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Hennessy-Milner and Van Benthem for Instantial Neighbourhood Logic

Abstract. We investigate bisimulations for instantial neighbourhood logic and an ω-indexed collection of its fragments. For each of these logics we give a Hennessy-Milner theorem and a Van Benthem-style characterisation theorem.

Keywords: Modal logic, Instantial neighbourhood logic, Bisimulation, Hennessy-Milner theorem, Van Benthem theorem.

1. Introduction

Bisimulations are important tools in the study of modal logics. They provide a structural notion of semantic equivalence: bisimilar worlds satisfy precisely the same logical formulae. If the converse is also true then the logical language is powerful enough to distinguish non-bisimilar states. This is called the *Hennessy-Milner property* [28]. Moreover, Van Benthem's theorem states that normal modal logic can be viewed as the (Kripke) bisimulation-invariant fragment of first-order logic. This was originally proven in [4], see also [11, §2.6], and can be viewed as saying that modal logic provides effective syntax to describe bisimulation-invariant first-order properties. Bisimulations have other uses as well; for example, they provide an equivalence relation between process graphs [32,34], and serve as extensional equality in non-wellfounded set theory [2].

Following Hennessy and Milner's theorem and Van Benthem's theorem for normal modal logic, similar results have been derived for a wide variety of (modal and non-modal) logics, each with its own appropriate notion of bisimulation. These include monotone modal logic [26], neighbourhood logic [27], fragments of XPath [1,14,22], (bi-)intuitionistic logic [3,25,33,35], modal μ-calculi (within monadic second order logics) [21,29], and PDL (within weak chain logic) [12].

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The logic at the centre of attention in the current paper is *instantial neighbourhood logic (INL)*, first introduced in [8]. It provides a system of modal logic for reasoning about neighbourhood models, in which modalities give existential information about what kind of states occur in a neighbourhood of a current state. Specifically, modal formulae are of the form $\Box(\varphi_1, \ldots, \varphi_n; \psi)$ and are true at a state x if x has a neighbourhood a such that each of the φ_i is true at some state in a, and ψ is true at every state in a. Motivations for investigating INL range from topology to games to modelling notions of evidence.

In recent years, various aspects of INL have been studied, including its canonical rules and formulae [39], proof theory [41,42], duality theory [9], and definability [43]. Furthermore, several interesting restrictions and variations of INL are discussed in [8, Section 7], and dynamic extensions have been studied in [6,7]. However, surprisingly little is known about the bisimulations. While the notion of an INL-bisimulation has been defined in [8], only the class of finite models has been shown to have the Hennessy-Milner property, and a Van Benthem-style characterisation theorem is still missing.

This is the gap we are closing in this paper. We study INL-bisimulations and provide several Hennessy-Milner classes that all extend the class of finite models. Using this, we prove a Van Benthem-style characterisation theorem, identifying INL as the bisimulation-invariant fragment of a two-sorted first-order language interpreted over a suitable class of first-order structures. Furthermore, we adapt these results to sublanguages of INL where modal formulae are of the form $\Box(\varphi_1, \ldots, \varphi_k; \psi)$ for some arbitrary but fixed natural number k. We obtain Hennessy-Milner properties and a Van Benthem-style characterisation for each of these sublanguages of INL.

Related work Hansen, Kupke and Pacuit investigated Hennessy-Milner classes for *neighbourhood logic*, the extension of propositional classical logic with a free unary modality [27]. Like INL, this logic can be interpreted in neighbourhood models. They also proved a Van Benthem theorem, identifying the logic as the (neighbourhood) bisimulation-invariant fragment of a two-sorted first-order logic interpreted in certain class $\mathbf{N}$ of first-order structures corresponding to neighbourhood models. When proving the characterisation theorem for INL in Section 7 below, we embed INL in the same two-sorted first-order logic (of course using a different translation of the modalities), and make use of their characterisation of $\mathbf{N}$.

In [38], Schröder, Litak and Pattinson investigated Van Benthem theorems for coalgebraic logics. They characterised coalgebraic logics with finite similarity type as the behavioural equivalence-invariant fragment of corresponding *coalgebraic predicate logics* [31]. INL can be viewed as a coalgebraic logic but with infinite similarity type, so that general results do not apply.

Structure of the paper In Section 2 we briefly recall the language and semantics of INL, as well as the definition of an INL-bisimulation. We also define ultrafilter extensions for instantial neighbourhood models, whose definition is straightforward but has not appeared in the literature before.

In Section 3 we commence our study of INL-bisimulations by comparing them to Kripke bisimulations. Specifically, we recall two ways of embedding normal modal logic into INL and the corresponding translation of Kripke models to instantial neighbourhood models, and show that these translations preserve and reflect both logical equivalence and bisimilarity.

Subsequently, in Section 4 we derive Hennessy-Milner properties for INL-bisimulation using analogues of the notions of *image-finiteness* and *modal saturation* (see e.g. [11, Section 2]). We shed a topological light on these notions in Section 5, where we elucidate the connection between modal saturation and the (double) Vietoris endofunctor on the category Top of topological spaces and continuous functions. Thereafter, in Section 6 we derive a different Hennessy-Milner class using the notion of *populated* models, which is related to the definition of descriptive instantial neighbourhood frames from [9]. Moreover, we give a bisimilarity-somewhere-else result.

The Van Benthem-style characterisation theorem for INL is proven in Section 7. Finally, in Section 8 we explain how the Hennessy-Milner and Van Benthem-style results can be adapted to certain sublangages of INL.

2. Instantial Neighbourhood Models

In this section we recall the language of instantial neighbourhood logic, its interpreting structures, and the definitions of (INL-)bisimulations and ultrafilter extensions. Most of these have appeared in the literature before, in [8] and [9]. The only exception is the definition of ultrafilter extensions and the subsequent lemma (Definition 2.12 and Lemma 2.13), which were communicated at the presentation accompanying [9] given at the 15th Workshop on Coalgebraic Methods in Computer Science in 2020.

Language, frames and models The language $\mathcal{INL}$ of instantial neighbourhood logic is generated by the grammar

$$\varphi ::= p \mid \top \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box(\varphi_1, \dots, \varphi_n; \psi)$$

where p ranges over a (potentially infinite) set Prop of proposition letters. We abbreviate $\bot, \varphi \vee \psi$ and $\varphi \to \psi$ as usual, and in case $n = 0$ we write $\Box\psi$ rather than $\Box(\,;\psi)$ or $\Box(\emptyset;\psi)$. These formulae can be interpreted in models based on neighbourhood frames, defined next.

Definition 2.1. An *instantial neighbourhood frame* is a pair (X, N) of a set X and a function $N : X \to \mathcal{PP}X$, where $\mathcal{PP}X$ denotes the double powerset of X. An *instantial neighbourhood model* is a tuple $\mathcal{M} = (X, N, V)$ consisting of an instantial neighbourhood frame (X, N) and a valuation $V : \text{Prop} \to \mathcal{P}X$ of the proposition letters. We define $m_{n,\Box} : (\mathcal{P}X)^{n+1} \to \mathcal{P}X$ by

$$m_{n,\Box}(a_1, \dots, a_n, b) = \{x \in X \mid \exists w \in N(x) \text{ s.t. } w \cap a_i \neq \emptyset \ \forall i, \text{ and } w \subseteq b\}.$$

A set $d \in m_{n,\Box}(a_1, \dots, a_n, b)$ is said to *witness* the tuple $(a_1, \dots, a_n; b)$.

The *truth set* of an $\mathcal{INL}$-formula φ in $\mathcal{M}$ is defined recursively via $[\![p]\!]^{\mathcal{M}} = V(p)$, $[\![\top]\!]^{\mathcal{M}} = X$, $[\![\neg\varphi]\!]^{\mathcal{M}} = X \setminus [\![\varphi]\!]^{\mathcal{M}}$, $[\![\varphi_1 \wedge \varphi_2]\!]^{\mathcal{M}} = [\![\varphi_1]\!]^{\mathcal{M}} \cap [\![\varphi_2]\!]^{\mathcal{M}}$, and

$$[\![\Box(\varphi_1, \dots, \varphi_n; \psi)]\!]^{\mathcal{M}} = m_{n,\Box}([\![\varphi_1]\!]^{\mathcal{M}}, \dots, [\![\varphi_n]\!]^{\mathcal{M}}, [\![\psi]\!]^{\mathcal{M}}).$$

We write $\mathcal{M}, x \Vdash \varphi$ if $x \in [\![\varphi]\!]^{\mathcal{M}}$. Two states x, x' in models $\mathcal{M}$ and $\mathcal{M}'$ are called *logically equivalent* if they satisfy precisely the same $\mathcal{INL}$-formulae, and this is denoted by $\mathcal{M}, x \leftrightsquigarrow_{\mathcal{INL}} \mathcal{M}', x'$, or by $\mathcal{M}, x \leftrightsquigarrow \mathcal{M}', x'$ if no confusion is likely.

Indeed, instantial neighbourhood models are simply neighbourhood models. These can also be used to interpret *neighbourhood logic*: the extension of classical propositional logic with a single unary modality [18,27]. When used to interpret $\mathcal{INL}$ the morphisms between them are defined differently.

Definition 2.2. A *bounded morphism* between instantial neighbourhood frames $\mathcal{F} = (X, N)$ and $\mathcal{F}' = (X', N')$ is a function $f : X \to X'$ such that

$$N'(f(x)) = \{f[a] \mid a \in N(x)\}$$

for all $x \in X$, where $f[a]$ denotes the direct image of $a \subseteq X$ under f. A *bounded morphism* between instantial neighbourhood models $\mathcal{M} = (X, N, V)$ and $\mathcal{M}' = (X', N', V')$ is a bounded morphism f between the underlying frames that additionally satisfies $V = f^{-1} \circ V'$. We write INF and INM for the categories of instantial neighbourhood frames and models, and their respective notions of bounded morphism.

Bounded morphisms preserve truth of $\mathcal{INL}$-formulae [8, Corollary 2.10].

It is sometimes helpful to take a coalgebraic perspective. Coalgebras can be used as to provide uniform treatment of a wide variety of modal logics see e.g. [30,36]. In Sect. 5 we use them to describe modal saturation using general frames and the Vietoris functor.

