mathcal{N}w, w\mathcal{R}v, w : \top \rhd A, \Gamma \Rightarrow \Delta}{\mathcal{N}w, w\mathcal{R}u, u : \top, v : A, w\mathcal{R}v, w : \top \rhd A, \Gamma \Rightarrow \Delta}\ W}{w\mathcal{R}v, w : \top \rhd A, \Gamma \Rightarrow \Delta}\ L\rhd$$

$$\dfrac{\dfrac{\dfrac{w\mathcal{R}u, \mathcal{N}w, \Gamma \Rightarrow \Delta, u : A}{w\mathcal{R}u, u : \top, w\mathcal{R}w, \mathcal{N}w, \Gamma \Rightarrow \Delta, u : A}\ W \quad \dfrac{}{w\mathcal{R}w, \mathcal{N}w, \Gamma \Rightarrow \Delta, w : \top \rhd A, w : \top}}{w\mathcal{R}w, \mathcal{N}w, \Gamma \Rightarrow \Delta, w : \top \rhd A}\ R\rhd}{\mathcal{N}w, \Gamma \Rightarrow \Delta, w : \top \rhd A}\ Ref$$

The case of rules $L/R\diamonddot$ is similar and can thus be omitted. ■

LEMMA 27. (Invertibility) *Each rule of* G3.STn *is height-preserving invertible: if*

$$\frac{\Gamma' \Rightarrow \Delta' \quad (\Gamma'' \Rightarrow \Delta'')}{\Gamma \Rightarrow \Delta}$$

is an instance of a rule of G3.STn *and* $\Gamma \Rightarrow \Delta$ *has a* G3.STn*-derivation of height* m *then also* $\Gamma' \Rightarrow \Delta'$ *(and* $\Gamma'' \Rightarrow \Delta''$*) has a* G3.STn*-derivation whose height is bounded by* m*.*

Proof. The proof is by induction on the height of the derivation of $\Gamma \Rightarrow \Delta$. The propositional cases are standard and can thus be omitted. All non-logical rules as well as rule $L\rhd$ w.r.t. both premisses and rule $R\rhd$ w.r.t. its right premiss are height-preserving invertible because of the hp-admissibility of weakening. Finally, the height-preserving invertibility of $R\rhd$ w.r.t. its left premiss can be proved by the same strategy adopted for the propositional rules, possibly applying an hp-admissible instance of the rule of substitution. ■

Theorem 28. (Contraction) *The following rule of contraction is hp-*G3.STn*-admissible:*

$$\frac{\Pi, \Pi, \Gamma \Rightarrow \Delta, \Sigma, \Sigma}{\Pi, \Gamma \Rightarrow \Delta, \Sigma}\ C$$

Proof. By induction on the height of the derivation $\mathcal{D}$ of $\Pi, \Pi, \Gamma \Rightarrow \Delta, \Sigma, \Sigma$. We assume that one of Π and Σ is a singleton and the other is empty. We have three cases according to whether (i) none, or (ii) exactly one, or (iii) both instances of the formula we are contracting are principal in the last rule inference in $\mathcal{D}$ which will be denoted by *Rule*.

In case (i) we permute the instance of contraction with that of *Rule* thus getting one (two) instance(s) of contraction that is (are) hp-admissible by IH. In case (ii) we start by applying hp-invertibility w.r.t. the non-principal instance of the contraction-formula to the premiss(es) of the last rule instance *Rule* in $\mathcal{D}$, then we apply an (hp-admissible) instance of substitution and the inductive hypothesis to it (them). We conclude by an instance of *Rule*. Case (iii) arises only when two instances of wRw are principal in an instance of *Trans*. In this case we apply the inductive hypothesis twice to the premiss of *Trans* and we are done. ■

Theorem 29. (Cut) *The following rule of Cut is* G3.STn*-admissible:*

$$\frac{\Gamma \Rightarrow \Delta, w : A \quad w : A, \Pi \Rightarrow \Sigma}{\Pi, \Gamma \Rightarrow \Delta, \Sigma}\ Cut$$

Proof. We use the standard Dragalin-style proof: we consider an uppermost instance of *Cut* and we have a principal induction on the weight of the cut-formula $w : A$ and a sub-induction on the sum of the heights of the derivations of the premisses of the instance of *Cut* under consideration (*cut-height*, for short). It is convenient to divide the proof into three exhaustive cases according to whether (i) some premiss of *Cut* is an initial sequent; (ii) the cut-formula is not principal in the last step of the derivation of some premiss of *Cut*; (iii) the cut-formula is principal in the last step of the derivations of both premisses.

$$\dfrac{\dfrac{\overset{\vdots\,\mathcal{D}_{11}}{\mathcal{N}w, w\mathcal{R}v_1, v_1 : B, w\mathcal{R}v \Rightarrow v_1 : C} \qquad \overset{\vdots\,\mathcal{D}_{12}}{\mathcal{N}w, w\mathcal{R}v \Rightarrow w : B \rhd C, v : B}}{\mathcal{N}w, w\mathcal{R}v \Rightarrow w : B \rhd C}\;{\scriptstyle R\rhd} \qquad \dfrac{\overset{\vdots\,\mathcal{D}_{21}}{\mathcal{N}w, w\mathcal{R}u_1, u_1 : B, w\mathcal{R}u, w : B \rhd C \Rightarrow u : B} \qquad \overset{\vdots\,\mathcal{D}_{22}}{\mathcal{N}w, w\mathcal{R}u_1, u : C, u_1 : B, w\mathcal{R}u, w : B \rhd C \Rightarrow}}{w : B \rhd C, w\mathcal{R}u \Rightarrow}\;{\scriptstyle L\rhd}}{\mathcal{N}w, w\mathcal{R}v, w\mathcal{R}u \Rightarrow}\;{\scriptstyle Cut}$$

is transformed into the following derivation (where Γ^k stands for k copies of the multiset Γ):

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{\vdots\,\mathcal{D}_{12}}{\mathcal{N}w, w\mathcal{R}v \Rightarrow v : B, w : B \rhd C} \quad \overset{\vdots\,\mathcal{D}_{2}}{w : B \rhd C, w\mathcal{R}u \Rightarrow}}{\mathcal{N}w, w\mathcal{R}u, w\mathcal{R}v \Rightarrow v : B}\;{\scriptstyle (\dagger)} \quad \dfrac{\overset{\vdots\,\mathcal{D}_{1}}{\mathcal{N}w, w\mathcal{R}v \Rightarrow w : B \rhd C} \quad \dfrac{\overset{\vdots\,\mathcal{D}_{21}}{\mathcal{N}w, w : B \rhd C, u_1 : B, w\mathcal{R}u_1, w\mathcal{R}u \Rightarrow u : B}}{w : B \rhd C, \mathcal{N}w, v : B, w\mathcal{R}v, w\mathcal{R}u \Rightarrow u : B}\;{\scriptstyle S}}{v : B, (\mathcal{N}w)^2, w\mathcal{R}u, (w\mathcal{R}v)^2 \Rightarrow u : B}\;{\scriptstyle (\dagger)}}{(\mathcal{N}w)^3, (w\mathcal{R}u)^2, (w\mathcal{R}v)^3 \Rightarrow u : B}\;{\scriptstyle (\ddagger)} \quad \dfrac{\dfrac{\dfrac{\overset{\vdots\,\mathcal{D}_{11}}{\mathcal{N}w, w\mathcal{R}v_1, w\mathcal{R}v, v_1 : B \Rightarrow v_1 : C}}{\mathcal{N}w, w\mathcal{R}u, w\mathcal{R}v, u : B \Rightarrow u : C}\;{\scriptstyle S} \quad \dfrac{\overset{\vdots\,\mathcal{D}_{1}}{\mathcal{N}w, w\mathcal{R}v \Rightarrow w : B \rhd C} \quad \dfrac{\overset{\vdots\,\mathcal{D}_{22}}{\mathcal{N}w, w : B \rhd C, u : C, w\mathcal{R}u_1, w\mathcal{R}u, u_1 : B \Rightarrow}}{\mathcal{N}w, w : B \rhd C, u : C, (w\mathcal{R}u)^2, u : B \Rightarrow}\;{\scriptstyle S}}{u : C, (\mathcal{N}w)^2, w\mathcal{R}v, (w\mathcal{R}u)^2, u : B \Rightarrow}\;{\scriptstyle (\dagger)}}{\dfrac{(\mathcal{N}w)^3, (u : B)^2, (w\mathcal{R}v)^2, (w\mathcal{R}u)^3 \Rightarrow}{u : B, \mathcal{N}w, w\mathcal{R}v, w\mathcal{R}u \Rightarrow}\;{\scriptstyle C}}\;{\scriptstyle Cut\ (\ddagger)}}{\dfrac{(\mathcal{N}w)^4, (w\mathcal{R}v)^4, (w\mathcal{R}u)^3 \Rightarrow}{\mathcal{N}w, w\mathcal{R}v, w\mathcal{R}u \Rightarrow}\;{\scriptstyle C}}\;{\scriptstyle Cut\ (\ddagger)}$$

Figure 2. Transformation for principal cuts with cut-formula $w : A \rhd B$

In case (i) it is easy to show that also $\Pi, \Gamma \Rightarrow \Delta, \Sigma$ has a cut-free derivation. In case (ii) we can permute the instance of *Cut* upward in the derivation of the premiss where the cut-formula is not principal, thus obtaining a derivation with an instance of *Cut* that is admissible by the sub-induction hypothesis. If we are in case (iii) and the principal operator of the cut-formula is a propositional one, then the proof is standard and can thus be omitted.

The transformation showing the admissibility of *Cut* for the final case of a cut-formula of shape $w : B \rhd C$ principal in both premisses is given in Figure 2. For reasons of space, we consider instances of rules $L\rhd$ and $R\rhd$ without contexts (if we have contexts, their duplications are handled in the final step by contraction). In the transformed derivations we have some instances of *Cut* that are admissible by IH (those marked with (†) have lesser cut-height, and those marked with (‡) have a cut formula of lower weight). ■

8. Completeness and Decidability of G3.STn

8.1. Soundness

The soundness of G3.STn will be proved by introducing a notion of validity for sequents that simulates the semantic notion of consequence and then by showing that each rule of G3.STn preserves validity w.r.t. models for STn.

Definition 30. Given a model $\mathfrak{M} = \langle W, R, N, V\rangle$ a $\mathfrak{M}$-*realisation* is a function $\sigma : \mathbb{W} \longrightarrow W$ mapping each world-label to a world of the model $\mathfrak{M}$.

2. The *truth of a* $\mathcal{L}_{\rhd}^{\mathbb{W}}$*-formula* E *under a* $\mathfrak{M}$*-realisation* σ, $\sigma \models E$, is thus defined:
   - $\sigma \models w\mathcal{R}v$ iff $\sigma(w)R\sigma(v)$
   - $\sigma \models \mathcal{N}w$ iff $N(\sigma(w))$
   - $\sigma \models w : A$ iff $\models^{\mathfrak{M}}_{\sigma(w)} A$
3. A sequent $\Gamma \Rightarrow \Delta$ is *realised* by σ if the fact that all formulas in Γ are true under σ implies that some formula in Δ is true under σ.
4. A sequent $\Gamma \Rightarrow \Delta$ is STn-*valid* if it is realised by each $\mathfrak{M}$-realisation with $\mathfrak{M}$ a model for STn.

Theorem 31. (Soundness) *If a sequent is* G3.STn*-derivable then it is* STn*-valid.*

Proof. By induction on the height of the derivation $\mathcal{D}$ of $\Gamma \Rightarrow \Delta$. The proof for the base case, the propositional ones, and those for the non-logical rules are like those in [13] and can thus be omitted.