Definition 2.3. Let $\mathcal{T}$ be an endofunctor on a category C. A $\mathcal{T}$-*coalgebra* is a pair (C, γ) of an object $C \in \mathsf{C}$ together with a morphism $\gamma : C \to \mathcal{T}C$ in C. A $\mathcal{T}$-*coalgebra morphism* from (C, γ) to (C', γ') is a morphism $f : C \to C'$ in C such that $\gamma' \circ f = \mathcal{T}f \circ \gamma$. The collection of $\mathcal{T}$-coalgebras and $\mathcal{T}$-coalgebra morphisms forms a category again, which is denoted by $\mathsf{Coalg}(\mathcal{T})$.

The category of instantial neighbourhood frames can be modelled as a category of coalgebras, i.e., $\mathsf{INF} \cong \mathsf{Coalg}(\mathcal{P}\mathcal{P})$, where $\mathcal{P} : \mathsf{Set} \to \mathsf{Set}$ is the covariant powerset function. The difference between neighbourhood logic and *instantial* neighbourhood logic is then displayed in the fact that the interpreting structures for the former are given by $\mathsf{Coalg}(\breve{\mathcal{P}}\breve{\mathcal{P}})$, where $\breve{\mathcal{P}}$ is the *contravariant* powerset functor on Set. This coalgebraic perspective on neighbourhood logic is pointed out in, and used throughout [27].

Bisimulations We introduce the notion of B-*exhaustive* pairs of sets.

Definition 2.4. Let $B \subseteq X \times X'$ be a relation. Then we call a pair (a, a') with $a \subseteq X$ and $a' \subseteq X'$ B-*exhaustive* if $a' \subseteq B[a]$ and $a \subseteq B^{-1}[a']$.

This definition is obtained by reversing the inclusions in the definition of so-called B-*coherent* pairs of sets, introduced in [27, Definition 2.1]. Spelling out the definition of B-exhaustive sets we get: (a, a') is B-exhaustive if every $y' \in a'$ is B-related to some $y \in a$, and vice versa. Thus we can reformulate the definition of an (INL-)bisimulation [8, Definition 2.5] as follows.

Definition 2.5. Let $\mathcal{M} = (X, N, V)$ and $\mathcal{M}' = (X', N', V')$ be instantial neighbourhood models. A relation $B \subseteq X \times X'$ is an *(INL-)bisimulation* if for all $(x, x') \in B$ we have:

(B_1) $x \in V(p)$ if and only if $x' \in V'(p)$, for all $p \in \text{Prop}$;

(B_2) If $a \in N(x)$ then there exists $a' \in N'(x')$ s.t. (a, a') is B-exhaustive;

(B_3) If $a' \in N'(x')$ then there exists $a \in N(x)$ s.t. (a, a') is B-exhaustive.

Two states $x \in X$ and $x' \in X'$ are called *(INL-)bisimilar* if there exists a bisimulation linking them, notation: $\mathcal{M}, x \leftrightharpoons_{\mathcal{INL}} \mathcal{M}', x'$, or $\mathcal{M}, x \leftrightharpoons \mathcal{M}', x'$ if there is no danger of confusion.

An easy verification shows that INL-bisimulations are closed under arbitrary unions. As a consequence, the relation of bisimilarity between two models is itself a bisimulation. Furthermore, as expected we have:

Theorem 2.6. *([8], Theorem 2.7) Let* $\mathcal{M} = (X, N, V)$ *and* $\mathcal{M}' = (X', N', V')$ *be two instantial neighbourhood models. Then for all* $x \in X$ *and* $x' \in X'$*:*

$$\mathcal{M}, x \leftrightharpoons \mathcal{M}', x' \quad \textit{implies} \quad \mathcal{M}, x \leftrightsquigarrow \mathcal{M}', x'.$$

The converse of Theorem 2.6 is not true in general. In fact, the converse being true is viewed as a special property a class of models can have.

Definition 2.7. A class $\mathcal{C} \subseteq \mathsf{INM}$ of instantial neighbourhood models is said to be a *Hennessy-Milner class*, or to possess the *Hennessy-Milner property* if for all models $\mathcal{M}, \mathcal{M}' \in \mathcal{C}$ and states $x \in \mathcal{M}$ and $x' \in \mathcal{M}'$,

$$\mathcal{M}, x \leftrightsquigarrow \mathcal{M}', x' \quad \text{iff} \quad \mathcal{M}, x \leftrightharpoons \mathcal{M}', x'.$$

For finite models (i.e. models based on finite sets) we have:

Theorem 2.8. *([8], Theorem 3.1) The class of finite instantial neighbourhood models has the Hennessy-Milner property.*

General frames and ultrafilter extensions We also make use of the notions of general and descriptive frames, introduced in [9]. We recall these, and define ultrafilter extensions of instantial neighbourhood models.

Definition 2.9. A *general (instantial neighbourhood) frame* is given by a tuple (X, N, A) such that $(X, N) \in \mathsf{INF}$ and $A \subseteq \mathcal{P}X$ is a collection of *admissible subsets* containing $\emptyset$ and X that is closed under Boolean operations and under $m_{n,\Box}$ from Definition 2.1 (for all $n \in \omega$). It is called:

- *differentiated* if for any two distinct states $x, y \in X$ there exists an $a \in A$ such that $x \in a$ and $y \notin a$;
- *compact* if $\bigcap A' \neq \emptyset$ for any $A' \subseteq A$ with the finite intersection property;
- *crowded* if for all $x \in X$ and $d \subseteq X$ such that $d \notin N(x)$ we can find $a_1, \ldots, a_n, b$ such that d witnesses $(a_1, \ldots, a_n; b)$ while no $d' \in N(x)$ witnesses $(a_1, \ldots, a_n; b)$.

A *descriptive INL-frame* is a general INL-frame that is differentiated, compact and crowded. A general or descriptive frame (X, N, A) can be turned into a model by equipping it with an *admissible valuation* $V : \mathrm{Prop} \to A$.

A *general frame morphism* from (X, N, A) to (X', N', A') is a bounded morphism $f : (X, N) \to (X', N')$ such that $f^{-1}(a') \in A$ for all $a' \in A'$. We write $\mathsf{G\text{-}INF}$ for the category of general frames and general frame morphisms, and $\mathsf{D\text{-}INF}$ for its full subcategory of descriptive frames.

It was proven in [9] that the category of descriptive frames is isomorphic to the category of coalgebras of the *double Vietoris functor* on Stone, the category of Stone spaces and continuous functions. We recall the definition of the Vietoris functor, and refer to [40] for a detailed account of the connection of the Vietoris functor and normal modal logic.

DEFINITION 2.10. For a topological space (X, τ), let $\mathcal{K}(X, \tau)$ denote the collection of compact subsets of X. The *Vietoris hyperspace* $\mathcal{V}(X, \tau)$ of (X, τ) is obtained by equipping $\mathcal{K}(X, \tau)$ with the topology generated by

$$\boxdot a = \{c \in \mathcal{K}(X, \tau) \mid c \subseteq a\}, \qquad \diamondcdot a = \{c \in \mathcal{K}(X, \tau) \mid a \cap c \neq \emptyset\},$$

where a ranges over τ. The assignment $\mathcal{V}$ extends to an endofunctor on Top by defining its action on a continuous function $f : (X, \tau) \to (X', \tau')$ via $\mathcal{V}f : \mathcal{V}(X, \tau) \to \mathcal{V}(X', \tau') : c \mapsto f[c]$. Its restriction to Stone is denoted $\mathcal{V}_{\mathsf{st}}$.

THEOREM 2.11. *([9], Theorem 5.1) We have* $\mathsf{D\text{-}INF} \cong \mathsf{Coalg}(\mathcal{V}_{st}\mathcal{V}_{st})$.

Proof sketch. The isomorphism on objects is obtained as follows. If (X, N, A) is a descriptive frame and τ_A is the topology on X generated by the basis A, then $\gamma_N : (X, \tau_A) \to \mathcal{V}\mathcal{V}(X, \tau_A) : x \mapsto \{c \in \mathcal{K}(X, \tau_A) \mid c \in N(x)\}$ gives a $\mathcal{V}\mathcal{V}$-coalgebra. Conversely, a $\mathcal{V}\mathcal{V}$-coalgebra $\gamma : (X, \tau) \to \mathcal{V}\mathcal{V}(X, \tau)$ gives rise rise to the descriptive frame (X, N_γ, A) where A is the collection of clopen subsets of (X, τ) and $N_\gamma(x) = \{c \subseteq X \mid \overline{c} \in \gamma(x)\}$. ■

The key idea to take away from this proof sketch is the intuition that in a descriptive frame (X, N, A) the collection of neighbourhoods $N(x)$ of a state x is determined uniquely by the closed subsets of (X, τ_A) it contains.

The algebraic semantics of INL is given by Boolean algebras with instantial operators (BAIOs) [9, Definition 3.1]. It was proven in *op. cit.* that the category of BAIOs and homomorphisms is dually equivalent to $\mathsf{D\text{-}INF}$. Therefore one can define the *ultrafilter extension* of an instantial neighbourhood model $\mathcal{M}$ as the instantial neighbourhood model underlying the descriptive model dual to the complex algebra (cf. Example 3.2 in [9]) of $\mathcal{M}$. Concretely this means:

DEFINITION 2.12. The *ultrafilter extension* of an instantial neighbourhood model $\mathcal{M} = (X, N, V)$ is defined as the tuple $uf\mathcal{M} = (\widehat{X}, \widehat{N}, \widehat{V})$, where $\widehat{X}$ is the set of ultrafilters of X, $\widehat{V}$ is defined by $\widehat{V}(p) = \{\mathfrak{u} \in \widehat{X} \mid p \in \mathfrak{u}\}$, and

$$\widehat{N}(\mathfrak{u}) = \{d \subseteq \widehat{X} \mid \text{for all } n \in \omega \text{ and } a_1, \dots, a_n, b \subseteq X :$$
$$(d \cap \widetilde{a}_i \neq \emptyset \text{ for all } i \text{ and } d \subseteq \widetilde{b}) \text{ iff } m_{n,\Box}(a_1, \dots, a_n; b) \in \mathfrak{u}\},$$

where $\widetilde{a_i} = \{\mathfrak{u} \in \widehat{X} \mid a_i \in \mathfrak{u}\}$ and similar for $\widetilde{b}$.