Suppose the last rule instance applied in $\mathcal{D}$ is the following instance of $L\rhd$:

$$\dfrac{\mathcal{N}w, w\mathcal{R}u, u : A, w\mathcal{R}v, w{:}A \rhd B, \Gamma \Rightarrow \Delta, v : A \quad \mathcal{N}w, w\mathcal{R}u, u : A, v : B, w\mathcal{R}v, w : A{\rhd}B, \Gamma \Rightarrow \Delta}{w\mathcal{R}v, w{:}A \rhd B, \Gamma \Rightarrow \Delta}\ {\scriptstyle L\rhd,\, u \text{ fresh}}$$

and that σ realises all formulas in $w\mathcal{R}v, w : A \rhd B, \Gamma$. From the fact that $\models_{\sigma(w)} A \rhd B$ we obtain that (i) Nw; (ii) some $t \in W$ is such that $\sigma(w)Rt$ and $\models_t A$; and (iii) for all $v \in W$, wRv implies $\models_v A \supset B$. Let τ be defined like σ except for the world-label u that is mapped to the world t. Fact (i) implies that τ realises $\mathcal{N}w$. By fact (ii) we obtain that $\tau(w)R\tau(u)$ and $\models_{\tau(u)} A$—i.e., τ realises $w\mathcal{R}u$ and $u : A$. Given that τ realises also all formulas in $w\mathcal{R}v, w : A \rhd B, \Gamma$, by induction on the left premiss we know that τ realises some formula in $\Delta, v : A$. In the former case we are done since, by the freshness of u, we can conclude that σ realises some formula in Δ. If τ realises $v : A$ then we have that $\models_{\tau(v)} A$ and this, together with fact (iii), implies that $\models_{\tau(v)} B$—i.e., τ realises $v : B$. By induction on the right premiss we conclude that τ, and hence σ, realises some formula in Δ.

Suppose the last rule instance applied in $\mathcal{D}$ is the following instance of $R\rhd$:

$$\dfrac{w\mathcal{R}u, u : A, \mathcal{N}w, w\mathcal{R}v, \Gamma \Rightarrow \Delta, u : B \quad \mathcal{N}w, w\mathcal{R}v, \Gamma \Rightarrow \Delta, w : A \rhd B, v : A}{\mathcal{N}w, w\mathcal{R}v, \Gamma \Rightarrow \Delta, w : A \rhd B}\ {\scriptstyle R\rhd,\, u \text{ fresh}}$$

and that σ realises all formulas in $\mathcal{N}w$, $w\mathcal{R}v$, Γ. By induction on the right premiss σ realises some formula in $\Delta, w : A \rhd B, v : A$. In the two former cases we are done. Suppose that σ realises $v : A$. We consider a τ that is like σ but it maps the world-label u on the world $\sigma(v)$. By induction on the left premiss we obtain that $\tau \models u : B$. By the freshness of the world-label u we have that for all $v \in W$, $\tau(w)Rv$ implies $\models_v A \supset B$. We conclude that both τ and σ realise $w : A \rhd B$. ■

8.2. Completeness

We are now going to show that the calculus G3ST.n is complete for validity by embedding the axiomatic calculus STn into G3.STn: we show that $\vdash_{\mathsf{STn}} A$ implies G3.STn $\vdash \mathcal{N}w \Rightarrow w : A$.

Lemma 32. *If A is an axiom of* STn *that is not the axiom* TID *or if A is an instance of* K*, *then the sequents $\Rightarrow w : A$ and $\mathcal{N}w \Rightarrow w : A$ are* G3.STn*−derivable. The sequent $\mathcal{N}w \Rightarrow w : \top \rhd \top$ is* G3STn*-derivable.*

PROOF. The sequent $\Rightarrow w : A$, for A an instance of K* or of an axiom that is not TID, can be shown to be derivable by applying root-first the rules of G3.STn as well as rules $L/R\boxdot$ and $L/R\dot{\Diamond}$. The derivability of $\mathcal{N}w \Rightarrow w : A$ follows by an instance of weakening. $\mathcal{N}w \Rightarrow w : \top \rhd \top$ is easily shown to be G3.STn-derivable. ■

LEMMA 33. *The restricted rule of necessitation is admissible in* G3.STn*:*

$$\dfrac{\Rightarrow w : A}{\mathcal{N}w \Rightarrow w : \boxdot A}\ rN^*$$

PROOF. We have the following derivation:

$$\dfrac{\dfrac{\dfrac{\Rightarrow w : A}{\Rightarrow u : A}\ {\scriptstyle S}}{\mathcal{N}w, w\mathcal{R}u \Rightarrow u : A}\ {\scriptstyle W}}{\mathcal{N}w \Rightarrow w : \boxdot A}\ {\scriptstyle R\boxdot}$$

■

Observe that Lemma 33 does not work when A is $\top \rhd \top$ since $\Rightarrow w : \top \rhd \top$ is not a derivable sequent.

LEMMA 34. *The axiomatic rule* rN* *of restricted necessitation is* G3.STn-*admissible.*

PROOF. By Lemma 32 we have that if A is an axiom of STn that differs from TID or if it is an instance of K*, then the sequent $\Rightarrow w : A$ is G3.STn-derivable. By Lemma 33 we conclude that the sequent $\mathcal{N}w \Rightarrow w : \boxdot A$ is also G3.STn-derivable. ■

LEMMA 35. *The rule* rLLE *is* G3.STn-*admissible and* LLE *is* G3.ST4(5)-*admissible.*

PROOF. We have the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow w : B \supset A}{\Rightarrow v : B \supset A}\ {\scriptstyle S}}{v : B \Rightarrow v : A}\ {\scriptstyle Le.27}}{v : B, w\mathcal{R}v, \mathcal{N}w, \ldots \Rightarrow v : C, v : A, w : A}\ {\scriptstyle W} \quad \dfrac{}{v : C, \mathcal{N}w, \ldots \Rightarrow \ldots, v : C}\ {\scriptstyle Le.23}}{w\mathcal{R}v, v : B, \mathcal{N}w, w : A \rhd C, \ldots \Rightarrow w : A, v : C}\ {\scriptstyle L\rhd} \quad \dfrac{\dfrac{\dfrac{\Rightarrow w : A \supset B}{\Rightarrow u : A \supset B}\ {\scriptstyle S}}{u : A \Rightarrow u : B}\ {\scriptstyle Le.27}}{u : A, \ldots \Rightarrow \ldots, u : B}\ {\scriptstyle W}}{w\mathcal{R}u, u : A, \mathcal{N}w, w\mathcal{R}w, w : A \rhd C \Rightarrow w : B \rhd C, w : A}\ {\scriptstyle R\rhd} \quad \mathcal{S}}{w\mathcal{R}w, w : A \rhd C \Rightarrow w : B \rhd C}\ {\scriptstyle L\rhd}}{w : A \rhd C \Rightarrow w : B \rhd C}\ {\scriptstyle Ref}}{\Rightarrow w : (A \rhd C) \supset (B \rhd C)}\ {\scriptstyle R\supset}$$

where $\mathcal{S}$ is like the sequent on its left (with $w : C$ in the antecedent in place of $w : A$ in the succedent) and it has an analogous derivation. ■