If $\mathcal{M} = (X, N, V)$ is an instantial neighbourhood model, then there is a natural counterpart in $uf\mathcal{M}$ for every state $x \in X$, given by $\widehat{x} = \{a \subseteq X \mid x \in a\} \in \widehat{X}$. This gives rise to a map $\eta_X : X \to \widehat{X} : x \mapsto \widehat{x}$. It follows from an induction on the structure of φ that $[\![\varphi]\!]^{uf\mathcal{M}} = \widetilde{[\![\varphi]\!]^{\mathcal{M}}}$, so:

LEMMA 2.13. *Let* $\mathcal{M} = (X, N, V)$ *be an instantial neighbourhood model. Then for all* $x \in X$ *we have* $\mathcal{M}, x \leftrightsquigarrow uf\mathcal{M}, \widehat{x}$.

3. Kripke Bisimulation Versus INL-Bisimulation

Kripke models [11, Definition 1.19] give rise to instantial neighbourhood models in several ways. In this section we investigate two such ways, corresponding to two ways of embedding normal modal logic into $\mathcal{INL}$. Apart from elucidating the connection between Kripke bisimulations and INL-bisimulations, this will aid us with the construction of counterexamples in Section 4.

Write $\mathcal{ML}$ for the language of normal modal logic, given by the grammar

$$\varphi ::= p \mid \top \mid \neg\varphi \mid \varphi \wedge \varphi \mid \boxtimes\varphi,$$

where p ranges over a set Prop of proposition letters. We use the symbol $\boxtimes$ rather than $\Box$ to distinguish it from $\mathcal{INL}$-formulae, and write $\diamondsuit = \neg\boxtimes\neg$. Since $\boxtimes$ and $\diamondsuit$ are interdefinable, we may choose either one of them as primitive connective. As apparent already in the next definition, both choices of primitive connective can be intuitive, and we will use them interchangeably.

DEFINITION 3.1. Define translations $\delta_1, \delta_2 : \mathcal{ML} \to \mathcal{INL}$ recursively via $\delta_i(p) = p$, $\delta_i(\top) = \top$, $\delta_i(\neg\varphi) = \neg\delta_i(\varphi)$, $\delta_i(\varphi \wedge \psi) = \delta_i(\varphi) \wedge \delta_i(\psi)$, and

$$\delta_1(\boxtimes\varphi) = \Box\delta_1(\varphi), \qquad \delta_2(\diamondsuit\varphi) = \Box(\delta_2(\varphi); \top).$$

The translation δ_2 is called δ in [8, Section 6], and δ_1 corresponds to τ in *loc. cit.* and is used to translate *monotone* modal logic into $\mathcal{INL}$. It is well known that $\mathcal{ML}$-formulae can be interpreted in *Kripke models* [11, Definition 1.19, 1.20]. With the relevant notion of bounded morphism [11, Definition 2.7], sometimes called a p-morphism, Kripke models form the category KM.

As announced, we consider two ways of embedding KM into INM.

DEFINITION 3.2. For a Kripke model $\mathcal{K} = (X, R, V)$ define $N_1, N_2 : X \to \mathcal{PP}X$ via

$$N_1(x) = \big\{\{y \in X \mid xRy\}\big\}$$
$$N_2(x) = \big\{\{y\} \mid y \in X \text{ and } xRy\big\}$$

and set $\theta_1\mathcal{K} = (X, N_1, V)$ and $\theta_2\mathcal{K} = (X, N_2, V)$. Both θ_1 and θ_2 extend to functors $\mathsf{KM} \to \mathsf{INM}$ by acting as the identity on morphisms.

The following proposition and theorem reveal the connection between the logical translations δ_1, δ_2, and the model translations θ_1 and θ_2.

Proposition 3.3. *Let $\mathcal{K} = (X, R, V)$ be Kripke model. Then for all $x \in X$ and $\varphi \in \mathcal{ML}$ we have*

$$\mathcal{K}, x \Vdash \varphi \quad \textit{iff} \quad \theta_1\mathcal{K}, x \Vdash \delta_1(\varphi) \quad \textit{iff} \quad \theta_2\mathcal{K}, x \Vdash \delta_2(\varphi).$$

Proof. This follows from induction on the structure of φ. ■

Theorem 3.4. *Let $\mathcal{K} = (X, R, V)$ and $\mathcal{K}' = (X', R', V')$ be Kripke models. Then for all $x \in X$ and $x' \in X'$ we have*

$$\mathcal{K}, x \leftrightsquigarrow_{\mathcal{ML}} \mathcal{K}', x' \quad \textit{iff} \quad \theta_1\mathcal{K}, x \leftrightsquigarrow_{\mathcal{INL}} \theta_1\mathcal{K}', x' \quad \textit{iff} \quad \theta_2\mathcal{K}, x \leftrightsquigarrow_{\mathcal{INL}} \theta_2\mathcal{K}', x'.$$

Proof. It follows from Proposition 3.3 that both the middle and right statement imply the left one. For the converse, it suffices to prove that for each formula $\varphi \in \mathcal{INL}$ there exists a formula $\psi \in \mathcal{ML}$ such that $[\![\varphi]\!]^{\theta_i\mathcal{K}} = [\![\delta_i(\psi)]\!]^{\theta_i\mathcal{K}}$ for all $\mathcal{K} \in \mathsf{KM}$. Indeed, then we have

$$\begin{aligned} \theta_i\mathcal{K}, x \Vdash \varphi \quad &\text{iff} \quad \theta_i\mathcal{K}, x \Vdash \delta_i(\psi) &&\text{iff} \quad \mathcal{K}, x \Vdash \psi \\ &\text{iff} \quad \mathcal{K}', x' \Vdash \psi &&\text{iff} \quad \theta_i\mathcal{K}', x' \Vdash \delta_i(\psi) \quad \text{iff} \quad \theta_i\mathcal{K}', x' \Vdash \varphi. \end{aligned}$$

We focus on θ_1, leaving θ_2 to the reader. Define $\beta : \mathcal{INL} \to \mathcal{ML}$ recursively via $\beta(p) = p$, $\beta(\top) = \top$, $\beta(\neg\varphi) = \neg\beta(\varphi)$, $\beta(\varphi \wedge \psi) = \beta(\varphi) \wedge \beta(\psi)$ and

$$\beta(\varphi_1, \dots, \varphi_n; \psi) = \diamondsuit\varphi_1 \wedge \dots \wedge \diamondsuit\varphi_n \wedge \Box\psi.$$

Then it follows from a routine induction on the structure of φ, using the fact that every state has only one neighbourhood, that $\theta_1\mathcal{K}, x \Vdash \varphi$ iff $\theta_1\mathcal{K}, x \Vdash \delta_1(\beta(\varphi))$ for any Kripke model $\mathcal{K}$, as required. ■

Kripke bisimulations [11, Definition 2.16] relate to INL-bisimulations:

Theorem 3.5. *Let $\mathcal{K} = (X, R, V)$ and $\mathcal{K}' = (X', R', V')$ be two Kripke models and $B \subseteq X \times X'$ a relation. Then the following are equivalent:*

1. *B is a Kripke bisimulation between $\mathcal{K}$ and $\mathcal{K}'$;*
2. *B is an INL-bisimulation between $\theta_1\mathcal{K}$ and $\theta_1\mathcal{K}$;*
3. *B is an INL-bisimulation between $\theta_2\mathcal{K}$ and $\theta_2\mathcal{K}$.*

Proof. (1) $\Leftrightarrow$ (2) and (1) $\Leftrightarrow$ (3) follow from unravelling the definitions. ■

As a consequence, Hennessy-Milner classes for normal modal logic carry over to Hennessy-Milner classes for $\mathcal{INL}$, and vice versa.

Corollary 3.6. *If $\mathcal{C} \subseteq \mathsf{KM}$ is a Hennessy-Milner class for $\mathcal{ML}$ with respect to Kripke bisimulation, then $\theta_1\mathcal{C} = \{\theta_1\mathcal{K} \mid \mathcal{K} \in \mathcal{C}\} \subseteq \mathsf{INM}$ and $\theta_2\mathcal{C} = \{\theta_2\mathcal{K} \mid \mathcal{K} \in \mathcal{C}\} \subseteq \mathsf{INM}$ are Hennessy-Milner classes for INL with INL-bisimulation.*

Proof. Combine Theorems 3.4 and 3.5. ∎

4. Hennessy-Milner Classes

We extend the Hennessy-Milner theorem for finite models to several larger classes of models. We begin by defining *image-finite* frames and models, and in Proposition 4.3 we prove that these form a Hennessy-Milner class. Subsequently, with the proof of this proposition in mind, we define a notion of *modal saturation* suitable for instantial neighbourhood logic. Modally saturated models then form a Hennessy-Milner class by design.

Definition 4.1. An instantial neighbourhood frame (X, N) is called *image-finite* if every state has a finite number of neighbourhoods, and each of these neighbourhoods is finite. An instantial neighbourhood model is called image-finite if it is based on an image-finite frame.

Remark 4.2. Since $\mathsf{INF} \cong \mathsf{Coalg}(\mathcal{P}\mathcal{P})$, we can also use general coalgebraic methods the arrive at a notion of image-finiteness. Namely, image-finite $\mathcal{P}\mathcal{P}$-coalgebras correspond to coalgebras for some *finitary version* of $\mathcal{P}\mathcal{P}$, see e.g. [27, §4.2] for details. The finitary version of $\mathcal{P}\mathcal{P}$ is given by $\mathcal{P}_\omega\mathcal{P}_\omega$, where $\mathcal{P}_\omega : \mathsf{Set} \to \mathsf{Set}$ is the finitary powerset functor. An easy verification shows that the full subcategory of INF of image-finite frames is isomorphic to $\mathsf{Coalg}(\mathcal{P}_\omega\mathcal{P}_\omega)$, so that the Definition 4.1 and the coalgebraic definition coincide.

We now prove that the image-finite models form a Hennessy-Milner class. While this follows from general (coalgebraic) results in [37] (akin to Remark 4.16 of [27]), we give a direct proof because it serves as inspiration for the notion of modal saturation in Definition 4.5 below.

Proposition 4.3. *The collection of image-finite instantial neighbourhood models has the Hennessy-Milner property.*

Proof. Let $\mathcal{M} = (X, N, V)$ and $\mathcal{M}' = (X', N', V')$ be image-finite models. We prove that the relation $B \subseteq X \times X'$ of logical equivalence is a bisimulation. Clause (B_1) holds trivially. We focus on (B_2), (B_3) being similar.

Let $(x, x') \in B$ and $a \in N(x)$. Suppose towards a contradiction that no $a' \in N'(x')$ makes (a, a') B-exhaustive. Then for each $a' \in N'(x')$ either:

- There exists a $y' \in a'$ such that $(y, y') \notin B$ for all $y \in a$; or
- There exists a $y \in a$ such that $(y, y') \notin B$ for all $y' \in a'$.

In the first case, we can pick a formula $\psi_{a'}$ such that $y' \nVdash \psi_{a'}$ and $y \Vdash \psi_{a'}$ for all $y \in a$. Indeed, since $(y, y') \notin B$ for all $y \in a$, there must be a formula χ_y for each $y \in a$ such that $y' \nVdash \chi_y$ and $y \Vdash \chi_y$. Since a is assumed to be finite we can then set $\psi_{a'} = \bigvee_{y \in a} \chi_y$. Likewise, in the second case we can find a formula $\varphi_{a'}$ such that $y \Vdash \varphi_{a'}$ and $y' \nVdash \varphi_{a'}$ for all $y' \in a'$. Write Ψ for the set formulae of the form $\psi_{a'}$, and Φ for the set of formulae of the form $\varphi_{a'}$. (We pick these such that every $a' \in N'(x')$ occurs as a subscript in either Ψ or Φ precisely once.) Then

$$\mathcal{M}, x \Vdash \Box(\Phi; \bigwedge \Psi)$$

because each of the $\varphi \in \Phi$ are true at some state in a and each $\psi \in \Psi$ is true anywhere in a. On the other hand, for each $a' \in N'(x')$ we either have a $\varphi \in \Phi$ that is satisfied nowhere in a', or a $\psi \in \Psi$ such that $a' \nsubseteq [\![\psi]\!]^{\mathcal{M}'}$. So

$$\mathcal{M}', x' \nVdash \Box(\Phi; \bigwedge \Psi),$$

contradicting the assumed logical equivalence of x and x'. ∎

The next examples show that we cannot drop the condition that neighbourhoods are finite, nor that states have a finite number of neighbourhoods.

Example 4.4. Consider the Kripke frames $\mathcal{K}$ and $\mathcal{K}'$ given in Figure 1, where $\mathcal{K}$ has an infinite branch and $\mathcal{K}'$ does not. Equip both with a valuation that sends every proposition letter to the empty set. The roots are denoted by x and x', respectively. Then $\mathcal{K}, x \leftrightsquigarrow_{\mathcal{ML}} \mathcal{K}', x'$ while $\mathcal{K}, x \not\rightleftharpoons_{\mathcal{ML}} \mathcal{K}', x'$, see e.g. [11, Example 2.23]. By Theorem 3.5 and 3.4 we have

$$\theta_1\mathcal{K}, x \leftrightsquigarrow \theta_1\mathcal{K}', x' \quad \text{while} \quad \theta_1\mathcal{K}, x \not\rightleftharpoons \theta_1\mathcal{K}', x'.$$

By construction all states in $\theta_1\mathcal{K}$ and $\theta_1\mathcal{K}'$ have only finitely many neighbourhoods (exactly one per state). This shows that we cannot drop the condition that neighbourhoods be finite from the assumptions of Proposition 4.3.

Likewise $\theta_2\mathcal{K}, x \leftrightsquigarrow \theta_2\mathcal{K}', x'$ while $\theta_2\mathcal{K}, x \not\rightleftharpoons \theta_2\mathcal{K}', x'$. Now all neighbourhoods of states in $\theta_2\mathcal{K}$ and $\theta_2\mathcal{K}$ are finite. So we cannot drop the condition that states have only a finite number of neighbourhoods from Proposition 4.3.

In order to find a suitable definition of modal saturation for instantial neighbourhood models, let us examine the proof of Proposition 4.3. If the models $\mathcal{M}$ and $\mathcal{M}'$ in the proof are *not* image-finite then we encounter two problems:

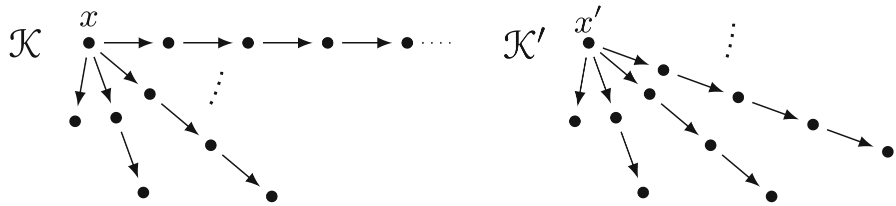

Figure 1. States x and x' are logically equivalent w.r.t. $\mathcal{ML}$, but not Kripke bisimilar

1. First, the construction of the formulae $\varphi_{a'}$ and $\psi_{a'}$ is in danger, because it relies on the fact that every neighbourhood is finite.
2. Second, assuming we *can* construct said $\varphi_{a'}$ and $\psi_{a'}$, we potentially obtain infinite sets Φ and Ψ, because states are no longer restricted to having a finite amount of neighbourhoods.

The first issue can be solved by a condition that resembles the definition of saturation for normal modal logic [11, §2.5]. That is, we require that every neighbourhood a of any state satisfies: If $\Phi \subseteq \mathcal{INL}$ is a family of subsets and every finite $\Phi' \subseteq_\omega \Phi$ is satisfied at some state in a, then there is a state $y \in a$ satisfying all of Φ. This can be rephrased as: If $\Phi \subseteq \mathcal{INL}$ is such that every state of a satisfies some $\varphi \in \Phi$, then there exists a finite subset $\Phi' \subseteq_\omega \Phi$ such that every state of a satisfies some $\varphi \in \Phi'$. Then, when constructing $\varphi_{a'}$ and $\psi_{a'}$, it suffices to only take a finite join of separating formulae χ_y.

Thus, with this saturation of neighbourhoods, we can find (potentially infinite) sets of formulae Φ and Ψ as in the proof of Proposition 4.3. For all finite subsets $\Phi' \subseteq \Phi$ and $\Psi' \subseteq \Psi$ we have

$$\mathcal{M}, x \Vdash \Box(\Phi'; \bigwedge \Psi').$$

So for the proof to go through, it suffices to find a single pair of finite Φ' and Ψ' such that $\mathcal{M}', x' \nVdash \Box(\Phi'; \bigwedge \Psi')$. This motivates item (3) below.

Definition 4.5. Let $\mathcal{M} = (X, N, V)$ be an instantial neighbourhood model.

1. We call a subset $a \subseteq X$ *saturated* if every set Φ of $\mathcal{INL}$-formulae that is finitely satisfiable in a, is also satisfiable in a.
2. $\mathcal{M}$ is called *locally saturated* if every $a \in \bigcup_{x \in X} N(x)$ is saturated.
3. We call a state $x \in X$ *neighbourhood saturated* if for all sets Φ, Ψ of formulae it satisfies the following condition:

If $x \Vdash \Box(\Phi'; \bigwedge \Psi')$ for all finite $\Phi' \subseteq \Phi$ and $\Psi' \subseteq \Psi$,
then there is a single neighbourhood $a \in N(x)$ witnessing this.

That is, there is a neighbourhood $a \in N(x)$ such that $a \cap [\![\varphi]\!]^{\mathcal{M}} \neq \emptyset$ for all $\varphi \in \Phi$ and $a \subseteq [\![\psi]\!]^{\mathcal{M}}$ for all $\psi \in \Psi$.

4. $\mathcal{M}$ is called *globally saturated* if each state is neighbourhood saturated.
5. Finally, $\mathcal{M}$ is *modally saturated* if it is both locally and globally saturated.

We have already sketched how the notions of local and global saturation resolve the problems that arise from dropping image-finiteness in the proof of Proposition 4.3. Thus, by design we obtain the following result.

THEOREM 4.6. *The collection of modally saturated instantial neighbourhood models forms a Hennessy-Milner class.*

5. A Topological Perspective

We take a topological perspective on modal saturation. This exposes the connection between modally saturated models and the Vietoris functor $\mathcal{V}$: $\mathsf{Top} \to \mathsf{Top}$ (from Definition 2.10). Specifically, we identify certain general frames as corresponding to $\mathcal{VV}$-coalgebras, and prove that models based on these general frames are precisely the modally saturated models.

DEFINITION 5.1. Let $\mathcal{M} = (X, N, V)$ be an instantial neighbourhood model. We define τ_V to be the topology on X generated by the clopen base

$$\{[\![\varphi]\!]^{\mathcal{M}} \mid \varphi \in \mathcal{INL}\}.$$

This topology allows us to reformulate saturation (Definition 4.5(1)) via compactness. Recall that a subset a of a topological space (X, τ) is compact if every open cover has a finite subcover.

LEMMA 5.2. *Let $\mathcal{M} = (X, N, V)$ be an instantial neighbourhood model. Then a set $a \subseteq X$ is saturated if and only if it is compact in (X, τ_V).*

Thus $\mathcal{M} = (X, N, V)$ is locally saturated iff $\bigcup_{x \in X} N(x) \subseteq \mathcal{V}(X, \tau_V)$. In this case we can characterise neighbourhood saturation as follows.