LEMMA 36. *The rule* rRW *is* G3.STn-*admissible and* RW *is* G3.ST4(5)-*admissible.*

Proof. We have the following derivation:

$$\dfrac{\dfrac{\dfrac{\overline{v : C, \ldots \Rightarrow \ldots, v : C}\;{\scriptstyle Le.23} \quad \dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow w : A \supset B}{\Rightarrow v : A \supset B}\;{\scriptstyle S}}{v : A \Rightarrow v : B}\;{\scriptstyle Le.27}}{v : A, \mathcal{N}w, \ldots \Rightarrow w : C, v : B}\;{\scriptstyle W}}{w\mathcal{R}v, v : C, w\mathcal{R}u, u : A, \mathcal{N}w, w\mathcal{R}w, w : C \rhd A \Rightarrow w : C, v : B}\;{\scriptstyle L\rhd} \quad \overline{u : C, \ldots \Rightarrow \ldots, u : C}\;{\scriptstyle Le.23}}{w\mathcal{R}u, u : C, \mathcal{N}w, w\mathcal{R}w, w : C \rhd A \Rightarrow w : C \rhd B, w : C}\;{\scriptstyle R\rhd} \quad \mathcal{S}}{\dfrac{\dfrac{w\mathcal{R}w, w : C \rhd A \Rightarrow w : C \rhd B}{w : C \rhd A \Rightarrow w : C \rhd B}\;{\scriptstyle Ref}}{\Rightarrow w : (C \rhd A) \supset (C \rhd B)}\;{\scriptstyle R\supset}}\;{\scriptstyle L\rhd}$$

where $\mathcal{S}$ is like the sequent of its left (with $w : A$ in the antecedent in place of $w : C$ in the succedent) and it has a similar derivation. ■

Lemma 37. *The rule of Modus Ponens* MP *is admissible in* G3.STn*.*

Proof. An immediate consequence of the admissiblity of *Cut*. ■

Lemma 38. *Becker's rule* BR* *is admissible in* G3.STn*.*

Proof. We have the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{N}w \Rightarrow w : \boxdot(A \supset B) \quad \overline{w : \boxdot(A \supset B), w : \boxdot A \Rightarrow w : \boxdot B}}{\mathcal{N}w, w : \boxdot A \Rightarrow w : \boxdot B}\;{\scriptstyle Cut}}{\mathcal{N}u, u : \boxdot A \Rightarrow u : \boxdot B}\;{\scriptstyle S}}{u : A, \mathcal{N}u, u\mathcal{R}u, \mathcal{N}w, w\mathcal{R}u, u : \boxdot A \Rightarrow u : \boxdot B}\;{\scriptstyle W}}{u\mathcal{R}u, \mathcal{N}w, w\mathcal{R}u, u : \boxdot A \Rightarrow u : \boxdot B}\;{\scriptstyle L\Box}}{\mathcal{N}w, w\mathcal{R}u, u : \boxdot A \Rightarrow u : \boxdot B}\;{\scriptstyle Ref}}{\mathcal{N}w, w\mathcal{R}u \Rightarrow u : \boxdot A \supset \boxdot B}\;{\scriptstyle R\supset}}{\mathcal{N}w \Rightarrow w : \boxdot(\boxdot A \supset \boxdot B)}\;{\scriptstyle R\boxdot}$$

where the rightmost leaf is G3.STn-derivable. ■

Theorem 39. (Completeness) *If A is valid w.r.t. models for the logic* STn *then the sequent $\mathcal{N}w \Rightarrow w : A$ is* G3.STn*-derivable.*

Proof. The axiomatic calculus STn is complete for validity, see Section 4.2 Theorem 17 and Section 5 Corollary 20. By Lemmas 32–38 we know that $\vdash_{\mathsf{STn}} A$ implies G3.STn $\vdash \mathcal{N}w \Rightarrow w : A$ and we conclude that G3.STn is complete for validity. ■

Corollary 40. *The axiomatic calculi* ST2 – ST5 *are sound.*

Proof. To prove Theorem 39 we have shown that (the sequent expressing) each axiom of STn is derivable in G3.STn and that each rules of STn is admissible in G3.STn. Thus the soundness of STn is implied by that of G3.STn (Theorem 31). ■

8.3. Decidability

We are now going to show that the calculus G3.ST2 is decidable by presenting a terminating proof-search procedure that outputs a finite countermodel for each underivable sequent. It would be possible to extend this result to the calculi G3.ST3–G3.ST5, but the presence of rule *Trans* complicates the proof. It forces indeed the use of a loop-checker since the interaction of *Trans* with the rules for $\rhd$ might generate infinite branches where some formulas are introduced multiples times, cf. [13]. Note that the terminating sequent calculus for S2 presented in [17] can be seen as a terminating sequent calculus for ST2. Nevertheless, the procedure presented below is better behaved in that it also allows the direct construction of a finite countermodel for each underivable sequent.

We begin by sketching the direct Tait-Schütte-Takeuti-style direct proof of completeness for G3.ST2 − G3.ST5. Notice that as a corollary we also have a semantical proof of the admissibility of *Cut*.