LEMMA 5.3. *A state x in a locally saturated instantial neighbourhood model $\mathcal{M} = (X, N, V)$ is neighbourhood saturated iff $N(x)$ is compact in $\mathcal{V}(X, \tau_V)$.*

This is a compactness property again! So a state x in a locally saturated model $\mathcal{M} = (X, N, V)$ is neighbourhood compact if and only if $N(x) \in \mathcal{VV}(X, \tau_V)$. One naturally wonders whether N can be conceived of as a *continuous* function from (X, τ_V) to $\mathcal{VV}(X, \tau_V)$ when $\mathcal{M}$ is modally saturated. This turns out to be the case, but before we can prove this we need the following characterisation of the topology on $\mathcal{VV}(X, \tau_V)$. This lemma extends [9, Proposition 4.3].

Lemma 5.4. *Let (X, τ) be a topological space. Then the topology of $\mathcal{V}\mathcal{V}(X, \tau)$ is generated by the subbase*

$$\boxbar(a_1, \dots, a_n; b) = \{W \mid \exists w \in W \text{ s.t. } w \cap a_i \neq \emptyset \text{ for all } i, \text{ and } w \subseteq b\}$$
$$\diamondbar(a_1, \dots, a_n; b) = \{W \mid \forall w \in W \text{ either } w \subseteq a_i \text{ for some } i, \text{ or } w \cap b \neq \emptyset\}$$

where the a_i and b range over τ.

Proof. Let $\bar{\tau}$ be the topology on the set underlying $\mathcal{V}\mathcal{V}(X, \tau)$ generated by $\boxbar(a_1, \dots, a_n; b)$ and $\diamondbar(a_1, \dots, a_n; b)$. Then we have $\bar{\tau} \subseteq \tau$ because

$$\boxbar(a_1, \dots, a_n; b) = \diamonddot(\diamonddot a_1 \cap \cdots \cap \diamonddot a_n \cap \boxdot b) \in \tau,$$
$$\diamondbar(a_1, \dots, a_n; b) = \boxdot(\boxdot a_1 \cup \cdots \cup \boxdot a_n \cup \diamonddot b) \in \tau.$$

To prove that $\tau \subseteq \bar{\tau}$, we need to show that $\boxdot A \in \bar{\tau}$ and $\diamonddot A \in \bar{\tau}$ for every open subset A of $\mathcal{V}(X, \tau)$. Let $A = \bigcup_{i \in I}(\diamonddot a_{i,1} \cap \cdots \cap \diamonddot a_{i,n_i} \cap \boxdot b_i)$, be an arbitrary open subset in $\mathcal{V}(X, \tau)$, where I is some index set, the $a_{i,j}, b_i \in \tau$ and $n_i \in \omega$ for each i. Since $\diamonddot$ distributes over unions we then have

$$\diamonddot A = \bigcup_{i \in I} \diamonddot(\diamonddot a_{i,1} \cap \cdots \cap \diamonddot a_{i,n_i} \cap \boxdot b_i) = \bigcup_{i \in I} \boxbar(a_{i,1}, \dots, a_{i,n_i}; b_i)$$

so it remains to show that $\boxdot A \in \bar{\tau}$. We have $W \in \boxdot A$ iff $W \subseteq \bigcup_{i \in I}(\diamonddot a_{i,1} \cap \cdots \cap \diamonddot a_{i,n_i} \cap \boxdot b_i)$, and since elements of $\mathcal{V}\mathcal{V}(X, \tau)$ are compact we have

$$\boxdot A = \bigcup_{I' \subseteq_\omega I} \left(\boxdot \bigcup_{i \in I'} (\diamonddot a_{i,1} \cap \cdots \cap \diamonddot a_{i,n_i} \cap \boxdot b_i) \right).$$

Consequently, it suffices to prove $\boxdot \bigcup_{i \in I'}(\diamonddot a_{i,1} \cap \cdots \cap \diamonddot a_{i,n_i} \cap \boxdot b_i) \in \tau$ for finite I'. Using distributivity, we can rewrite this as

$$\bigcup_{i \in I'} (\diamonddot a_{i,1} \cap \cdots \cap \diamonddot a_{i,n_i} \cap \boxdot b_i) = \bigcap_{j \in J} (\diamonddot c_j \cup \boxdot d_{j,1} \cup \cdots \cup \boxdot d_{j,m_j})$$

where $c_j, d_{j,k} \in \tau$. Since $\boxdot$ distributes over intersections we then have

$$\begin{aligned} \boxdot \bigcup_{i \in I'} (\diamonddot a_{i,1} \cap \cdots \cap \diamonddot a_{i,n_i} \cap \boxdot b_i) &= \bigcap_{j \in J} \boxdot(\diamonddot c_j \cup \boxdot d_{j,1} \cup \cdots \cup \boxdot d_{j,m_j}) \\ &= \bigcap_{j \in J} \diamondbar(d_{j,1}, \dots, d_{j,m_j}; c_j). \end{aligned}$$

This proves $\boxdot A \in \bar{\tau}$, and hence $\tau \subseteq \bar{\tau}$. ∎

Proposition 5.5. *An instantial neighbourhood model $\mathcal{M} = (X, N, V)$ is modally saturated iff N is a continuous function $(X, \tau_V) \to \mathcal{V}\mathcal{V}(X, \tau_V)$.*

Proof. It follows from Lemma 5.2 and 5.3 that N is well defined iff $\mathcal{M}$ is modally saturated. So it suffices to prove that N is continuous whenever $\mathcal{M}$ is modally saturated. This follows from Lemma 5.4 and the fact that

$$N^{-1}\big(\boxminus\big([\![\varphi_1]\!]^{\mathcal{M}}, \dots, [\![\varphi_n]\!]^{\mathcal{M}}; [\![\psi]\!]^{\mathcal{M}}\big)\big) = m_N\big([\![\Box(\varphi_1, \dots, \varphi_n; \psi)]\!]^{\mathcal{M}}\big)$$

is clopen in (X, τ_V) and hence $N^{-1}(\diamondminus([\![\varphi_1]\!]^{\mathcal{M}}, \dots, [\![\varphi_n]\!]^{\mathcal{M}}; [\![\psi]\!]^{\mathcal{M}})) = X \setminus N^{-1}([\![\Box(\neg\varphi_1, \dots, \neg\varphi_n; \neg\psi)]\!]^{\mathcal{M}})$ is in τ as well. ■

Finally, we characterise modally saturated models as the models based on certain *general* frames. If (X, N, A) is a general frame, then we let τ_A be the topology on X generated by the clopen base $\{a \subseteq X \mid a \in A\}$. Inspired by Lemma 5.2 and 5.3 we define modal saturation of a general frame as follows.

Definition 5.6. A general frame (X, N, A) is called *modally saturated* if N is a well-defined continuous function from (X, τ_A) to $\mathcal{V}\mathcal{V}(X, \tau_A)$.

Clearly, every modally saturated model $\mathcal{M} = (X, N, V)$ can be viewed as coming from a modally saturated general frame. Indeed, we can simply take $A = \{[\![\varphi]\!]^{\mathcal{M}} \mid \varphi \in \mathcal{INL}\}$. It turns out that the converse is true as well, that is, every model based on a modally saturated general frame is modally saturated. This justifies calling the frames from Definition 5.6 modally saturated: they are precisely the general frames underlying modally saturated models!

Theorem 5.7. *An instantial neighbourhood model* $\mathcal{M} = (X, N, V)$ *is modally saturated if and only if there exists* $A \subseteq \mathcal{P}X$ *such that* (X, N, A) *is a modally saturated general frame and* $V : \mathrm{Prop} \to A$ *is an admissible valuation.*

Proof. The direction from left to right we have already seen. So suppose (X, N, A) is a modally saturated general frame and $V : \mathrm{Prop} \to A$ is an admissible valuation. Then the topology τ_V on X generated by V is contained in τ_A. As a consequence, every $a \in \bigcup_{x \in X} N(x)$ is compact in τ_V, and $N(x)$ is compact in $\mathcal{V}(X, \tau_V)$. So N is a well-defined map from (X, τ_V) to $\mathcal{V}\mathcal{V}(X, \tau_V)$, and this implies that (X, N, V) is modally saturated by Proposition 5.5. ■

6. Populated Models and Bisimilarity-Somewhere-Else

In [9] it was proven that the category of Boolean algebras with instantial operators (BAIOs) is dually equivalent to $\mathsf{Coalg}(\mathcal{V}_{\mathsf{st}}\mathcal{V}_{\mathsf{st}})$. However, rather than simply defining descriptive frames as general frames (X, N, A) such that

(X, τ_A) is a Stone space and N a function from (X, τ_A) to $\mathcal{V}_{\mathsf{st}}\mathcal{V}_{\mathsf{st}}(X, \tau_A)$, additional (non-compact) neighbourhoods are added to the collection of compact neighbourhoods of each state. Specifically, $b \subseteq X$ is defined to be a neighbourhood of $x \in X$ if its closure $\overline{b}$ is in $N(x)$, see Theorem 2.11.

It was shown in [9, Theorem 6.4] that the class of models based on crowded descriptive frames has the Hennessy-Milner property. Guided by this fact, we investigate to what extend one can add non-compact neighbourhoods as neighbourhoods of states, thus generalising *loc. cit.* Furthermore, we use ultrafilter extensions to derive a bisimilarity-somewhere-else result (Theorem 6.6).

Definition 6.1. Let $\mathcal{M} = (X, N, V)$ be an instantial neighbourhood model. If $a \subseteq X$ then we write $\overline{a}$ for the closure of a in (X, τ_V). A neighbourhood model (X, N, V) is called *populated* if for all $x \in X$ and $a \subseteq X$ we have $a \in N(x)$ iff $\overline{a} \in N(x)$. It is called *compact* if (X, τ_V) is a compact space.

Remark 6.2. Compactness and populatedness can be expressed via general frames, akin to Theorem 5.7. We leave the details to the reader.

Call $\mathcal{M} = (X, N, V)$ *closed* if every neighbourhood $a \in \bigcup_{x \in X} N(x)$ is closed in τ_V. Then a populated model $\mathcal{M} = (X, N, V)$ is determined uniquely by the closed model $\mathcal{M}^- = (X, N^-, V)$, where $N^-(x) = \{\overline{b} \mid b \in N(x)\}$. Conversely, every closed model $\mathcal{M} = (X, N, V)$ gives rise to a populated model $\mathcal{M}^+ = (X, N^+, V)$ via $N^+(x) = \{b \subseteq X \mid \overline{b} \in N(x)\}$.