Definition 41. A branch in a proof search in the system G3.STn from (the root-sequent) $\Gamma \Rightarrow \Delta$ is *saturated* if, for every rule R of G3.STn, if the principal formulas of R occur in the branch, the formulas introduced by one of the premises of R also occur in the branch. In detail, a saturated branch from $\Gamma \Rightarrow \Delta$ has to satisfy the following conditions (we omit some of them and we use $\uparrow \Gamma$ ($\uparrow \Delta$) to denote the multiset union of all antecedents (succedents) occurring in that branch):

- (Ax) There is no sentential variable p such that $w : p \in \uparrow \Gamma \cap \uparrow \Delta$.
- (L$\rhd$) If $w\mathcal{R}v$ and $w : A \rhd B$ are in $\uparrow \Gamma$, then $\mathcal{N}w$ and, for some u, $w\mathcal{R}u$ and $u : A$ are in $\uparrow \Gamma$; moreover $v : A$ is in $\uparrow \Delta$ or $v : B$ is in $\uparrow \Gamma$.
- (R$\rhd$) If $\mathcal{N}w$ and $w\mathcal{R}v$ are in $\uparrow \Gamma$ and $w : A \rhd B$ is in $\uparrow \Delta$, then if $v : A$ is not in $\uparrow \Delta$, there is some u such that $w\mathcal{R}u$ and $u : A$ are in $\uparrow \Gamma$ and $u : B$ is in $\uparrow \Delta$.
  (*Ref*) for each w occurring in $\uparrow \Gamma \cup \uparrow \Delta$ the relational atom $w\mathcal{R}w$ occurs in $\uparrow \Gamma$.

Definition 42. Given a sequent $\Gamma \Rightarrow \Delta$ we build a G3.STn proof search tree by applying all possible rules of the calculus G3.STn. To run through each rule stepwise, we fix a counter. At step 1 we apply all possible instances of the rule L$\neg$, at step 2 all instances of the rule R$\neg$ and so forth. Without loss of generality, we assume that an instance of rule *Ref* or *Norm* can be applied only w.r.t. labels occurring in the branch under consideration. There are 11, 12, or 13 different steps, depending on the number of rules of calculus under consideration, and at step 11, 12, or 13 + 1 we repeat step 1.

If the construction ends we obtain a derivation otherwise we obtain an infinite tree. In the latter case, by König's Lemma, the tree has an infinite branch. It is easy to see that this branch is saturated and, therefore, it allows to extract a countermodel.

THEOREM 43. *Given a saturated branch $\mathcal{B}$ in a proof search tree for $\Gamma \Rightarrow \Delta$ built according to the rules of system* G3.STn*, we can extract a countermodel $\mathcal{M}$ for the endsequent that is based on a model for* STn*.*

PROOF. Given a saturated branch $\mathcal{B}$ from $\Gamma \Rightarrow \Delta$, we define the following countermodel: $\mathfrak{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{N}, \mathcal{V} \rangle$ such that:

- $\mathcal{W}$ is the set of all world labels occurring in $\uparrow \Gamma \cup \uparrow \Delta$.
- $\mathcal{N}$ is the set of all labels w such that $\mathcal{N}w \in \uparrow \Gamma$.
- $w\mathcal{R}u$ if and only if $w\mathcal{R}u$ occurs in $\uparrow \Gamma$.
- $\mathcal{V}(p)$ is the set of all worlds w such that $w : p$ occurs in $\uparrow \Gamma$.

Notice that $\mathcal{V}$ is well defined by the saturation condition *Ax* and by the saturation condition *Ref* we know that the accessibility relation is reflexive. We define the realisation ρ such that $\rho(w) \equiv w$. We claim that:

1. If $w : A$ is in $\uparrow \Gamma$, then $\rho \models w : A$.
2. If $w : A$ is in $\uparrow \Delta$, then $\rho \not\models w : A$.

The proof, which is by simultaneous induction on the weight of A, is omitted for the sake of brevity.

To conclude, we have to argue that the model defined above is a model for STn. This is an immediate consequence of saturation under the non-logical rules of G3.STn. ■

THEOREM 44. (Completeness) *For every formula A (and $\models$ w.r.t. relational models in the appropriate class):*

$$\models A \text{ if and only if } \mathsf{G3.STn} \vdash Nw \Rightarrow w : A$$

COROLLARY 45. *The rule of Cut is semantically admissible.*

We have thus shown that the procedure given in Def. 42 outputs a countermodel for each underivable sequent and a derivation for each derivable one). We are now ready to prove that G3.ST2 is decidable by showing how to modify the procedure given in Def. 42 in order to avoid the construction of infinite branches in a proof search tree for an underivable sequent. Given that a branch of a G3.ST2 proof search tree can be infinite because of an unbounded number of application of the rules *Ref*, $L \rhd$, and $R \rhd$, we have to show how to bound the number of applications of these rules.

LEMMA 46. *The rule* Ref *needs not be instantiated more than once on the same label in every branch in a proof search.*

PROOF. By height-preserving admissibility of contraction. ■

In order to obtain a termination result we need to show that the number of labels introduced in a proof search is finite. This will be done by bounding the number of instances of rules $L \rhd$ and $R \rhd$, which are the only rules introducing new labels.

We say that v is an (*non-trivial*) *immediate successor* of w in the branch $\mathcal{B}$ of a proof search tree if wRv occurs in $\mathcal{B}$ (and $w \neq v$). Moreover, let $R^*_{\mathcal{B}}$ denote the transitive closure relation of immediate successor. As it is easy to check, $R^*_{\mathcal{B}}$ defines a tree which does not contain cycles except the reflexive ones.

THEOREM 47. *Each label in a branch $\mathcal{B}$ of a proof search tree of an endsequent $\Gamma \Rightarrow \Delta$ has only a finite number of immediate successors.*

PROOF. The non-trivial immediate successors of a label can be introduced only by applications of the rules $L \rhd$ and $R \rhd$. The subformulas of the formula A (which occur in a proof search) are finite, therefore if there were infinite immediate successors there would be more than one application of one of the above mentioned rules to the same principal labelled formulas. We show that every derivation can be transformed into a derivation of the same height in which every branch contains at most one application of such rules to the same principal formulas. We detail one case of rule $L \rhd$ as an example.

$$
\dfrac{\dfrac{\mathcal{N}w, w\mathcal{R}u_2, u_2 : A, \mathcal{N}w, w\mathcal{R}u_1, u_1 : A, w\mathcal{R}v, w : A \rhd B, \Gamma' \Rightarrow \Delta', v : A, v : A \quad \mathcal{S}_2}{\mathcal{N}w, w\mathcal{R}u_1, u_1 : A, w\mathcal{R}v, w : A \rhd B, \Gamma' \Rightarrow \Delta', v : A}{\scriptstyle L\rhd} \\ \vdots\, \mathcal{D} \\ \mathcal{N}w, w\mathcal{R}u_1, u_1 : A, w\mathcal{R}v, w : A \rhd B, \Gamma \Rightarrow \Delta, v : A \qquad \mathcal{S}}{w\mathcal{R}v, w : A \rhd B, \Gamma \Rightarrow \Delta}{\scriptstyle L\rhd}
$$

We transform the derivation as follows:

$$
\dfrac{\dfrac{\dfrac{\mathcal{N}w, w\mathcal{R}u_2, u_2 : A, \mathcal{N}w, w\mathcal{R}u_1, u_1 : A, w\mathcal{R}v, w : A \rhd B, \Gamma' \Rightarrow \Delta', v : A, v : A}{\mathcal{N}w, w\mathcal{R}u_1, u_1 : A, \mathcal{N}w, w\mathcal{R}u_1, u_1 : A, w\mathcal{R}v, w : A \rhd B, \Gamma' \Rightarrow \Delta', v : A, v : A}{\scriptstyle S}}{\mathcal{N}w, w\mathcal{R}u_1, u_1 : A, w\mathcal{R}v, w : A \rhd B, \Gamma' \Rightarrow \Delta', v : A}{\scriptstyle C} \\ \vdots\, \mathcal{D} \\ \mathcal{N}w, w\mathcal{R}u_1, u_1 : A, w\mathcal{R}v, w : A \rhd B, \Gamma \Rightarrow \Delta, v : A \qquad \mathcal{S}}{w\mathcal{R}v, w : A \rhd B, \Gamma \Rightarrow \Delta}{\scriptstyle L\rhd}
$$

The application of the hp-admissible rules of substitution and contraction does not introduce new applications of L$\rhd$ (this is easily checked). ■

As a consequence, the tree of immediate successors is finitely branching. It remains to show that in every branch the length of a chain of labels is finite. This depends on the fact that the relation defined by the tree of immediate successors in a proof search is intransitive. In particular, a label *sees* only its immediate successors and itself (by reflexivity). Therefore the length of a branch is determined by the number of modal operators occurring in a formula.

THEOREM 48. *Every chain of labels in a branch in a proof search for the sequent* $\Gamma \Rightarrow \Delta$ *is finite.*

PROOF. Given a chain of labels in a branch in a proof search for the sequent $\Gamma \Rightarrow \Delta$ and a label u in the chain, every non-trivial immediate successor of u is introduced (bottom-up) by the application of one of either $L \rhd$ or $R \rhd$ to a formula B labelled by u. However, since every label sees only its immediate successors, every label introduced by the analysis of $u : B$ will label only formulas of lesser weight. Since by definition the weight of each formula is finite, the chain is finite. ■

THEOREM 49. *The proof search for a sequent* $\Gamma \Rightarrow \Delta$ *in the system* G3.ST2 *terminates.*

PROOF. The proof is immediate because in every branch the number of labels generated is finite. ■

COROLLARY 50. *The relation* G3.ST2 $\vdash \Gamma \Rightarrow \Delta$ *is decidable.*

PROOF. By Theorem 43 we can extract a countermodel out of a saturated branch and by Theorem 49 we know that every saturated branch can be made finite, so we get the finite model property and the decidability of the system. ■

9. Conclusion

In this article, we considered a strengthening of strict implication by imposing that the antecedent is possible: $A \rhd B$ iff $\Box(A \supset B)$ and $\Diamond A$. The connective $\rhd$ is known as 'super-strict implication'. Logics of super-strict implication are *Boethian* and paradox-free versions of Lewis' logics of strict implication that have been introduced semantically in [4,5]. In this paper we provided two proof-theoretic characterisations of normal and non-normal logics of super-strict implication—in a Hilbert-style proof system and in a labelled sequent calculus. The logics ST2–ST5 investigated here can be thought of as super-strict companions to C. I. Lewis' logics S2–S5 of strict implication. First, we showed that our Hilbert-style axiomatisations of ST2–ST5 are sound and complete for the respective semantics by simulating and backsimulating complete axiomatic calculi for the strict implication companions S2–S5. Next, we showed that our labelled sequent calculi have good structural properties: all rules are height-preserving admissible, the structural rules of weakening and contraction are height-preserving admissible, and the rule of cut is syntactically admissible. We also showed that each of these calculi is sound and complete and that the calculus expressing ST2 admits for terminating proof-search and is hence decidable. We further proved that strong Boethius thesis (BT), the headache of consequential implication [20], cannot be added to a normal system of super-strict implication without creating inconsistency. However, a sensible weakening of BT—by the possibility assumption of the antecedent conditional—termed 'possibilistic BT (pBT)' here, is valid in normal reflexive semantics.

There is an obvious way to strengthen super-strict implication $\rhd$. Namely by imposing that the negation of the consequent is also possible: $A \blacktriangleright B$ iff $A \rhd B$ and $\Diamond \neg B$. This connective has been investigated semantically [4] (termed *strong super-strict implication*) and axiomatically [27] (termed *implicative conditional*).[25] There are some interesting similarities, as well as differences. Both connectives invalidate the paradoxes of strict implication—Antilogical Antecedent (AA) and Tautological Consequent (TC). But they do so in different ways. Whereas $\rhd$ validates the negation of the first paradox of strict implication NAA ($\neg(\bot \rhd B)$), $\blacktriangleright$ also validates the negation of the second paradox NTC ($\neg(A \blacktriangleright \top)$). Both connectives are weakly transitive (wTR), validated weak Boethius Thesis (wBT) and Aristotle's thesis (AT), but invalidate the principles of Right Weakening (RW), Identity (ID), Simplification (SI), and Strengthening the Antecedent (SA), as well as strong Boethius Thesis (BT). But, contrary to $\rhd$, $\blacktriangleright$ also validates Contraposition (C) and Aristotle's Second Thesis. Whereas $\Box$ is expressible in the language $\mathcal{L}_{\rhd}$ of super-strict implication, it is not always expressible in the language $\mathcal{L}_{\blacktriangleright}$ of strong super-strict implication. In [27] a Hilbert-style axiomatisation of $\blacktriangleright$ was given and shown to be sound and complete for reflexive normal Kripke models. Yet non-normal versions of $\blacktriangleright$ are also available [5]. There remain thus at least three open questions for $\blacktriangleright$, inspired by the present work: (1) Is it possible to provide an axiomatisation of $\blacktriangleright$ without using $\Box$? (2) What about axiomatisations of $\blacktriangleright$ for non-normal semantics? (3) How do systems for $\blacktriangleright$ behave when we add BT or sensible weakenings?