If (X, N, V) is populated then $(N^-)^+ = N$, and if (X, N, V) is closed then $(N^+)^- = N$, so these assignment form a bijective correspondence. In fact, this correspondence does not affect truth of formulae.

Lemma 6.3. *Let $\mathcal{M} = (X, N, V)$ be a populated model. Then for all $x \in X$ and $\varphi \in \mathcal{INL}$ we have $\mathcal{M}, x \Vdash \varphi$ iff $\mathcal{M}^-, x \Vdash \varphi$.*

Proof. This follows from induction on the structure of φ. By definition of τ_V, truth sets of formulae are clopen, and therefore the case $\varphi = p \in \text{Prop}$ is immediate. The cases $\varphi = \top$, $\varphi = \neg\varphi_1$ and $\varphi = \varphi_1 \wedge \varphi_2$ are straightforward. The induction step for $\varphi = \Box(\varphi_1, \ldots, \varphi_n; \psi)$ follows from the fact that for all $b \subseteq X$ we have $b \cap [\![\varphi_i]\!]^{\mathcal{M}} \neq \emptyset$ iff $\overline{b} \cap [\![\varphi_i]\!]^{\mathcal{M}} \neq \emptyset$ and $b \subseteq [\![\psi]\!]^{\mathcal{M}}$ iff $\overline{b} \subseteq [\![\psi]\!]^{\mathcal{M}}$. ■

Despite this fact, the relation of logical equivalence between a populated model and its underlying closed model need not be a bisimulation.

Example 6.4. Let X be the interval $[0, 1]$ and $N(x) = \{X\}$ for all $x \in X$. Assume a countably infinite number of proposition letters indexed by

$\mathbb{Q} \cap [0, 1]$, and let $V(p_q) = [q, 1]$. Then $\mathcal{I} = (X, N, V)$ is a closed instantial neighbourhood model, (X, τ_V) is compact, and since X is the only neighbourhood of a state $\mathcal{I}$ is locally saturated. Moreover, the fact that each state has only one neighbourhood implies global saturation, so $\mathcal{I}$ is modally saturated. Let $\mathcal{I}^+ = (X, N^+, V)$ be the corresponding populated model. Then the half-open interval $(0, 1] \in N^+(x)$ for all $x \in X$ because its closure is X.

The relation of logical equivalence between $\mathcal{I}$ and $\mathcal{I}^+$ is simply the diagonal $\Delta_X \subseteq X \times X$. To see that this is not a bisimulation, let x be any state in X and consider $(0, 1] \in N^+(x)$. The pair $\big([0, 1], (0, 1]\big)$ is not Δ_X-exhaustive, because $\Delta_X^{-1}[(0, 1]] = (0, 1] \neq [0, 1]$. As $[0, 1]$ is the only neighbourhood in $N(x)$ this proves that (B_3) fails, and therefore Δ_X is not a bisimulation.

However, under the additional assumption of compactness and global saturation, populated models form their own Hennessy-Milner class. (Global saturation of populated models is defined as in Definition 4.5(4).)

THEOREM 6.5. *The class of compact and globally saturated populated models is a Hennessy-Milner class.*

PROOF. If $\mathcal{M}_1 = (X_1, N_1, V_1)$ and $\mathcal{M}_2 = (X_2, N_2, V_2)$ are compact and globally saturated populated models, then their underlying closed models $\mathcal{M}_1^-$ and $\mathcal{M}_2^-$ are modally saturated. So the relation B of logical equivalence is a bisimulation between $\mathcal{M}_1^-$ and $\mathcal{M}_2^-$. It can be shown to be a bisimulation between $\mathcal{M}_1$ and $\mathcal{M}_2$ as well: If $(x_1, x_2) \in B$ and $b_1 \in N_1(x_1)$ then $\overline{b}_1 \in N_1^-(x_1)$, so there must be a $c_2 \in N_2^-(x_2)$ such that $(\overline{b}_1, c_2)$ is B-exhaustive. Setting $b_2 = B[b_1] \cap c_2$ yields a neighbourhood such that (b_1, b_2) is B-exhaustive, and it can be shown that $\overline{b}_2 = c_2$ so that $b_2 \in N_2(x_2)$. ■

Finally we prove a bisimilarity-somewhere-else theorem, stating that logical equivalence between two states in two models can be verified by checking bisimilarity between two states in certain related models. The rôle of related model can be played either by the ultrafilter extensions or their underlying closed models. Recall that the ultrafilter extension of a model $\mathcal{M} = (X, N, V)$ is denoted by $uf\mathcal{M}$. Denote the closed model underlying $uf\mathcal{M}$ by $\overline{uf}\mathcal{M}$.

THEOREM 6.6. *Let $\mathcal{M} = (X, N, V)$ and $\mathcal{M}' = (X', N', V')$ be two instantial neighbourhood models. Then*

$$\mathcal{M}, x \leftrightsquigarrow \mathcal{M}', x' \quad \textit{iff} \quad uf\mathcal{M}, \widehat{x} \leftrightharpoons uf\mathcal{M}', \widehat{x}' \quad \textit{iff} \quad \overline{uf}\mathcal{M}, \widehat{x} \leftrightharpoons \overline{uf}\mathcal{M}', \widehat{x}'.$$

PROOF. The ultrafilter extension $uf\mathcal{M} = (\widehat{X}, \widehat{N}, \widehat{V})$ of $\mathcal{M}$ is compact because the topology $\tau_{\widehat{V}}$ is a sub-topology of a compact topology. Furthermore, it is populated as a consequence of [9, Lemma 5.4] and it is globally saturated because the underlying closed model is globally saturated (which in turn follows from the fact that $uf\mathcal{M}$ arises from a $\mathcal{V}_{\mathsf{st}}\mathcal{V}_{\mathsf{st}}$-coalgebra). The same goes for $uf\mathcal{M}'$, so that the left "iff" follows from Lemma 2.13 and Theorem 6.5.

Since the (closed) frames underlying $\overline{uf}\mathcal{M}$ and $\overline{uf}\mathcal{M}'$ are given by $\mathcal{V}_{\mathsf{st}}\mathcal{V}_{\mathsf{st}}$-coalgebras, Theorem 5.7 shows that models based on them are modally saturated. The right "iff" then follows from Lemma 2.13 and 6.3, and Theorem 4.6. ∎

7. Characterisation Theorem

Van Benthem's characterisation theorem for normal modal logic states that normal modal logic is the Kripke bisimulation-invariant fragment of first-order logic [5]. In this section we prove an INL-counterpart of this theorem. Our proof resembles the one for normal modal logic in [11, Theorem 2.68].

We use a translation of $\mathcal{INL}$ into a two-sorted first-order language denoted by $\mathcal{FOL}$, previously used in e.g. [13,23,26,27]. In particular, [27] also deals with neighbourhood models, albeit to interpret a different modality, and we can make use of their result characterising those two-sorted first-order structures that correspond to neighbourhood models (recalled after Lemma 6.5 below).

The two sorts in this language are denoted by s and n, and intuitively contain the **s**tates and **n**eighbourhoods of a model. To avoid notational clutter, we omit the type of the elements and agree to denote elements of sort s with $x, y, z, \ldots$, and elements of sort n by a, b, c. These two sorts are related via binary relations N, relating elements of sort s to elements of sort n, and E, relating elements of sort n to elements of sort s. The intuitive reading of $x\mathsf{N}a$ is "a is a neighbourhood of x" and $a\mathsf{E}x$ is intended to mean "x is an element of a." Furthermore, we have a unary predicate P_p for every propositional variable p. So $\mathcal{FOL}$ is generated by the following grammar:

$$\varphi ::= (x = y) \mid (a = b) \mid \mathsf{P}_p x \mid x\mathsf{N}a \mid a\mathsf{E}x \mid \neg\varphi \mid \varphi \wedge \varphi \mid \exists x\varphi \mid \exists a\varphi,$$

where $p \in \mathrm{Prop}$, x, y range over s, and a, b range over n. We abbreviate $\vee, \rightarrow, \forall$ as usual.

Definition 7.1. The standard translation $\mathrm{st}_x : \mathcal{INL} \to \mathcal{FOL}$ is defined recursively via $\mathrm{st}_x(\top) = (x = x)$, $\mathrm{st}_x(p) = \mathsf{P}_p x$, $\mathrm{st}_x(\neg\varphi) = \neg\,\mathrm{st}_x(\varphi)$, $\mathrm{st}_x(\varphi \wedge$

$\psi) = \mathrm{st}_x(\varphi) \wedge \mathrm{st}_x(\psi)$, and

$$\begin{aligned}\mathrm{st}_x(\Box(\varphi_1, \ldots, \varphi_n; \psi)) = \exists a\big(x\mathsf{N}a \wedge &\exists y(a\mathsf{E}y \wedge \mathrm{st}_y(\varphi_1)) \\ &\wedge \cdots \\ &\wedge \exists y(a\mathsf{E}y \wedge \mathrm{st}_y(\varphi_n)) \\ &\wedge \forall y(a\mathsf{E}y \to \mathrm{st}_y(\psi))\big)\end{aligned}$$

Intuitively, the clause for $\Box(\varphi_1, \ldots, \varphi_n; \psi)$ reads: "A state x satisfies $\Box(\varphi_1, \ldots, \varphi_n; \psi)$ if there exists an a of type 'neighbourhood' such that

- a is a neighbourhood of x;
- for each of the φ_i there a state y such that $y \in a$ and y satisfies φ_i;
- every state y that is in a satisfies ψ."

Let us make this intuition more precise by investigating the first-order structures of $\mathcal{FOL}$, and relating them to our instantial neighbourhood models.

Definition 7.2. A first-order structure for $\mathcal{FOL}$ is a tuple of the form $\mathfrak{M} = (D^{\mathsf{s}}, D^{\mathsf{n}}, \{P_p \mid p \in \mathrm{Prop}\}, N, E)$, where D^{s} and D^{n} are sets, $P_p \subseteq D^{\mathsf{s}}$, $N \subseteq D^{\mathsf{s}} \times D^{\mathsf{n}}$ and $E \subseteq D^{\mathsf{n}} \times D^{\mathsf{s}}$. Truth of $\varphi \in \mathcal{FOL}$ in a structure $\mathfrak{M}$ is defined as expected, and is denoted by $\mathfrak{M} \models \varphi$.