[25] Also suggested by Priest [21], Gomes [6], and Lenzen [10].

In the future, we plan to extend the results presented here to the super-strict companion of Lewis' system $\mathsf{S1}$ and to give a proof-theoretic characterisation of logics of super-strict implication by means of an internal sequent calculus such as hypersequents or nested sequents.

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A Deduction Theorem for $\supset$ in $\mathsf{S2}$ and $\mathsf{ST2}$

LEMMA 51. (Deduction Theorem) *If* $A \vdash_{\mathsf{S2}} B$ *then* $\vdash_{\mathsf{S2}} A \supset B$.

PROOF. Suppose $A \vdash_{\mathsf{S2}} B$, where without loss of generality we can assume that A is not an axiom of $\mathsf{S2}$. Then there is a sequence of formulas $A_1, \ldots, A_n$, such that $A_n = B$ and for each $i \in \{1, \ldots, n\}$ A_i is either (1) A and thus not an axiom, or (2) an axiom, or (3) obtained by one of the three rules from previous formulas A_j [and A_k] for $j, k < i$. We show by induction on the length of the proof that then there is a proof of $A \supset A_i$, without the assumption A, i.e. $\vdash_{\mathsf{S2}} A \supset A_i$.

1. Suppose A_i is A and thus not an axiom. But $A \supset A$ is PT. Since $A_i = A$ by assumption, we have a proof of $A \supset A_i$ without assuming A.

2. Suppose A_i is an axiom. But $A_i \supset (A \supset A_i)$ is PT. Hence by MP and the axiom A_i, we obtain $A \supset A_i$. Thus we have a proof of $A \supset A_i$ without assuming A.

3. Suppose A_i is obtained by one of the rules from previous formulas, and assume as induction hypothesis (IH) that the property holds for these.

MP: If A_i is obtained by MP, then there are previous formulas $A_k, A_j = (A_k \supset A_i)$ such that $k, j < i$. By IH there is a proof of $A \supset A_k$ and of $A \supset (A_k \supset A_i)$ without assuming A. We can then construct a proof of $A \supset A_i$ without assuming A as follows. $(A \supset A_k) \supset ((A \supset (A_k \supset A_i)) \supset (A \supset A_i))$ is PT. But we have a proof of $A \supset A_k$ and a proof of $A \supset (A_k \supset A_i)$, both without assuming A. Thus by two applications of MP, we obtain a proof of $A \supset A_i$ without assuming A.

rN. If A_i is obtained by rN, then there is a previous formula A_j in the proof which is derivable without assumption and, such that $A_i = \Box A_j$, $j < i$ and A_j is PT, T or K. We thus have a proof of A_i without assuming A. But $A_i \supset (A \supset A_i)$ is PT. Hence by MP we have a proof of $A \supset A_i$ without assuming A.

BR. If A_i is obtained by BR, then there is a previous formula $A_j = \Box(C \supset D)$ in the proof which is derivable without assumption and such that $A_i = \Box(\Box C \supset \Box D)$ and $j < i$. Then we have a proof of A_i without assuming A. By the same reasoning as for rN, we obtain a proof of $A \supset A_i$ without assuming A. ■

LEMMA 52. (Deduction Theorem) *The deduction theorem (for $\supset$) holds in* ST2*.*

PROOF. Similar proof as in Lemma 51. Suppose $A \vdash_{\mathsf{ST2}} B$ (assuming that A is not an axiom of ST2). Then there is a sequence of formulas $A_1, \ldots, A_n$, such that $A_n = B$ and for each $i \in \{1, \ldots, n\}$ A_i is either (1) A, or (2) an axiom, or (3) obtained by one of the four rules from previous formulas A_j [and A_k] for $j, k < i$. We show by induction on the length of the proof that then $\vdash_{\mathsf{ST2}} A \supset A_i$. Case 1 and 2 are treated by the same reasoning as in Lemma 51.

3. Suppose A_i is obtained by one of the rules from previous formulas, and assume as induction hypothesis (IH) that the property holds for these.

MP: Same reasoning as in Lemma 51.

rLLE. If A_i is obtained by LLE, then there is a previous formula $A_j = (B \equiv C)$ in the proof which is derivable without assumption and, such that $A_i = (B \rhd D) \supset (C \rhd D)$, $j < i$ and A_j is PT. Then we have a proof of A_i without assuming A. Since $A_i \supset (A \supset A_i)$ is PT, applying MP we obtain a proof of $A \supset A_i$ without assuming A.

rRW. Same reasoning as for rLLE.

rN*. Same reasoning as for rN (noting that we can prove K* without assumption).

BR*. Same reasoning as for rLLE (or for BR in Lemma 51). ∎

References

[1] Anderson, A.R., and N. Belnap, *Entailment. The Logic of Relevance and Necessity* vol. 1. Princeton University Press, Princeton, 1975.

[2] Cresswell, M., and G.H. Hughes, *A New Introduction to Modal Logic*, Routledge, London, 1996.

[3] Crupi, V., and A. Iacona, The evidential conditional, *Erkenntnis* https://doi.org/10.1007/s10670-020-00332-2, 2020.

[4] Gherardi, G., and E. Orlandelli, Super-strict implications, *Bulletin of the Section of Logic* 50(1):1–34. https://doi.org/10.18778/0138-0680.2021.02, 2021.