We write $\varphi(x)$ if x is a free variable of type s in φ. In this case, we write $\mathfrak{M} \models \varphi[s]$ if φ is true in $\mathfrak{M}$ when $s \in D^{\mathsf{s}}$ is assigned to x. Similarly define $\varphi(a)$ and $\mathcal{M} \models \varphi[n]$ if a is a free variable of type n and $n \in D^{\mathsf{n}}$.

If Ψ is a set of $\mathcal{FOL}$-formulae and $\mathfrak{M}$ is a two-sorted first-order structure, then we write $\mathfrak{M} \models \Psi$ if $\mathfrak{M} \models \psi$ for all $\psi \in \Psi$. We denote the semantic consequence relation of a class $\mathbf{K}$ of $\mathcal{FOL}$-models by $\models_{\mathbf{K}}$. Finally, a set $\Phi(v)$ of formulae with free variable v of either type is said to be $\mathbf{K}$-consistent if there exists an $\mathfrak{M} \in \mathbf{K}$ and a u of the same type as v in $\mathfrak{M}$ such that $\mathfrak{M} \models \Phi[u]$.

Every instantial neighbourhood model $\mathcal{M}$ gives rise to a two-sorted first-order structure as follows [27, Definition 5.1].

Definition 7.3. Given an instantial neighbourhood model $\mathcal{M} = (X, N, V)$, define the first-order model $\mathcal{M}^\circ = (D^{\mathsf{s}}, D^{\mathsf{n}}, \{P_p \mid p \in \mathrm{Prop}\}, R_N, R_\ni)$ via

- $D^{\mathsf{s}} = X$ and $D^{\mathsf{n}} = \bigcup_{x \in X} N(x)$;
- $P_p = V(p)$ for all $p \in \mathrm{Prop}$;
- $R_N = \{(x, a) \mid x \in X \text{ and } a \in N(x)\}$;
- $R_\ni = \{(a, x) \mid a \in D^{\mathsf{n}} \text{ and } x \in a\}$.

This always gives a first-order structre in the sense of Definition 7.2. Moreover:

LEMMA 7.4. *For all instantial neighbourhood models* $\mathcal{M} = (X, N, V)$*, states* $w \in X$*, and INL-formulae* φ *we have*

$$\mathcal{M}, w \Vdash \varphi \quad \textit{iff} \quad \mathcal{M}^\circ \models \mathrm{st}_x(\varphi)[w].$$

PROOF. By induction on the structure of φ. The only non-trivial case is for $\square$, which can be read off from the definition of the standard translation. ■

While every instantial neighbourhood model gives rise to a two-sorted first-order structure, not every first-order model is of such a form. It was proven in [27, Proposition 5.4] that a first-order model is isomorphic to a model of the form $\mathcal{M}^\circ$ iff it satisfies NAX, which consists of the following axioms:

$$\forall a \exists x (x \mathsf{N} a), \qquad \forall a \forall b ((\forall x (a \mathsf{E} x \leftrightarrow b \mathsf{E} x)) \to a = b).$$

It is clear that every first-order model of the form $\mathcal{M}^\circ$ satisfies NAX. Conversely, if $\mathfrak{M} = (D^{\mathsf{s}}, D^{\mathsf{n}}, \{P_p \mid p \in \mathrm{Prop}\}, N, E)$ is a two-sorted first-order structure satisfying NAX, then the model $\mathfrak{M}_\circ = (D^{\mathsf{s}}, N, V)$, with

$$N(s) = \big\{a \subseteq D^{\mathsf{s}} \mid \exists n \in D^{\mathsf{n}} \text{ s.t. } sNn \text{ and } \{t \in D^{\mathsf{s}} \mid nEt\} = a\big\}$$

and $V(p) = \{s \in D^{\mathsf{s}} \mid \mathfrak{M} \models \mathsf{P}_p[s]\}$ is such that $\mathfrak{M} \cong (\mathfrak{M}_\circ)^\circ$ [27, Proposition 5.4].

Before proceeding to the characterisation theorem, we discuss ω-saturated structures [17]. Let $\mathfrak{M} = (D^{\mathsf{s}}, D^{\mathsf{n}}, \{P_p \mid p \in \mathrm{Prop}\}, N, E)$ be a $\mathcal{FOL}$-model, $X \subseteq D^{\mathsf{s}}$ and $A \subseteq D^{\mathsf{n}}$. The (X, A)-expansion $\mathcal{FOL}[X, A]$ of $\mathcal{FOL}$ is the language obtained from $\mathcal{FOL}$ by adding constants $\underline{x}, \underline{a}$ for each $x \in X$ and $a \in A$, and is interpreted in $\mathfrak{M}$ by requiring that $\underline{x}$ is interpreted as x, and $\underline{a}$ as a. The $\mathcal{FOL}$-model $\mathfrak{M}$ is called *ω-saturated* if for all finite $X \subseteq_\omega D^{\mathsf{s}}$ and $A \subseteq_\omega D^{\mathsf{n}}$ and every collection $\Gamma(v)$ of $\mathcal{FOL}[X, A]$-formulae with free variable v of type either s or n the following holds: If $\Gamma(x)$ is finitely satisfiable in $\mathfrak{M}$, then it is satisfiable in $\mathfrak{M}$. Using e.g. ultraproducts, one can show that every $\mathcal{FOL}$-model has an ω-saturated elementary extension [17]. Moreover, if $\mathfrak{M} \in \mathbf{N}$ then its ω-saturated elementary extension is also in $\mathbf{N}$, since validity of NAX is preserved under elementary extensions.

In order to use ω-saturated structures to prove the characterisation theorem, we show that the instantial neighbourhood model corresponding to any ω-saturated $\mathcal{FOL}$-model in $\mathbf{N}$ is modally saturated.

LEMMA 7.5. *Suppose* $\mathfrak{M} \in \mathbf{N}$ *is* ω*-saturated. Then* $\mathfrak{M}_\circ$ *is modally saturated.*

Proof. Let $\mathfrak{M}_\circ = (X, N, V)$. First we show that $\mathfrak{M}_\circ$ is locally saturated. Let $a \subseteq X$ be the neighbourhood of some world w. Then a corresponds to some domain element $a^\circ \in D^{\mathsf{n}}$. If Φ is finitely satisfiable in a, then the set

$$\mathscr{A} = \{\underline{a}^\circ \mathsf{E} x\} \cup \{\mathrm{st}_x(\varphi) \mid \varphi \in \Phi\}$$

of $\mathcal{FOL}[\{a^\circ\}]$-formulae is finitely satisfiable in $\mathfrak{M}$. Since $\mathfrak{M}$ is ω-saturated, it follows that $\mathscr{A}$ is satisfiable in $\mathfrak{M}$, and therefore Φ is satisfiable in a.

For neighbourhood saturation, let $w \in X$ and let Φ and Ψ be sets of $\mathcal{INL}$-formulae. Suppose that $x \Vdash \Box(\Phi'; \bigwedge \Psi')$ for all $\Phi' \subseteq_\omega \Phi$ and $\Psi' \subseteq_\omega \Psi$. Let $w^\circ \in D^{\mathsf{s}}$ correspond to $w \in X$. Then the following set of $\mathcal{FOL}[\{w^\circ\}]$-formulae with free variable u of type n is finitely satisfiable:

$$\mathscr{B} = \{w^\circ \mathsf{N} u\} \cup \{\exists y(u \mathsf{E} y \wedge \mathrm{st}_y(\varphi)) \mid \varphi \in \Phi\} \cup \{\forall y(u \mathsf{E} y \rightarrow \mathrm{st}_y(\psi)) \mid \psi \in \Psi\}.$$

(This is witnessed by the fact that $\mathfrak{M}$ satisfies $\mathrm{st}_x(\Box(\Phi'; \bigwedge \Psi'))[w]$ for all $\Phi' \subseteq_\omega \Phi$ and $\Psi' \subseteq_\omega \Psi$.) Again, ω-saturation of $\mathfrak{M}$ implies that the set $\mathscr{B}$ satisfiable in $\mathfrak{M}$. Therefore there must exist a neighbourhood a of w such that $a \cap [\![\varphi]\!] \neq \emptyset$ for all $\varphi \in \Phi$ and $a \subseteq [\![\psi]\!]$ for all $\psi \in \Psi$. ■

We now have all the ingredients to prove the Van Benthem-style characterisation theorem for instantial neighbourhood logic.

Theorem 7.6. *Let $\alpha(x)$ be a $\mathcal{FOL}$-formula. Over the class $\mathbf{N}$ the following are equivalent:*

1. $\alpha(x)$ is equivalent to the translation of an $\mathcal{INL}$-formula;

2. $\alpha(x)$ is invariant under INL-bisimulation.

Proof. The implication (1) ⇒ (2) follows from Theorem 2.6. To prove the converse implication, assume that $\alpha(x)$ is invariant under bisimulations. Consider the set of modal consequences of α:

$$\mathrm{MOC}_{\mathbf{N}}(\alpha) = \{\mathrm{st}_x(\varphi) \mid \varphi \in \mathcal{INL} \text{ and } \alpha(x) \models_{\mathbf{N}} \mathrm{st}_x(\varphi)\}.$$

It suffices to show that $\mathrm{MOC}_{\mathbf{N}}(\alpha) \models_{\mathbf{N}} \alpha(x)$, because then compactness entails existence of a finite subset $\Gamma(x) \subseteq \mathrm{MOC}_{\mathbf{N}}(\alpha)$ such that $\Gamma(x) \models_{\mathbf{N}} \alpha(x)$, so that over $\mathbf{N}$ $\alpha(x)$ is equivalent to $\bigwedge \Gamma(x)$, which is the standard translation of a formula in $\mathcal{INL}$. So let $\mathfrak{M} \in \mathbf{N}$ and assume $\mathfrak{M} \models \mathrm{MOC}_{\mathbf{N}}(\alpha)[s]$. We need to show that $\mathfrak{M} \models \alpha(x)[s]$. Set

$$T(x) = \{\mathrm{st}_x(\varphi) \mid \mathfrak{M}_\circ, s \Vdash \varphi\}.$$

We claim that $T(x) \cup \{\alpha(x)\}$ is consistent. Suppose not, then by compactness there exists a finite subset $T_0(x) \subseteq T(x)$ such that $\alpha(x) \models_{\mathbf{N}} \neg \bigwedge_{i=1}^{n} T_0(x)$. Consequently $\neg \bigwedge T_0(x) \in \mathrm{MOC}_{\mathbf{N}}(\alpha)$. But this implies $\mathfrak{M} \models \neg \bigwedge T_0(x)[s]$, which contradicts the assumption that $\mathfrak{M} \models T_0(x)[s]$.