[5] Gherardi, G., and E. Orlandelli, Non-normal super-strict implications, in A. Indrzejczak, and M. Zawidzki, (eds.), *Proceedings of the 10th International Conference on Non-Classical Logics. Theory and Applications*, EPTCS, Sydney, https://doi.org/10.4204/EPTCS.358.1, 2022, pp. 1–11.

[6] Gomes, G., Concessive conditionals without *Even if* and nonconcessive conditionals with *Even if*, *Acta Analytica* 35:1–21, 2020.

[7] Günther, M., A connexive conditional, *Logos & Episteme* 13(1):55–63, 2022.

[8] Kripke, S.A., Semantical analysis of modal logic II. Non-normal modal propositional calculi, in J.W. Addison, A. Tarski, and L. Henkin, (eds.), *The Theory of Models: Proceedings of the 1963 International Symposium at Berkeley*, North Holland, Amsterdam, 1965, pp. 206–220.

[9] Lemmon, E.J., New foundations for Lewis modal systems, *The Journal of Symbolic Logic* 22(2):176–186, 1957.

[10] Lenzen, W., The third and fourth stoic accounts of conditionals, in M. Blicha and I. Sedlár, (eds.), *The Logica Yearbook 2020*, College Publications, Rickmansworth, 2020, pp. 127–146.

[11] Lewis, C.I., and C.H. Langford, *Symbolic Logic*, Century Co, New York, 1932.

[12] Lewis, D., *Counterfactuals*, Harvard University Press, Cambridge, MA, 1973.

[13] Negri, S., Proof analysis in modal logic, *Journal of Philosophical Logic* 34:507–544, https://doi.org/10.1007/s10992-005-2267-3, 2005.

[14] Negri, S., and J. von Plato, Meaning in use, in H. Wansing, (ed.), *Dag Prawitz on Proofs and Meaning*, Springer, Cham, 2001, pp. 239–57.

[15] Nelson, E.J., Intensional relation, *Mind* 39: 440–453, https://doi.org/10.1093/mind/XXXIX.156.440, 1930.

[16] Odintsov, S., D., Skurt, and H. Wansing, Connexive variants of modal logics over FDE, in O. Arieli, and A. Zamansky, (eds.), *Arnon Avron on Semantics and Proof Theory of Non-Classical Logics. Outstanding Contributions to Logic*, Springer, Cham, https://doi.org/10.1007/978-3-030-71258-7_13, 2021, pp. 295–318.

[17] Ohnishi, M., Gentzen decision procedures for Lewis's systems $S2$ and $S3$, *Osaka Mathematical Journal* 13(1):125–137 1961.

[18] Ohnishi, M., von Wright-Anderson's decision procedures for Lewis's systems $S2$ and $S3$, *Osaka Mathematical Journal* 13(1):139–142, https://doi.org/10.18910/9565, 1961.

[19] Orlandelli, E., and M. Tesi, Labelled sequent calculi for logics of strict implication, in D.F. Duque, and A. Palmigiano, (eds.), *Advances in Modal Logic 2022*, College Publications, London, 2022, pp. 625–642.

[20] Pizzi, C., and T. Williamson, Strong Boethius' thesis and consequential implication, *Journal of Philosophical Logic* 26:569–588, https://doi.org/10.1023/A:1004230028063 1997.

[21] Priest, G., Negation as cancellation, and connexive logic, *Topoi* 18:14–148, https://doi.org/10.1023/A:1006294205280, 1999.

[22] Raidl, E., Completeness for counter-doxa conditionals: using ranking semantics, *The Review of Symbolic Logic* 12(4):1–31, https://doi.org/10.1017/S1755020318000199, 2019.

[23] Raidl, E., Strengthened conditionals, in B. Liao, and Y. Wáng, (eds.), *Context, Conflict and Reasoning. Logic in Asia: Studia Logica Library* Springer, Singapore, https://doi.org/10.1007/978-981-15-7134-3_11, 2020, pp. 139–155.

[24] Raidl, E., Definable conditionals, *Topoi* 40(1):87–105, https://doi.org/10.1007/s11245-020-09704-3, 2021.

[25] Raidl, E., Three conditionals: contraposition, difference-making and dependency, in: M. Blicha, and I. Sedlar, (eds.) *The Logica Yearbook 2020*, College Publications, London, 2021, pp. 201–217.

[26] Raidl, E., Neutralization, Lewis' doctored conditional, or another note on "A connexive conditional", *Logos & Episteme* 14(1) 2023. 2023, pp. 101–118, https://doi.org/10.5840/logosepisteme20231415

[27] Raidl, E., and G. Gomes, The implicative conditional, *Journal of Philosophical Logic*, 1–36 2022, forthcoming.

[28] Raidl, E., A. Iacona, and V. Crupi, The logic of the evidential conditional, *The Review of Symbolic Logic*, 1–13, https://doi.org/10.1017/S1755020321000071, 2021.

[29] Routley, R., V. Plumwood, R.K. Meyer, and R.T. Brady, *Relevant Logics and Their Rivals*, Ridgeview, Atascardero, CA, 1982.

[30] Stalnaker, R.C., A theory of conditionals, in N. Rescher, (ed.), *Studies in Logical Theory*, Basil Blackwell, Hoboken, 1968, pp. 98–112.

[31] Tesi, M., Labelled sequent calculi for Lewis' non-normal propositional modal logics. *Studia Logica* 109(4):725–757, https://doi.org/10.1007/s11225-020-09924-z, 2021.

[32] von Fintel, K., The presupposition of subjunctive conditionals, in U. Sauerland, and O. Percus, (eds.), *The Interpretive Tract*: MIT Working Papers in Linguistics 25:29–44, 1998.

[33] von Fintel, K., Counterfactuals in a Dynamic Context, in Kenstowicz, M., (ed.), *Ken Hale: A Life in Language*, MIT Press, Cambridge, MA, 2001, pp. 123–152.

[34] Wansing, H., and H. Omori, A note on "a connexive conditional", *Logos & Episteme* XIII(3):325–328, 2022.

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