Thus we can find a first-order structure $\mathfrak{N} \in \mathbf{N}$ and an element t of sort s in $\mathfrak{N}$ such that $\mathfrak{N} \models T(x) \cup \{\alpha(x)\}[t]$. Furthermore, we have

$$\mathfrak{M} \models \mathrm{st}_x(\varphi)[s] \quad \text{iff} \quad \mathfrak{N} \models \mathrm{st}_x(\varphi)[t] \quad \text{for all } \varphi \in \mathcal{INL}. \tag{1}$$

Now let $\mathfrak{M}^*$ and $\mathfrak{N}^*$ be two ω-saturated elementary extensions of $\mathfrak{M}$ and $\mathfrak{N}$, respectively, and let s^* and t^* be the images of s and t. Then by Lemma 7.5 $(\mathfrak{M}^*)_\circ$ and $(\mathfrak{N}^*)_\circ$ are modally saturated, and (1) and Theorem 4.6 entail that $(\mathfrak{M}^*)_\circ, s^*$ and $(\mathfrak{N}^*)_\circ, t^*$ are bisimilar. Finally, as $\mathfrak{N} \models \alpha(x)[t]$ we have $\mathfrak{N}^* \models \alpha(x)[t^*]$, and since $\alpha(x)$ is assumed to be invariant under bisimulations we get $\mathfrak{M}^* \models \alpha(x)[s^*]$. Invariance of truth of formulae under elementary embeddings then gives $\mathfrak{M} \models \alpha(x)[s]$, as desired. ■

8. Adaptation to Bounded Instantial Neighbourhood Logic

In this final section we adapt the results above to *bounded INL*, obtained from $\mathcal{INL}$ by limiting the number of instances in modal formulae [8, §7.3].

Definition 8.1. For $k \in \omega$, the language $\mathcal{INL}_k$ is the extension of classical propositional logic with operators of the form $\Box(\varphi_1, \ldots, \varphi_k; \psi)$.

Modal formulae with less than k instances can be viewed as formulae in $\mathcal{INL}_k$ by adding $\top$ for the remaining instances. Instantial neighbourhood models serve as semantics for $\mathcal{INL}_k$ via the clauses of Definition 2.1, and logical equivalence is denoted by $\leftrightsquigarrow_k$. In particular, the case $k = 0$ yields monotone modal logic interpreted in neighbourhood models, with monotonicity of the modality built into the definition of its interpretation (rather than the model, as in [26, Definition 3.5]).

We extend [8, Definition 7.7] to the following notion of k-bisimulation.

Definition 8.2. A *k-bisimulation* between instantial neighbourhood models $\mathcal{M} = (X, N, V)$ and $\mathcal{M}' = (X', N', V')$ is a relation $B \subseteq X \times X'$ such that for all $(x, x') \in B$ we have:

kB_1 $x \in V(p)$ if and only if $x' \in V'(p)$, for all $p \in \mathrm{Prop}$;

kB_2 If $a \in N(x)$ and $y_1, \ldots, y_k \in a$, then there exists $a' \in N'(x')$ such that $a' \subseteq B[a]$ and $y_1, \ldots, y_k \in B^{-1}[a']$;

kB_3 If $a' \in N'(x')$ and $y'_1, \ldots, y'_k \in a'$, then there exists $a \in N(x)$ such that $a \subseteq B^{-1}[a']$ and $y'_1, \ldots, y'_k \in B[a]$.

Write $\leftrightharpoons_k$ for the induced notion of k-bisimilarity.

While clearly every INL-bisimulation is also a k-bisimulation for any $k \in \omega$, the converse need not be true. This is witnessed by Example 8.4 below. As announced, we can again prove the Hennessy-Milner property for the class of modally saturated models, with modal saturation defined as in Definition 4.5.

THEOREM 8.3. *Let $\mathcal{M} = (X, N, V)$ and $\mathcal{M}' = (X', N', V')$ be two modally saturated instantial neighbourhood models and $x \in X, x' \in X'$. Then*

$$\mathcal{M}, x \leftrightsquigarrow_k \mathcal{M}', x' \quad \textit{iff} \quad \mathcal{M}, x \leftrightharpoons_k \mathcal{M}', x'.$$

PROOF. We claim that the relation B of logical equivalence between $\mathcal{M}$ and $\mathcal{M}'$ is a k-bisimulation. The first clause is obvious. We focus on proving (kB_2). Item (kB_3) can be proven symmetrically.

Let $a \in N(x)$ and $y_1, \dots, y_k \in a$ and suppose there exists no $a' \in N'(x')$ with $a' \subseteq B[a]$ and $y_1, \dots, y_k \in B^{-1}[a']$. Then for each $a' \in N'(x')$, either

1. There exists $z' \in a'$ such that $z' \notin B[a]$, i.e., $(z, z') \notin B$ for all $z \in a$; or
2. $y_i \notin B^{-1}[a']$ for one of the y_i.

In the first case we can pick a formula ψ such that $a \subseteq [\![\psi]\!]^{\mathcal{M}}$ and $\mathcal{M}', z' \nVdash \psi$. Let Ψ be the set of all ψ that arise in such a way. In the second case, we can find a formula φ_i such that $\mathcal{M}, y_i \Vdash \varphi_i$ while $\mathcal{M}', y' \nVdash \varphi_i$ for all $y' \in a'$. Let Φ_i denote all φ_i that arise in such a manner. (The index runs from 1 to k and indicates which of the y_i is used to find φ_i.) Then the set $N'(x')$ of neighbourhoods of x', conceived of as a subset of $\mathcal{V}(X, \tau_V)$, is covered by

$$\bigcup_{\varphi_1 \in \Phi_1} \boxdot [\![\neg \varphi_1]\!]^{\mathcal{M}'} \cup \cdots \cup \bigcup_{\varphi_k \in \Phi_k} \boxdot [\![\neg \varphi_k]\!]^{\mathcal{M}'} \cup \bigcup_{\psi \in \Psi} \Diamonddot [\![\neg \psi]\!]^{\mathcal{M}'}.$$

Since $N'(x')$ is assumed to be compact, we can find a finite subcover indexed by finite sets $\Phi_i' \subseteq \Phi_i$ and $\Psi' \subseteq \Psi$. We then arrive at a contradiction because $\mathcal{M}, x \Vdash \Box(\bigwedge \Phi_1', \dots, \bigwedge \Phi_k'; \bigwedge \Psi')$ while $\mathcal{M}', x' \Vdash \Box(\bigwedge \Phi_1', \dots, \bigwedge \Phi_k'; \bigwedge \Psi')$. ■

Call a relation B between instantial neighbourhood models an ω-bisimulation if it is a k-bisimulation for every $k \in \omega$. Since $\mathcal{INL} = \bigcup_{k \in \omega} \mathcal{INL}_k$ Theorems 4.6 and 8.3 yield that on modally saturated models, ω-bisimilarity and $\mathcal{INL}$-logical equivalence coincide (so we can also characterise $\mathcal{INL}$ as the ω-bisimulation invariant fragment of $\mathcal{FOL}$ on the class $\mathbf{N}$). However, they do not coincide in general, as the following example shows.

EXAMPLE 8.4. Let $X = [0, 1]$ and define $N_1, N_2 : X \to \mathcal{PP}X$ by

$$N_1(x) = \{a \subseteq X \mid a \text{ is countably infinite}\} \cup \{X\}$$
$$N_2(x) = \{a \subseteq X \mid a \text{ is countably infinite}\}$$

for all $x \in X$. Let $V(p) = \emptyset$ for all $p \in \text{Prop}$ and define models $\mathfrak{I}_1 = (X, N_1, V)$ and $\mathfrak{I}_2 = (X, N_2, V)$. Then the diagonal $\Delta_X \subseteq X \times X$ is an ω-bisimulation between $\mathfrak{I}_1$ and $\mathfrak{I}_2$, but not an INL-bisimulation.

Defining the standard translation $\mathcal{INL}_k \to \mathcal{FOL}$ as the restriction of the one for $\mathcal{INL}$ (Definition 7.1), analogously to Theorem 7.6 we obtain:

Theorem 8.5. *Let $\alpha(x)$ be an $\mathcal{FOL}$-formula. Then over the class* **N** *the following are equivalent:*

1. *$\alpha(x)$ is equivalent to the translation of an $\mathcal{INL}_k$-formula;*
2. *$\alpha(x)$ is invariant under k-bisimulation.*

9. Conclusion

We have identified several Hennessy-Milner classes and proved a Van Benthem-style characterisation theorem for instantial neighbourhood logic, and for an ω-indexed family of its sublogics. There are several appealing directions for further research.

First, it would be interesting to see whether the results in this paper can be adapted to accommodate other variations of INL or its semantics, like the topological interpretation [8, Section 7.2] or dynamic versions of INL [6,7].

Second, one could investigate a suitable notion of *simulation* for INL. Such a notion would probably only preserve a certain negation-free fragment of the language. The interplay of existential and universal information in INL-modalities turns finding a suitable definition of simulation and the fragment it preserves into a non-trivial task. This is related to [20,24].

Finally, the conclusion of [9] speculates about a *positive* and a *geometric* version of INL. It would be fascinating to examine these further and investigate their relation with (normal) positive modal logic [15,16,19] and (normal) geometric modal logic [10]. Subsequently, one could think about suitable notions of simulation and bisimulation for these logics.

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